Mathematical Methods II — Natural Sciences Tripos, Part IB
Dr S.J. Cowley — Lent Term 2020

Example Sheet 2: Laplace and Poisson's equations

1. The function $\Phi(x, y)$, defined in the semi-infinite strip $0 \leqslant x \leqslant 1$, $y \geqslant 0$, satisfies $\nabla^2\Phi = 0$ (two-dimensional Laplace equation) subject to the boundary conditions

$$\Phi = \begin{cases} 0 & x = 0 \text{ and } x = 1 \\ x(1-x) & y = 0 \\ 0 & y = \infty \end{cases}$$

Use the method of separation of variables in Cartesian coordinates to find $\Phi(x, y)$. Hence find $\int_0^1 dx\, (\partial\Phi/\partial y)_{y=0}$, leaving your answer as an infinite series.

2. A solute with diffusivity κ, of concentration $\Phi(r, \phi)$ in plane polar coordinates, both surrounds and fills a cylinder of radius a. The boundary of the cylinder is permeable and the solute diffuses through it in such a way that the flux has normal component $F \cos 2\phi$ at $r = a$, where F is a constant. Assuming a steady state, find the distribution of solute in the interior of the cylinder. Why does your answer include an arbitrary constant? How is it possible for there to be a constant flux of solute through the boundary of the cylinder and yet for the concentration within to remain fixed?

3. A single-valued potential $\Phi(r, \phi)$, in plane polar coordinates, satisfies $\nabla^2\Phi = 0$ inside a cylindrical annulus $a < r < b$. On $r = b$, $\Phi = 0$ for all ϕ. On $r = a$, $\Phi = 0$ for $-\pi < \phi \leqslant 0$ and $\Phi = 1$ for $0 < \phi \leqslant \pi$. Find $\Phi(r, \phi)$ inside the annulus.

4. The axis of a solid cylinder of radius a coincides with the z-axis of cylindrical polar coordinates (r, ϕ, z). Heat flows past the cylinder such that the temperature T inside and outside, i.e. for $r < a$ and $r > a$, is independent of z and satisfies $\nabla^2 T = 0$. Find T subject to the following boundary conditions: T finite at $r = 0$ and $T \sim Gr\cos\phi$ as $r \to \infty$, where G is a constant; T continuous at $r = a$ but $\frac{\partial T}{\partial r}(a^+, \phi) = \beta \frac{\partial T}{\partial r}(a^-, \phi)$ for constant β $(0 < \beta < 1)$. What is the physical meaning of the constants G and β? [*a^+ denotes the limit $r \to a$ from $r > a$ and a^- denotes the limit $r \to a$ from $r < a$.*]

5. Find the solution to $\nabla^2\Psi = 0$ inside a sphere of radius a for axisymmetric Ψ, i.e. $\partial\Psi/\partial\varphi = 0$ in spherical polar coordinates (r, θ, φ), subject to the boundary condition that $\Psi = 1 + \cos\theta + \cos^2\theta$ on the surface of the sphere. Include a derivation of any general solution that you use.

6. In the presence of an electric charge density $\rho_q(\mathbf{r})$, the electrostatic potential $\Phi(\mathbf{r})$ satisfies $\nabla^2\Phi = -\rho_q/\epsilon_0$. In spherical polar coordinates (r, θ, φ), the charge density is

$$\rho_q = \begin{cases} r^{-1}\cos\theta & 0 < r < a \\ 0 & r \geqslant a \end{cases}$$

Why must both Φ and $\partial\Phi/\partial r$ be everywhere continuous despite the discontinuity in ρ_q at $r = a$? By writing Φ as a function of r times a *suitably chosen* function of θ, and assuming it to be independent of φ, find Φ subject to the boundary conditions that it be finite at $r = 0$ and zero at $r = \infty$. [*You may wish to start by proving that the general solution to the differential equation* $r^2R'' + 2rR' - 2R = \alpha r$, *where* α *is a constant, is* $R = Ar + Br^{-2} + (\alpha/3)\, r\ln r$.]

7. Show, for each of the following three cases, that the solution of the given equation for scalar field Φ in volume V, with surface S, is unique subject to the boundary conditions of the specified type.

   (i) Laplace equation: $\nabla^2\Phi = 0$ with normal derivative $\partial\Phi/\partial n$ specified on S (Neumann boundary conditions) *and* Φ specified at one point.

   (ii) Yukawa equation: $\nabla^2\Phi = m^2\Phi$ for real non-zero constant m, with Φ specified on S (Dirichlet boundary conditions).

   (iii) Yukawa equation subject to Neumann boundary conditions.

8. The quarter-plane $x > 0, y > 0$ is occupied by a single source of heat of strength Q positioned at the point (x_0, y_0). At $x = 0$ there is a plane conducting wall held at temperature T_0. At $y = 0$ there is an insulated wall across which no heat can flow (i.e. the heat flux normal to the wall must vanish). Write down the equation and boundary conditions satisfied by a time-independent temperature field $T(x, y)$, and use the method of images to find it. Hence show that the magnitude of the heat flux across the wall at the point $(0, y)$ is

$$\frac{Qx_0}{\pi}\left\{\frac{1}{x_0^2 + (y - y_0)^2} + \frac{1}{x_0^2 + (y + y_0)^2}\right\}.$$

   Calculate the *total* heat radiated across the wall at $x = 0$ and comment on the result.

9. Use the method of images in plane polar coordinates to find the Green's function for the Laplacian (∇^2), with Dirichlet boundary conditions, in the following two-dimensional domains:

   (i) disc: $0 \leqslant r < a\,,$

   (ii) half-disc: $0 < r < a\,, \quad 0 < \theta < \pi\,.$

10. Define the Green's function $G(\mathbf{r}, \mathbf{r}')$ for the three-dimensional Laplacian with Dirichlet boundary conditions, acting on functions defined in a volume V with bounding surface S. Given that $\nabla^2 u = 0$ in V and that $u = f$ on S, show that

$$u(\mathbf{r}') = \int_S f(\mathbf{r}) \frac{\partial G}{\partial n} (\mathbf{r}, \mathbf{r}') \, dS \,,$$

where $\partial/\partial n$ denotes differentiation along the outward normal on S. What is the analogous equation for two space dimensions?

Show that a solution of $\nabla^2_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$ in two dimensions is $G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}'|$. Hence find the Green's function, with Dirichlet boundary conditions, for the two-dimensional Laplacian in the half-plane $-\infty < x < \infty$, $y > 0$. Use it to solve the Laplace equation for u in this region with boundary conditions that $u \to 0$ as $r \to \infty$, $u = 0$ on $y = 0$ for $|x| > 1$, and $u = 1$ on $y = 0$ for $|x| \leqslant 1$.

11. A thin disc of uniform density and total mass M lies in the plane $z = 0$ of *cylindrical* polar coordinates (r, ϕ, z) and occupies the region $r \leqslant a$ of this plane. Use the integral expression for a solution of Poisson's equation to find the gravitational potential $\Phi(r, z)$ on the axis of symmetry $r = 0$. Assuming that $|z| \gg a$, expand your result as a power series accurate to $\mathcal{O}(z^{-5})$.

Now write down the general solution of Laplace's equation in *spherical* polar coordinates valid at large distance r_s from the origin. By comparing this with your previous expansion, find an expression for Φ that is valid off the axis of symmetry. [*Recall that* $P_n(\cos\theta) = 1$ *when* $\cos\theta = 1$.]

12. State a version of Green's identity applicable to a plane surface S bounded by a closed curve C. Use this identity, and the Green's function from Qus. 9(i), to show that the solution of $\nabla^2 \Phi = 0$ within the disc of radius a, i.e. $r < a$ in plane polar coordinates (r, ϕ), subject to the boundary condition that $\Phi = \Psi(\phi)$ on the boundary at $r = a$, is

$$\Phi(r, \phi) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\Psi(\phi') \, d\phi'}{a^2 - 2ar\cos(\phi - \phi') + r^2} \,.$$

13. Use the result of Qus. 12 to show that in $0 \leqslant r < 1$, and for $F(r, \phi, \phi') \equiv 1 - 2r\cos(\phi - \phi') + r^2$,

$$\begin{aligned} \int_0^{2\pi} \frac{d\phi'}{F(r,\phi,\phi')} &= \frac{2\pi}{1 - r^2} \,, \\ \int_0^{2\pi} \frac{\sin(\phi') \, d\phi'}{F(r,\phi,\phi')} &= \frac{2\pi r \sin\phi}{1 - r^2} \,, \\ \int_0^{2\pi} \frac{\cos^2(\phi') \, d\phi'}{F(r,\phi,\phi')} &= \frac{2\pi r^2 \cos^2\phi}{1 - r^2} + \pi \,. \end{aligned}$$

This example sheet is available on `Moodle`*. Hints and answers will also be posted there. Comments/corrections to* `S.J.Cowley@maths.cam.ac.uk`*.*