Mathematical Methods III
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Examples Sheet 2

1. Let z be a complex number and consider the transformation $z \mapsto az + b$, where a and b are complex and $|a| = 1$. Thinking about z as a vector in the complex plane, this transformation can be understood to be a rotation and a translation of the vector z. Convince yourself that these transformations form a group under the operation of composition. Show that the 2×2 complex matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

form a faithful representation. Show that, within the 2-dimensional space of column vectors with two entries, there is an invariant subspace.

2. Show that the symmetries of a regular tetrahedron in 3-dimensional space, including reflections, form a group isomorphic to the permutation group S_4. Show that the symmetry group without reflections, i.e. the rigid rotations of a tetrahedron, is isomorphic to the alternating group A_4, the subgroup of S_4 consisting of even permutations only.

3. Use the permutation $\begin{pmatrix} 1\,2\,3\,4\,5 \\ a\,b\,c\,d\,e \end{pmatrix}$ to show that the permutations with cycle decompositions $(123)(45)$ and $(abc)(de)$ are in the same conjugacy class within S_5. Generalize this example to show that two non-identity permutations are in the same conjugacy class within S_5 if and only if their cycle decompositions have the same cycle shape. Deduce that there are 7 conjugacy classes in S_5.

4. Let S_3 be the permutation group on 3 objects. Show that $|S_3| = 6$ and that S_3 is isomorphic to D_3, the symmetry group of an equilateral triangle.

   By considering the action of S_3 in permuting the components of a vector $(a, b, c)^T$ in 3-dimensional space $\mathbb{R}^3$, or otherwise, show that the 3×3 unit matrix, together with

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

   provides a 3-dimensional faithful representation of S_3. Explain the statement that the matrices are the representatives of the permutations $(2\ 3)$, $(3\ 1)$, $(1\ 2)$, $(1\ 2\ 3)$ and $(1\ 3\ 2)$ in the order shown. Show that the representation is reducible by verifying that the vectors $(a, a, a)^T$ form a 1-dimensional invariant subspace. Find a further 2-dimensional invariant subspace.

   Consider the matrix

$$S = \begin{pmatrix} \alpha & 0 & 2\beta \\ \alpha & \beta\sqrt{3} & -\beta \\ \alpha & -\beta\sqrt{3} & -\beta \end{pmatrix} .$$

For nonzero α and β, and E representing the matrices above, the matrix products $S^{-1}ES$ are equal to

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&\frac{1}{2}&-\frac{\sqrt{3}}{2}\\0&-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}, \begin{pmatrix}1&0&0\\0&\frac{1}{2}&\frac{\sqrt{3}}{2}\\0&\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}, \begin{pmatrix}1&0&0\\0&-\frac{1}{2}&\frac{\sqrt{3}}{2}\\0&-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}, \begin{pmatrix}1&0&0\\0&-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\0&\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}$$

respectively. [Feel free to check one or two.] How are these transformed matrices related to the irreducible representations of S_3? What goes wrong if $\alpha = 0$ or $\beta = 0$?

5. Consider the following mappings from D_4 (the symmetry group of a square) into or onto C_2, with C_2 represented as $\{1, -1\}$:

$$\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1,\ 1,\ 1,\ 1,\ -1,\ -1,\ -1,\ -1\}$$
$$\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1,\ -1,\ 1,\ -1,\ -1,\ -1,\ 1,\ 1\}$$
$$\{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1,\ -1,\ 1,\ -1,\ 1,\ -1,\ -1,\ 1\}$$

in the order displayed. Show that the first two are homomorphisms but that the last is not. Verify that the kernels of the first two mappings are all normal subgroups of D_4, and that the kernel of the last mapping is not.

6. Let G be an Abelian group with $|G|$ elements. Show that each element of G forms a conjugacy class by itself. Deduce that there are $|G|$ one-dimensional representations of G and no other irreducible representations. Find the one-dimensional representations of the cyclic group C_n.

7. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be unit vectors in the plane separated by an angle of 120°, Δ the equilateral triangle with vertices $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3 = -(\mathbf{e}_1 + \mathbf{e}_2)$ and D_3 the symmetry group of Δ. Calculate the matrices of the two-dimensional irreducible representation of D_3 by considering the action on vectors in the plane, taking $\mathbf{e}_1$ and $\mathbf{e}_2$ as basis vectors. Show that the traces of these matrices agree with those in the character table of D_3. Verify that the orthogonality theorem is satisfied, i.e. that

$$\sum_g d^{(\alpha)}(g)_{ij}\, d^{(\alpha)}(g^{-1})_{kl} = \frac{|G|}{n_\alpha}\, \delta_{il}\delta_{jk}\,.$$

8. Let D be a unitary representation of a finite group G and $\{\chi(g) : g \in G\}$ the character of D. Show that

$$\frac{1}{|G|}\sum_g \chi(g)^*\chi(g)$$

is a positive integer, equal to 1 if and only if D is irreducible.

9. Let $D_1 : G \to GL(n, \mathbb{C})$ be a representation of group G. (Recall $GL(n, \mathbb{C})$ is the group of $n \times n$ invertible, complex matrices.) Define $\mathbf{D}_2(g) = [\mathbf{D}_1(g^{-1})]^\dagger$. Show that D_2 is a representation.

   Suppose that W is an invariant subspace of $\mathbb{C}^n$ with respect to D_2. Let $W^\perp$ be the vector space of vectors orthogonal to W, and show that $W^\perp$ is an invariant subspace of $\mathbb{C}^n$ with respect to D_1. Finally show that if D_1 is irreducible, then D_2 must also be irreducible.

Please e-mail me at M.Wingate@damtp.cam.ac.uk with any comments, especially any errors.