Example Sheet 3: Cartesian Tensors

1. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a set of orthonormal basis vectors. Confirm that the vectors

$$\mathbf{e}'_1 = \frac{1}{2}\left(\mathbf{e}_1 + \sqrt{3}\mathbf{e}_3\right), \quad \mathbf{e}'_2 = \mathbf{e}_2\,, \quad \mathbf{e}'_3 = \frac{1}{2}\left(\mathbf{e}_3 - \sqrt{3}\mathbf{e}_1\right)$$

are orthonormal and hence define a new frame. If $\mathbf{v} = \sqrt{3}\,\mathbf{e_1} + 5\,\mathbf{e_2} + 2\,\mathbf{e_3}$, find its components, v'_i, in the new frame.

The moments of inertia of a rigid body, with respect to its principal axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, are 1, 2 and 3. Find the inertia tensor I' in the frame defined by $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3)$.

With respect to a third (orthonormal) frame $(\mathbf{e}''_1, \mathbf{e}''_2, \mathbf{e}''_3)$, the vector $\mathbf{e}_1$ has components $\frac{1}{\sqrt{2}}(1, 1, 0)$ and the vector $\mathbf{e}_2$ has components $\frac{1}{\sqrt{3}}(1, -1, 1)$. Find the inertia tensor I'' in this frame. Compute the trace of the square of I'' and verify that it agrees with the trace of the square of I; why must this be so?

[*The moments of inertia are the diagonal entries of the inertia tensor I in the frame provided by the (orthonormal) principal axes; the off-diagonal entries are zero in this frame.*]

2. A right circular *solid* cylinder of uniform mass density ρ has radius a and length $\sqrt{3}\,a$. Identify the principal axes of this rigid body and in the frame defined by these axes find its inertia tensor about its centre of mass. Comment on your result.

3. The components of a vector $\mathbf{F}$ are related to those of vectors $\mathbf{J}$ and $\mathbf{H}$ by an equation of the form $F_i = T_{ijk}J_jH_k$ for arbitrary $\mathbf{J}$ and $\mathbf{H}$. Starting from the transformation law for the components of a vector, deduce that the coefficients T_{ijk} are the components of a 3rd-order tensor.

4. Show that any second-order tensor T may be expressed, uniquely, as the sum of a symmetric tensor S with zero trace, an isotropic tensor I and an antisymmetric tensor A. Exhibit this decomposition for the matrix

$$T = \begin{pmatrix} 3 & -1 & 5 \\ 1 & 0 & 5 \\ 1 & -5 & 3 \end{pmatrix}$$

5. Let A be an antisymmetric 2nd-order tensor, with components A_{ij} in a frame in which the vector $\mathbf{x}$ has components x_i. Show that the vector $A\mathbf{x}$, with components $A_{ij}x_j$, may be written as $\boldsymbol{\omega} \times \mathbf{x}$ for some vector $\boldsymbol{\omega}$ determined by A. Now show that the tensor $B = A^2$ is symmetric, and find its eigenvalues in terms of $\boldsymbol{\omega}$.

6. The electrical conductivity tensor σ determines the electric current density $\mathbf{J}$ that flows in response to a uniform and constant applied electric field $\mathbf{E}$, according to the formula $J_i = \sigma_{ij} E_j$. The electrical conductivity tensor of a crystal is measured to have entries

$$\sigma_{ij} = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

in a laboratory frame. Show that there is a direction in which no current flows, and find that direction. The rate of energy dissipation per unit volume is given by $\mathbf{E} \cdot \mathbf{J}$. For $|\mathbf{E}|^2 = 1$, find the minimum and maximum possible energy dissipation rates.

7. The eigenvalues of the conductivity tensor σ of an anisotropic crystal are α, β and β. Write down the tensor σ in the frame in which it is diagonal and identify the axis of symmetry due to the repeated eigenvalue. A particular crystal is grown such that it takes the form of a long thin wire of length L and cross-sectional area A, the direction of the wire making an angle θ with the axis of symmetry of the crystal. Show that the wire's resistance to a current passing along it is

$$R = (L/A)(\alpha^{-1} \cos^2 \theta + \beta^{-1} \sin^2 \theta) \,.$$

8. Let M be a second-order tensor with components M_{ij} in one frame and components M'_{ij} in another frame. Explain (i) why $\epsilon_{ijk} \det M = \epsilon_{lmn} M_{il} M_{jm} M_{kn}$ and (ii) why ϵ_{ijk} are the components of a 3rd-order isotropic tensor. (iii) Deduce that $\det M' = \det M$, and hence that $\det M$ is a scalar.

9. A 4th-order tensor has components

$$c_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk} \,,$$

where α, β and γ are constants. Show that this defines an isotropic tensor. It is, in fact, the most general 4th-order isotropic tensor, and you may assume this for the rest of the question.

In an isotropic fluid the pressure p and the (2nd order) stress tensor σ are related by $p = -\frac{1}{3}\sigma_{ii}$. Given that the deviatoric stress tensor s is defined by

$$s_{ij} = \sigma_{ij} + p\delta_{ij} \,,$$

show that it has zero trace. In "Newtonian" fluids the deviatoric stress tensor is linearly related to the strain tensor e, defined for a fluid of velocity field $\mathbf{v}(\mathbf{x})$, by

$$e_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) .$$

Show that the relationship between σ and e must take the form

$$\sigma_{ij} = 2\mu \left(e_{ij} - \frac{1}{3} \delta_{ij} e_{kk} \right) - p\delta_{ij} \,,$$

for some constant μ (known as the coefficient of viscosity). Given that $\mu > 0$, show further (by considering an appropriate frame, or otherwise) that the scalar $s_{ij} e_{ij}$ is non-negative.

10. A second-order tensor T is defined by $T_{ij} = \delta_{ij} + \epsilon_{ijk}x_k$ where x_k are the (cartesian) components of a position vector $\mathbf{x}$. Compute the following surface integrals, where in each case the surface of integration is the unit sphere $|\mathbf{x}| = 1$:

$$(i) \quad \iint \mathbf{x}\, dS\,, \qquad (ii) \quad \iint T_{ij}\, dS\,, \qquad (iii) \quad \iint T_{ij}T_{jk}\, dS\,.$$

11. Use suffix notation to establish the following vector identities for any scalar field Φ and vector field $\mathbf{F}$:

$$(i) \quad \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\Phi) \equiv \mathbf{0}\,, \qquad (ii) \quad \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) \equiv 0\,, \qquad (iii) \quad \boldsymbol{\nabla}\times(\Phi\mathbf{F}) = \Phi\boldsymbol{\nabla}\times\mathbf{F} + \boldsymbol{\nabla}\Phi\times\mathbf{F}$$

Show also that

$$(iv) \quad \frac{1}{2}\boldsymbol{\nabla}\left(|\mathbf{F}|^2\right) - \mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{F}) \equiv (\mathbf{F}\cdot\boldsymbol{\nabla})\,\mathbf{F}\,.$$

Is this identity still true in non-cartesian coordinate systems?

12. A conductor in a magnetic field $\mathbf{B}$ carries a current density $\mathbf{J} = \mu^{-1}\boldsymbol{\nabla}\times\mathbf{B}$, where μ is the permeability. The mechanical force $\mathbf{F}$ per unit volume acting on the conductor is $\mathbf{F} = \mathbf{J}\times\mathbf{B}$. Using the fact that $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, show (for cartesian coordinates x_i) that

$$F_i = \frac{\partial S_{ij}}{\partial x_j}\,, \qquad S_{ij} = \mu^{-1}\left(B_iB_j - \frac{1}{2}|\mathbf{B}|^2\delta_{ij}\right)\,.$$

This example sheet is available on `Moodle`*. Hints and answers will also be posted there. Comments/corrections to* `S.J.Cowley@maths.cam.ac.uk`*.*