3P1c **Quantum Field Theory: Example Sheet 3** Michaelmas 2019

Corrections and suggestions should be emailed to B.C.Allanach@damtp.cam.ac.uk*. Starred questions may if you wish be handed in to your supervisor for feedback prior to the class.*

1. The chiral representation of the Clifford algebra is

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} .$$

Show that these indeed satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$. Find a unitary matrix U such that $(\gamma')^\mu = U\gamma^\mu U^\dagger$, where $(\gamma')^\mu$ form the Dirac representation of the Clifford algebra

$$(\gamma')^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \quad , \quad (\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} .$$

2. Show that if $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, then

$$[\gamma^\mu\gamma^\nu \, , \gamma^\rho\gamma^\sigma] = 2\eta^{\nu\rho}\gamma^\mu\gamma^\sigma - 2\eta^{\mu\rho}\gamma^\nu\gamma^\sigma + 2\eta^{\nu\sigma}\gamma^\rho\gamma^\mu - 2\eta^{\mu\sigma}\gamma^\rho\gamma^\nu .$$

Show further that $S^{\mu\nu} \equiv \frac{1}{4}[\gamma^\mu \, , \gamma^\nu] = \frac{1}{2}(\gamma^\mu\gamma^\nu - \eta^{\mu\nu})$. Use this to confirm that the matrices $S^{\mu\nu}$ form a representation of the Lie algebra of the Lorentz group.

3. Using just the algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ (that is to say without resorting to any particular representation of the gamma matrices), and defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{p} = p_\mu\gamma^\mu$ and $S^{\mu\nu} \equiv \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, prove the following results (these are useful when calculating cross-sections or decay widths involving spinor fields):

   (a) $\mathrm{Tr}\gamma^\mu = 0$

   (b) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$

   (c) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$

   (d) $(\gamma^5)^2 = 1$

   (e) $\mathrm{Tr}\gamma^5 = 0$

   (f) $\not{p}\not{q} = 2p \cdot q - \not{q}\not{p} = p \cdot q + 2S^{\mu\nu}p_\mu q_\nu$

   (g) $\mathrm{Tr}(\not{p}\not{q}) = 4p \cdot q$

   (h) $\mathrm{Tr}(\not{p}_1 \ldots \not{p}_n) = 0$ if n is odd

   (i) $\mathrm{Tr}(\not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) = 4[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)(p_2 \cdot p_4)]$

   (j) $\mathrm{Tr}(\gamma^5 \not{p}_1 \not{p}_2) = 0$

   (k) $\gamma_\mu \not{p} \gamma^\mu = -2\not{p}$

   (l) $\gamma_a \not{p}_1 \not{p}_2 \gamma^a = 4p_1 \cdot p_2$

   (m) $\gamma_\mu \not{p}_1 \not{p}_2 \not{p}_3 \gamma^\mu = -2\not{p}_3 \not{p}_2 \not{p}_1$

   (n) $\mathrm{Tr}(\gamma^5 \not{p}_1 \not{p}_2 \not{p}_3 \not{p}_4) = 4i\,\epsilon_{\mu\nu\rho\sigma}\,p_1^\mu\,p_2^\nu\,p_3^\rho\,p_4^\sigma$

4*. The plane-wave solutions to the Dirac equation are

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \text{ and } v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix},$$

where $\sigma^\mu = (1,\vec{\sigma})$ and $\bar{\sigma}^\mu = (1,-\vec{\sigma})$ and ξ^s, with $s \in \{1,2\}$, is a basis of orthonormal two-component spinors, satisfying $(\xi^r)^\dagger \cdot \xi^s = \delta^{rs}$. Show that

$$\begin{aligned} u^r(\vec{p})^\dagger \cdot u^s(\vec{p}) &= 2p_0\delta^{rs} \\ \bar{u}^r(\vec{p}) \cdot u^s(\vec{p}) &= 2m\delta^{rs} \end{aligned} \tag{1}$$

and similarly,

$$\begin{aligned} v^r(\vec{p})^\dagger \cdot v^s(\vec{p}) &= 2p_0\delta^{rs} \\ \bar{v}^r(\vec{p}) \cdot v^s(\vec{p}) &= -2m\delta^{rs}. \end{aligned} \tag{2}$$

Show also that the orthogonality condition between u and v is

$$\bar{u}^s(\vec{p}) \cdot v^r(\vec{p}) = 0,$$

while taking the inner product using † requires an extra minus sign

$$u^s(\vec{p})^\dagger \cdot v^r(-\vec{p}) = 0. \tag{3}$$

5. Using the same notation as Question 4, show that

$$\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m, \tag{4}$$

$$\sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p}) = \not{p} - m, \tag{5}$$

where, rather than being contracted, the two spinors on the left-hand side are placed back to back to form a 4×4 matrix.

6. The Fourier decomposition of the Dirac field operator $\psi(x)$ and the hermitian conjugate field $\psi^\dagger(\vec{x})$ is given by

$$\begin{aligned} \psi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1}^{2}\left[b^s_{\vec{p}}u^s(\vec{p})e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right], \\ \psi^\dagger(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1}^{2}\left[b^{s\dagger}_{\vec{p}}u^s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + c^{s}_{\vec{p}}v^s(\vec{p})^\dagger e^{i\vec{p}\cdot\vec{x}}\right]. \end{aligned} \tag{6}$$

The creation and annihilation operators are taken to satisfy

$$\begin{aligned} \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} &= (2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q}), \\ \{c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} &= (2\pi)^3\delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{q}), \end{aligned}$$

with all other anticommutators vanishing. Show that these imply that the field and its conjugate field satisfy the anti-commutation relations

$$\begin{aligned} \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \{\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = 0, \\ \{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} &= \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y}). \end{aligned}$$

Note: the calculation is very similar to that for the bosonic field, but at some point you will need to make use of the identities Eqs. (4),(5).

7*. Using the results of Question 6, show that the quantum Hamiltonian

$$H = \int d^3x \; \bar{\psi}(-i\gamma^i \partial_i + m)\psi$$

can be written, after normal ordering, as

$$H = \int \frac{d^3p}{(2\pi)^3} \, E_{\vec{p}} \sum_{r=1}^{2} \left[b_{\vec{p}}^{r\dagger} b_{\vec{p}}^{r} + c_{\vec{p}}^{r\dagger} c_{\vec{p}}^{r} \right] .$$

Note: the calculation is very similar to that of the bosonic field. This time you will need to make use of the identities in Eqs. (1), (2) and (3).

8. A fermionic Yukawa theory has the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i \not{\partial} - \mu)\psi + \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - g\bar{\psi}\psi\phi.$$

Show that the differential cross-section in the centre of mass frame for nucleon-nucleon scattering ($\psi\psi \to \psi\psi$) *including the masses* m and μ is

$$\frac{d\sigma}{dt} = \frac{|g|^4}{16\pi s(s-4\mu^2)} \left[\frac{(u-4\mu^2)^2}{(u-m^2)^2} + \frac{(t-4\mu^2)^2}{(t-m^2)^2} + \frac{1}{2}\frac{(s-4\mu^2)^2 - (u-4\mu^2)^2 - (t-4\mu^2)^2}{(u-m^2)(t-m^2)} \right] .$$

9. Consider the theory of a fermion ψ and a real scalar ϕ with Lagrangian density

$$\mathcal{L} = \bar{\psi}(i \not{\partial} - m)\psi + \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}\mu^2 \phi^2 - \lambda \bar{\psi}\gamma^\mu \psi \partial_\mu \phi.$$

Draw and write momentum-space Feynman rules for the interactions of the theory. What is the mass dimension of λ?

What is the spin averaged/summed cross-section for $\psi\bar{\psi} \to \psi\psi$?

What is the tree-level width for the decay $\phi \to \psi\bar{\psi}$, assuming that $\mu > 2m$?