Example Sheet 1: Sturm-Liouville Theory and Variational Methods

1. Put the following operators in Sturm-Liouville form by finding an appropriate weight function,

$$\text{(i)} \quad \tilde{\mathcal{L}} = -\frac{d^2}{dx^2} - \frac{d}{dx}\,, \qquad \text{(ii)} \quad \tilde{\mathcal{L}} = -\frac{d^2}{dx^2} - \frac{1}{x}\frac{d}{dx}\,.$$

2. The function $y(x)$ satisfies the equation $y'' + 2\gamma y' + (\gamma^2 + n^2)\,y = 0$, for real constant γ and integer n. Show that the boundary conditions $y(0) = y(\pi) = 0$ allow a solution y_n, for each n, that is unique up to multiplication by a constant. Why is it sufficient to consider only positive integers? Put the equation for y into Sturm-Liouville form and hence find the function $w(x)$ such that

$$\int_0^\pi y_m(x)\,y_n(x)\,w(x)\,dx = 0 \qquad \text{for} \quad m \neq n\,.$$

3. Show that the operator

$$\mathcal{L} = -\left(1 - x^2\right)\frac{d^2}{dx^2} + 2x\frac{d}{dx}$$

is self-adjoint, for inner product with unit weight function, when acting on functions $y(x)$ that are finite at $x = -1$ and $x = 1$. The eigenfunctions of $\mathcal{L}$ with normalization $y(1) = 1$ are the Legendre Polynomials $P_\ell(x)$ with eigenvalues $\ell(\ell+1)$ for non-negative integer ℓ. Verify that

$$P_0 = 1\,, \quad P_1 = x\,, \quad P_2 = \tfrac{1}{2}\left(3x^2 - 1\right)\,, \quad P_3 = \tfrac{1}{2}(5x^3 - 3x)\,,$$

and that these eigenfunctions are orthogonal. Using the orthogonality of eigenfunctions with distinct eigenvalues, show that

$$\int_{-1}^{1} \left(1 - x^2\right) P'_m(x) P'_n(x)\,dx = 0 \qquad \text{for} \quad m \neq n\,.$$

Find the solution of the equation $\mathcal{L}y = x^3$, with the above boundary conditions, in terms of P_1 and P_3. What happens if x^3 is replaced by $x^2 - k$ with (i) $k = \frac{1}{3}$ and (ii) $k \neq \frac{1}{3}$?

4. The general real 4th order linear differential operator acting on functions of the real variable x may be written as

$$\mathcal{L} = p(x)\frac{d^4}{dx^4} + q(x)\frac{d^3}{dx^3} + r(x)\frac{d^2}{dx^2} + s(x)\frac{d}{dx} + t(x)$$

for real functions p, q, r, s, t. Find necessary conditions on these functions such that $\mathcal{L}$ is self-adjoint for inner product with (i) unit weight function, and (ii) arbitrary weight function $w(x)$. Is there always a choice of w that makes $\mathcal{L}$ self-adjoint?

5. Find the eigenvalues of the differential operator $\mathcal{L} = -d^2/dx^2 + 1$ acting on functions $y(x)$ subject to the boundary conditions $y(0) = y'(\pi) = 0$. Write down the Green function for $\mathcal{L}$ in terms of its orthonormal eigenfunctions. Find the eigenfunction expansion of the function $f(x) = x(2\pi - x)$. Hence find a solution, as a sum over eigenfunctions, of the equation $y'' - y = x(x - 2\pi)$ subject to the above boundary conditions.

6. Find the function $y(x)$ which extremizes the functional

$$F[y] = \int_0^a \left(y + y'^2\right) dx$$

subject to the boundary conditions $y(0) = y(a) = 0$.

7. A soap film is bounded by two circular wires at $r = a$, $z = \pm b$ in cylindrical polar coordinates (r, θ, z). Given that the soap surface is cylindrically symmetric, show that the equation of the surface of minimal area is

$$r = c \cosh\left(z/c\right)$$

where c satisfies the condition $a/c = \cosh\left(b/c\right)$. Show graphically that this condition has no solution for c if b/a is larger that a certain critical ratio. What happens to the soap surface as b/a is increased from below this ratio to above it?

8. Show from first principles that the functional $I[x] = \int_{t_1}^{t_2} f\left(t, x, \dot{x}, \ddot{x}\right) dt$ is extremized, for variations with both $x(t)$ and $\dot{x}(t)$ fixed at t_1 and t_2, by solutions of the equation

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial f}{\partial \ddot{x}}\right) = 0\,.$$

Hence find the function $x(t)$ with $x(1) = 1$, $\dot{x}(1) = -2$, $x(2) = 1/4$ and $\dot{x}(2) = -1/4$ that minimises $\int_1^2 t^4 \left\{\ddot{x}(t)\right\}^2 dt$.

9. State Fermat's principle governing the paths traced by light rays in a medium. A horizontally stratified medium has refractive index $\mu(z) = \sqrt{a - bz}$, where z is height and a and b are positive constants. Prove that the path of a light ray within a vertical plane in this medium is an inverted parabola. Show further that all such parabolas have their directrix in the plane $z = a/b$. [*The directrix of a parabola in the standard form* $y^2 = 4\alpha x$ *is the line* $x = -\alpha$.]

10. Given that the refractive index $\mu(\mathbf{r})$ of some material equals $|\boldsymbol{\nabla} f|$ for some function $f(\mathbf{r})$, show that the optical path length $\int_A^B \mu\, d\ell$ between points A and B in the material is no less than $f(B) - f(A)$, with equality if and only if the path is orthogonal to the family of surfaces of constant f. Deduce that such 'orthogonal' trajectories satisfy Fermat's principle.

11. A particle of unit mass moves in a plane, with polar coordinates (r, θ), under the influence of a central force derived from a potential $V(r)$. Write down the action functional for this problem and use Hamilton's principle to find differential equations for $r(t)$ and $\theta(t)$. What is the physical interpretations of these equations? Given that a particle's trajectory is $r = a \sin\theta$ for some constant a, deduce that there exists a constant E such that $V - E \propto r^{-4}$.

12. The temperature of a point on the unit sphere $x^2+y^2+z^2=1$ is given by $T(x,y,z) = 1+xyz$. Find the points of maximum and minimum temperature on the sphere. What are their temperatures?

13. A coal box, in the shape of a cuboid, is to be placed flush against a wall so that only its top, front and two ends are visible. How should the height h and the depth d be chosen so as to minimize the visible surface area A under the constraint that the box has a given volume V?

14. Extremize the functional

$$F[y] = \int_0^a \left(y'^2 + yy' + 2y\right) dx$$

with the boundary conditions $y(0) = y(a) = 0$ and subject to the constraint $\int_0^a y\, dx = \ell$, where ℓ is a constant.

15. An area A of a field is enclosed by a length ℓ of flexible fencing with its ends attached a distance a apart on a straight wall, where $a < \ell < \frac{1}{2}\pi a$. What shape maximizes A?

16. A Sturm-Liouville eigenvalue problem for functions $y(x)$ satisfying $y(\pm 1) = 0$ has the equation

$$-\left(1+x^2\right) y'' - 2xy' = \lambda y\,.$$

Use the Rayleigh-Ritz method with trial function $y_1 = 1 - x^2$ to obtain an upper bound on the lowest eigenvalue λ. Show that a better bound is obtained from the trial function $y_2 = \cos\left(\pi x/2\right)$. Explain how a further improvement could be achieved by considering y_1 and y_2 in combination. [$\int_{-1}^{1} x^2 \sin^2\left(\pi x/2\right) dx = 1/3 + 2/\pi^2$]

17. The differential equation governing small transverse displacements $y(x)$ of a string with fixed endpoints at $x = 0$ and $x = \pi$ is

$$y'' + \omega^2 f(x) y = 0$$

where ω is the angular frequency of the vibration and f is a positive function. Show that the allowed values of ω^2 are given by the stationary values of the ratio of functionals $\Omega^2 = F/G$, where $F = \int_0^\pi \left(y'\right)^2 dx$ and $G = \int_0^\pi f y^2 dx$. Use this fact to find an approximate value for the angular frequency of the fundamental mode when $f = 1 + \epsilon \sin x$, assuming $\epsilon \ll 1$.

18. Show that $\psi_0 = \exp(-\frac{1}{2}x^2)$ is an eigenfunction of the operator

$$\mathcal{L} = -\frac{d^2}{dx^2} + (x^2 - 1)$$

acting on functions $\psi(x)$ for which $\psi \to 0$ as $|x| \to \infty$, and find the corresponding eigenvalue λ_0. This is in fact the lowest eigenvalue.

Use the Rayleigh-Ritz method with trial function

$$\tilde{\psi}_0 = \begin{cases} b(a^2 - x^2) & |x| < a \\ 0 & |x| \geq a \end{cases}$$

where a and b are adjustable parameters, to obtain an approximation $\tilde{\lambda}_0$ to λ_0. Comment on the sign of $\tilde{\lambda}_0 - \lambda_0$.

This example sheet is available on `Moodle`. *Hints and answers will also be posted there. Comments/corrections to* `S.J.Cowley@maths.cam.ac.uk`.