Mathematical Tripos: Part III PM

Contents

0 Perturbation Methods (16 Lectures)

0.1 Introduction

- 24 lectures prised into 16.
- Any corrections and suggestions should be emailed to me at `S.J.Cowley@maths.cam.ac.uk`.
- Closed book examination. *Likely* rubric:

  Attempt no more than TWO questions.
  There are THREE questions in total.
  The questions carry equal weight.

- **Books**
  - Hinch, *Perturbation methods.*
  - Van Dyke, *Perturbation methods in fluid mechanics.*
  - Kevorkian & Cole, *Perturbation methods in applied mathematics.*
  - Bender & Orszag, *Advanced mathematical methods for scientists and engineers.*
- **Philosophy**
  - Many physical processes are described by equations that cannot be solved analytically.
  - One approach is to solve the equations numerically; however, often there exists a 'small' parameter, ε, e.g.
    - * in low Mach number flows $\varepsilon = M = \frac{u}{c}$, where u is the fluid velocity and c is the speed of sound;
    - * in fast flows $\varepsilon = \frac{1}{Re}$, where Re is the Reynolds number.
  - We can use the smallness of ε to simplify the equations, and then find analytic (or simpler numerical) solutions.
- Primarily interested in differential equations, but a number of the ideas can be illustrated for algebraic equations and/or integrals. We will use algebraic equations to motivate some of the ideas.
- The only pre-requisites are (a) a course in 'Sums' (i.e. a competency to perform moderately messy calculations), and (b) an ability to solve simple differential equations and evaluate simple integrals (e.g. using integration by parts).

Include Olver's paradox as an example?

1 Algebraic Equations

1.1 Regular Expansions and Iteration

Consider
$$x^2 + \varepsilon x - 1 = 0\ . \tag{1.1}$$

Exact solution:
$$x = -\frac{1}{2}\varepsilon \pm \left(1 + \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}}\ .$$

If $|\varepsilon| < 2$, then can expand in a convergent series:
$$x = \begin{cases} 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 - \frac{1}{128}\varepsilon^4 + \cdots \\ -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \frac{1}{128}\varepsilon^4 + \cdots \end{cases}$$

Since the series is convergent for $|\varepsilon| < 2$, for small ε we can increase the accuracy by taking more terms. We have

solved the equation and then **approximated the solution**

However, we cannot always solve the equation exactly, so can we

approximate and then **solve the equation?**

1.1.1 Iterative method (liked by Pure Mathematicians)

Based on
$$x_{n+1} = g(x_n)\,.$$

Suppose $x_n = x^* + \delta_n$ where $x^* = g(x^*)$. Then by Taylor Series
$$\delta_{n+1} = g'(x^*)\,\delta_n + \mathcal{O}\left(\delta_n^2\right)\ .$$

If we have a good guess, so that $|\delta_n|$ is small, this is convergent if
$$|g'(x^*)| < 1\ .$$

Rearrange (1.1):
$$x^2 = 1 - \varepsilon x\ .$$

For the root near $x = 1$ try
$$\begin{aligned} x_{n+1} &= (1 - \varepsilon x_n)^{\frac{1}{2}} \\ x_0 &= 1 \\ x_1 &= (1-\varepsilon)^{\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \cdots \\ x_2 &= \left(1 - \varepsilon\,(1-\varepsilon)^{\frac{1}{2}}\right)^{\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \frac{1}{8}\varepsilon^3 + \cdots \\ x_3 &= \left(1 - \varepsilon\left(1 - \varepsilon\,(1-\varepsilon)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\ &= 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + 0 + \mathcal{O}\left(\varepsilon^4\right) + \cdots \end{aligned}$$

Hard work for the higher terms — also, how many terms are correct?

1.1.2 Expansion method

For $\varepsilon = 0$, the roots are $x = \pm 1$. For the root near $x = 1$ try

$$x(\varepsilon) = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots$$

Substitute into equation (1.1):

$$\begin{array}{llllllll} & 1 + 2\varepsilon x_1 & + 2\varepsilon^2 x_2 + \varepsilon^2 x_1^2 & + 2\varepsilon^3 x_3 + 2\varepsilon^3 x_1 x_2 & + \cdots & \\ & \;\; + \varepsilon & + \varepsilon^2 x_1 & + \varepsilon^3 x_2 & + \cdots & \\ -1 & & & & & = 0 \end{array}$$

Equate powers of ε:

$$\begin{array}{lrll} \varepsilon^0 : & 1 - 1 & = 0 & \\ \varepsilon^1 : & 2x_1 + 1 & = 0 \quad , & x_1 = -\tfrac{1}{2} \\ \varepsilon^2 : & 2x_2 + x_1^2 + x_1 & = 0 \quad , & x_2 = \tfrac{1}{8} \\ \varepsilon^3 : & 2x_3 + 2x_1 x_2 + x_2 & = 0 \quad , & x_3 = 0 \end{array}$$

Easier than the iterative method for higher terms, but you need to guess the expansion correctly.

1.2 Singular Perturbations and Rescaling

Consider

$$\varepsilon x^2 + x - 1 = 0 \, . \tag{1.2}$$

$$\begin{array}{lcl} \varepsilon = 0 & : & \text{one solution} \\ \varepsilon \neq 0 & : & \text{two solutions} \end{array}$$

The limit process $\varepsilon \to 0$ is said to be *singular*.

Exact solution: $\dfrac{-1 \pm (1 + 4\varepsilon)^{\frac{1}{2}}}{2\varepsilon}$.

Expansion for $|\varepsilon| < \frac{1}{4}$:

$$x = \begin{cases} 1 - \varepsilon + 2\varepsilon^2 - 5\varepsilon^3 + \cdots \\ -\frac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + \cdots \end{cases} \tag{1.3}$$

The singular (i.e. extra) root $\to \mp\infty$ as $\varepsilon \to 0\pm$.

1.2.1 Iterative method

(a) For the non-singular root try

$$x_{n+1} = 1 - \varepsilon x_n^2 \, .$$

(b) For the singular root, we need to keep the 'εx^2' term as a major player. The leading order approximation is

$$\varepsilon x^2 + x \approx 0 \, ;$$

so try rearranging (1.2) to

$$x_{n+1} = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x_n} \, .$$

Exercise. Confirm (1.3) by iteration.

Note that in (b)

$$x_{n+1} = g(x_n) \, , \qquad \text{where} \qquad g(x) = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x} \, .$$

Hence

$$g'(x) = -\frac{1}{\varepsilon x^2} \, , \qquad \left| g'\left(-\frac{1}{\varepsilon}\right) \right| = \varepsilon < 1 \quad \text{if} \quad 0 < \varepsilon < 1 \, .$$

1.2.2 Expansion method

For one root try

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \ , \tag{1.4a}$$

and for the other try

$$x = \frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \cdots \ . \tag{1.4b}$$

Substitute (1.4b) into (1.2):

$$\begin{array}{rcccl} & \dfrac{x_{-1}^2}{\varepsilon} + 2x_{-1}x_0 + \varepsilon\left(x_0^2 + 2x_{-1}x_1\right) + \cdots & & & \\ + & \dfrac{x_{-1}}{\varepsilon} + \qquad x_0 + \qquad\qquad\qquad \varepsilon x_1 + \cdots & & & \\ & - \qquad 1 & & & = 0 \end{array}$$

Equate powers

$$\begin{array}{lrclcccccc} \varepsilon^{-1}: & x_{-1}^2 + x_{-1} & = 0 & ; & x_{-1} & = & 0 & , & -1 \\ \varepsilon^{0}: & (2x_{-1}+1)\,x_0 - 1 & = 0 & ; & x_0 & = & 1 & , & -1 \\ \varepsilon: & x_0^2 + 2x_{-1}x_1 + x_1 & = 0 & ; & x_1 & = & -1 & , & 1 \\ & & & & & & \uparrow & & \uparrow \\ & & & & & & (1.3\text{a}) & & (1.3\text{b}) \end{array}$$

1.2.3 Rescaling before expansion

How do you decide on the expansion if you do not know the solution?

Seek rescaling[s] to convert the singular equation into a regular equation. Try

$$x = \ \delta(\varepsilon)\, X$$
$$\text{need to choose suitable } \delta \nearrow \qquad \nwarrow \text{strictly order ‘unity’; say } X = \text{ord}(1).$$

(1.2) becomes

$$\varepsilon\delta^2 X^2 + \delta X - 1 = 0 \ .$$

Consider the possibilities for different choices of δ ($|\varepsilon| \ll 1$):

$\delta \ll 1$:			small	$+$	small	$-$	1	$=$	0	$\divideontimes$
$\delta = 1$:			small	$+$	X	$-$	1	$=$	0	regular root
$1 \ll \delta \ll \frac{1}{\varepsilon}$:	$\frac{\text{LHS}}{\delta}$	$=$	small	$+$	X	$+$	small	$=$	0	$\divideontimes$
										(since $X = \text{ord}(1)$)
$\delta = \frac{1}{\varepsilon}$:	$\frac{\text{LHS}}{\delta}$	$=$	X^2	$+$	X	$+$	small	$=$	0	singular root
$\delta \gg \frac{1}{\varepsilon}$:	$\frac{\text{LHS}}{\varepsilon\delta^2}$	$=$	X^2	$+$	small	$+$	small	$=$	0	$\divideontimes$

The distinguished choices are therefore:

$$\begin{array}{lccl} \delta = 1: & \varepsilon X^2 + X - 1 = 0 & ; & X = 1 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots \\ \delta = \frac{1}{\varepsilon}: & X^2 + X - \varepsilon = 0 & ; & X = -1 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots \end{array}$$

01/01

1.3 Non Integral Powers

Inter alia, double roots can cause problems. Consider, with $\varepsilon > 0$,

$$(1-\varepsilon)\,x^2 - 2x + 1 = 0\ . \tag{1.5}$$

When $\varepsilon = 0$, there is a double root at $x = 1$. Try an expansion:

$$x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

then

$$\begin{array}{rcrcrcl} & 1 & + & 2\varepsilon x_1 & + & \varepsilon^2\left(2x_2 + x_1^2\right) & + \cdots \\ & & - & \varepsilon & - & \varepsilon^2\,(2x_1) & \\ - & 2 & - & 2\varepsilon x_1 & - & 2\varepsilon^2 x_2 & + \cdots \\ + & 1 & & & & & \quad = \quad 0 \end{array}$$

and equating powers of ε:

$$\begin{array}{lrll} \varepsilon^0: & 1 - 2 + 1 & = 0 & \\ \varepsilon^1: & 2x_1 - 1 - 2x_1 & = 0 & \divideontimes \end{array}$$

We need 'εx_1' to be larger.

From the exact solution:

$$x = \frac{1 \pm \varepsilon^{\frac{1}{2}}}{1-\varepsilon}\,,$$

we see that we should have expanded in powers of $\varepsilon^{\frac{1}{2}}$:

$$x = 1 + \varepsilon^{\frac{1}{2}} x_{\frac{1}{2}} + \varepsilon x_1 + \varepsilon^{\frac{3}{2}} x_{\frac{3}{2}} + \cdots$$

$$\begin{array}{rcrcrcrcl} & 1 & + & 2\varepsilon^{\frac{1}{2}} x_{\frac{1}{2}} & + & 2\varepsilon x_1 & + & \varepsilon x_{\frac{1}{2}}^2 & \\ & & & & - & \varepsilon & & & \\ - & 2 & - & 2\varepsilon^{\frac{1}{2}} x_{\frac{1}{2}} & - & 2\varepsilon x_1 & & & \\ + & 1 & & & & & & & \quad = \quad 0 \end{array}$$

This time on equating powers of ε we see that

$$\begin{array}{lrll} \varepsilon^0: & 1 - 2 + 1 & = 0 & \\ \varepsilon^{\frac{1}{2}}: & 2x_{\frac{1}{2}} - 2x_{\frac{1}{2}} & = 0 & \textbf{no information} \\ \varepsilon^1: & 2x_1 + x_{\frac{1}{2}}^2 - 1 - 2x_1 & = 0 & x_{\frac{1}{2}} = \pm 1 \end{array}$$

We must work to $\mathcal{O}(\varepsilon)$ to obtain the solution to $\mathcal{O}(\varepsilon^{\frac{1}{2}})$.

From the original equation

$$(x-1)^2 = \varepsilon x^2\ ,$$

we see that, since the roots are near $x = 1$ when $\varepsilon \ll 1$, a change in the ordinate by $\mathrm{ord}(\varepsilon)$ changes the position of the root by $\mathrm{ord}(\varepsilon^{\frac{1}{2}})$.

In general we must derive (guess) the expansion required, e.g. try

$$\begin{aligned} x(\varepsilon) \quad &= \quad 1 + \delta_1(\varepsilon)\, x_1 + \delta_2(\varepsilon)\, x_2 + \cdots \\ & \qquad\quad 1 \gg \delta_1 \gg \delta_2 \gg \cdots \\ & \qquad\quad x_j = \operatorname{ord}(1). \end{aligned}$$

Substitute into (1.5):

$$\begin{array}{cccccccccccccc} & 1 & + & 2\delta_1 x_1 & + & 2\delta_2 x_2 & + & \cdots & + & \delta_1^2 x_1^2 & + & \cdots & + & 2\delta_1\delta_2 x_1 x_2 \; + \; \cdots \\ & & & & - & \varepsilon & & & - & 2\varepsilon\delta_1 x_1 & + & \cdots & & \\ - & 2 & - & 2\delta_1 x_1 & - & 2\delta_2 x_2 & + & \cdots & & & & & & \\ + & 1 & & & & & & & & & & & = & 0 \end{array}$$

The leading order terms are $\quad \delta_1^2 x_1^2 \quad$ and $\quad -\varepsilon$.
Hence take $\quad \delta_1 = \varepsilon^{\frac{1}{2}}$
allow x_1 to absorb any multiple roots.

Exercise. Show that the choices $\delta_1^2 \gg \varepsilon$, or $\delta_1^2 \ll \varepsilon$, lead to a $\divideontimes$.

Cancelling off these two terms, the leading-order terms become

$$2\delta_1\delta_2 x_1 x_2 \qquad \text{and} \qquad -\,2\varepsilon\delta_1 x_1 \;.$$

Repeating the argument $\Rightarrow \delta_2 = \varepsilon$ (and $x_2 = 1$).

1.4 Logarithms

Solve

$$x e^{-x} = \varepsilon \;. \tag{1.6}$$

One root is close to $x = \varepsilon$, the other root is between

$$x = \ln \frac{1}{\varepsilon} \qquad\qquad (x e^{-x} = \varepsilon \ln \tfrac{1}{\varepsilon} > \varepsilon)$$

and

$$x = 2 \ln \frac{1}{\varepsilon} \qquad (x e^{-x} = 2\varepsilon^2 \ln \tfrac{1}{\varepsilon} < \varepsilon, \text{ for } \varepsilon \text{ small}).$$

Note: doubling x reduces the e^{-x} factor by an order of magnitude.

The expansion method is unclear, so try the iteration scheme. Consider a rearrangement that emphasises the e^{-x} factor:

$$e^x = \frac{x}{\varepsilon}$$

so try

$$x_{n+1} = \log\frac{1}{\varepsilon} + \log x_n \ .$$

Then

$$\begin{aligned}
x_0 &= \log\frac{1}{\varepsilon} \\
x_1 &= \underbrace{\log\frac{1}{\varepsilon}}_{L_1} + \underbrace{\log\log\frac{1}{\varepsilon}}_{L_2} \\
x_2 &= L_1 + \log(L_1 + L_2) \\
&= L_1 + L_2 + \frac{L_2}{L_1} - \frac{L_2^2}{2L_1^2} + \frac{L_2^3}{3L_1^3} + \cdots \\
x_3 &= L_1 + \log\left(L_1 + L_2 + \frac{L_2}{L_1} - \frac{L_2^2}{2L_1^2} + \frac{L_2^3}{3L_1^3} + \cdots\right) \\
&= L_1 + L_2 + \frac{L_2}{L_1} + \frac{-\frac{1}{2}L_2^2 + L_2}{L_1^2} + \frac{\frac{1}{3}L_2^3 - \frac{3}{2}L_2^2}{L_1^3} + \cdots
\end{aligned}$$

The iterative method can give more than one term per iteration.

Numerical disaster. Percentage errors for the truncated series:

ε	L_1	L_2	L_2/L_1	$-L_2^2/2L_1^2$	L_2/L_1^2
10^{-1}	36%	12%	2%	4%	0.03%
10^{-3}	24%	3%	0.02%	0.04%	0.04%
10^{-5}	19%	1%	0.04%	0.1%	0.001%

Do not separate terms like $-L_2^2/2L_1^2$ & L_2/L_1^2.

A very small ε is needed before this is tolerably accurate.

Check convergence.

$$\begin{aligned}
x_{n+1} &= g(x_n) \\
g(x) &= \log\frac{1}{\varepsilon} + \log x \\
g'(x) &= \frac{1}{x} \\
g'(x^*) &\approx \frac{1}{\log\frac{1}{\varepsilon}}
\end{aligned}$$

$\uparrow$ need ε very small for $|g'| \ll 1$.

01/19
01/20
(long)

2 Asymptotic Approximations

2.1 Convergence and Asymptoticness

An expansion $\sum_{n=0}^{\infty} f_n(z)$ converges for a fixed z if, given $\varepsilon > 0$, $\exists\ N(z, \varepsilon)$ s.t.

$$\left| \sum_{\ell}^{m} f_n(z) \right| < \varepsilon \qquad \forall\ \ell, m > N\ .$$

Convergent series can be useful analytically, but hopeless in practice. For instance, consider

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt\ .$$

We know that

$$e^{-t^2} = \sum_{0}^{\infty} \frac{\left(-t^2\right)^n}{n!}$$

is analytic in the entire complex plane. Hence we have uniform convergence on any bounded part of the plane $\Rightarrow$ we can integrate term by term:

$$\operatorname{erf}(z) \;=\; \underbrace{\tfrac{2}{\sqrt{\pi}} \textstyle\sum_0^\infty \tfrac{(-)^n z^{2n+1}}{(2n+1)n!}}_{\downarrow\text{also has } \infty \text{ radius of convergence}}\ .$$

To obtain an accuracy of 10^{-5} we need

8 terms up to $z = 1$
16 terms up to $z = 2$
31 terms up to $z = 3$
75 terms up to $z = 5$

However, intermediate terms can be large $\Rightarrow$ problems due to round-off error on computers.

An alternative for large z is to proceed as follows. First rewrite the integral:

$$\operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt\ .$$

Then repeatedly integrate by parts:

$$\begin{aligned}
\int_z^\infty e^{-t^2}\, dt &= \int_z^\infty \left(-\frac{1}{2t}\right) d\left(e^{-t^2}\right) \\
&= \frac{e^{-z^2}}{2z} - \int_z^\infty \frac{1}{2t^2} e^{-t^2}\, dt \\
&\;\;\vdots \\
&= \left(1 - \frac{1}{2z^2} + \frac{1.3}{\left(2z^2\right)^2} - \frac{1.3.5}{\left(2z^2\right)^3}\right) \frac{e^{-z^2}}{2z} + R_5
\end{aligned}$$

where

$$\begin{aligned}
R_5 &= \int_z^\infty \frac{105}{16} \frac{e^{-t^2}}{t^8}\, dt = \int_z^\infty \frac{105}{32\, t^9}\, d\left(-e^{-t^2}\right) \\
&\leqslant \frac{105}{32 z^9} \int_z^\infty d\left(-e^{-t^2}\right) = \frac{105}{32} \frac{e^{-z^2}}{z^9}\ .
\end{aligned}$$

The series in z^{-1} is divergent (due to the odd factorial in the numerator), but the truncated series is useful, e.g. 10^{-5} accuracy with 3 terms for $z = 2.5$
2 terms for $z = 3$.

"First term is essentially the answer, while subsequent terms are minor corrections."

Problem: What if the leading term is not sufficiently accurate (e.g. in reality ε is not sufficiently small)? Adding a few extra terms *may* help, but there is a limit to the number of useful extra terms if the series diverges as $N \to \infty$ at fixed ε. It is not sensible to include extra terms once they stop decreasing in magnitude. By suitable truncation, one can obtain exponential accuracy (see §3.1 and the first example
01/04 sheet).

2.2 Definitions

The expansion $\sum_0^N f_n(\varepsilon)$ is an *asymptotic approximation* of $f(\varepsilon)$ as $\varepsilon \to 0$, if $\forall\, m \leqslant N$,

$$\frac{\sum_0^m f_n(\varepsilon) - f(\varepsilon)}{f_m(\varepsilon)} \to 0 \qquad \text{as } \varepsilon \to 0$$

i.e. the remainder is less than the last included term.

If we can let $N \to \infty$ (in principle) then we have an *asymptotic expansion.*

If $f_n = a_n \varepsilon^n$, then we have an *asymptotic power series*; however we frequently need more general expansions involving terms like ε^α, $\left(\ln \frac{1}{\varepsilon}\right)^{-1}$, etc. We write these as

$$\sum_{n=0}^{N} a_n \delta_n(\varepsilon) \tag{2.1}$$

where the δ_n form an asymptotic sequence:

$$\frac{\delta_{n+1}}{\delta_n} \to 0 \qquad \text{as } \varepsilon \to 0\ .$$

Note that sometimes we need to restrict to one sector of the complex ε plane to keep the δ_n single valued.

Often ε is real and positive. A useful set of asymptotic functions are then Hardy's logarithm–exponential functions obtained by a finite number of $+, -, *, /, \exp$ & $\log$ operations, with all intermediate quantities real.

This class has the property that it can be ordered, i.e. either $f(\varepsilon) = \mathrm{o}\,(g(\varepsilon))$, or $g(\varepsilon) = \mathrm{o}\,(f(\varepsilon))$ or
02/01 $f(\varepsilon) = \mathrm{ord}\,(g(\varepsilon))$.

2.3 Uniqueness and Manipulation

If f can be expanded asymptotically for a given asymptotic sequence, then the expansion is unique. For if the expansion exists it has the form

$$f(\varepsilon) \sim \sum_n a_n \delta_n(\varepsilon)\ ,$$

then by construction

$$\begin{aligned} a_0 &= \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\delta_0(\varepsilon)} \\ a_n &= \lim_{\varepsilon \to 0} \left\{ \frac{f(\varepsilon) - \sum_0^{n-1} a_m \delta_m}{\delta_n} \right\}\ . \end{aligned}$$

However, a single function can have different asymptotic expansions for different sequences:

$$\begin{aligned}
\tan(\varepsilon) &\sim \varepsilon + \frac{1}{3}\varepsilon^3 + \frac{2}{15}\varepsilon^5 + \cdots \\
&\sim \sin\varepsilon + \frac{1}{2}(\sin\varepsilon)^3 + \frac{3}{8}(\sin\varepsilon)^5 + \cdots \\
&\sim \varepsilon\cosh\sqrt{\frac{2}{3}}\varepsilon + \frac{31}{270}\left(\varepsilon\cosh\sqrt{\frac{2}{3}}\varepsilon\right)^5 + \cdots .
\end{aligned}$$

Part of the 'art' of obtaining an effective asymptotic solution is choosing the most appropriate asymptotic sequence.

Worse: two functions can have the same asymptotic expansion:

$$\begin{aligned}
\exp\varepsilon &\sim \sum_0^\infty \frac{\varepsilon^n}{n!} \qquad \text{as } \varepsilon \to 0 \\
\exp\varepsilon + \exp\left(-\frac{1}{\varepsilon}\right) &\sim \sum_0^\infty \frac{\varepsilon^n}{n!} \qquad \text{as } \varepsilon \searrow 0 .
\end{aligned}$$

Exercise. Does $f = x^2 + e^{-x^2(1-\sin x)}$ have an asymptotic expansion as $x \to \infty$?

- Asymptotic expansions can be added, multiplied and divided to produce asymptotic expansions for the sum, product and quotient (if necessary one may need to enlarge the asymptotic sequence).
- If appropriate, one can try to substitute an asymptotic expansion into another – but care is needed, e.g. if

$$f(z) = e^{z^2}, \qquad z(\varepsilon) = \frac{1}{\varepsilon} + \varepsilon$$

then

$$\begin{aligned}
f(z(\varepsilon)) &= \exp\left[\frac{1}{\varepsilon^2} + 2 + \varepsilon^2\right] \\
&\sim e^{1/\varepsilon^2} e^2 \left\{1 + \varepsilon^2 + \frac{\varepsilon^4}{2} + \cdots\right\},
\end{aligned}$$

but if we just work to leading order

$$\begin{aligned}
z &\sim \frac{1}{\varepsilon} \\
f(z) &\not\sim e^{1/\varepsilon^2}
\end{aligned}$$

$\uparrow$missing e^2

The leading-order approximation in z is inadequate for the leading-order approximation in $f(z)$.

- Integration w.r.t. ε of asymptotic expansions is allowed term-by-term producing the correct result.
- Differentiation is not allowed in principle because $\mathcal{O}$ and o estimates do not survive differentiation. For instance:

(a)

$$\begin{aligned}
f &= e^{ix^2} = \mathcal{O}(1) \qquad \text{as } x \to \infty \\
\frac{df}{dx} &= 2ixe^{ix^2} = \mathcal{O}(x) \qquad \text{as } x \to \infty
\end{aligned}$$

(b)

$$\begin{aligned}
f &= 1 + e^{-1/x^2}\sin\left(e^{1/x^2}\right) \sim 1 + \cdots \qquad \text{as } x \to 0 \\
\frac{df}{dx} &= \underbrace{-\frac{2}{x^3}\cos\left(e^{1/x^2}\right)} + \frac{2}{x^3}e^{-1/x^2}\sin\left(e^{1/x^2}\right)
\end{aligned}$$

No asymptotic expansion as $x \to 0$.

(c)

01/03

$$f = t^2 + t\sin t \sim t^2\,, \quad f' = (2+\cos t)t + \sin t \not\sim 2t \quad \text{as} \quad t \to \infty.$$

However:

(i) If $f'(x)$ exists and is integrable, and $f(x) \sim \sum_{n=0}^{N} a_n x^n$ as $x \to 0$, then

$$f' \sim \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{as } x \to 0.$$

(ii) If $f(z)$ is analytic in $\theta_1 \leqslant \arg z \leqslant \theta_2$, $0 < |z| < R$ and

$$f \quad \sim \quad \sum_{n=0}^{\infty} a_n z^n \qquad \text{as } z \to 0 \ (\theta_1 \leqslant \arg z \leqslant \theta_2)$$

then

$$f' \quad \sim \quad \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \text{as } z \to 0 \ (\theta_1 \leqslant \arg z \leqslant \theta_2).$$

(iii) There are lots more special cases. For instance, consider asymptotic expansions of solutions to differential equations.

Suppose that y is the solution to

$$y'' + qy = 0 \tag{2.2}$$

where q has an asymptotic expansion as $x \to 0$.

Assume y has an asymptotic expansion as $x \to 0$;

then from (2.2) y'' has an asymptotic expansion (multiplication OK)
thus y' has an asymptotic expansion (integration OK)
thus y has an asymptotic expansion (integration OK) .

Hence if y has an asymptotic expansion, the equation ensures that its differentials have asymptotic expansions (the *proof* that y has an asymptotic expansion in the first place is often tricky).

2.4 Parametric Expansions

For functions of two (or more) variables, e.g. $f(x, \varepsilon)$ (as might arise in solutions to pdes, etc.), we make the obvious generalisation of (2.1) to allow the a_n to be functions of x:

$$f(x,\varepsilon) \sim \sum_{n=0}^{N} a_n(x)\delta_n(\varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{2.3}$$

If the approximation is asymptotic as $\varepsilon \to 0$ for each x, then it is called a Poincaré, or classical, asymptotic approximation.

The above pointwise asymptoticness may not be uniform in x, e.g. it may require $\varepsilon < x$ (restrictive as $x \to 0$). Such problems sometimes need a further extension:

$$f(x,\varepsilon) \sim \sum_n a_n(x,\varepsilon)\delta_n(\varepsilon) \tag{2.4}$$
$$\text{e.g. } a_n(x,\varepsilon) = b_n\left(\tfrac{x}{\varepsilon}\right).$$

Uniqueness extends to (2.3), but not to (2.4), etc.

3 Integral Methods

3.1 Elementary Examples

Example 1. Rewrite an integral so that we can use a Taylor series. For instance:

$$I = \int_x^\infty e^{-t^4}\,dt \qquad \text{as} \qquad x \to 0\ .$$

Then

$$\begin{aligned} I &= \int_0^\infty e^{-t^4}\,dt - \int_0^x e^{-t^4}\,dt \\ &= \Gamma(5/4) - \int_0^x \sum_{n=0}^\infty \frac{(-t^4)^n}{n!}\,dt \\ &= \Gamma(5/4) - \sum_{n=0}^\infty \frac{(-)^n x^{4n+1}}{(4n+1)n!}\ . \end{aligned}$$

Example 2. Use a Taylor series even when we cannot! For instance:

$$I = \int_0^\infty \frac{e^{-t}}{x+t}\,dt \qquad \text{as} \qquad x \to \infty\ .$$

Then

$$\begin{aligned} I &= \frac{1}{x}\int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{-1} dt \\ &= \frac{1}{x}\int_0^\infty e^{-t}\left(1-\frac{t}{x}+\frac{t^2}{x^2}-\frac{t^3}{x^3}+\dots\right) dt \\ &= \frac{1}{x}\left(1-\frac{1!}{x}+\frac{2!}{x^2}-\frac{3!}{x^3}+\dots\right)\ . \qquad \uparrow\text{dubious, since invalid for } t > x. \end{aligned}$$

$\uparrow$Divergent

Estimate the remainder using

$$1-\frac{t}{x}+\frac{t^2}{x^2}+\dots+\left(-\frac{t}{x}\right)^{m-1} = \frac{1-\left(-\frac{t}{x}\right)^m}{1+\frac{t}{x}}\ .$$

Then

$$I = \frac{1}{x}\sum_{n=0}^{m-1}\int_0^\infty \left(-\frac{t}{x}\right)^n e^{-t}\,dt + R_m(x)\ ,$$

where

$$R_m(x) = \frac{1}{x^{m+1}}\int_0^\infty \frac{(-t)^m e^{-t}}{(1+\frac{t}{x})}\,dt\ ,$$

and

$$|R_m(x)| \leqslant \frac{1}{|x^{m+1}|}\int_0^\infty t^m e^{-t}\,dt = \frac{m!}{x^{m+1}}\ .$$

Hence

02/22
$$I = \frac{1}{x}\left(1-\frac{1}{x}+\frac{2!}{x^2}+\dots+\frac{m!}{(-x)^m}+\mathcal{O}\left(\frac{(m+1)!}{x^{m+1}}\right)\right)$$

Truncate the series when the remainder has the smallest bound, i.e. stop one before smallest term when $x \sim m$. The error when we truncate is then (after using Stirling's formula)

03/01
02/04
02/19
$$|R_m| \sim \frac{x!}{x^{x+1}} \sim \frac{(2\pi)^{1/2}e^{-x}}{x^{1/2}}\ ,$$

02/20 i.e. the error is exponentially small for large x (so the 'dubious' step wasn't too bad).

3.2 Integration by Parts

Integrals of the form $\int f(t)g(t)\,dt$ can be integrated by parts and **may** so yield asymptotic expansions; one automatically obtains the remainder.

Example 1. See §2.1 for $\operatorname{erf}(z)$.

Example 2. Consider the *exponential integral*

$$E_1(x) \equiv \int_x^\infty \frac{e^{-t}}{t}\,dt = e^{-x}\int_0^\infty \frac{e^{-t}\,dt}{x+t}\;.$$

Then integrating by parts

$$\begin{aligned} E_1(x) &= \left[-\frac{e^{-t}}{t}\right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2}\,dt \\ &= \frac{e^{-x}}{x}\left(1-\frac{1}{x}+\frac{2!}{x^2}+\ldots+\frac{m!}{(-x)^m}\right)+R_m(x)\;, \end{aligned}$$

where

$$R_m(x) = (-)^{m+1}(m+1)!\int_x^\infty \frac{e^{-t}}{t^{m+2}}\,dt\,.$$

Hence

$$|R_m(x)| \leqslant \frac{(m+1)!e^{-x}}{x^{m+2}}\,,$$

and as in §3.1, the remainder is asymptotically smaller than the retained terms on truncation with $m \sim x$.

Example 3. The sine and cosine integrals.

$$\begin{aligned} -\operatorname{Ci}(x) - i\operatorname{si}(x) &= -\operatorname{Ci}(x) + i\left(\frac{\pi}{2} - \operatorname{Si}(x)\right) \equiv \int_x^\infty \frac{e^{it}\,dt}{t} \\ &= -\frac{e^{ix}}{ix}\left(1+\frac{1}{ix}+\frac{2!}{(ix)^2}+\ldots+\frac{m!}{(ix)^m}\right)+R_m(x)\;, \end{aligned}$$

where

$$R_m(x) = i(m+1)!\int_x^\infty \frac{e^{it}\,dt}{(it)^{m+2}}\;.$$

If we proceed to estimate the remainder as before

$$|R_m| \leqslant (m+1)!\int_x^\infty \frac{dt}{t^{m+2}} = \frac{m!}{x^{m+1}} = \mathcal{O}(\text{last term})\;,$$

so this does not demonstrate asymptoticness. We seek an improved error estimate by integrating by parts:

$$R_m = \left[\frac{(m+1)!\,e^{it}}{(it)^{m+2}}\right]_x^\infty + i(m+2)!\int_x^\infty \frac{e^{it}\,dt}{(it)^{m+3}}\;,$$

and then we can demonstrate that the remainder is asymptotically smaller than the retained terms:

$$|R_m| \leqslant \frac{(m+1)!}{x^{m+2}} + \frac{(m+1)!}{x^{m+2}} = \mathcal{O}\left(\frac{1}{x^{m+2}}\right)\;.$$

3.3 Integrals with Algebraic Parameter Dependence

Example 1. Consider the integral

$$I(\varepsilon) = \int_0^1 \frac{1}{(x+\varepsilon)^{\frac{1}{2}}}\,dx = 2\left(\sqrt{1+\varepsilon}-\sqrt{\varepsilon}\right).$$

The leading-order ($\varepsilon \to 0$) estimate is just

$$I(0) = \underbrace{\int_0^1 \frac{1}{x^{\frac{1}{2}}}\,dx}_{\substack{\text{global contribution from}\\ \text{all of integration range}}} = 2\ .$$

In order to obtain an improved estimate one *cannot* expand $(1+\varepsilon/x)^{-1/2}$ throughout the range as

$$(1+\varepsilon/x)^{-1/2} = 1 - \varepsilon/2x + \dots\ ,$$

since for $0 \leqslant x \ll \varepsilon$ the expansion is not convergent.[1] Further, we note that when $x = \operatorname{ord}(\varepsilon)$, the integrand is $\operatorname{ord}\left(\varepsilon^{-1/2}\right) \Rightarrow$ contribution to the integral for this range of x will be $\operatorname{ord}\left(\varepsilon^{-1/2}\cdot\varepsilon\right)$, i.e. $\operatorname{ord}\left(\varepsilon^{1/2}\right)$.

To account for this correction, one could subtract the leading-order estimate exactly; then

$$I = 2 + \underbrace{\int_0^1 \left[\frac{1}{(x+\varepsilon)^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}}\right] dx}_{\substack{x=\operatorname{ord}(\varepsilon),\ \text{integrand } = \operatorname{ord}\left(\varepsilon^{-1/2}\right),\ \text{contribution to } \int = \operatorname{ord}\left(\varepsilon^{1/2}\right)\\ x=\operatorname{ord}(1),\ \text{integrand } = \operatorname{ord}(\varepsilon),\ \text{contribution to } \int = \operatorname{ord}(\varepsilon)}}$$

The major contribution is from near $x = 0$ so, as in §0, try the scaling $x = \varepsilon\xi$ ($\xi = \operatorname{ord}(1)$); then

$$\begin{aligned} I &= 2 + \varepsilon^{\frac{1}{2}} \int_0^{\frac{1}{\varepsilon}\approx\infty} \left[\frac{1}{(1+\xi)^{\frac{1}{2}}} - \frac{1}{\xi^{\frac{1}{2}}}\right] d\xi \\ &\approx 2 - 2\varepsilon^{\frac{1}{2}} \end{aligned}$$

Further corrections can be obtained by now subtracting out this contribution, but this method is tedious
02/03 and difficult! There must be a better way.

Alternative 1: Solve a differential equation. Let

$$J(x) = \int_0^x \frac{1}{(q+\varepsilon)^{\frac{1}{2}}}\,dq\ .$$

Then we need to find $J(1)$. This can be done by solving the differential equation

$$\frac{dJ}{dx} = \frac{1}{(x+\varepsilon)^{\frac{1}{2}}}$$

subject to the initial condition $J(0) = 0$. We will discover how to do this in §5.

Alternative 2: Divide & Conquer. In this method we split the range of integration. Split $[0,1]$ at $x = \delta$ where $\varepsilon \ll \delta \ll 1$, and then use Taylor series when we can use Taylor series:

$$\begin{aligned} I &= \int_0^\delta \frac{dx}{(x+\varepsilon)^{\frac{1}{2}}} + \int_\delta^1 \frac{dx}{(x+\varepsilon)^{\frac{1}{2}}} \\ &= \varepsilon^{\frac{1}{2}} \int_0^{\delta/\varepsilon} \frac{d\xi}{(1+\xi)^{\frac{1}{2}}} + \int_\delta^1 \frac{1}{x^{\frac{1}{2}}}\left(1 - \frac{\varepsilon}{2x} + \frac{3\varepsilon^2}{8x^2} + \dots\right) dx \\ &= 2\varepsilon^{\frac{1}{2}}\left(\left(\frac{\delta}{\varepsilon}+1\right)^{\frac{1}{2}} - 1\right) + \left(2 - 2\delta^{\frac{1}{2}}\right) + \left(\varepsilon - \frac{\varepsilon}{\delta^{\frac{1}{2}}}\right) + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^{\frac{3}{2}}}, \varepsilon^2\right) \\ &= 2\delta^{\frac{1}{2}} + \frac{\varepsilon}{\delta^{\frac{1}{2}}} - 2\varepsilon^{\frac{1}{2}} + 2 - 2\delta^{\frac{1}{2}} + \varepsilon - \frac{\varepsilon}{\delta^{\frac{1}{2}}} + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^{\frac{3}{2}}}, \varepsilon^2\right) \\ &= 2 - 2\varepsilon^{\frac{1}{2}} + \varepsilon + \mathcal{O}\left(\frac{\varepsilon^2}{\delta^{\frac{3}{2}}}, \varepsilon^2\right)\ . \end{aligned}$$

[1] And in this case there is no exponentially small multiplier.

Remarks.

- Since δ is arbitrary, all terms containing a δ must cancel.
- The error term is definitely small if $\varepsilon^{\frac{2}{3}} \ll \delta \ll 1$.
- To organise the algebra it is sometimes helpful to tie δ to ε, e.g.

$$\delta = K\varepsilon^{\frac{3}{4}} \ ,$$

and then the answer must be independent of K.

Example 2. Suppose that we wish to estimate the integral

$$I(m,\varepsilon) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(1-m^2\cos^2\theta\right)^2\sin^2\theta+\varepsilon^2}\,d\theta \qquad 0<m<\infty\ ,$$

for $0<\varepsilon\ll 1$. It turns out that there are three cases to consider: $0<m<1$; $|m-1|\ll 1$; $m>1$.

(a) $\quad 0<m<1$

θ	integrand	contribution to $\int$
ord(1)	ord(1)	ord(1)
ord(ε)	ord(1)	ord(ε)

$\uparrow \left(1-m^2\cos^2\theta\right)^2\sin^2\theta \sim \varepsilon^2$

We will find the solution correct to $\mathcal{O}\left(\varepsilon^2\right)$; to this end let $0<\varepsilon\ll\delta\ll 1$. Then

$$\begin{aligned}
I &= \varepsilon\int_0^{\frac{\delta}{\varepsilon}} \frac{\sin^2(\varepsilon u)}{\left(1-m^2\cos^2(\varepsilon u)\right)^2\sin^2(\varepsilon u)+\varepsilon^2}\,du + \int_\delta^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(1-m^2\cos^2\theta\right)^2\sin^2\theta+\varepsilon^2}\,d\theta \\
&= \varepsilon\int_0^{\frac{\delta}{\varepsilon}} \frac{u^2\,du}{\left(1-m^2\right)^2u^2+1} + \int_\delta^{\frac{\pi}{2}} \frac{1}{\left(1-m^2\cos^2\theta\right)^2}\,d\theta + \mathcal{O}\left(\varepsilon^2\right) \\
&= \varepsilon\left[\frac{(1-m^2)u-\tan^{-1}\left((1-m^2)u\right)}{(1-m^2)^3}\right]_0^{\frac{\delta}{\epsilon}} + \frac{(2-m^2)\pi}{4(1-m^2)^{\frac{3}{2}}} - \int_0^\delta \frac{d\theta}{\left(1-m^2\cos^2\theta\right)^2} + \mathcal{O}\left(\varepsilon^2\right) \\
&\qquad\qquad \text{via a } \tan\theta = t = (1-m^2)^{\frac{1}{2}}\tan\psi \text{ substitution} \\
&= \frac{\delta}{(1-m^2)^2} - \frac{\varepsilon\pi}{2(1-m^2)^3} + \frac{(2-m^2)\pi}{4(1-m^2)^{\frac{3}{2}}} - \frac{\delta}{(1-m^2)^2} + \mathcal{O}\left(\varepsilon^2,\delta^2,\frac{\varepsilon^2}{\delta}\right) \\
&\qquad\qquad \text{since } \arctan\left(\tfrac{1}{\Delta}\right) \sim \tfrac{\pi}{2}-\Delta
\end{aligned}$$

$$I = \underbrace{\frac{(2-m^2)\pi}{4(1-m^2)^{\frac{3}{2}}}}_{\text{global}} - \underbrace{\frac{\varepsilon\pi}{2(1-m^2)^3}}_{\text{local}} + \dots \tag{3.1}$$

04/01
03/22

Note that this is a non-uniform approximation as $m\to 1$. There is a loss of ordering of the series solution when

$$\frac{1}{(1-m^2)^{\frac{3}{2}}} \sim \frac{\varepsilon}{(1-m^2)^3}$$

i.e. when

$$(1-m^2)\sim\varepsilon^{\frac{2}{3}} \quad\text{and}\quad I\sim\frac{1}{\varepsilon}\,.$$

(b) This suggests that when $|m-1| \ll 1$, we should introduce a scaled parameter: viz.

$$m = 1 - \varepsilon^{\frac{2}{3}}\lambda \,. \tag{3.2a}$$

First let us examine the local contribution from near $\theta = 0$ (since on the basis of the estimates above it will be leading order). Put $\theta = \varepsilon^{\beta} u$, then

$$\left(1 - m^2\cos^2\theta\right)^2 \sin^2\theta + \varepsilon^2 = \left(\varepsilon^{2\beta}u^2 + 2\varepsilon^{\frac{2}{3}}\lambda\right)^2 \varepsilon^{2\beta}u^2 + \varepsilon^2 + \dots$$

All leading order terms balance if $\beta = \frac{1}{3}$, i.e.

$$\theta = \varepsilon^{\frac{1}{3}} u \,. \tag{3.2b}$$

This is referred to as a *distinguished scaling*.

As a first guess, let us assume that this is the scaling in θ to consider. Then

$$\begin{aligned} &\theta = \mathrm{ord}(\varepsilon^{\frac{1}{3}}); \quad \text{integrand } = \mathrm{ord}\left(\varepsilon^{\frac{2}{3}}/\varepsilon^2\right); \quad \text{contribution to } \textstyle\int = \mathrm{ord}\,(1/\varepsilon) \\ &\theta = \mathrm{ord}(1) \;\; ; \quad \text{integrand } = \mathrm{ord}\,(1) \qquad ; \quad \text{contribution to } \textstyle\int = \mathrm{ord}\,(1) \end{aligned}$$

The 'local' contribution dominates. Hence introduce $\varepsilon^{\frac{1}{3}} \ll \delta \ll 1$, and split the integral:

$$\begin{aligned} I &= \int_0^{\delta} \dots \, d\theta + \int_{\delta}^{\frac{\pi}{2}} \dots \, d\theta \\ &\sim \frac{1}{\varepsilon}\int_0^{\delta\varepsilon^{-\frac{1}{3}}} \frac{u^2\,du}{\left(u^2+2\lambda\right)^2 u^2 + 1} \sim \frac{1}{\varepsilon} f(\lambda) \end{aligned}$$

where

$$f(\lambda) = \int_0^{\infty} \frac{u^2\,du}{\left(u^2+2\lambda\right)^2 u^2 + 1} \,.$$

03/19

Hence for a given λ (or equivalently m), we have a leading order asymptotic estimate. However, we should check that as $\lambda \to \infty$, we obtain the same estimate as in (a). In particular, when $\lambda \gg 1$

$$\begin{aligned} &u = \mathrm{ord}(1) \;\;, \text{ integrand } = \mathrm{ord}\left(1/\lambda^2\right), \text{ contribution to } \textstyle\int = \mathrm{ord}\left(1/\lambda^2\right) \\ &u = \mathrm{ord}(\lambda^{\frac{1}{2}}), \text{ integrand } = \mathrm{ord}\left(1/\lambda^2\right), \text{ contribution to } \textstyle\int = \mathrm{ord}\left(1/\lambda^{\frac{3}{2}}\right) \end{aligned}$$

This suggests that the largest contribution will come from where $v = \lambda^{-\frac{1}{2}}u = \mathrm{ord}(1)$. Hence estimate f in this range:

$$f(\lambda) \sim \frac{1}{\lambda^{\frac{3}{2}}}\int_0^{\infty}\frac{dv}{\left(2+v^2\right)^2} = \frac{\pi}{4\,(2\lambda)^{\frac{3}{2}}} \,,$$

and

$$I \sim \frac{\pi}{4\varepsilon\,(2\lambda)^{\frac{3}{2}}} \sim \underbrace{\frac{\pi}{4\,(1-m^2)^{\frac{3}{2}}}}_{\downarrow\text{agrees with (3.1) for } m \approx 1} \tag{3.3}$$

03/20

We might also be interested in the other limit, i.e. $\lambda \to -\infty$. This estimate is a little more tricky, since $\left(u^2 + 2\lambda\right)$ can now have a zero (when $|\lambda| \gg 1$, this term normally dominates the denominator). First we test for a significant contribution from near this zero by introducing a scaled coordinate, say w:

$$u = (-2\lambda)^{\frac{1}{2}} + (-\lambda)^{\gamma} w \,.$$

Then

$$\begin{aligned} 1 + u^2\left(u^2+2\lambda\right)^2 &\sim 1 + (-2\lambda)\left(2\,(-2\lambda)^{\frac{1}{2}}\,(-\lambda)^{\gamma} w + \dots\right)^2 \\ &\sim 1 + 16\lambda^2(-\lambda)^{2\gamma}w^2 + \dots \,. \end{aligned}$$

There is a distinguished scaling (that ensures the scaled integral is convergent) for the choice $\gamma = -1$; in that case the contribution to the integral from near the zero can be estimated as follows:

$$u = (-2\lambda)^{\frac{1}{2}} + \mathrm{ord}\,(1/|\lambda|)\,;\ \text{integrand} = \mathrm{ord}\,(|\lambda|/1)\,;\ \text{contribution to } \int = \mathrm{ord}(1)\ .$$

This is a much larger contribution than we found in (3.3) for $\lambda \gg 1$.

In order to estimate the contribution set

$$u \quad = \quad (-2\lambda)^{\frac{1}{2}} + \frac{w}{(-\lambda)} \tag{3.4}$$

then

$$f(\lambda) \quad = \quad \int_{-2^{1/2}(-\lambda)^{3/2}\approx-\infty}^{\infty} \frac{(-2\lambda + \dots)\,dw}{(-\lambda)\,[1+16w^2\dots]} \sim \frac{\pi}{2}\ .$$

Hence as $\lambda \to -\infty$, the value of the integral tends to a large constant, viz.

$$I \sim \frac{\pi}{2\varepsilon}\ . \tag{3.5}$$

(c) Finally consider the case when $m > 1$.

The limit $\lambda \to -\infty$ (i.e. $0 < (m-1) \ll 1$) suggests that the main contribution will be local, and will come from the region close to the point where

$$m^2 \cos^2\theta = 1\ .$$

Define

$$\theta_m = \cos^{-1}\left(\frac{1}{m}\right) \qquad\qquad \left(0 < \theta_m < \frac{\pi}{2}\right)\ .$$

In order to deduce the coordinate scaling that is appropriate close to θ_m, we note from (3.2a) and (3.4) that the 'inner' scaling for $0 < m-1 \ll 1$ can be written in the form

$$\theta = \varepsilon^{\frac{1}{3}} u = \varepsilon^{\frac{1}{3}}\left((-2\lambda)^{\frac{1}{2}} + \frac{w}{(-\lambda)}\right) = (2(m-1))^{\frac{1}{2}} + \frac{\varepsilon w}{(m-1)} \sim \theta_m + \frac{2\varepsilon w}{\theta_m^2}\ .$$

This suggests that for $(m-1) = \mathcal{O}(1)$ we might guess the scaling

$$\theta \quad = \quad \theta_m + \varepsilon t\ ,$$

in which case

$$\left(1 - m^2\cos^2\theta\right)^2 \sin^2\theta + \varepsilon^2 \quad \sim \quad 4\varepsilon^2 m^2 \sin^4\theta_m t^2 + \dots + \varepsilon^2$$

and

$$\begin{aligned} I \quad &\sim \quad \int_{-\frac{1}{\varepsilon}\theta_m\approx-\infty}^{\frac{1}{\varepsilon}\left(\frac{\pi}{2}-\theta_m\right)\approx+\infty} \frac{\varepsilon\sin^2(\theta_m+\varepsilon t)\ dt}{\varepsilon^2\left(4m^2t^2\sin^4\theta_m + 1\right) + \dots} \\ &\sim \quad \frac{1}{\varepsilon}\cdot\frac{\pi}{2m} \end{aligned} \tag{3.6}$$

05/01
03/04
We note that (3.6) agrees with (3.5) in the limit $m \to 1$.

3.4 Logarithms

As an illustrative example, consider integrals of the form

$$\int_0^a f(x,\varepsilon)\,dx \quad \text{with} \quad f(x,\varepsilon) = \begin{cases} \mathrm{ord}\,(\varepsilon^{-\alpha}) & x = \mathrm{ord}(\varepsilon) \\ x^{-\alpha} & \varepsilon \ll x \ll 1 \\ \mathrm{ord}(1) & x = \mathrm{ord}(1). \end{cases}$$

e.g.

$$f = \frac{1}{(x+\varepsilon)^\alpha}\frac{1}{1+x}\ .$$

There are three possibilities for the leading-order contribution depending on the value of α:

(i) $\alpha < 1$. Dominant contribution from $x = \text{ord}(1)$, e.g. with $\alpha = \frac{1}{2}$:

$$\int_0^\infty \frac{dx}{(x+\varepsilon)^{\frac{1}{2}}(1+x)} \sim \int_0^\infty \frac{dx}{x^{\frac{1}{2}}(1+x)} \ .$$

(ii) $\alpha > 1$. Dominant contribution from $x = \text{ord}(\varepsilon)$, e.g. with $\alpha = \frac{3}{2}$:

$$\int_0^\infty \frac{dx}{(x+\varepsilon)^{\frac{3}{2}}(1+x)} \sim \int_0^\infty \frac{d\xi}{\varepsilon^{\frac{1}{2}}(1+\xi)^{\frac{3}{2}}} \qquad (x = \varepsilon\xi) \ .$$

(iii) $\alpha = 1$. Dominant contribution *not* from $x = \text{ord}(\varepsilon)$ or $x = \text{ord}(1)$ but from the *intermediate* region between. Easiest to see by using *divide and conquer*, and splitting the integration region, e.g. with $\varepsilon \ll \delta \ll 1$:

$$\begin{aligned}
\int_0^\infty \frac{dx}{(x+\varepsilon)(1+x)} &= \int_0^{\frac{\delta}{\varepsilon}} \frac{d\xi}{(1+\xi)(1+\varepsilon\xi)} + \int_\delta^\infty \frac{dx}{(x+\varepsilon)(1+x)} \\
&= \Big[\log(1+\xi) - \varepsilon\left[\xi - \log(1+\xi)\right] + \ldots\Big]_0^{\frac{\delta}{\varepsilon}} \\
&\qquad + \left[\log\left(\frac{x}{x+1}\right) + \frac{\varepsilon}{x} - \varepsilon\log\left(\frac{x+1}{x}\right) + \ldots\right]_\delta^\infty \\
&\sim (1+\varepsilon)\left(\log\delta - \log\varepsilon\right) + \frac{\varepsilon}{\delta} + \ldots \\
&\qquad - \log\delta - \frac{\varepsilon}{\delta} - \varepsilon\log\delta + \ldots \\
&\sim (1+\varepsilon)\,\log\left(\frac{1}{\varepsilon}\right) + \underset{\substack{\uparrow \\ \text{'fortunate' ord(1) cancellation}}}{\ldots} \ .
\end{aligned}$$

03/16

3.5 Integrals with Exponential Power Dependence

General case: limit as $\lambda \to \infty$ of integrals of type

$$I(\lambda) = \underset{\text{paths in } \mathbb{C}}{\int_a^b} e^{\lambda\phi(z;\lambda)} \underset{\substack{\uparrow \\ \text{'weak' algebraic} \\ \text{dependence on } \lambda}}{f(z;\lambda)}\, dz \ .$$

Initially assume a, b, λ, ϕ, f, and the path of the integral are real. Then we estimate the integral by assuming that the major contribution comes from close to the point where ϕ is largest (and the integrand is exponentially largest).

There are different cases to consider depending on whether the maximum of ϕ is at an end point (Watson's Lemma), or in the interior of the integration range (Laplace's Method).

3.5.1 Watson's Lemma

In this section we assume the maximum is at an end point, say wlog $z = a$. We also assume that ϕ is monotonic decreasing function of z (so $\phi' < 0$). Write

$$x = \phi(a;\lambda) - \phi(z;\lambda) \quad , \quad F(x;\lambda) = -\frac{f(z;\lambda)}{\phi'(z;\lambda)} e^{\lambda\phi(a;\lambda)} \quad , \quad c = \phi(a;\lambda) - \phi(b;\lambda) > 0 \ ,$$

then

$$I(\lambda) = \int_0^c e^{-\lambda x} F(x;\lambda)\, dx \ .$$

Assume that F is analytic in some sector S of the complex plane, and that as $x \to 0$,

$$F(x;\lambda) \sim \sum_{k=0}^{N} a_k x^{\alpha_k} \qquad -1 < \alpha_0 < \alpha_1 < \ldots \ . \tag{3.7a}$$

Also assume that c is in S, that F is bounded in S, and that, for simplicity, $F(x;\lambda) \equiv F(x)$; the changes when this is not the case are straightforward, but somewhat messy, since the a_k are now functions of λ and themselves need to be expanded in λ when $\lambda \gg 1$. Then

$$\int_0^c e^{-\lambda x} F(x)\, dx \sim \sum_{k=0}^{N} a_k \frac{\Gamma(\alpha_k+1)}{\lambda^{\alpha_k+1}} \,. \tag{3.7b}$$

Unlectured Proof. For a given $\varepsilon > 0$, $\exists\ \delta(\varepsilon)$ s.t.

$$\left| F(x) - \sum_{k=0}^{N} a_k x^{\alpha_k} \right| < \varepsilon \left| x^{\alpha_N} \right| \qquad \forall\ x \text{ in } S \text{ with } |x| < \delta. \tag{3.8}$$

Split the range of the integral at $\lambda^{-1} \ll \delta(\varepsilon) \ll 1$; then

$$I = \int_0^\delta e^{-\lambda x} F(x)\, dx + \int_\delta^c e^{-\lambda x} F(x)\, dx \ = I_1 + I_2 \,.$$

First note that I_2 is exponentially small as $\lambda \to \infty$:

$$I_2 = \int_\delta^c e^{-\lambda x} F(x)\, dx < F_{max} \frac{e^{-\lambda\delta}}{\lambda} \,, \quad \text{where } F_{max} = \max_{x\in S} |F(x)| \,.$$

Further, consider the difference between I_1 and the asymptotic series (3.7b), then from (3.8)

$$\begin{aligned}
\left| I_1 - \sum_{k=0}^{N} a_k \lambda^{-\alpha_k-1}\Gamma(\alpha_k+1) \right| &= \left| \int_0^\delta e^{-\lambda x} F(x)\, dx - \sum_{k=0}^{N} a_k \left(\int_0^\delta dx + \int_\delta^\infty dx \right) x^{\alpha_k} e^{-\lambda x} \right| \\
&\leqslant \left| \varepsilon \int_0^\delta e^{-\lambda x} \left| x \right|^{\alpha_N} \, dx \right| + \left| \sum_{k=0}^{N} a_k \int_\delta^\infty x^{\alpha_k} e^{-\lambda x}\, dx \right| \\
&\leqslant \varepsilon \left| \frac{\Gamma(\alpha_N+1)}{\lambda^{\alpha_N+1}} \right| + \left| e^{-(\lambda-1)\delta} \right| \int_\delta^\infty \sum_{k=0}^{N} \left| a_k x^{\alpha_k} \right| \left| e^{-x} \right| \, dx \,.
\end{aligned}$$

Hence as $\lambda \to \infty$,

$$\text{error} = \mathcal{O}\left(\frac{\varepsilon}{|\lambda^{\alpha_N+1}|}, \exp \right) \,.$$

This proves the result since ε can be arbitrarily small (and λ arbitrarily large).

This proof can be extended to the cases when

- $|F(x)| < K e^{mx}$ for $K, m > 0$;
- λ is complex (by deforming the integration contour so that $x\lambda$ is real).

03/03
04/19
04/22

<u>How to obtain a practical answer</u>

The introduction of the coordinate x is not always simple. If all that is required is a few leading-order terms, then it is possible to proceed as follows (for $\phi'(a) < 0$). Assume, for simplicity, $f(z;\lambda) \equiv f(z)$ and $\phi(z;\lambda) \equiv \phi(z)$, where the more general case generally leads to more Taylor expansions. Then, for $\lambda \gg 1$, expand close to $z = a$:

$$\begin{aligned}
I &= \int_a^b e^{\lambda\phi(z)} f(z)\, dz \\
&= \int_0^{\lambda^\beta(b-a)} f\left(a + \frac{t}{\lambda^\beta}\right) \exp\left(\lambda\phi\left(a + \frac{t}{\lambda^\beta}\right)\right) \frac{dt}{\lambda^\beta} \,, \quad \text{where } z = a + \frac{t}{\lambda^\beta} \text{ for some } \beta > 0 \\
&= \int_0^{\lambda^\beta(b-a)} \left[f(a) + \frac{t}{\lambda^\beta} f'(a) + \dots \right] \exp\left(\lambda\phi(a) + \frac{t}{\lambda^{\beta-1}}\phi'(a) + \frac{t^2}{2\lambda^{2\beta-1}}\phi''(a) + \dots \right) \frac{dt}{\lambda^\beta} \\
&= \int_0^{\lambda(b-a)\approx\infty} \left[f(a) + \frac{t}{\lambda} f'(a) + \dots \right] e^{\lambda\phi(a)} e^{t\phi'(a)} \left[1 + \frac{t^2}{2\lambda}\phi''(a) + \dots \right] \frac{dt}{\lambda} \,, \quad \text{choosing } \beta = 1 \\
&\approx \frac{e^{\lambda\phi(a)}}{\lambda} \left[-\frac{f(a)}{\phi'(a)} + \frac{1}{\lambda}\left(\frac{f'(a)}{[\phi'(a)]^2} - \frac{\phi''(a) f(a)}{[\phi'(a)]^3} \right) + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \right] + \exp \,.
\end{aligned}$$

Summary. This approach works since the major asymptotic contribution comes from near the maximum of ϕ, courtesy of the strong exponential decay of the integrand; the choice of β is made to pick out this
04/20 contribution.

3.5.2 Intermediate maximum (Laplace's method)

Again consider

$$I = \int_a^b e^{\lambda\phi(x)} f(x)\, dx\,,$$

where the generalisation to $\phi \equiv \phi(x;\lambda)$ and $f \equiv f(x;\lambda)$, for algebraic λ dependence, is more messy than conceptual. Suppose that

(a) $\max\limits_{x\in[\,a,b\,]} \phi = \phi(c)$;

(b) $a < c < b$, $\phi'(c) = 0, \quad \phi''(c) < 0$;

Similar to above, assume that the major contribution to the integral comes from when the integrand is close to maximal, and introduce a scaled co-ordinate of the form

$$x = c + \frac{t}{\lambda^\beta}.$$

Then expanding $\lambda\phi$ close to the maximum at $x = c$ we obtain

$$\begin{aligned}\lambda\phi(x) &\sim \lambda\phi(c) + (x-c)\lambda\phi'(c) + \tfrac{1}{2}(x-c)^2\lambda\phi''(c) + \tfrac{1}{6}(x-c)^3\lambda\phi'''(c) + \dots\\ &\sim \lambda\phi(c) + \tfrac{1}{2}t^2\lambda^{1-2\beta}\phi''(c) + \tfrac{1}{6}t^3\lambda^{1-3\beta}\phi'''(c) + \dots\end{aligned}$$

The choice $\beta = \frac{1}{2}$ ensures that the decay of the exponential occurs over an ord(1) scaled distance t.

It follows that

$$\begin{aligned}
I &= \frac{1}{\lambda^{\frac12}} \int_{(a-c)\lambda^{\frac12}}^{(b-c)\lambda^{\frac12}} f\left(c + \frac{t}{\lambda^{\frac12}}\right) \exp\left(\lambda\phi\left(c + \frac{t}{\lambda^{\frac12}}\right)\right) dt\\
&= \frac{1}{\lambda^{\frac12}} \int_{\lambda^{\frac12}(a-c)}^{\lambda^{\frac12}(b-c)} \left(f(c) + \frac{t}{\lambda^{\frac12}} f'(c) + \dots\right) \exp\left(\lambda\phi(c) + \frac{t^2}{2}\phi''(c) + \frac{t^3}{6\lambda^{\frac12}}\phi'''(c) + \dots\right) dt\\
&\approx \frac{1}{\lambda^{\frac12}} \int_{-\infty}^{\infty} f(c) e^{\lambda\phi(c)} e^{\frac12 t^2\phi''(c)} \left(1 + \mathcal{O}\left(\lambda^{-\frac12}\right)\right) dt + \exp\\
&\approx \left(\frac{2\pi}{-\lambda\phi''(c)}\right)^{\frac12} f(c) e^{\lambda\phi(c)} + \dots\ .
\end{aligned}$$

06/01

Example: Stirling's Formula. Consider

$$\Gamma(\lambda) = \int_0^\infty e^{-x} x^{\lambda-1}\, dx = \int_0^\infty \frac{e^{-x}}{x} e^{\lambda\log x}\, dx \qquad \text{as } \lambda\to\infty$$

$$\text{Then} \quad f(x) = \frac{e^{-x}}{x}, \quad \phi(x) = \log x$$

$$\max\phi(x) = \infty \quad \text{for } 0 < x < \infty.$$

The method seems invalid! Instead, use the 'generalisation'

$$\Gamma(\lambda) = \int_0^\infty \frac{1}{x} \exp\big(\underbrace{-x + \lambda\log x}\big)\, dx$$

$$\phi(x) = \log x - x/\lambda$$
$$\phi'(x) = 1/x - 1/\lambda\,, \quad \phi' = 0 \text{ at } x = \lambda\,.$$

Let $x = \lambda s$.

$$\begin{aligned}\Gamma(\lambda) &= \int_0^\infty \frac{ds}{s}\, \exp\bigl(-\lambda s + \lambda \log \lambda + \lambda \log s\bigr) \\ &= \lambda^\lambda \int_0^\infty \frac{ds}{s}\, \exp\bigl(-\lambda\,(s - \log s)\bigr)\end{aligned}$$

$$\begin{array}{lcl} f(s) = \frac{1}{s} & , & \phi(s) = \log s - s \\ \phi' = \frac{1}{s} - 1 & , & c = 1 \\ \phi'' = -\frac{1}{s^2} & , & \phi''(c) = -1 \end{array}$$

04/16

$$\Gamma(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{\frac{1}{2}} \lambda^\lambda e^{-\lambda} + \dots .$$

3.5.3 Stationary phase

Let $\phi(x) = i\psi(x)$, with $\psi(x)$ real. Consider

$$I(x) = \int_a^b f(x) e^{i\lambda\psi(x)}\, dx$$

Generalised Fourier Integral

Riemann-Lebesgue Lemma. If $\int_a^b |f(x)|\, dx$ exists, then

$$\int_a^b f(x) e^{i\lambda x}\, dx \to 0 \qquad \text{as} \qquad \lambda \to \infty\,.$$

Generalised Riemann-Lebesgue Lemma. If

(a) $|f(x)|$ is integrable;
(b) $\psi(x)$ is continuously differentiable ($\psi'(x) = 0$ is OK at isolated points);
(c) $\psi(x)$ is not constant on any sub-interval,

then

$$I(x) \to 0 \qquad \text{as} \qquad \lambda \to \infty\,.$$

$\psi' \neq 0$ on $[\,a, b\,]$. In this case integrate by parts (note problems if $\psi' = 0$):

$$\begin{aligned}I(x) &= \left[\frac{f}{i\lambda\psi'} e^{i\lambda\psi}\right]_a^b - \int_a^b \left(\frac{f}{i\lambda\psi'}\right)' e^{i\lambda\psi}\, dx \\ &= \frac{i}{\lambda}\left[\frac{f(a)}{\psi'(a)} e^{i\lambda\psi(a)} - \frac{f(b)}{\psi'(b)} e^{i\lambda\psi(b)}\right] + \frac{i}{\lambda} \underbrace{\int_a^b \left(\frac{f}{\psi'}\right)' e^{i\lambda\psi}\, dx}_{J}\ .\end{aligned}$$

J satisfies the conditions for the generalised Riemann-Lebesgue Lemma if $f(x)/\psi'(x)$ is smooth; hence to leading order

$$I(x) \sim \frac{i}{\lambda}\left[\frac{f(a)}{\psi'(a)} e^{i\lambda\psi(a)} - \frac{f(b)}{\psi'(b)} e^{i\lambda\psi(b)}\right]\ .$$

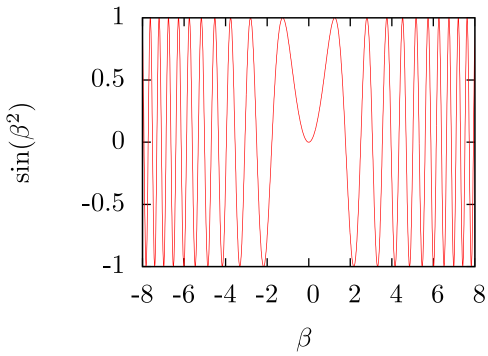
04/04 *Remark.* If we can continue to integrate by parts, we can obtain higher-order terms.

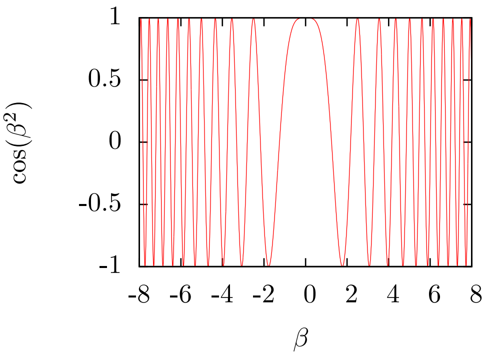
$\psi' = 0$ on $[\,a, b\,]$. Assume a unique zero at $x = c$:

$$\psi'(c) = 0, \qquad \psi''(c) \neq 0\ .$$

Since cancellation is much reduced near $x = c$, try a local scaling

$$x = c + \frac{y}{\lambda^\beta}\ .$$





$$\begin{aligned}
I(x) &= \int_{(a-c)\lambda^\beta}^{(b-c)\lambda^\beta} f\left(c+\frac{y}{\lambda^\beta}\right)\exp\left(i\lambda\psi\left(c+\frac{y}{\lambda^\beta}\right)\right)\frac{dy}{\lambda^\beta} \\
&= \int_{(a-c)\lambda^\beta\approx-\infty}^{(b-c)\lambda^\beta\approx\infty}\left[f(c)+\frac{y}{\lambda^\beta}f'(c)+\dots\right]e^{\left[i\lambda\psi(c)+\frac{i}{2}\psi''(c)y^2\lambda^{1-2\beta}+\frac{i}{6}\psi'''(c)y^3\lambda^{1-3\beta}+\dots\right]}\frac{dy}{\lambda^\beta}
\end{aligned}$$

again choose $\beta=\frac{1}{2}$ for the distinguished limit

$$= \frac{f(c)}{\lambda^{\frac{1}{2}}}e^{i\lambda\psi(c)}\int_{-\infty}^{\infty}\exp\left(\frac{i\psi''(c)y^2}{2}\right)dy\left(1+\mathcal{O}\left(\lambda^{-\frac{1}{2}}\right)\right)$$

substitute $y=\left(\dfrac{2}{|\psi''(c)|}\right)^{\frac{1}{2}}t\ ,\ \ s=\operatorname{sgn}\left[\psi''(c)\right]$

$$\sim \left(\frac{2}{\lambda\,|\psi''(c)|}\right)^{\frac{1}{2}} f(c)e^{i\lambda\psi(c)}\underbrace{\int_{-\infty}^{\infty}e^{ist^2}\,dt}_{\pi^{\frac{1}{2}}e^{is\pi/4}\text{ by contour deformation}}$$

$$\sim \left(\frac{2\pi}{\lambda\,|\psi''(c)|}\right)^{\frac{1}{2}} f(c)\exp\left(i\lambda\psi(c)+i\operatorname{sgn}\left[\psi''(c)\right]\frac{\pi}{4}\right)\ .$$

↑ leading order; next order approximation can come from end points, etc.

One can tighten up the 'proof' by changing variables at the start:

$$\psi(x)=\psi(c)+\tfrac{1}{2}\psi''(c)Y^2\ .$$

05/19

3.5.4 Steepest descents

This is a method for estimating integrals (for large $|\lambda|$) of the form

$$I=\int_C f(z)e^{\lambda\phi(z)}\,dz\ ,$$

where C is an integration path in the complex z-plane, f and ϕ are analytic functions of z. In principle λ may be complex, but wlog ϕ can then be redefined so that we can take λ to be real. There is a straightforward extension to $f(z)\equiv f(z;\lambda)$ and $\phi(z)\equiv\phi(z;\lambda)$.

(a) The idea is to deform the contour and then use Watson's Lemma **or** Laplace's Method.

First some notation. Let $\phi=u+iv$.

Then (i) $u_x=v_y$, $u_y=-v_x$ Cauchy Riemann
(ii) $\nabla^2 u=0=\nabla^2 v$.

05/22

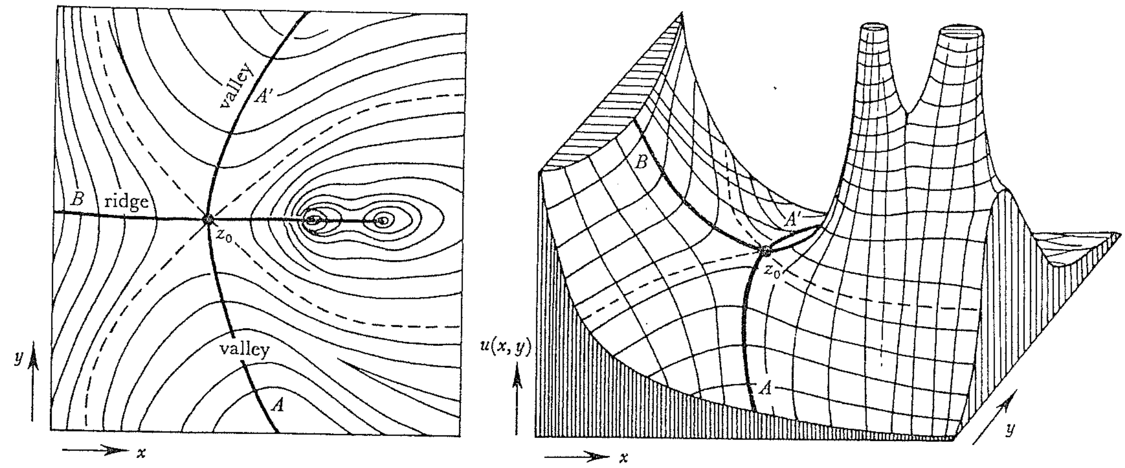


Figure 3.1: Plot of the tomography of the surface $u = \operatorname{Re}\phi(z;\lambda)$ near the saddle point z_0 for a typical function $\phi(z;\lambda)$. The heavy solid curves follow the centres of the ridges and valleys from the saddle point (i.e. lines of constant v), and the dashed curves follow level contours, $u = u(x_0, y_0) =$ constant. The curve AA' is the path of steepest descent. Source: *Mathematical Methods of Physics*, by Jon Mathews and Robert L. Walker.

(b) From stationary phase we have seen that rapid oscillations can cause cancellation. This makes estimation of the integral difficult and in particular means that the dominant contribution to I may not come from the part of C where $\operatorname{Re}(\lambda\phi(z;\lambda)) = \lambda u$ is largest. We eliminate such oscillations by choosing an integration path with

$$\operatorname{Im}(\phi) = v = \text{constant} .$$

The Cauchy-Riemann equations imply that

$$\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = 0 . \tag{3.9}$$

Thus the $v =$ constant contours are $\parallel$ to $\boldsymbol{\nabla} u$. It follows that the $v =$ constant contours are *paths of steepest ascent/descent* of u. [Note that we need the steepest descent path to obtain 'all' terms of
04/03 05/20 the series.]

(c) The major contribution to the integral I then comes from close to the 'highest' point (w.r.t. u) on the integration path.[2] If the 'highest' point is at the end of the integration path, then Watson's Lemma is most likely to be appropriate (but see below for a case when Laplace's method is needed), while if the 'highest' point is in the middle of the integration path then Laplace's Method is likely to be needed.

(d) *A constraint on interior maxima.* For the case of an interior 'highest' point on the integration path, i.e. a maximum, we will have at the maximum

$$\hat{\mathbf{s}}.\boldsymbol{\nabla} u = 0 ,$$

where $\hat{\mathbf{s}}$ is a unit vector in the direction of the integration path. Further, since the integration path is a line of constant v, it is perpendicular to $\boldsymbol{\nabla} v$, i.e.

$$\hat{\mathbf{s}}.\boldsymbol{\nabla} v = 0 .$$

It follows from (3.9), and the fact the integration path is two-dimensional, that at a turning point

$$|\boldsymbol{\nabla} u| = 0 .$$

[2] Those of you who already know about the method of steepest descents need to remember this — do not just go for the turning points!

Hence we conclude from the Cauchy-Riemann equations, that a maximum on an integration path that is a steepest descent contour can only occur at points where

$$\phi'(z) = 0\,.$$

Further, since $\nabla^2 u = 0$, from the maximum modulus principle, these points can only be saddles in the surface $u(x, y)$, i.e.

$$\Delta \equiv u_{xx}u_{yy} - (u_{xy})^2 = -(u_{xx})^2 - (u_{xy})^2 \leqslant 0\ .$$

07/01
05/16 Example. Find an asymptotic expansion for

$$I = \int_0^1 e^{i\lambda z^2}\,dz \qquad \text{as} \qquad \lambda \to \infty\,.$$

The leading-order approximation can be obtained by a stationary phase calculation near $z = 0$. To obtain a full expansion try to use steepest descent contours. From above

$$\begin{aligned} \phi = iz^2 = i(x^2 - y^2) - 2xy\ , \\ u = -2xy, \quad v = x^2 - y^2\ . \end{aligned}$$

Hence

steepest contours through	$z = 0$:	$v = 0,$	$x = \pm y,$	$u = \mp 2y^2$
S.D. contour through	$z = 0$:		$x = +y,$	$u = -2y^2$
			$z = (1+i)y,$	$iz^2 = -2y^2$
steepest contours through	$z = 1$:	$v = 1,$	$x = \pm\sqrt{1+y^2},$	$u = \mp 2y\sqrt{1+y^2}$
S.D. contour through	$z = 1$:		$x = \sqrt{1+y^2},$	$u = -2y\sqrt{1+y^2}$
			$z = \sqrt{1+y^2} + iy,$	$iz^2 = i - 2y\sqrt{1+y^2}$

The contribution from C_2 vanishes as $y_{\max} \to \infty$; thus

$$\begin{aligned} I &= \int_{C_1} e^{i\lambda z^2}\,dz + \int_{C_3} e^{i\lambda z^2}\,dz \\ &= (1+i)\int_0^\infty e^{-2\lambda y^2}\,dy - \frac{i}{2}\int_0^\infty \frac{e^{i\lambda}e^{-\lambda s}\,ds}{(1+is)^{\frac{1}{2}}} \qquad \text{substitute } iz^2 = i - s \\ &= \left(\frac{\pi}{4\lambda}\right)^{\frac{1}{2}} e^{\frac{i\pi}{4}} - \frac{ie^{i\lambda}}{2}\int_0^\infty ds\,e^{-\lambda s}\sum_{n=0}^{\infty}\frac{(-is)^n\Gamma\left(n+\frac{1}{2}\right)}{n!\,\Gamma\left(\frac{1}{2}\right)} \\ &\sim \left(\frac{\pi}{4\lambda}\right)^{\frac{1}{2}} e^{\frac{i\pi}{4}} + \frac{e^{i\lambda}}{2}\sum_{n=0}^{\infty}\left(\frac{-i}{\lambda}\right)^{n+1}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\ . \end{aligned}$$

Changing Steepest Descent Contours

It is often necessary to change from one steepest descent contour comprising part of the integration path to another steepest descent contour. As the above example illustrates, two such steepest descent contours should be joined in regions of the complex plane where the real part of the exponent, i.e. u, is asymptotically smaller than its maximum value.

The Local Contribution from a Saddle

We have adopted the approach that you choose steepest descent contours, and then look for maxima of u. If there are maxima in the interior of the path, then we have seen that they occur at a saddle points of $u(x, y)$.

An alternative view is that you deform the integration path so that u is as small as possible. If there is an interior maximum of u on the path, then it will occur at a saddle.

Either way we need to evaluate the contribution to the path in the neighbourhood of a saddle, which wlog we take to be at $z = z_s$. Close to this point

$$\lambda\phi(z) \sim \lambda\phi(z_s) + \lambda(z - z_s)\phi'(z_s) + \tfrac{1}{2}\lambda(z - z_s)^2\phi''(z_s) + \tfrac{1}{6}\lambda(z - z_s)^3\phi'''(z_s) + \dots \ .$$

As in Laplace's method introduce a rescaling such that $\lambda(z - z_s)^2 = \mathrm{ord}(1)$:

$$z = z_s + \frac{w}{\lambda^{\frac{1}{2}}}\,,$$

where we have assumed $\lambda > 0$. Then

$$\begin{aligned} \lambda\phi(z) &\sim \lambda\phi(z_s) + \tfrac{1}{2}\phi''(z_s)w^2 + \mathcal{O}\left(\lambda^{-\frac{1}{2}}\right) \\ \int_C f(z)e^{\lambda\phi(z)}\,dz &= \int_C f(z_s)e^{\lambda\phi(z_s)+\frac{1}{2}\phi''(z_s)w^2}\left(1 + \mathcal{O}\left(\lambda^{-\frac{1}{2}}\right)\right)\frac{dw}{\lambda^{\frac{1}{2}}} \\ &\sim f(z_s)e^{\lambda\phi(z_s)}\left(\frac{-2\pi}{\lambda\phi''(z_s)}\right)^{\frac{1}{2}} + \dots \qquad \text{by using } \eta = \left(-\tfrac{1}{2}\phi''(z_s)\right)^{\frac{1}{2}} w, \end{aligned}$$

where we have evaluated the integral using Laplace's method on the steepest descent path (by a suitable rotation of the contour C), and the choice of sign of the square root depends both on the rotation and the direction of traversed along the contour. That there is a dominant local contribution from close to the saddle is 'confirmed' by the fact that the integral is convergent as $|w| \to \infty$.

Note also that while it is not strictly necessary to choose the steepest descent path at the final stage (we just need a path that goes downhill), the steepest descent path *is* necessary to obtain 'all' terms of the
06/17 series.

3.5.5 The Airy function and Stokes phenomenon

The Airy function is defined as

$$\mathrm{Ai}(\lambda) = \frac{1}{2\pi i}\int_C e^{\lambda z - \frac{1}{3}z^3}\,dz\,, \tag{3.10}$$

where C starts from ∞ with $\arg(z) = -2\pi/3$ and ends at ∞ with $\arg(z) = +2\pi/3$. Define, consistent with our earlier notation,

$$\lambda\phi = \Phi = \lambda z - \tfrac{1}{3}z^3\,.$$

Thus there are saddles at

$$z_s = \pm\lambda^{\frac{1}{2}}\,, \quad \text{and} \quad e^{\Phi(z_s)} = e^{\pm\frac{2}{3}\lambda^{\frac{3}{2}}}\,.$$

First consider $\lambda \to +\infty$. Then we see from the contours of $\mathrm{Re}(\lambda z - \frac{1}{3}z^3)$ for $\arg\lambda = 0$ in figures 3.2 and 3.3, that it is only necessary to pass over the [lower] left-hand saddle in order to traverse the ridge separating the end points of integration.

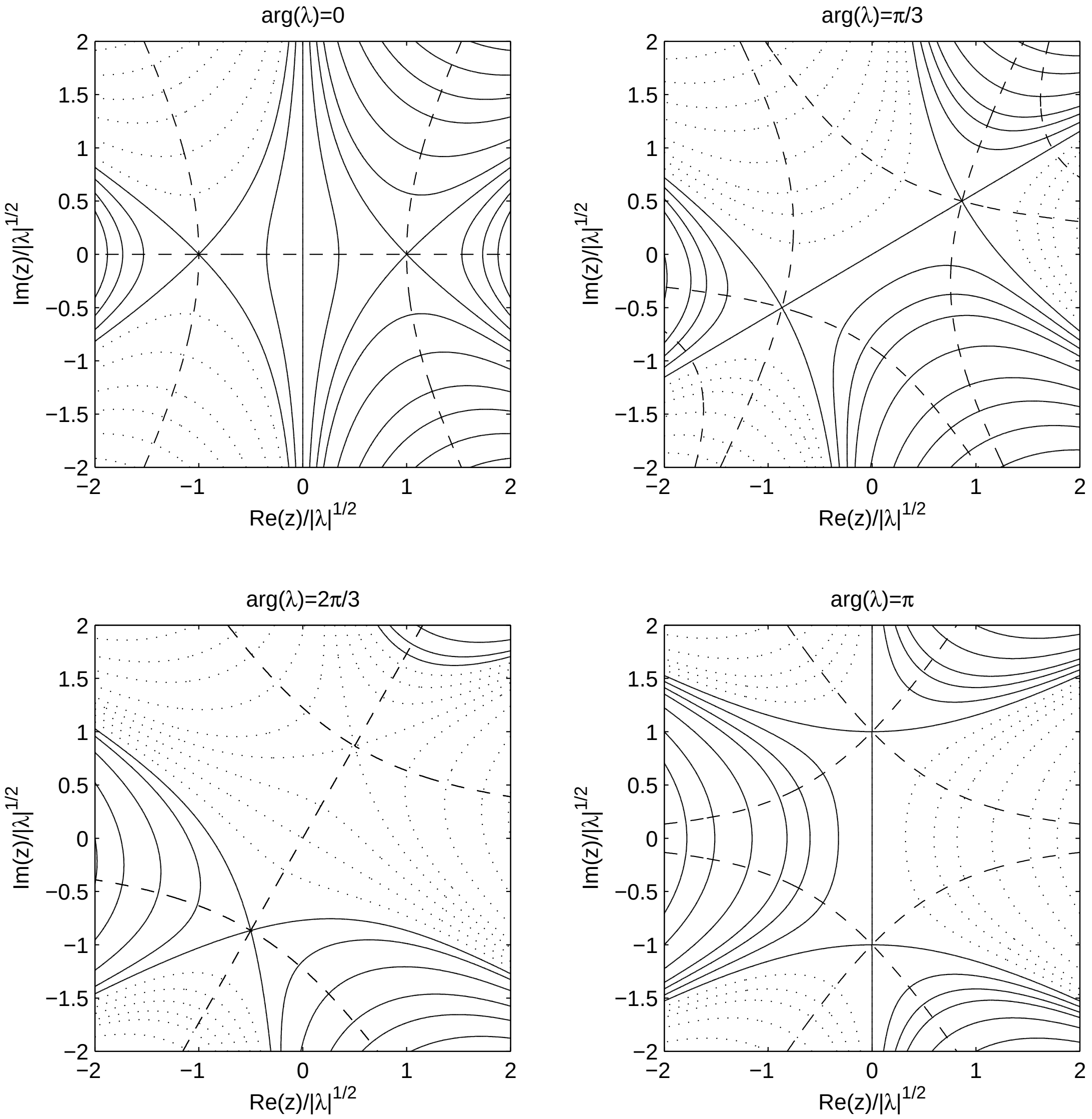


Figure 3.2: Contours of $\mathrm{Re}(3\lambda z - z^3)$ (solid and dotted, where solid/dotted is higher/lower than the operational saddle), and $\mathrm{Im}(3\lambda z - z^3)$ (dashed).

Seek a local contribution from near the saddle. Write:

$$\begin{aligned} z &= -\lambda^{\frac{1}{2}} + i\lambda^{\beta} w \\ \lambda z - \tfrac{1}{3}z^3 &= -\tfrac{2}{3}\lambda^{\frac{3}{2}} - \lambda^{\frac{1}{2}+2\beta} w^2 + i\tfrac{1}{3}\lambda^{3\beta} w^3\,. \end{aligned}$$

To apply Laplace's method choose $\beta = -\frac{1}{4}$, then

$$\begin{aligned} \mathrm{Ai}(\lambda) &= \frac{1}{2\pi\lambda^{\frac{1}{4}}}\int_C e^{-\frac{2}{3}\lambda^{\frac{3}{2}}} e^{-w^2}\left(1 + \frac{iw^3}{\lambda^{\frac{3}{4}}} - \frac{w^6}{18\lambda^{\frac{3}{2}}} + \dots\right) dw \\ &\sim \frac{e^{-\frac{2}{3}\lambda^{\frac{3}{2}}}}{2\pi^{\frac{1}{2}}\lambda^{\frac{1}{4}}}\left(1 - \frac{5}{48\lambda^{\frac{3}{2}}} + \dots\right). \end{aligned} \tag{3.11a}$$

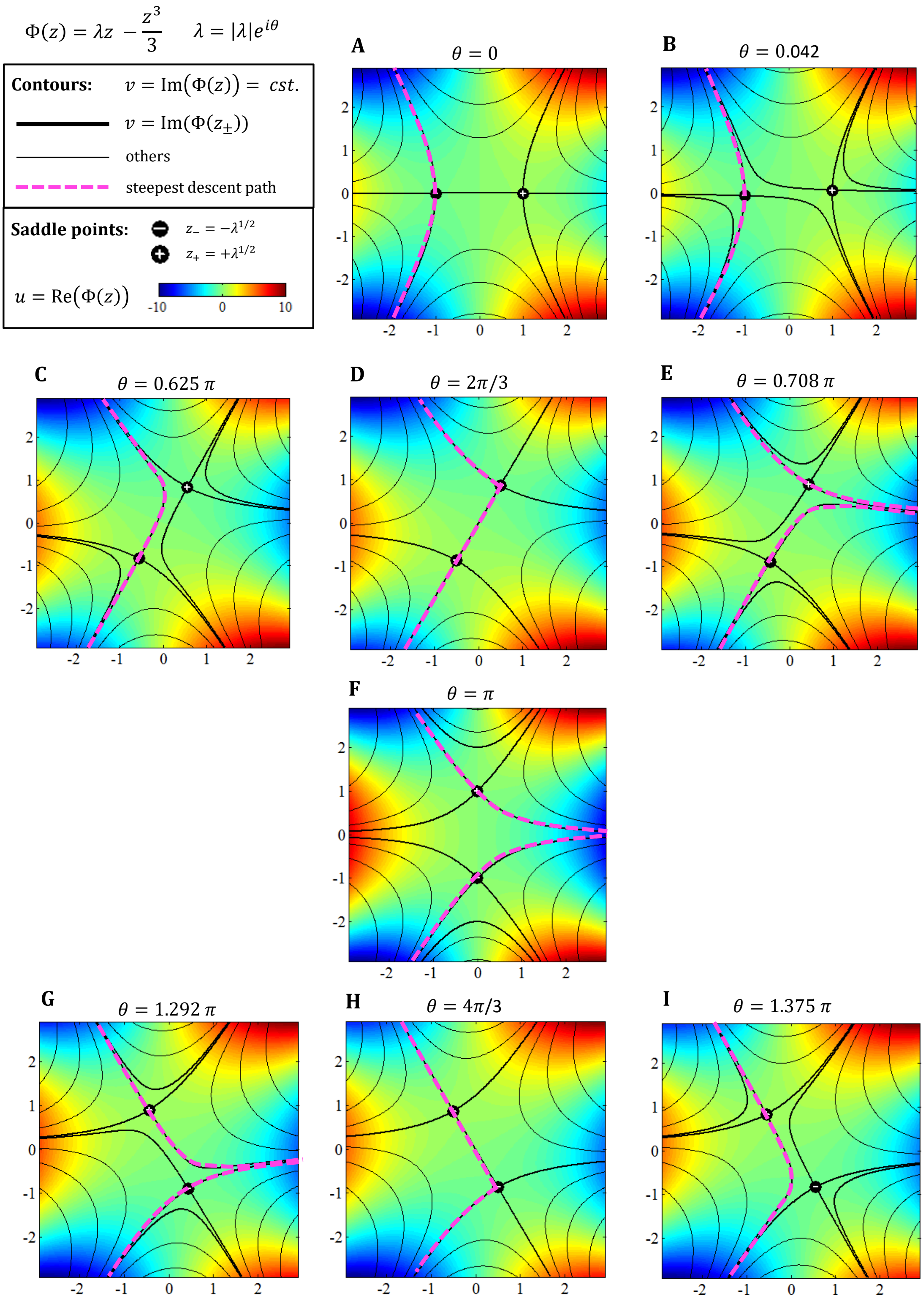


Figure 3.3: Contours of $\mathrm{Re}(3\lambda z - z^3)$ in colour

By brute force higher order terms can be obtained:

05/04
06/19

$$\mathrm{Ai}(\lambda) \sim \frac{e^{-\frac{2}{3}\lambda^{\frac{3}{2}}}}{2\pi^{\frac{1}{2}}\lambda^{\frac{1}{4}}} \sum_{r=0}^{\infty} Y_r \,, \quad \text{where} \quad Y_r = \frac{\Gamma(r+\frac{1}{6})\Gamma(r+\frac{5}{6})}{2\pi\xi^r\Gamma(r+1)} \quad \text{and} \quad \xi = -\tfrac{4}{3}\lambda^{\frac{3}{2}} \,. \tag{3.11b}$$

Consider next complex values of λ, and in particular for what values of $\arg(\lambda)$ result (3.11a) remains valid. From above we have that the positions of the saddles, z_s, rotate anti-clockwise. Further, the saddles swap dominance at

$$\arg(\lambda) = \frac{\pi}{3} + \frac{2}{3}n\pi \ .$$

However, to go from the valley at $\infty\, e^{-2\pi i/3}$ to the valley at $\infty\, e^{2\pi i/3}$ it is only necessary to go over the left-hand saddle up to $\arg\lambda = \pi$. Hence we deduce that (3.11a) remains valid for $\arg\lambda < \pi$.

For $\arg\lambda = \pi$ we need to go over both saddles. Hence

$$\begin{aligned} \mathrm{Ai} \quad &\sim \quad \frac{e^{-\frac{2}{3}\lambda^{\frac{3}{2}}}}{2\pi^{\frac{1}{2}}\lambda^{\frac{1}{4}}} + \text{c.c.} \\ &\sim \quad \frac{1}{(-\lambda)^{\frac{1}{4}}\,\pi^{\frac{1}{2}}} \sin\left(\frac{2}{3}(-\lambda)^{\frac{3}{2}} + \frac{\pi}{4}\right) , \end{aligned} \tag{3.11c}$$

where c.c. stands for complex conjugate. For $\pi < \arg\lambda < 5\pi/3$ we need to go through the other saddle, but (3.11c) is still an asymptotic approximation; in fact (3.11c) is correct for $|\arg\lambda - \pi| < 2\pi/3$.

This is an example of *Stokes phenomenon*, since (3.11a) and (3.11c) are distinct expressions (note that
06/16 (3.11c) certainly is not valid for $\arg\lambda = 0$).
06/18
06/20

3.6 Stokes Phenomena in the Complex Plane

Suppose that $f(z)$ is analytic, with say an isolated singularity at $z = z_0$, where, say, $z_0 = 0$. If $z^a f(z)$ is regular for some a, then $z^a f(z)$ has a power series that converges. This suggests that if an asymptotic power series is divergent, then the divergence must be associated with, say, an essential singularity, in which case the asymptotic series could only be valid in a sector of angle $< 2\pi$. This suggests that a single function may possess several asymptotic expansions, each restricted to a different sector; this is referred to as the *Stokes phenomenon*, as illustrated by the Airy function.

As a further example consider

$$\begin{aligned} \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt &= 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\,dt \\ &\sim 1 - \frac{e^{-z^2}}{\sqrt{\pi}z} \qquad \text{as } z \to \infty,\ z \text{ real} \ . \end{aligned}$$

One can extend this approximation into the complex plane as long as the contour for

$$\frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\,dt$$

is kept in the sector where $e^{-z^2} \to 0$ as $z \to \infty$. Hence

$$\operatorname{erf} z \sim 1 - \frac{e^{-z^2}}{\sqrt{\pi}z} \qquad \text{as } z \to \infty,\ |\arg z| < \pi/4. \tag{3.12a}$$

But erf is an odd function, so

$$\operatorname{erf} z \sim -1 - \frac{e^{-z^2}}{\sqrt{\pi}z} \qquad \text{as } z \to \infty,\ 3\pi/4 < |\arg z| < 5\pi/4. \tag{3.12b}$$

For $\pi/4 < \arg z < 3\pi/4$ we can integrate the defintion of the error function, i.e. $\operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$, by parts to show that

$$\operatorname{erf} z \sim -\frac{e^{-z^2}}{\sqrt{\pi}z} \qquad \text{as } z \to \infty,\ \pi/4 < |\arg z| < 3\pi/4. \tag{3.12c}$$

We now have three different asymptotic expansions for erf z. This is because, while erf is analytic everywhere in the finite complex plane, there is a non-analytic essential singularity at ∞.

3.6.1 Terminology

- The line where a term that is sub-dominant (i.e. much smaller) in one sector becomes comparable with a term that is dominant in that sector, is called an *anti-Stokes line* by some (e.g. Stokes, physicists and some mathematicians), and a *Stokes lines* by others (e.g. Bender & Orszag). For the error function the *anti-Stokes* lines are at

$$\arg z = (2n+1)\pi/4 \ .$$

- The lines where the leading behaviours of the two terms are most unequal are called *Stokes lines* by some (e.g. Stokes, physicists and some mathematicians), and a *anti-Stokes lines* by others (e.g. Bender & Orszag). In the case above the Stokes lines are at

$$\arg z = n\pi/2 \ .$$

  Stokes lines are important since the coefficient of the sub-dominant term can jump at them.

06/15

3.7 What Happens At Stokes Lines?

If we concentrate on the steepest descent paths, then in the case of the Airy function there is a change in topology of the integration path when $\arg\lambda = 2\pi/3$. The aim of this section is both to demonstrate that
07/17 the sub-dominant exponentially small term is 'turned on' here, and to understand the 'turn on' process.

3.7.1 The Airy function

Lemma. We first need a lemma. Consider the integral $I(\sigma, n)$ defined for real integer n and $Re(\sigma) > 0$ by

$$I(\sigma, n) = \int_0^\infty \frac{t^{n-1}\exp(\sigma(1-t))}{1-t}\,\mathrm{d}t\,, \tag{3.13}$$

where the contour of integration is chosen, based on the hindsight that we are doing a 'turn-on' problem, to pass just above the pole at $t = 1$.

First we note that, by expanding $(1-t)^{-1}$ as a binomial, (3.13) can be formally expressed as a [divergent] series (see (4.4) below for 'justification' of this):

$$\begin{aligned} I &= \int_0^\infty \mathrm{d}t\; t^{n-1}\exp(\sigma(1-t))\sum_{p=0}^\infty t^p \\ &= e^\sigma\sum_{p=0}^\infty \sigma^{-n-p}\int_0^\infty \mathrm{d}s s^{n+p-1}e^{-s} \\ &= e^\sigma\sum_{r=n}^\infty \frac{\Gamma(r)}{\sigma^r}\,. \end{aligned} \tag{3.14}$$

Next, we seek an asymptotic expansion of I in the limit as $n\to\infty$, for the [inspired] choice

$$\sigma \sim n + i\mu n^{\frac{1}{2}} + \nu + \ldots\,, \tag{3.15}$$

where $\mu = \mathcal{O}(1)$ and $\nu = \mathcal{O}(1)$.

One way possible way forward is to note that the exponent in (3.13), $\phi = \sigma(1-t) + (n-1)\log t$, is stationary at

$$t = \frac{n-1}{\sigma} \sim \frac{n-1}{n + i\mu n^{\frac{1}{2}} + \nu} \sim 1 - \frac{i\mu}{n^{\frac{1}{2}}} + \ldots\,, \tag{3.16}$$

and then to proceed using steepest descents. Alternatively, we note that

$$\begin{aligned}\frac{\partial I}{\partial \sigma} &= \int_0^\infty t^{n-1}\exp(\sigma(1-t))\,\mathrm{d}t \\ &= \frac{e^\sigma}{\sigma^n}\int_0^\infty u^{n-1}e^{-u}\mathrm{d}u \\ &= \frac{e^\sigma}{\sigma^n}\,\Gamma(n)\,.\end{aligned}$$

Hence from using Stirling's formula

$$\begin{aligned}\frac{\partial I}{\partial \mu} &\sim in^{\frac12}\,\frac{e^n e^{i\mu n^{\frac12}}e^\nu}{(n+i\mu n^{\frac12}+\nu)^n}\,\frac{(2\pi)^{\frac12}n^n e^{-n}}{n^{\frac12}} \\ &\sim i(2\pi)^{\frac12}\exp\left(-\tfrac12\mu^2\right)\,.\end{aligned}$$

We now wish to integrate this expression; for this we need a boundary condition. As noted in (3.16), the exponent in (3.13) has a stationary point at,

$$t \sim 1 - \frac{i\mu}{n^{\frac12}} + \dots\,.$$

Hence as $\mu \to -\infty$ the stationary point moves further and further above the pole at $t = 1$, whereas as μ passes through 0 a contribution will be picked up from the pole. We can show, say using a steepest descents estimate, that $I \to 0$ as $\mu \to -\infty$ (*exercise*: do this); it follows that

$$\begin{aligned}I(\sigma,n) &\sim i(2\pi)^{\frac12}\int_{-\infty}^{\mu}\exp\left(-\tfrac12 t^2\right)\mathrm{d}t \\ &\sim i\pi\Big(1+\mathrm{erf}(\mu/\sqrt2)\Big)\,. \qquad (3.17)\end{aligned}$$

□

With this lemma in our armoury, consider the full asymptotic series for the Airy function (see (3.10)),

$$\mathrm{Ai}(\lambda) = \frac{1}{2\pi i}\int_C e^{\lambda z - \frac13 z^3}\,dz,$$

when $|\lambda| \gg 1$ and $|\arg(\lambda)| < \pi$. Recall from (3.11b) that this is given by

$$\mathrm{Ai}(\lambda) \sim \frac{1}{2\lambda^{\frac14}\pi^{\frac12}}\exp\left(\tfrac12\xi\right)\sum_{r=0}^{\infty} Y_r\,,$$

where

$$Y_r = \frac{\Gamma(r+\frac16)\Gamma(r+\frac56)}{2\pi\xi^r\Gamma(r+1)} \quad \text{and} \quad \xi = -\tfrac43\lambda^{\frac32}\,.$$

We aim to estimate this when the asymptotic expansion is optimal. We note that Y_r is a minimum when

$$(r+1)\,\xi \sim \left(r+\tfrac16\right)\left(r+\tfrac56\right) \quad \text{i.e. when} \quad r \sim \xi\,.$$

Let

$$n = \mathrm{int}(|\xi|) + 1\,, \qquad (3.18)$$

and write

$$\mathrm{Ai}(\lambda) = \frac{1}{2\lambda^{\frac14}\pi^{\frac12}}\exp\left(\tfrac12\xi\right)\sum_{r=1}^{n-1} Y_r + R_n\,,$$

where

$$R_n = \frac{1}{2\lambda^{\frac{1}{4}}\pi^{\frac{1}{2}}} \exp\left(\tfrac{1}{2}\xi\right) \sum_{r=n}^{\infty} Y_r \,.$$

Next we need an estimate for R_n. Using Stirling's formula we can show that

$$Y_r \to \frac{\Gamma(r)}{2\pi\xi^r} \quad \text{as} \quad r \to \infty \,.$$

Hence from the lemma, in particular (3.14), we deduce that

$$R_n \sim \frac{1}{4\lambda^{\frac{1}{4}}\pi^{\frac{3}{2}}} e^{-\frac{1}{2}\xi} I(\xi, n) \,. \tag{3.19}$$

Finally, we consider values of ξ which have small argument. Specifically, write

$$\arg(\xi) = \frac{\mu}{|\xi^{\frac{1}{2}}|} \,,$$

so that

$$\xi = |\xi| \exp\left(\frac{i\mu}{|\xi^{\frac{1}{2}}|}\right) \sim |\xi| + i\mu|\xi|^{\frac{1}{2}} + \mathcal{O}(1) \,. \tag{3.20}$$

Thence from (3.15), (3.17), (3.18) and (3.19) it follows that[a]

$$R_n \sim \frac{i \exp(-\frac{1}{2}\xi)}{4\lambda^{\frac{1}{4}}\pi^{\frac{1}{2}}} \Big(1 + \operatorname{erf}(\mu/\sqrt{2})\Big) \,. \tag{3.21}$$

Since $\xi = -\frac{4}{3}\lambda^{\frac{3}{2}}$, we can interpret this result as saying that within an $\mathcal{O}(|\xi|^{-\frac{1}{2}})$ angle, i.e. an $\mathcal{O}(|\lambda|^{-\frac{3}{4}})$ angle, of $\arg\lambda = \pm\frac{2\pi}{3}$, the sub-dominant exponentially small term is 'turned on' by an error function. This is why Stokes lines are more important than *anti*-Stokes lines. We note that as $\mu \to \infty$ then (3.21) is the contribution from the sub-dominant saddle point in figures 3.2 and 3.3.

Asymptotics beyond all orders. In order to see the sub-dominant exponentially small term 'turn on', it was not sufficient to consider just the algebraic asymptotic expansion. We needed a clever trick to look beyond the infinite number of algebraic terms; this is an example of asymptotics beyond all orders. We will return to this topic later.

07/16
07/18
07/19
07/20

[a] The choice of the contour going above the pole at $t = 1$ means that the remainder, R_n, 'turns on' as μ increases; if the contour had been chosen beneath the pole then the remainder would have 'turned off' as μ increased.

4 Summation Of Series By 'Magic'

How do we sum series? E.g. how do we find the value of

$$S_n = \sum_{r=0}^{n} a_r \quad \text{as} \quad n \to \infty \ .$$

For instance what are the sums of

(a) $\quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\ ,$

(b) $\quad 1 - 1 + 1 - 1 + \dots\ ,$

(c) $\quad 1 + 2 + 4 + 8 + \dots\ .$

For starters note that in the case of example (b)

$$\begin{aligned}\lim_{n\to\infty} S_n \equiv S &= 1 - 1 + 1 - 1 + \dots \\ &= 1 - (1 - 1 + 1 - \dots) \\ &= 1 - S\ ;\end{aligned}$$

hence we might guess that

$$S = \tfrac{1}{2}\,.$$

More generally, we might expect that the value of the sum depends on the definition of the sum. We will consider a number of different 'magical methods' (most of which are based on analytical continuation), most of which, reassuringly, come up with the same answer.

4.1 Cesàro Sums

$$S = \lim_{n\to\infty} \frac{S_0 + S_1 + \dots + S_n}{n+1}\ .$$

For example (b):

$$\begin{aligned} S_n &= \tfrac{1}{2}(1 + (-)^n)\,, \\ S &= \lim_{n\to\infty} \frac{1 + 0 + 1 + 0 + \dots}{n+1} = \tfrac{1}{2}\,. \end{aligned}$$

4.2 Euler Sums

Define

$$f(x) = \sum_{r=0}^{\infty} a_r x^r\ .$$

Suppose that this series is convergent for $|x| < 1$; then, based on the idea analytic continuation, define the Euler sum to be

$$S = \lim_{x\to 1-} f(x)\ .$$

For instance:

(a)
$$\begin{aligned} a_r &= (-)^r\,, \\ f(x) &= \sum_{r=0}^{\infty} (-)^r x^r = \frac{1}{1+x}\ , \end{aligned}$$
and so $1 - 1 + 1 - 1 + \ldots = f(1) = \frac{1}{2}$ (again).

(b)
$$\begin{aligned} a_r &= 2^r\ , \\ f(x) &= \sum_{r=0}^{\infty} (2x)^r = \frac{1}{1-2x}\ , \end{aligned}$$
and so $1 + 2 + 4 + 8 + \cdots = f(1) = -1$.

(c)
$$\begin{aligned} a_r &= r\ , \\ f(x) &= \sum_{r=0}^{\infty} r x^r = \frac{x}{(1-x)^2}\ , \end{aligned}$$
16/13 and so the Euler sum of $1 + 2 + 3 + 4 + \ldots$ is not defined.

4.3 Borel Sums

If the coefficients a_n grow too fast, then Euler summation is not applicable. However, the power series may still have meaning as an asymptotic series. Define

$$\phi(x) = \sum_{r=0}^{\infty} \frac{a_r x^r}{r!}\ ,$$

and let

$$\begin{aligned} B(x) &= \int_0^\infty e^{-t} \phi(xt) dt \\ &= \sum_{r=0}^{\infty} \frac{a_r}{r!} \int_0^\infty (xt)^r e^{-t} dt \\ &= \sum_{r=0}^{\infty} a_r x^r\,, \end{aligned}$$

by Watson's lemma (or by playing *fast-and-loose* with the interchange of the summation and integration). We define the *Borel sum* to be:

$$S = \sum_{r=0}^{\infty} a_r = \lim_{x \to 1-} B(x)\ .$$

4.3.1 An example: the Stieltjes series

The [divergent] Stieltjes series is given by

$$f(x) = \sum_{r=0}^{\infty} (-)^r r!\, x^r\ , \qquad a_r = (-)^r r!\,,$$

with $x = 1$. Adopting the above method we write

$$\phi(x) = \sum_{r=0}^{\infty} (-)^r x^r = \frac{1}{1+x}\ ,$$

and

$$B(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\, dt\ .$$

Hence

$$0! - 1! + 2! - 3! + \dots \quad = \int_0^\infty \frac{e^{-t}}{1+t} dt.$$

4.3.2 Summation in the Borel plane

There is another way of looking at Borel sums. Suppose instead that we have a series

$$\psi(x) = \sum_{r=1}^{\infty} \frac{a_r}{x^r}\,, \tag{4.1}$$

with $a_r \propto r!$ as $x \to \infty$. Let $\mathcal{L}$ be the Laplace operator, with inverse $\mathcal{L}^{-1}$. We adopt a normalisation so that

$$\mathcal{L}\{t^{r-1}\} = \int_0^\infty e^{-xt} t^{r-1}\, dt = \frac{(r-1)!}{x^r}\,. \tag{4.2}$$

Then, analytically continuing in the Borel "t" plane,

$$\begin{aligned} \psi(x) &= \mathcal{L}\,\mathcal{L}^{-1} \sum_{r=1}^{\infty} \frac{a_r}{x^r} \\ &= \mathcal{L}\left\{ \sum_{r=1}^{\infty} \frac{a_r t^{r-1}}{(r-1)!} \right\} \\ &= \int_0^\infty e^{-xt} \sum_{r=1}^{\infty} \frac{a_r t^{r-1}}{(r-1)!}\, dt \\ &= \int_0^\infty e^{-xt} \varphi(t)\, dt\,, \end{aligned} \tag{4.3a}$$

where $\varphi(t)$ is given by the series (which is convergent for small t)

$$\varphi(t) = \sum_{r=1}^{\infty} \frac{a_r t^{r-1}}{(r-1)!}\,. \tag{4.3b}$$

4.3.3 An example

Suppose that $a_r = 0$ for $r = 1, \dots, n-1$, and that

$$a_r = (r-1)! \quad \text{for} \quad r = n, n+1, \dots$$

Then, using analytical continuation for $|t| \geqslant 1$,

$$\varphi(t) = \sum_{r=n}^{\infty} t^{r-1} = \frac{t^{n-1}}{1-t}\,,$$

and hence

$$\psi(x) = \sum_{r=n}^{\infty} \frac{(r-1)!}{x^r} = \int_0^\infty e^{-xt} \frac{t^{n-1}}{1-t}\, dt\,, \tag{4.4}$$

where the integration contour is assumed to pass just above the pole at $t = 1$. The sum $I(\sigma, n) = e^{\sigma}\psi(\sigma)$ in equation (3.14), in § 3.7.1 on Stokes lines of the Airy function, is thus a Borel sum (and relies on ideas
16/07 of analytical continuation in the Laplace transform plane).
16/08
07/15
08/17

4.4 Padé Approximants

Suppose we only know partial sums. Let

$$\sum_{r=0}^{N+M} a_r x^r = \frac{\sum_{n=0}^{N} A_n x^n}{\sum_{m=0}^{M} B_m x^m} = P_M^N(x) \ .$$

Often if

$$f(x) = \sum_{r=0}^{\infty} a_r x^r \ ,$$

then

$$P_M^N(x) \to f(x) \quad \text{as} \quad N, M \to \infty \ ,$$

even if $\sum_{r=0}^{\infty} a_r x^r$ is divergent.

1. If $a_r = 1$, then
$$P_N^N(x) = \frac{1}{1-x} \qquad \text{exact !}$$

2. Stieltjes series, $a_r = (-)^r r!$
$$\begin{aligned} P_5^5(1) &= 0.59738\dots \qquad && \text{11 terms} \\ P_{10}^{10}(1) &= 0.59638\dots && \text{21 terms} \\ B(1) &= 0.59635\dots \ . && \end{aligned}$$

Padé Approximants work because they put

- poles near poles,
- a cluster of poles at essential singularities,
- sequences of poles and zeros along branch cuts.

4.5 Continued Fractions

A variation of the Padé method of summing power series. Define

$$F_N(x) = \cfrac{c_0}{1 + \cfrac{c_1 x}{1 + c_2 x} \\ \qquad \ddots \\ \qquad\qquad \cfrac{c_{N-1} x}{1 + c_N x}}$$

There are fast numerical methods for the evaluation of continued fractions.

4.6 Shanks' Transformation

Suppose

$$S_n = \sum_{r=0}^{n} a_r = A + BC^n \ ,$$

then from eliminating A, B and C,

$$S(S_n) = S_n - \frac{(S_{n+1} - S_n)(S_n - S_{n-1})}{(S_{n+1} - S_n) - (S_n - S_{n-1})} \ .$$

This can be applied repeatedly, e.g. $S(S(S_n))$, to remove higher transients. For instance, consider

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots = 0.693147\ldots$$

Partial Sums	1-Shanks	2-Shanks	3-Shanks
1			
0.5			
0.833	0.7000		
0.583	0.6905		
0.783	0.6944	0.693277	
0.617	0.6924	0.693106	
0.760	0.6936	0.693163	0.693149

4.7 Richardson Extrapolation

Suppose instead

$$S_n \sim Q_0 + \frac{Q_1}{n} + \frac{Q_2}{n^2} + \frac{Q_3}{n^3} + \ldots \quad \text{as} \quad n \to \infty \,.$$

Calculate the $N+1$ partial sums $S_n, S_{n+1}, \ldots, S_{n+N}$. Then it is possible to show that

$$Q_0 = \sum_{k=0}^{N} \frac{S_{n+k}(n+k)^N(-)^{k+N}}{k!(N-k)!} \,.$$

4.8 Other Methods

- Neville tables;
- Domb-Sykes plots (to find the nearest singularity);
- Euler transformations;
- etc.

Unlectured 08/17

5 Matched Asymptotic Expansions (MAEs)

Matched asymptotic expansions are mainly used for solving *singular perturbation problems* that arise when finding solutions to differential equations. MAEs are often needed when the highest-order derivative is multiplied by a small parameter, say ε, where $0 < \varepsilon \ll 1$ henceforth. We will apply them primarily to ODEs, but they are equally applicable to PDEs.

5.1 Regular Perturbation Problems: An Example

$$y'' + 2\varepsilon y' + (1+\varepsilon^2)y = 1 \ , \qquad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0 \quad \text{where, as noted above,} \quad 0 < \varepsilon \ll 1 \, .$$

5.1.1 Exact solution

$$\begin{aligned} y &= \frac{1}{1+\varepsilon^2}\left[1 - e^{-\varepsilon(x-\pi/2)}\sin x - e^{-\varepsilon x}\cos x\right] \\ &= (1 - \sin x - \cos x) + \varepsilon\left[\left(x - \frac{\pi}{2}\right)\sin x + x\cos x\right] \\ &\qquad -\varepsilon^2\left[1 - \cos x - \sin x + \tfrac{1}{2}\left(x - \frac{\pi}{2}\right)^2\sin x + \tfrac{1}{2}x^2\cos x\right] + \dots \ . \end{aligned}$$

5.1.2 Perturbation solution

Try

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots \ .$$

Then

$$\begin{array}{lll} \varepsilon^0 : & y_0'' + y_0 = 1 \ , & y_0(0) = 0, \quad y_0\left(\frac{\pi}{2}\right) = 0 \ , \\ \varepsilon^1 : & y_1'' + y_1 = -2y_0' \ , & y_1(0) = 0, \quad y_1\left(\frac{\pi}{2}\right) = 0 \ , \\ \varepsilon^2 : & y_2'' + y_2 = -2y_1' - y_0 \ , & y_2(0) = 0, \quad y_2\left(\frac{\pi}{2}\right) = 0 \ . \end{array}$$

Hence

$$\begin{aligned} y_0 &= 1 - \sin x - \cos x \ , \\ y_1 &= \left(x - \frac{\pi}{2}\right)\sin x + x\cos x \ , \\ y_2 &= -1 + \cos x + \sin x - \tfrac{1}{2}\left(x - \frac{\pi}{2}\right)^2\sin x - \tfrac{1}{2}x^2\cos x \ . \end{aligned}$$

08/01
05/03

5.2 Singular Perturbation: Example

$$\varepsilon y'' + y' = -e^{-x} \ , \qquad y(0) = 0, \quad y \to 0 \quad \text{as } x \to \infty.$$

5.2.1 Exact solution

$$y = \frac{e^{-x} - e^{-x/\varepsilon}}{1-\varepsilon} \ .$$

Limit $\varepsilon \to 0$, x fixed:

$$y \sim e^{-x}\left(1 + \varepsilon + \varepsilon^2 + \dots\right) \ . \tag{5.1}$$

This expansion satisfies the boundary condition as $x \to \infty$, but does not satisfy the boundary condition $y(0) = 0$.

The limit $\varepsilon \to 0$, with x fixed, is a non-uniform limit since

$$e^{-x/\varepsilon} \ll \varepsilon^m \quad \text{only if} \quad |x| \gg m\varepsilon \log \frac{1}{\varepsilon} \ ;$$

hence we cannot put $x = 0$ in (5.1).

For x small we obtain an asymptotic expansion by first setting $x = \varepsilon\xi$, and then expanding:

$$y \sim \left(1 - e^{-\xi}\right) + \varepsilon\left(1 - e^{-\xi} - \xi\right) + \varepsilon^2\left(1 - e^{-\xi} - \xi + \tfrac{1}{2}\xi^2\right) + \dots \ . \tag{5.2}$$

Now

$$y(0) = 0 + \varepsilon 0 + \varepsilon^2 0 + \dots \ ,$$

while

$$y \to 1 + \varepsilon(1-\xi) + \varepsilon^2\left(1 - \xi + \tfrac{1}{2}\xi^2\right) + \dots \qquad \text{as } \xi \to \infty.$$

'Outer' ($\varepsilon \to 0$, x fixed) expansion satisfies the $x \to \infty$ boundary condition,
'Inner' ($\varepsilon \to 0$, ξ fixed) expansion satisfies the $\xi = 0$ boundary condition.

08/16 *Exercise.* Put $x = \varepsilon^{\frac{1}{2}}\eta$ in (5.1) and expand to $\mathcal{O}(\varepsilon)$;
08/18 $\xi = \varepsilon^{-1}x = \varepsilon^{-\frac{1}{2}}\eta$ in (5.2) and expand to $\mathcal{O}(\varepsilon)$.
08/19 Compare the results.
08/20

5.2.2 Expansion solution

Outer Approximation. Pose a Poincaré expansion for x fixed ($\neq 0$) and $\varepsilon \to 0$:

$$y = \sum_{n=0}^{\infty} \varepsilon^n y_n(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) \dots \ .$$

Then, from substituting into the governing equation and equating terms with the same power of ε,

$$\begin{array}{lrl} O(\varepsilon^0): & y_0' = -e^{-x} \ , & y_0 = A_0 + e^{-x} \ , \\ O(\varepsilon^1): & y_0'' + y_1' = 0 \ , & y_1 = A_1 + e^{-x} \ , \\ O(\varepsilon^n): & y_{n-1}'' + y_n' = 0 \ , & y_n = A_n + e^{-x} \ . \end{array}$$

We wish to apply two boundary conditions at each order, but have only one unknown constant. From comparison with the exact solution we choose not to satisfy the boundary condition at $x = 0$. From applying the boundary condition as $x \to \infty$, it follows that $A_n = 0$, and

$$y = e^{-x}\left(1 + \varepsilon + \varepsilon^2 + \dots\right) \ . \tag{5.3}$$

This is in agreement with (5.1).

Inner Approximation. Since we wish to apply two boundary conditions, we need the $\varepsilon y''$ term to be important somewhere at leading order. Note that in a somewhat rough and ready sense

$$\left.\begin{array}{rcl} \varepsilon y'' & \sim & \dfrac{\varepsilon y}{(x-x_0)^2} \\ y' & \sim & \dfrac{y}{(x-x_0)} \end{array}\right\} \quad \text{this suggests rescaling for } (x - x_0) \sim \varepsilon.$$

Hence try

$$x = x_0 + \varepsilon\xi \ , \qquad y(x) = Y(\xi) = \sum_{n=0}^{\infty} \varepsilon^n Y_n(\xi) \, ,$$

where $Y(\xi)$ satisfies

$$\frac{1}{\varepsilon}\frac{d^2Y}{d\xi^2} + \frac{1}{\varepsilon}\frac{dY}{d\xi} = -e^{-\varepsilon\xi} \, .$$

From substituting into the governing equation it follows that

$$\begin{array}{lll} O(\varepsilon^{-1}): & Y_0'' + Y_0' = 0\,, & Y_0 = B_0 + C_0 e^{-\xi}\,, \\ O(\varepsilon^{0}): & Y_1'' + Y_1' = -e^{-x_0}\,, & Y_1 = B_1 + C_1 e^{-\xi} - \xi e^{-x_0}\,. \end{array}$$

Since we need to satisfy the boundary condition at $x = 0$, take $x_0 = 0$. Then

$$\left.\begin{array}{lcl} Y_0 & = & B_0\left(1 - e^{-\xi}\right)\,, \\ Y_1 & = & B_1\left(1 - e^{-\xi}\right) - \xi\,, \\ Y_2 & = & B_2\left(1 - e^{-\xi}\right) - \xi + \frac{1}{2}\xi^2\,, \\ & \dots\,. & \end{array}\right\} \tag{5.4}$$

Match. We have two asymptotic expansions valid in x fixed, i.e. (5.3), and ξ fixed, i.e. (5.4). They must represent the same function in the intermediate region

$$\varepsilon \ll x \ll 1\,, \text{ i.e. } 1 \ll \xi \ll \varepsilon^{-1}\,.$$

08/15

Forcing the two expansions to be identical determines the B_j. To this end introduce an 'intermediate variable', η, where

$$\eta = \frac{x}{\varepsilon^\alpha} = \frac{\varepsilon\xi}{\varepsilon^\alpha} \qquad \left(0 < \alpha < 1, \text{ e.g. } \alpha = \tfrac{1}{2}\right),$$

so that

$$x = \varepsilon^\alpha \eta\,, \quad \xi = \varepsilon^{\alpha - 1}\eta\,.$$

When $\eta = \operatorname{ord}(1)$, then as required $\varepsilon \ll x \ll 1$. Expand both outer and inner asymptotic expansions in powers of η:

$$\begin{array}{llllllll} \text{Outer:} & y & \sim & 1 & -\varepsilon^\alpha\eta & +\frac{1}{2}\varepsilon^{2\alpha}\eta^2 & +\frac{1}{6}\varepsilon^{3\alpha}\eta^3 & +\dots \\ & & & \boxed{1} & \boxed{2} & \boxed{3} & & \\ & & & +\varepsilon & -\varepsilon^{1+\alpha}\eta & +\varepsilon^{1+2\alpha}\frac{1}{2}\eta^2 & +\dots & \\ & & & \boxed{4} & \boxed{5} & & & \\ & & & +\varepsilon^2 & -\varepsilon^{2+\alpha}\eta & +\dots & & \\ & & & \boxed{6} & & & & \\ & & & +\dots & , & & & \end{array}$$

$$\begin{array}{lllllll} \text{Inner:} & y & \sim & B_0 & +\exp & & \\ & & & \boxed{1} & & & \\ & & & +\varepsilon B_1 & -\varepsilon^\alpha\eta & +\exp & \\ & & & \boxed{4} & \boxed{2} & & \\ & & & +\varepsilon^2 B_2 & -\varepsilon^{\alpha+1}\eta & +\frac{1}{2}\varepsilon^{2\alpha}\eta^2 & +\exp \\ & & & \boxed{6} & \boxed{5} & \boxed{3} & \\ & & & +\dots & . & & \end{array}$$

After reordering the expansions should be the same; hence

07/04
$$B_0 = 1\,, \qquad B_1 = 1\,, \qquad B_2 = 1\,.$$

Terms jump order when matching. This indicates that there are terms in the governing equation that, although small in one region, are to be treated as dominant in the next region.

$$x = \mathcal{O}(1) \qquad\qquad\qquad\qquad \xi = \mathcal{O}(1)$$

$$\underbrace{-\varepsilon\frac{d^2y}{dx^2}}_{\text{small}} = e^{-x} + \underbrace{\frac{dy}{dx} \qquad\qquad \frac{dy}{d\xi}}_{\text{common term}} + \frac{d^2y}{d\xi^2} = \underbrace{-\varepsilon e^{-\varepsilon\xi}}_{\text{small}}$$

Note that if the smallest retained terms, i.e. the $\mathcal{O}\left(\varepsilon^2\right)$ terms in both expansions, are to be bigger than the largest [small] ignored terms, i.e. the $\mathcal{O}\left(\varepsilon^{3\alpha}\right)$ terms in both expansions, then we require

$$\varepsilon^2 \gg \varepsilon^{3\alpha}, \quad \text{i.e.} \quad \tfrac{2}{3} < \alpha < 1\,.$$

If matching to higher order by, say, retaining the terms up-to $\mathcal{O}\left(\varepsilon^Q\right)$, then for the $\mathcal{O}\left(\varepsilon^{(Q+1)\alpha}\right)$ ignored terms to be formally smaller, we would require that

$$\varepsilon^Q \gg \varepsilon^{(Q+1)\alpha}, \quad \text{i.e.} \quad \frac{Q}{Q+1} < \alpha < 1\,.$$

13/06
12/13
12/14
09/17

5.3 Van Dyke's Matching Rule

This can be simpler than using an intermediate variable, but sometimes fails (beware of logs).

Notation

$$E_n y = \text{Outer limit } (x \text{ fixed, } \varepsilon \downarrow 0) \text{ of } y \text{ retaining } n \text{ terms} = \sum_{r=0}^{n-1} \varepsilon^r y_r(x)$$

$$H_m y = \text{Inner limit } (\xi \text{ fixed, } \varepsilon \downarrow 0) \text{ of } y \text{ retaining } m \text{ terms} = \sum_{r=0}^{m-1} \varepsilon^r Y_r(\xi)$$

Van Dyke's rule is

$$E_n H_m y = H_m E_n y\,.$$

↑ Take m terms of the inner expansion, re-express ξ in terms of x, and then take n terms of the resulting expansion.

Forcing equality determines the unknown constants. We illustrate this using our model problem:

$$\begin{aligned} E_2 H_2 y &= E_2\left(B_0\left(1 - e^{-\xi}\right) + \varepsilon B_1\left(1 - e^{-\xi}\right) - \varepsilon\xi\right) \\ &= E_2\left(B_0\left(1 - e^{-x/\varepsilon}\right) + \varepsilon B_1\left(1 - e^{-x/\varepsilon}\right) - x\right) \\ &= B_0 - x + \varepsilon B_1\,, \end{aligned}$$

$$\begin{aligned} H_2 E_2 y &= H_2\left(e^{-x} + \varepsilon e^{-x}\right) \\ &= H_2\left(e^{-\varepsilon\xi} + \varepsilon e^{-\varepsilon\xi}\right) \\ &= 1 - \varepsilon\xi + \varepsilon\,. \end{aligned}$$

Hence require

$$B_0 - x + \varepsilon B_1 = 1 - \underbrace{\varepsilon\xi}_{x} + \varepsilon\,,$$

and

$$B_0 = 1 = B_1\,.$$

09/01
06/03
13/08
Exercise. Do for general m and n.

5.4 The Choice of Scaling

There is no magic law that enables one to make the correct choice of scaling. However, there are tips.[3]

(a) First find 'the' regular solution:

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots \ .$$

If for some x it happens that, $\varepsilon y_1 \sim y_0$ or $\varepsilon^2 y_2 \sim \varepsilon y_1$ or ..., then the solution is no longer asymptotic. This often suggests a rescaling for x. For instance suppose that the regular-perturbation solution yields

$$y = 1 + \frac{2\varepsilon}{(x-x_0)^2} + \frac{7\varepsilon^2}{(x-x_0)^4} + \dots \ .$$

This breaks down when $(x - x_0) \sim \varepsilon^{\frac{1}{2}}$, which suggests that an appropriate rescaling would be $x = x_0 + \varepsilon^{\frac{1}{2}}\xi$.

(b) Look at the equation and see if one can predict the scaling from there, i.e. seek distinguished limits. For instance consider the problem

$$(x + \varepsilon y)\frac{dy}{dx} + y = 1 \ , \qquad y(1) = 2 \ .$$

This has the leading-order (i.e. $\varepsilon = 0$) solution

$$x\frac{dy_0}{dx} + y_0 = 1 \ , \quad y_0 = 1 + \frac{1}{x} \ . \tag{5.5}$$

Now, using(5.5), compare the size of the terms in the equation:

$$\underbrace{x\left(\frac{y}{x}\right) \quad ; \quad \frac{\varepsilon y^2}{x}}_{\text{comparable when } y \sim \frac{x}{\varepsilon}.} \quad ; \quad y \quad ; \quad 1$$

Hence the neglected term is comparable with the largest retained term when $\frac{1}{x} \sim \frac{x}{\varepsilon}$, i.e. when
13/07 $x \sim \varepsilon^{\frac{1}{2}}$.

5.5 Where is the 'Inner Layer'?

The 'inner layer' could be anywhere! *One* way to try and track it down is to look at regular solution and see where it breaks down. However, this method does not always work, as illustrated by the following examples.

Example 1. Consider the problem

$$\varepsilon y'' - y = 0 \ , \quad y(0) = y(1) = 1 \ .$$

For $\varepsilon > 0$ this has solution

$$y = \left(\frac{1 - e^{-1/\varepsilon^{\frac{1}{2}}}}{1 - e^{-2/\varepsilon^{\frac{1}{2}}}}\right)\left[e^{-x/\varepsilon^{\frac{1}{2}}} + e^{(x-1)/\varepsilon^{\frac{1}{2}}}\right] \ .$$

[3] In a forest, a fox bumps into a little rabbit, and inquires, 'Hi, what are you up to?'. 'I'm writing a dissertation on how rabbits eat foxes', says the rabbit. 'Come now rabbit, you know that's impossible', replies the fox. 'Well, follow me and I'll show you', says the rabbit. They both go into the rabbit's dwelling and after a while the rabbit emerges with a satisfied expression on his face.

Along comes a wolf who asks, 'Hello, what are you doing these days?'. 'I'm writing the second chapter of my thesis, on how rabbits devour wolves', says the rabbit. 'Are you crazy! Where is your academic honesty?' explodes the wolf. 'Come with me and I'll show you', says the rabbit. As before the rabbit comes out of his dwelling with a satisfied expression on his face, and with a diploma in his paw.

Switch to the rabbit's dwelling to find a huge lion sitting next to some bloody and furry remnants of the fox and the wolf. The moral: it's your supervisor that really counts.

The asymptotic solution is

$$y = 0 + \varepsilon\, 0 + \varepsilon^2\, 0 + \dots .$$

There are inner layers at both $x = 0$ and $x = 1$.

Now consider the case $\varepsilon < 0$. This has solution

$$y = \frac{\sin\left(x/\left|\varepsilon\right|^{\frac{1}{2}}\right) - \sin\left((x-1)/\left|\varepsilon\right|^{\frac{1}{2}}\right)}{\sin\left(1/\left|\varepsilon\right|^{\frac{1}{2}}\right)} .$$

In this case there are 'inner layers' everywhere.

What happens if $\sin\left(1/\left|\varepsilon\right|^{\frac{1}{2}}\right) = 0$?

Example 2.

$$\tfrac{1}{2}\varepsilon^2 f'' - f(f^2 - 1) = 0\,, \text{ with } f(\infty) = 1\ ,\ f(-\infty) = -1\ .$$

$$\varepsilon = 0\ :\qquad f(f^2-1) = 0\,,\ \text{hence } f = -1 \text{ or } 0 \text{ or } +1\,.$$

$$\varepsilon \neq 0\ :\qquad \textit{an} \text{ exact solution is } f = \tanh\left(\frac{x}{\varepsilon}\right)\ . \tag{5.6}$$

There is a inner layer in the interior of width $\mathcal{O}(\varepsilon)$. Within the 'inner layer'

09/16 $$\varepsilon^2 f'' \sim f(f^2-1)\ ,$$

09/18

09/19 i.e. the inner layer is confined to a region where $(x - x_0) \sim \varepsilon$.

09/20 *Exercise:* Is (5.6) unique?

09/22

5.6 Composite Expansions

The outer solution in (5.3) fails as $x \to 0$ due to the missing $e^{-x/\varepsilon}$ term.

The inner solution in (5.4) fails as $\xi \to \infty$ due to the missing $\frac{\varepsilon^n \xi^n}{n!}$ terms.

By correcting either one we can obtain a uniformly valid asymptotic expansion called a *composite expansion* — this is useful for real answers/comparison with experiment.

It takes little effort to obtain the composite when using Van Dyke's matching rule — just use the composite operator:

$$C_{nm}y = E_n y + H_m y - E_n H_m y \ .$$

Note:

$$\begin{aligned} E_n C_{nm} y &= E_n y \ , \\ H_m C_{nm} y &= H_m y \ . \end{aligned}$$

For the example we have been considering

$$\begin{aligned} C_{22}y &= \left(e^{-x} + \varepsilon e^{-x}\right) + \left(\left(1 - e^{-x/\varepsilon}\right) + \varepsilon\left(1 - e^{-x/\varepsilon}\right) - x\right) - 1 + x - \varepsilon \\ &= (1+\varepsilon)\left(e^{-x} - e^{-x/\varepsilon}\right) \ . \end{aligned}$$

(i) This is correct to $\mathcal{O}(\varepsilon)$. Such expansions tend to be accurate to $\mathcal{O}\left(\varepsilon^{\min(m,n)}\right)$.

(ii) The expansion is not of Poincaré form — so it is not unique.

The above *additive composition* is not always [most] effective, e.g. if there are exponents or singularities in the expansions. However, other rules exist, for instance the *multiplicative composition*:

$$C_{nm}y = \frac{E_n y H_m y}{E_n H_m y} \ .$$

Alternatively, suppose that F is a sufficiently smooth functional with an inverse, then a composite expansion can be defined by

$$C_{nm}y \quad = \quad F^{-1}\Big\{F(E_n y) + F(H_m y) - F(E_n H_m y)\Big\} \ .$$

09/15 Hence, additive composition corresponds to $F(x) = x$, while multiplicative composition corresponds to $F(x) = \log(x)$.

5.7 Matching Involving Logarithms

5.7.1 A model equation

We consider a model equation which can be thought of representing heat conduction outside a spherical cavity with a weak nonlinear heat source. The equation can be written in two forms. In the first form the small parameter ε occurs in the equation

$$f_{rr} + \left(\frac{n-1}{r}\right) f_r + \varepsilon f f_r = 0 \quad , \quad f(1) = 0, \quad f \to 1 \quad \text{as} \quad r \to \infty \ , \tag{5.7}$$

while in the second form, with $\rho = \varepsilon r$, ε occurs in one of the boundary conditions

$$f_{\rho\rho} + \left(\frac{n-1}{\rho}\right) f_\rho + f f_\rho = 0 \quad ; \quad f(\varepsilon) = 0, \quad f \to 1 \quad \text{as} \quad \rho \to \infty \ . \tag{5.8}$$

14/06
13/13

5.7.2 The case $n = 3$

First seek a regular expression (r fixed, $\varepsilon \downarrow 0$):

$$f(r, \varepsilon) \sim f_0(r) + \varepsilon f_2(r) + \dots \ .$$

Then from substituting into (5.7) we find that

$$\begin{aligned}
\varepsilon^0: &\qquad f_0'' + \frac{2}{r} f_0' = 0\,, \quad f_0(1) = 0, \quad f_0 \to 1 \quad \text{as} \quad r \to \infty; \\
&\text{with solution} \quad f_0 = 1 - \frac{1}{r}\,, \\
\varepsilon^1: &\qquad \frac{1}{r^2}\left(r^2 f_2'\right)' = -f_0 f_0'\,, \quad f_2(1) = 0, \quad f_2 \to 0 \quad \text{as} \quad r \to \infty.
\end{aligned}$$

On integrating and applying $f_2(1) = 0$, we obtain

$$f_2 = A_2\left(1 - \frac{1}{r}\right) - \ln r\left(1 + \frac{1}{r}\right) \ . \tag{5.9}$$

The boundary condition at ∞, i.e. $f_2 \to 0$ as $r \to \infty$, cannot be satisfied for any choice of A_2. As a result the expansion cannot be uniformly asymptotic at large r. In fact for $r \gg 1$

$$f_0'' \sim -\frac{2}{r^3} \quad , \quad \varepsilon f_0 f_0' \sim \frac{\varepsilon}{r^2}\,,$$

and hence the $\mathcal{O}(\varepsilon)$ term is no longer a small correction to the equation when

$$r = \mathcal{O}\left(\frac{1}{\varepsilon}\right) \ .$$

Remark. Unfortunately, trying to derive the scaling from balancing the first two term of the series, i.e. $\varepsilon \ln r \sim 1$, does not work. Scalings are a black art.

13/14
10/17 Since $\varepsilon f_2 \sim \varepsilon \ \ln(1/\varepsilon)$ when $r = \mathcal{O}\big(\varepsilon^{-1}\big)$, we try the asymptotic sequence

$$1, \quad \varepsilon \ln(1/\varepsilon), \quad \varepsilon, \ \dots \ .$$

Note that we can view the $\ln(1/\varepsilon)$ term as coming from the particular integral:

$$f_2 = -\int_0^r \frac{ds}{s^2} \underbrace{\int_0^s t^2 f_0(t) f_0'(t)\, dt}_{\sim\ s \text{ as } s \to \infty.}$$

10/01
07/03
08/04

Asymptotic expansion for r fixed and $\varepsilon \downarrow 0$. Try the Poincaré expansion:

$$f \sim f_0 + \varepsilon \ln(1/\varepsilon)\, f_1 + \varepsilon f_2 + \dots\,, \quad f_j(1) = 0\,. \tag{5.10}$$

Substitute into (5.7) and solve.

$O(\varepsilon^0)$. At leading order, as before

$$f_0 = \left(1 - \frac{1}{r}\right)\,.$$

$O(\varepsilon \ln(1/\varepsilon))$. At this order the same linear equation is obtained as for f_0, hence

$$f_1 = A_1 \left(1 - \frac{1}{r}\right)\,.$$

$O(\varepsilon)$. This order is the same as (5.9), viz.

$$f_2 = A_2 \left(1 - \frac{1}{r}\right) - \ln r \left(1 + \frac{1}{r}\right)\,.$$

The constants A_1 & A_2 are to be determined by matching.

Asymptotic expansion for ρ fixed, and $\varepsilon \downarrow 0$. Try the Poincaré expansion:

$$f \sim g_0 + \varepsilon \ln(1/\varepsilon)\, g_1(\rho) + \varepsilon\, g_2(\rho) + \dots\,, \tag{5.11a}$$

where from the outer boundary consition

$$g_0(\infty) = 1\,, \quad g_1(\infty) = g_2(\infty) = 0\,. \tag{5.11b}$$

g_0 satisfies the nonlinear equation (5.8), which we note is satisfied if g_0 is a constant. Since $f_0 \to 1$ as $r \to \infty$ and $g_0(\infty) = 1$ we guess that $g_0 = 1$. Then on substitution of (5.11a) into (5.8) we obtain the same equation for both g_1 and g_2:

$$g_j'' + \frac{2}{\rho} g_j' + g_j' = 0\,, \qquad \text{i.e.} \qquad \left(\rho^2 e^{\rho} g_j'\right)' = 0\,,$$

with solution

$$g_j = B_j \int_\rho^\infty \frac{e^{-\tau}}{\tau^2}\, d\tau\,, \qquad g_j(\infty) = 0\,.$$

14/08

Match by intermediate variable to fix A_1, A_2, B_1 & B_2. First observe that (e.g. by integrating by parts)

$$\int_\rho^\infty \frac{e^{-\tau}}{\tau^2}\, d\tau \quad \sim \quad \frac{1}{\rho} + (\ln\rho + \gamma - 1) - \tfrac{1}{2}\rho + \mathrm{o}(\rho) \quad \text{as} \quad \rho \to 0\,,$$

where $\gamma = -\int_0^\infty e^{-\tau} \log\tau d\tau$ is the Euler[-–Mascheroni] constant. Introduce $\eta = \varepsilon^\alpha r = \varepsilon^{\alpha-1}\rho$, with $0 < \alpha < 1$. Take the limit of η fixed, $\varepsilon \downarrow 0$:

$$\text{(5.10):} \quad f \sim \underset{\boxed{1}}{1} \quad \underset{\boxed{3}}{-\frac{\varepsilon^\alpha}{\eta}} \quad + \dots \quad \underset{\boxed{5}}{+\varepsilon \ln(1/\varepsilon)\, A_1} \quad -\frac{\varepsilon^{1+\alpha}}{\eta} \ln(1/\varepsilon)\, A_1 \quad + \dots$$

$$\underset{\boxed{6}}{+\varepsilon A_2} \quad -\frac{\varepsilon^{1+\alpha}}{\eta} A_2 \quad \underset{\boxed{5}}{-\alpha\varepsilon \ln(1/\varepsilon)} \quad \underset{\boxed{6}}{-\varepsilon \ln\eta} \quad -\alpha\frac{\varepsilon^{1+\alpha}}{\eta} \ln(1/\varepsilon) \quad -\frac{\varepsilon^{1+\alpha}}{\eta} \ln\eta \; + \dots$$

$$\text{(5.11a):} \quad f \sim \underset{\boxed{1}}{1} \quad \underset{\boxed{2}}{+\frac{\varepsilon^\alpha}{\eta} \ln(1/\varepsilon)\, B_1} \quad \underset{\boxed{4}}{+\varepsilon(\ln(1/\varepsilon))^2 B_1(\alpha - 1)} \quad +\varepsilon \ln(1/\varepsilon) B_1(\ln\eta + (\gamma - 1)) \; + \dots$$

$$\underset{\boxed{3}}{+\frac{\varepsilon^\alpha}{\eta} B_2} \quad \underset{\boxed{5}}{+\varepsilon \ln(1/\varepsilon) B_2(\alpha - 1)} \quad \underset{\boxed{6}}{+\varepsilon B_2(\ln\eta + (\gamma - 1))} \qquad + \dots$$

Make the expansions agree:

$$
\begin{array}{lll}
\varepsilon^0 : & 1 = 1 \ ; & \\
\varepsilon^\alpha \ \ln(1/\varepsilon) : & 0 = B_1 \ , & B_1 = 0 \ ; \\
\varepsilon^\alpha : & -1 = B_2 \ , & B_2 = -1 \ ; \\
\varepsilon(\ln(1/\varepsilon))^2 : & 0 = B_1 \ , & \text{consistent} \ ; \\
\varepsilon \ln(1/\varepsilon) : & A_1 - \alpha = (\alpha - 1)B_2 \ , & A_1 = 1 \ ; \\
\varepsilon : & A_2 - \ln\eta = B_2 \ln\eta + (\gamma - 1)B_2 \ , & A_2 = 1 - \gamma \ .
\end{array}
$$

Hence

$$r \text{ fixed:} \qquad f \sim \left(1 - \frac{1}{r}\right) + \varepsilon\ln(1/\varepsilon)\left(1 - \frac{1}{r}\right) + \varepsilon\left((1-\gamma)\left(1 - \frac{1}{r}\right) - \ln r\left(1 + \frac{1}{r}\right)\right) + \dots \ ,$$

14/07
10/16

$$\rho \text{ fixed:} \qquad f \sim 1 - \varepsilon\int_\rho^\infty \frac{e^{-\tau}}{\tau^2}\, d\tau + \dots \ .$$

Match using Van Dyke's rule. Identify E and H with the coordinates r and ρ respectively. Then

$$
\begin{aligned}
H_2E_2f &= H_2\left[\left(1 - \frac{1}{r}\right) + \varepsilon\ln(1/\varepsilon)\, A_1\left(1 - \frac{1}{r}\right)\right] \\
&= 1 + \varepsilon\ln(1/\varepsilon)\, A_1 \ , \\
E_2H_2f &= E_2\left[1 + \varepsilon\ln(1/\varepsilon)\, B_1\int_\rho^\infty e^{-\tau}\,\frac{d\tau}{\tau^2}\right] \\
&= 1 + \frac{B_1}{r}\ln(1/\varepsilon) - \varepsilon\ln^2(1/\varepsilon)\, B_1 + B_1\varepsilon\ln(1/\varepsilon)\,(\ln r + \gamma - 1) \ .
\end{aligned}
$$

If these two expansions are to agree then $B_1 = 0$ and $A_1 = 0$, which is ***incorrect***. The trouble is a $\ln\rho$ in the $O(\varepsilon)$ term when $\rho = \mathcal{O}(1)$ — this changes to a $\varepsilon\ln(1/\varepsilon)$ term in the intermediate scaling.

In general, terms like $(\ln r)^p$ lead to failures near to the diagonal where $|n - m| < p$. However, in general there is success sufficiently far from the diagonal, e.g.

$$
\begin{aligned}
H_3E_2f &= 1 + \varepsilon\ln(1/\varepsilon)\, A_1 - \frac{\varepsilon}{\rho} \ , \\
E_2H_3f &= E_2\left[1 + (\varepsilon\ln(1/\varepsilon)\, B_1 + \varepsilon B_2)\int_\rho^\infty \frac{e^{-\tau}}{\tau^2}\, d\tau\right] \\
&= 1 + \frac{1}{r}\,(B_1\ln(1/\varepsilon) + B_2) - \varepsilon\ln(1/\varepsilon)\big(\ln(1/\varepsilon)\, B_1 + B_2\big) \\
&\qquad + \varepsilon\ln(1/\varepsilon)\, B_1\,(\ln r + \gamma - 1) \ ;
\end{aligned}
$$

so $B_1 = 0$, $B_2 = -1$, and $A_1 = 1$ as before.

It is best to apply Van Dyke's rule (and composite expansions) only at changes in the power of ε:

$$
\begin{array}{ccccccc}
1 & \varepsilon\ln(1/\varepsilon) & \varepsilon & \varepsilon^2\ln^2(1/\varepsilon) & \varepsilon^2\ln(1/\varepsilon) & \varepsilon^2 & \dots \ . \\
\uparrow & & \uparrow & & & \uparrow &
\end{array}
$$

10/15
10/18
10/19
10/20
10/22

Because of the way that logarithmic terms jump order, apply Van Dyke's rule only at the arrowed orders: ***do not split logs!*** The mindless application of rules can be dangerous.

5.7.3 The case $n = 2$

In this case the governing equation is

$$f_{rr} + \frac{1}{r}f_r + \varepsilon f f_r = 0 \ , \quad f(1) = 0 \quad , \quad f \to 1 \quad \text{as } r \to \infty.$$

Try a regular expansion, $f \sim f_0 + \varepsilon f_1 + \ldots$; then

$$\varepsilon^0: \qquad f_0'' + \frac{1}{r} f_0' = 0 \quad , \quad f_0 = A_0 \ln r + C_0 \,.$$

No choice of A_0 or C_0 will satisfy the boundary conditions both at $r = 1$ and as $r \to \infty$. Choose to satisfy the boundary condition at $r = 1$, i.e. set $C_0 = 0$. At next order

$$\varepsilon^1: \qquad f_1'' + \frac{1}{r} f_1' = -f_0 f_0' \,, \quad f_1 = A_1 \ln r + C_1 - A_0^2 \left(r \ln r - 2r + 2 \right) \,.$$

Again satisfy the boundary condition at $r = 1$ (i.e. $f_1(1) = 0$) – this time by setting $C_1 = 0$. Note that if $A_0 \neq 0$, then f_1 has even worse behaviour as $r \to \infty$ than f_0. By comparing where the expansion for f
14/13 becomes non-asymptotic, it follows that we should introduce $\rho = \varepsilon r$ as the stretched variable.

Note that when $r = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$,

$$f_0 \sim A_0 \ln(1/\varepsilon) \,.$$

Since $f_0 \sim 1$ as $\rho \to \infty$, this suggests trying $A_0 = \frac{1}{\ln(1/\varepsilon)}$, and the asymptotic sequence

$$1 \quad , \quad \frac{1}{\ln(1/\varepsilon)} \quad , \quad \frac{1}{\left(\ln(1/\varepsilon)\right)^2} \quad , \quad \ldots \,.$$

15/06

Remark. This asymptotic sequence is likely to have non-wonderful convergence properties.

Asymptotic expansion for r fixed and $\varepsilon \downarrow 0$. We propose the asymptotic expansion

$$f(r, \varepsilon) \sim 0 + \frac{f_1(r)}{\ln(1/\varepsilon)} + \frac{f_2}{\left(\ln(1/\varepsilon)\right)^2} + \ldots \,. \tag{5.12}$$

Then

$$f_n'' + \frac{1}{r} f_n' = 0 \,, \quad \text{and} \quad f_n = A_n \ln r \,.$$

Note that the $\varepsilon f f'$ term never enters into the expansion for $r = O(1)$.

Asymptotic expansion for ρ fixed and $\varepsilon \downarrow 0$. In this case we propose

$$f \sim 1 + \frac{g_1(\rho)}{\ln(1/\varepsilon)} + \frac{g_2(\rho)}{\left(\ln(1/\varepsilon)\right)^2} + \ldots \,. \tag{5.13}$$

Then

$$\begin{aligned}
& g_1'' + \left(\frac{1}{\rho} + 1\right) g_1' = 0 \,, \\
& g_1 = B_1 \int_\rho^\infty \frac{e^{-\tau}}{\tau} \, d\tau = B_1 E_1(\rho) \,; \\
& g_2'' + \left(\frac{1}{\rho} + 1\right) g_2' = -g_1 g_1' \,, \\
& g_2 = B_2 E_1(\rho) - B_1^2 \left(e^{-\rho} E_1(\rho) - 2E_1(2\rho) \right) \,.
\end{aligned}$$

Match using the intermediate variable

$$\eta = \varepsilon^\alpha r = \varepsilon^{\alpha - 1} \rho \qquad (0 < \alpha < 1)$$

and the asymptotic expansion

$$E_1(\rho) \to -\ln \rho - \gamma + \rho + \mathcal{O}\left(\rho^2\right) \quad \text{as} \quad \rho \to 0 \,.$$

Then

$$f \sim \frac{1}{\ln(1/\varepsilon)} A_1 \left(\alpha \ln(1/\varepsilon) + \ln \eta\right) + \frac{1}{\left(\ln(1/\varepsilon)\right)^2} A_2 \left(\alpha \ln(1/\varepsilon) + \ln \eta\right) + \dots \tag{5.14}$$

(5.12)

$$\begin{aligned} f \sim & 1 + \frac{B_1}{\ln(1/\varepsilon)} \left(-(\alpha-1)\ln(1/\varepsilon) - \ln\eta - \gamma + \varepsilon^{1-\alpha}\eta + \dots\right) \\ & + \frac{1}{\left(\ln(1/\varepsilon)\right)^2} \Big(B_2 \left[-(\alpha-1)\ln(1/\varepsilon) - \ln\eta - \gamma + \dots\right] \\ & \qquad + B_1^2 \left[-(\alpha-1)\ln(1/\varepsilon) - \gamma - \ln\eta - \ln 4 + \dots\right]\Big) . \end{aligned} \tag{5.15}$$

(5.13)

On equating equal orders of ε we find that (again noting that terms jump order)

$\ln^0(1/\varepsilon)$: $\quad \alpha A_1 = 1 - (\alpha - 1)B_1$,
— if this is true $\forall\alpha$ then $B_1 = -1$, $A_1 = 1$;

$\ln^{-1}(1/\varepsilon)$: $\quad A_1 \ln\eta + \alpha A_2 = - B_1 \left(\ln\eta + \gamma\right) - B_2 \left(\alpha - 1\right) - B_1^2 \left(\alpha - 1\right)$,
— if this is true $\forall\alpha,\eta$ then $B_2 = -(1+\gamma)$, $A_2 = \gamma$.

11/17

Match by Van Dyke's Rule (if you must). Put $\alpha = 1$ and $\eta = \rho$ in (5.14), and $\alpha = 0$ and $\eta = r$ in (5.15). Then Van Dyke's rule gives

$$\begin{array}{lll} E_1 H_1 = 1\,, & H_1 E_1 = 0\,, & \textbf{\textit{Contradictory}}\ ; \\ E_2 H_1 = 1\,, & H_1 E_2 = A_1\,, & A_1 = 1\,; \\ E_1 H_2 = 1 + B_1\,, & H_2 E_1 = 0\,, & B_1 = -1\,. \end{array}$$

Similarly

$$\left.\begin{aligned} E_2 H_2 &= 1 + B_1 - \frac{B_1}{\ln(1/\varepsilon)} \left(\ln r + \gamma\right) \\ H_2 E_2 &= A_1 \left(1 + \frac{\ln\rho}{\ln(1/\varepsilon)}\right) = \frac{A_1 \ln r}{\ln(1/\varepsilon)} \end{aligned}\right\} \textbf{\textit{Contradictory}}.$$

However

$$\begin{aligned} E_3 H_2 &= 1 + B_1 - \frac{B_1}{\ln(1/\varepsilon)} \left(\ln r + \gamma\right) \\ H_2 E_3 &= A_1 + \frac{1}{\ln(1/\varepsilon)} \left(A_1 \ln\rho + A_2\right) = \frac{A_1 \ln r}{\ln(1/\varepsilon)} + \frac{A_2}{\ln(1/\varepsilon)}\,, \end{aligned}$$

and hence

$$A_1 = 1\ , \quad B_1 = -1\ , \quad A_2 = \gamma\ .$$

11/01 As before, Van Dyke's rule works if $n \neq m$.

08/03
09/04
15/08
14/14

5.7.4 A 'terrible' problem

Consider the equation with $n = 2$ plus a new term:

$$f_{rr} + \frac{1}{r} f_r + f_r^2 + \varepsilon f f_r = 0\ , \quad f(1) = 0, \quad f \to 1 \quad \text{as} \quad r \to \infty\ .$$

First compare the size of terms using the solution calculated in §5.7.3:

$$\begin{array}{lllll} r = \text{ord}(1), & f \sim \dfrac{1}{\ln(1/\varepsilon)} & , \ f_r^2 \sim \left(\dfrac{1}{\ln(1/\varepsilon)}\right)^2, & f_{rr} \sim \dfrac{1}{\ln(1/\varepsilon)}\ , & \\ \rho = \text{ord}(1), & f \sim 1 & , \ f_\rho \sim \dfrac{1}{\ln(1/\varepsilon)} & , \ f_\rho^2 \sim \left(\dfrac{1}{\ln(1/\varepsilon)}\right)^2, & f_{\rho\rho} \sim \dfrac{1}{\ln(1/\varepsilon)}\ . \end{array}$$

From this comparison of terms we might expect a small perturbation to the previous answer.

Asymptotic expansion for r fixed. As in §5.7.3 propose the asymptotic expansion

$$f \sim \frac{1}{\ln(1/\varepsilon)}\, f_1 + \frac{1}{\big(\ln(1/\varepsilon)\big)^2}\, f_2 + \frac{1}{\big(\ln(1/\varepsilon)\big)^3}\, f_3 + \dots \,. \tag{5.16}$$

Then from substituting into the equation we find:

$$\begin{array}{llll} \ln^{-1}(1/\varepsilon): & f_1'' + \dfrac{1}{r} f_1' = 0 & , & f_1 = A_1 \ln r \,, \\ \ln^{-2}(1/\varepsilon): & f_2'' + \dfrac{1}{r} f_2' = -{f_1'}^2 & , & f_2 = A_2 \ln r - \tfrac{1}{2} A_1^2 \ln^2 r \,, \\ \ln^{-3}(1/\varepsilon): & f_3'' + \dfrac{1}{r} f_3' = -2 f_1' f_2' & , & f_3 = A_3 \ln r + \tfrac{1}{3} A_1^3 \ln^3 r - A_1 A_2 \ln^2 r \,. \end{array}$$

By induction, one can show that as $r \to \infty$,

$$f_n \sim (-)^n \left(-\frac{1}{n} A_1^n \ln^n r + A_1^{n-2} A_2 \ln^{n-1} r \right) \,,$$

and hence by summation that

$$f \sim \ln \left[1 + \left(\frac{A_1}{\ln(1/\varepsilon)} + \frac{A_2}{\big(\ln(1/\varepsilon)\big)^2} + \dots \right) \ln r \right] \qquad \text{as} \quad r \to \infty \,.$$

15/07
11/16

Lemma (for future reference). Instead of adopting the above approach, ignore §5.7.3 and assume

$$f = f_0 + \dots \,.$$

Then

$$f_0'' + \frac{1}{r} f_0' + {f_0'}^2 = 0 \quad \Rightarrow \quad f_0 = \ln\left(1 + A \ln r\right) \quad \text{if} \quad f_0(1) = 0 \,.$$

If $A = \mathcal{O}\left(\ln^{-1}(1/\varepsilon)\right)$, this suggests that the natural variable is, say,

$$t = \frac{\ln r}{\ln(1/\varepsilon)} \,. \qquad \square \tag{5.17}$$

Asymptotic expansion for ρ fixed. In this variable ε does not appear in the equation:

$$f_{\rho\rho} \quad + \tfrac{1}{\rho} f_\rho + f_\rho^2 + f f_\rho = 0 \,.$$

We pose the Poincaré expansion:

$$f \sim 1 + \frac{g_1}{\ln(1/\varepsilon)} + \frac{g_2}{\big(\ln(1/\varepsilon)\big)^2} + \dots \,. \tag{5.18}$$

Substitute, equate, etc:

$$\ln^{-1}(1/\varepsilon): \qquad g_1'' + \frac{1}{\rho} g_1' + g_1' = \frac{e^{-\rho}}{\rho} (\rho e^{\rho} g_1')' = 0 \,,$$

$$g_1 = B_1 \int_\rho^\infty \frac{e^{-\tau}}{\tau}\, d\tau = B_1 E_1(\rho) \,, \qquad \text{setting } g_1(\infty) = 0 \,.$$

$$\ln^{-2}(1/\varepsilon): \qquad \frac{e^{-\rho}}{\rho} (\rho e^{\rho} g_2')' = -{g_1'}^2 - g_1 g_1' \,,$$

$$g_2 = B_2 E_1(\rho) + B_1^2 \,\left(2E_1(2\rho) - \tfrac{1}{2} E_1^2(\rho) - e^{-\rho} E_1(\rho) \right) \,.$$

As $\rho \to 0$ we have

$$\begin{aligned} g_1 &\sim B_1 \left(-\ln\rho - \gamma \right) \,, \\ g_2 &\sim B_2 \left(-\ln\rho - \gamma \right) + B_1^2 \,\left(-\tfrac{1}{2} \ln^2 \rho - (\gamma + 1) \ln\rho - \tfrac{1}{2} \gamma^2 - \gamma - \ln 4 \right) \,. \end{aligned}$$

15/13

The leading-order behaviour as $\rho \to 0$, $g_2 \sim -\frac{1}{2} B_1^2 \ln^2 \rho$, comes from the balance

$$\frac{1}{\rho}(\rho g_2')' \sim -{g_1'}^2 \sim \frac{B_1^2}{\rho^2} \; .$$

Similarly we can show that for small ρ

$$g_3'' + \frac{1}{\rho} g_3' \sim -2 g_1' g_2' \sim -2B_1^3 \frac{\ln \rho}{\rho^2} - \big(2B_1^3(\gamma+1) + 2B_1 B_2\big) \frac{1}{\rho^2} \; ,$$
$$g_3 \sim -\tfrac{1}{3} B_1^3 \ln^3 \rho - \big(B_1^3(\gamma+1) + B_1 B_2\big) \ln^2 \rho \; .$$

By induction it is possible to conclude that as $\rho \to 0$

$$g_n \quad \sim \quad -\frac{1}{n} B_1^n \ln^n \rho - \big(B_1^n(\gamma+1) + B_1^{n-2} B_2\big) \ln^{n-1} \rho \; .$$

11/15

Match using the intermediate variable

$$\eta = \varepsilon^\alpha r = \rho \varepsilon^{\alpha-1} \; .$$

Then

(5.16) :
$$\begin{aligned} f \sim & \frac{1}{\ln(1/\varepsilon)} A_1 \; (\alpha \ln(1/\varepsilon) + \ln \eta) \\ & + \frac{1}{\ln^2(1/\varepsilon)} \left[-\tfrac{1}{2} A_1^2 \; (\alpha \ln(1/\varepsilon) + \ln \eta)^2 + \dots \right] + \dots \\ & + \frac{1}{\ln^n(1/\varepsilon)} \left[\frac{(-)^{n+1}}{n} A_1^n \; (\alpha \ln(1/\varepsilon) + \ln \eta)^n + \dots \right] + \dots \; ; \end{aligned}$$

(5.18) :
$$\begin{aligned} f \sim & 1 + \frac{B_1}{\ln(1/\varepsilon)} \left[-(\alpha-1)\ln(1/\varepsilon) - \ln \eta - \gamma + \dots \right] \\ & + \frac{1}{\ln^2(1/\varepsilon)} \left[-\frac{B_1^2}{2} \Big(((\alpha-1)\ln(1/\varepsilon) + \ln \eta)^2 + \dots \Big) + \dots \right] + \dots \\ & + \frac{1}{\ln^n(1/\varepsilon)} \left[-\frac{B_1^n}{n} \; ((\alpha-1)\ln(1/\varepsilon) + \ln \eta)^n + \dots \right] + \dots \; . \end{aligned}$$

Equate these two expansions. At leading order

$\ln^0(1/\varepsilon)$:
$$\alpha A_1 - \tfrac{1}{2}\alpha^2 A_1^2 + \tfrac{1}{3}\alpha^3 A_1^3 + \dots = 1 - B_1(\alpha - 1) - \tfrac{1}{2} B_1^2 \, (\alpha-1)^2 + \dots \; .$$

or from summing the series

$$\ln(1 + \alpha A_1) = 1 + \ln\left[1 - (\alpha-1) B_1\right] \; .$$

This must be true $\forall \alpha$, hence

$$e(1 + B_1) - 1 = 0 \quad , \quad A_1 + e B_1 = 0 \; ,$$

i.e.

$$B_1 = -\left(\frac{e-1}{e}\right) \quad , \quad A_1 = (e-1) \; .$$

Note that:

- in matching, an infinite number of terms jumped order — hence the need for general expressions
11/18 for f_n & g_n;
11/19
11/20
- hence there is no hope for Van Dyke's rule. ☹
11/22

Is there an easier way? Recall from earlier that a natural variable is

$$t = \frac{\ln r}{\ln(1/\varepsilon)} . \tag{5.19}$$

Note that

$$\{r=1\} \equiv \{t=0\} ,$$

and that

$$\rho = \mathrm{ord}(1) \quad \text{when} \quad r = \frac{k}{\varepsilon} \quad \text{for} \quad k = \mathrm{ord}(1) ,$$

i.e. when

$$t = \underset{\substack{\uparrow \\ \text{finite value}}}{1} + \frac{\ln k}{\ln(1/\varepsilon)} \quad \text{for} \quad k = \mathrm{ord}(1).$$

Let $\tau = 1 - t$, so that $\rho = \mathrm{ord}(1)$ when $\tau = \mathrm{ord}(\ln^{-1}(1/\varepsilon))$. Next substitute into the equation to obtain

$$f_{\tau\tau} + f_\tau^2 = -\ln(1/\varepsilon)\, e^{-\tau \ln(1/\varepsilon)} f f_\tau .$$

Seek a Poincaré expansion for $\tau > 0$ (so that the r.h.s. is 'exponentially' small):

$$f = f_0 + \frac{1}{\ln(1/\varepsilon)} f_1 + \dots , \tag{5.20}$$

then

$$f_{0\tau\tau} + f_{0\tau}^2 = 0 .$$

If we require $f_0(1) = 0$, then

$$f_0 = \log\left(1 + \alpha_0 (1-\tau)\right) . \tag{5.21a}$$

We need to match with the outer solution that is valid for $\rho = \mathrm{ord}(1)$, i.e. we need to match with the solution that is valid in the region where $\tau = \mathrm{ord}(\ln^{-1}(1/\varepsilon))$. Since

$$\tau = 1 + \frac{\ln 1/r}{\ln(1/\varepsilon)} = \frac{\ln 1/\rho}{\ln(1/\varepsilon)} ,$$

introduce

$$s = \ln\rho = -(\ln(1/\varepsilon))\tau$$

and seek an expansion

$$f = 1 + \frac{G_1(s)}{\ln(1/\varepsilon)} + \frac{G_2(s)}{\left(\ln(1/\varepsilon)\right)^2} + \dots . \tag{5.21b}$$

As before

$$\begin{aligned} G_1 &= B_1 \int_{e^s}^{\infty} \frac{e^{-u}}{u}\, du , \\ G_1 &\to B_1 \left(-s - \gamma + \dots\right) \qquad \text{as} \quad s \to -\infty. \end{aligned}$$

Now try matching by Van Dyke's rule using $s = -(\ln(1/\varepsilon))\tau$:

$$\begin{aligned} H_2E_1 f &= H_2\left[\log\left(1 + \alpha_0 + \frac{\alpha_0 s}{\ln(1/\varepsilon)}\right)\right] = \ln(1+\alpha_0) + \frac{\alpha_0 s}{(1+\alpha_0)\ln(1/\varepsilon)} , \\ E_1H_2 f &= E_1\left[1 + \frac{B_1}{\ln(1/\varepsilon)}\left((\ln(1/\varepsilon))\tau - \gamma + \dots\right)\right] = 1 + B_1\tau = 1 - \frac{B_1 s}{\ln(1/\varepsilon)} . \end{aligned}$$

Hence, as before,

$$\alpha_0 = e - 1 \quad , \quad B_1 = \frac{1-e}{e} .$$

16/06
15/14

5.8 Strained Coordinates

The method of strained co-ordinates is a better, but less general way, of solving certain singular perturbation problems. However, usually such problems can also be solved either by using *MAEs*, or by means of the method of *Multiple Scales*.

6 A Little More on Asymptotics Beyond All Orders

As we have seen in the case of Stokes lines, sometimes it is *not* sufficient to consider just the algebraic asymptotic expansion of a solution. This section is concerned with looking at further examples where exponentially small terms can play a key role. For a more general overview I recommend *The Devil's invention: asymptotic, super-asymptotic and hyper-asymptotic series* by John P. Boyd (Acta Applicandae, **56**, 1-98, 1999) which is available at

http://www-personal.engin.umich.edu/~jpboyd/boydactaapplicreview.pdf

6.1 More on What Happens at Stokes Lines

In §3.7 we looked at what happens near the Stokes line of the Airy function when $\arg\lambda = 2\pi/3$. In this section we return to the 'turn-on' of the sub-dominant exponentially small term at a Stokes line, but this time for the complementary error function.

6.1.1 The complementary error function

There are a number of ways of getting a handle on what happens at Stokes lines. In §3.7 we used Borel summation and and an integral estimate obtained using steepest descents in the complex plane. Here we will use a differential equations approach for model problem of the complementary error function:

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\,\mathrm{d}t\,. \tag{6.1}$$

From the first part of the course (see also (3.12a), (3.12b) and (3.12c))

$$\operatorname{erfc}(z) \sim \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{(-)^s (2s)!}{s!(4z^2)^s} \quad \text{for} \quad |\arg(z)| < \tfrac{3}{4}\pi\,, \tag{6.2a}$$

$$\operatorname{erfc}(z) \sim 2 + \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{(-)^s (2s)!}{s!(4z^2)^s} \quad \text{for} \quad |\arg(-z)| < \tfrac{3}{4}\pi\,. \tag{6.2b}$$

We note that erfc and '2' are solutions to the differential equation

$$w'' + 2zw' = 0\,. \tag{6.3}$$

Moreover, if we let

$$\operatorname{erfc}(z) = \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{s=0}^{N-1} \frac{(-)^s (2s)!}{s!(4z^2)^s} + R_N\,, \tag{6.4a}$$

then

$$R_N'' + 2zR_N' = -\frac{e^{-z^2}}{z\sqrt{\pi}} \frac{(-)^N (2N)!}{(N-1)!\,4^{N-1} z^{2N}}\,. \tag{6.4b}$$

Write $z = re^{i\theta}$ and consider the case of fixed r, so that

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{-ie^{-i\theta}}{r} \frac{\mathrm{d}}{\mathrm{d}\theta}\,. \tag{6.5}$$

12/17

Then (6.4b) becomes

$$-\frac{e^{-2i\theta}}{r^2}(R_N)_{\theta\theta} + i\left(\frac{e^{-2i\theta}}{r^2} - 2\right)(R_N)_\theta = -\frac{\exp\left(-r^2 e^{2i\theta} + i\pi N - (2N+1)i\theta\right)(2N)!}{\sqrt{\pi}(N-1)!4^{N-1}r^{2N+1}}\,. \tag{6.6}$$

Now assume that $|z| = r \gg 1$, then it is possible to show using Stirling's formula (see § 3.5.2) that the right-hand-side forcing is smallest when $N \sim r^2$. Guessing that the remainder, R_N, will be smallest then, we let

$$N = \mathrm{int}(r^2)\,, \quad r^2 = N + \alpha\,, \tag{6.7a}$$

then

$$\mathrm{RHS} \sim -\frac{8r}{\sqrt{2\pi}} \exp\left(-i\alpha(2\theta - \pi) - i\theta\right) \exp\left(-r^2\left(e^{2i\theta} + 1 + i(2\theta - \pi)\right)\right)\,. \tag{6.7b}$$

This has a local maximum when $\cos(2\theta) = -1$, i.e. when $\theta = \pm\pi/2$. Moreover we note that when
06/04 $\theta = \pm\pi/2$, then at leading order the RHS both stops oscillating and is independent of α.
12/16

On the basis of this try an asymptotic rescaling of the form

$$r = \frac{\rho}{\varepsilon}\,, \quad \theta = \frac{\pi}{2} + \delta\phi\,, \tag{6.8}$$

where $\varepsilon \ll 1$ and $\delta \ll 1$ (and for simplicity we have focused close to $\theta = +\frac{\pi}{2}$). Then, from (6.7a), $N = O(\varepsilon^{-2})$, and

$$\begin{aligned}\frac{\varepsilon^2}{\delta^2}\frac{e^{-2i\delta\phi}}{\rho^2}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}R_N \;-\; & \frac{i}{\delta}\left(\frac{\varepsilon^2\, e^{-2i\delta\phi}}{\rho^2} + 2\right)\frac{\mathrm{d}}{\mathrm{d}\phi}R_N \\ \sim\; & -\frac{8\rho i}{\varepsilon\sqrt{2\pi}} e^{(-2i\delta\phi\alpha - i\delta\phi)} \exp\left(\frac{\rho^2}{\varepsilon^2}\left(e^{2i\delta\phi} - 1 - 2i\delta\phi\right)\right) \\ \sim\; & -\frac{8\rho i}{\varepsilon\sqrt{2\pi}} \exp\left(-2\rho^2\phi^2\frac{\delta^2}{\varepsilon^2}\right)\,.\end{aligned} \tag{6.9}$$

There is a distinguished scaling when $\delta = \mathcal{O}(\varepsilon)$; for simplicity take $\delta = \varepsilon$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\phi}R_N \sim \frac{4\rho}{\sqrt{2\pi}} \exp\left(-2\rho^2\phi^2\right)\,, \tag{6.10}$$

and thus

$$R_N \sim A + \mathrm{erf}\left(\sqrt{2}\rho\phi\right)\,, \tag{6.11}$$

where A is a constant. We recall that

$$z = \frac{\rho}{\varepsilon}\exp\left(i\left(\frac{\pi}{2} + \varepsilon\phi\right)\right)\,,$$

and hence for 'matching' with (6.2a) we deduce that that we require that $R_N \to 0$ as $\phi \to -\infty$, i.e. we require $A = 1$. Thus

12/15
$$R_N \sim 1 + \mathrm{erf}\left(\sqrt{2}\rho\phi\right)\,. \tag{6.12}$$

We can interpret this result as saying that within an angle of $\mathcal{O}(|z|^{-1})$ of $\arg z = \pm\frac{\pi}{2}$, the sub-dominant

term is 'turned on' by an error function (as was the case for the Airy function). In fact the turn on by
12/18 an error function is generic.

This agrees with question 1 on Example Sheet 2. There you will find that

$$\operatorname{erfc}(z) = \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{s=0}^{N-1} \frac{(-)^s (2s)!}{s!(4z^2)^s} + R_N\,, \tag{6.13}$$

where

$$R_N = \frac{2}{\sqrt{\pi}} \frac{(-)^N (2N)!}{2^{2N} N!} \int_z^\infty \frac{e^{-t^2}}{t^{2N}} \mathrm{d}t\,. \tag{6.14}$$

On the basis of the above scaling (i.e. $\varepsilon = \delta = \mathrm{O}(N^{-\frac{1}{2}})$), take

$$z = N^{\frac{1}{2}} \exp\left(\frac{i\pi}{2} + \frac{i\psi}{N^{\frac{1}{2}}}\right). \tag{6.15}$$

For large N try applying the method of steepest descents to (6.14). The stationary point is found to occur at $t = iN^{\frac{1}{2}}$. Let $t = iN^{\frac{1}{2}} e^{iv/N^{\frac{1}{2}}}$, then

$$\begin{aligned} -t^2 - 2N\log t &= Ne^{2iv/N^{\frac{1}{2}}} - 2N\log(iN^{\frac{1}{2}}) - 2N^{\frac{1}{2}} iv \\ &\sim N(1 - \log N - i\pi) - 2v^2 + \ldots\,. \end{aligned} \tag{6.16}$$

Hence

$$\begin{aligned} \int_z^\infty \frac{e^{-t^2}}{t^{2N}} \mathrm{d}t &\sim -\int_\psi^{-\infty} \frac{(-)^N e^N}{N^N} e^{-2v^2} \mathrm{d}v \\ &\sim \frac{(-)^N e^N}{\sqrt{2} N^N} \int_{-\infty}^{\sqrt{2}\psi} e^{-w^2} \mathrm{d}w \\ &\sim \frac{(-)^N e^N}{2N^N} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(1 + \operatorname{erf}\left(\sqrt{2}\psi\right)\right), \end{aligned} \tag{6.17}$$

18/07
18/08 and thus, as before,

12/19
12/20

$$R_N \sim \left(1 + \operatorname{erf}\left(\sqrt{2}\psi\right)\right), \tag{6.18}$$

12/22 since $\rho\phi \sim \psi$.

6.2 A Model Equation (With Wider Implications)

Consider the asymptotic solution to

$$f_{yy} + \lambda^3(1 + iy)f = -\lambda^2, \qquad f \to 0 \text{ as } |y| \to \infty\,, \tag{6.19}$$

for large $|\lambda|$, and real y. Try

$$\lambda f = f_0 + \frac{f_1}{\lambda^3} + \frac{f_2}{\lambda^6} + \cdots = \sum_{n=0}^\infty \frac{f_n}{\lambda^{3n}}\,. \tag{6.20}$$

Then

$$f_0 = -\frac{1}{1 + iy}\,, \quad \text{and for } n = 0, 1, 2, \ldots \quad f_{n+1} = -\frac{f_n''}{(1 + iy)}\,.$$

Hence

$$f_1 = -\frac{2}{(1 + iy)^4}\,, \text{ etc.}\,.$$

Thus an asymptotic expansion can be found to all orders, irrespective of the sign of λ. Further, the expansion satisfies the boundary conditions as $|y| \to \infty$. However the expansion (6.20) is only valid $\forall y$ if
16/14 $\lambda \to -\infty$.

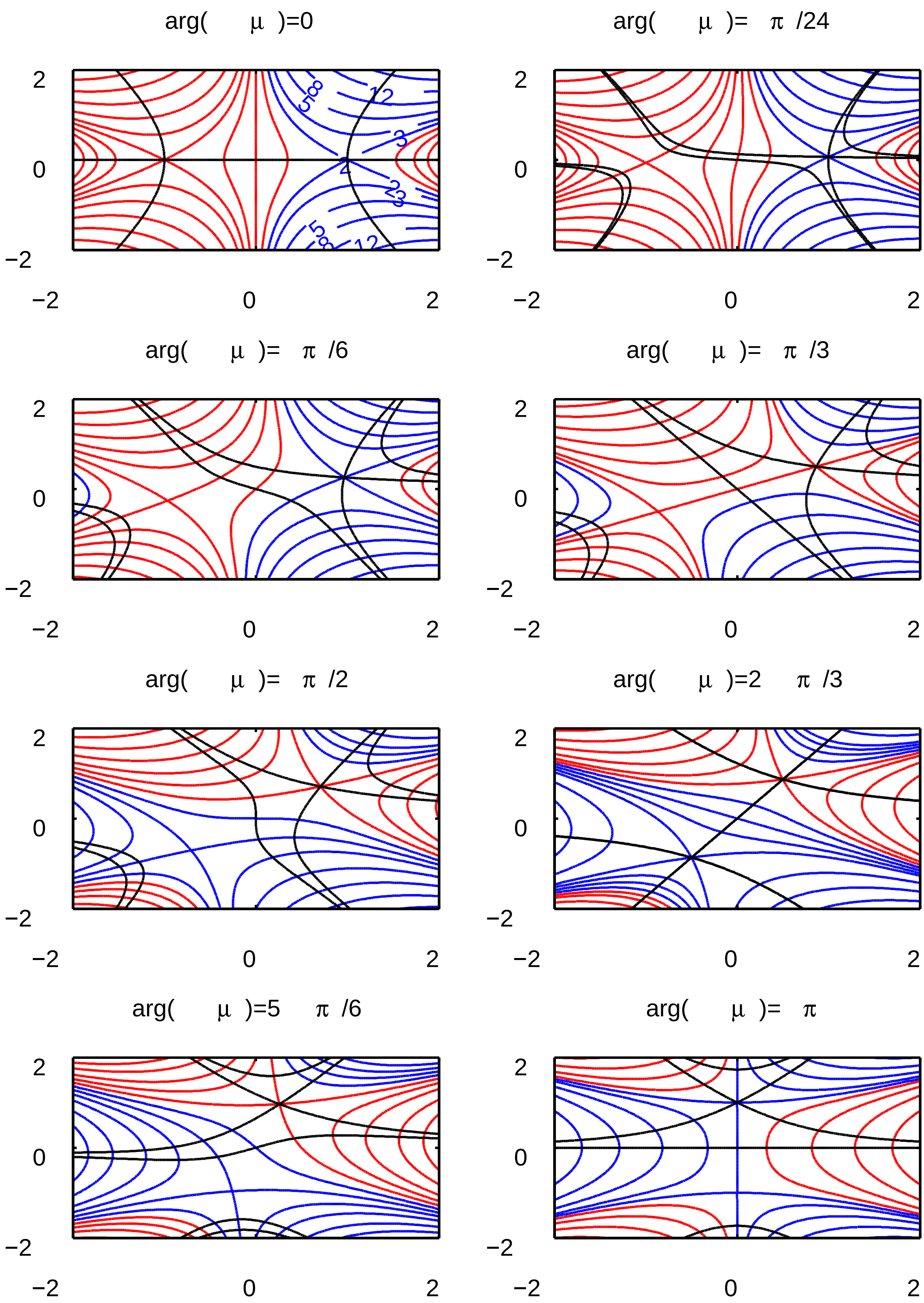


Figure 6.4: Contours of $\mathrm{Re}(3\mu z - z^3)$ (blue: high; red: low), and $\mathrm{Im}(3\mu z - z^3)$ (black).

To see this we start from the fact that the exact solution is

$$f(y,\lambda) = \int_{\mathcal{C}} \exp\left(\lambda(1+iy)z - \tfrac{1}{3}z^3\right) dz \,, \tag{6.21}$$

where $\mathcal{C}$ starts from $z = 0$ and extends to $z = \infty$ in the sector $|\arg(z)| < \pi/6$. For large $|\lambda|$, we can estimate the integral using steepest descents. In the figure 6.4 we plot contours of $\mathrm{Re}(3\mu z - z^3)$ and $\mathrm{Im}(3\mu z - z^3)$, where $\mu = \lambda(1+iy)$. There are two cases to consider.

$\lambda \to -\infty$. If $\lambda \to -\infty$, then $|\pi - \arg\mu| < \pi/2$, in which case we deduce from figure 6.4 that (6.20) is recovered by Watson's Lemma.

$\lambda \to +\infty$. However, if $\lambda \to +\infty$, then the asymptotic behaviour depends crucially on whether the Watson's lemma contribution from the end point at $z = 0$ is larger or smaller than the Laplace's method contribution from the saddle point. As indicated in figure 6.4, if $\pi/3 < |\arg\mu| < \pi/2$, i.e. if $|y| > \sqrt{3}$, then the Watson's lemma contribution dominates, and (6.20) is again recovered. However, if $|\arg\mu| < \pi/3$, i.e. if $|y| < \sqrt{3}$, then the Laplace's method contribution dominates and

$$f \sim \frac{\pi^{\frac{1}{2}}}{\lambda^{\frac{1}{4}}(1+iy)^{\frac{1}{4}}} \exp\left(\tfrac{2}{3}\lambda^{\frac{3}{2}}(1+iy)^{\frac{3}{2}}\right) \,; \tag{6.22}$$

this is exponentially large.

To understand this result, note that equation (6.19) has a turning point at

$$1 + iy = 0 \,.$$

Set

$$y = i + \left(\frac{i}{\lambda^3}\right)^{\frac{1}{3}} s \,,$$

then

$$f_{ss} - sf = -i^{\frac{2}{3}} \,.$$

The complementary function solutions to this equation are $\mathrm{Ai}(s)$ and $\mathrm{Bi}(s)$, which have *anti*-Stokes lines in the complex s-plane at

$$\arg s = -\frac{\pi}{3}, \frac{\pi}{3}, \pi \,.$$

We plot these *anti*-Stokes lines in the complex y-plane:

$\lambda > 0$ $\lambda < 0$

$\lambda > 0$

Hence when $\lambda > 0$, we see that since two *anti*-Stokes lines cross the real y axis, the solution that decays as $|y| \to \infty$ can be exponentially small as $\lambda \to \infty$ for $|y| > \sqrt{3}$, but exponentially large for $|y| < \sqrt{3}$. This is not possible when $\lambda < 0$, since only one *anti*-Stokes line crosses the real y-axis. Note that in the case when $\lambda > 0$, it is possible to get from $y = -\infty$ to $y = \infty$ without seeing the exponentially large solution, by deforming into the complex y-plane.

The idea of deforming into the complex plane to sidestep regions where the solution is exponentially large
10/04 has wider applications (e.g. eigenvalue problems in stability, nonlinear models of crystal growth).

17/13

13/16

13/17 6.3 A Model of Crystal Growth (Unlectured)

A simple geometric model of crystal growth is:

$$\varepsilon^2 \theta''' + \theta' = \cos\theta \qquad -\infty < s < \infty \; ; \tag{6.23}$$

ε	represents	surface tension;
s	—— $''$ ——	arclength along the solid-liquid interface;
$\theta(s,\varepsilon)$	—— $''$ ——	the angle between the local normal and the direction of propagation of the crystal.

A 'needle crystal' is a monotonic solution satisfying

$$\theta(s,\varepsilon) \to \pm\frac{\pi}{2} \quad \text{as} \quad s \to \pm\infty \; . \tag{6.24}$$

6.3.1 Regular perturbation

Try

$$\theta = \theta_0 + \varepsilon^2\theta_1 + \varepsilon^4\theta_2 + \ldots \; . \tag{6.25}$$

We fix the apex at $s=0$ by requiring that $\theta_j(0)=0$.

$$\varepsilon^0: \qquad \theta_0' - \cos\theta_0 = 0 \ ,$$
$$\theta_0 = -\frac{\pi}{2} + 2\tan^{-1}(e^s) \ ,$$
$$\theta_0 \to \pm\frac{\pi}{2} \quad \text{as} \quad s \to \pm\infty \ ,$$
θ_0 increases monotonically.

$$\varepsilon^2: \qquad \theta_1' + \sin\theta_0\,\theta_1 = -\theta_0''' \ ,$$
$$\theta_1 = (2\tanh s - s)\,\mathrm{sech}\, s \ ,$$
$$\theta_1 \to 0 \quad \text{as} \quad s \to \pm\infty \ .$$

$$\varepsilon^4: \qquad \theta_2 = \left(-\tfrac{1}{2}s^2\tanh s + 5s - 4s\,\mathrm{sech}^2 s - \tfrac{32}{3}\tanh s + \tfrac{50}{3}\tanh s\,\mathrm{sech}^2 s\right)\mathrm{sech}\, s \ ,$$
$$\theta_2 \to 0 \quad \text{as} \quad s \to \pm\infty \ .$$

It is possible to prove that: (a) $\theta_j(-s) = -\theta_j(s) \Rightarrow \theta_j''(0) = 0$,
(b) $\sum_0^N \varepsilon^{2n}\theta_j(s) \mp \pi/2 \to 0$ as $s \to \pm\infty$,
(c) the solution is monotonic for small ε.

18/14 Hence we appear to have a solution correct to all orders!

6.3.2 Too many boundary conditions

How many boundary conditions are implied by (6.24)? Suppose we linearise about $s=-\infty$ by setting
$$\theta = -\frac{\pi}{2} + \alpha e^{ms} \ .$$
We find that
$$\varepsilon^2 m^3 + m = 1$$
$$m = \begin{cases} 1 - \varepsilon^2 + \dots & \text{decays as } s \to -\infty \\ \pm\dfrac{i}{\varepsilon} - \frac{1}{2} + \dots & \textbf{grow} \text{ as } s \to -\infty. \end{cases}$$
Hence we have effectively imposed 2 boundary conditions as $s \to -\infty$. Similarly, we have imposed 2 boundary conditions as $s \to +\infty$.

Thus we have imposed 4 boundary conditions on a 3rd order ODE!

6.3.3 A well posed problem

Suppose that we just impose
$$\theta + \frac{\pi}{2} \to 0 \quad \text{as} \quad s \to -\infty \ . \tag{6.26}$$
Then a one-parameter family of solutions will exist. We fix the solution by requiring that
$$\theta(0;\varepsilon) = 0 \ . \tag{6.27}$$
The question is: 'Does this solution satisfy $(\theta - \frac{\pi}{2}) \to 0$ as $s \to +\infty$?'

Suppose that it does, then a second solution is
$$\Theta(s;\varepsilon) = -\theta(-s;\varepsilon) \ .$$
Θ and θ differ by at most a translation, hence θ is antisymmetric about *some* point. However, θ is monotonic, analytic and vanishes at $s=0$, thus
$$\theta(s;\varepsilon) \qquad \text{is antisymmetric about } s=0 \ .$$
We conclude that a needle crystal satisfies
$$\theta''(0;\varepsilon) = 0 \ . \tag{6.28}$$

18/06
19/13

6.3.4 Analytical continuation into the complex plane

We analytically continue solution into the complex s-plane; the continued solution still satisfies

$$\varepsilon^2\theta''' + \theta' = \cos\theta \ .$$

For future reference we note that if $\theta(s;\varepsilon)$ is antisymmetric, then

$$\theta(s;\varepsilon) = \sum_0^\infty a_n s^{2n+1} \ ,$$

and hence $\mathrm{Re}\,(\theta) = 0$ if s is pure imaginary.

Next we analytically extend the asymptotic expansion (6.25) into the complex s-plane. We note that this asymptotic expansion breaks down near

$$s = \pm(2n+1)\frac{i\pi}{2} \qquad n = 0, 1, 2, \dots \ ,$$

because $\operatorname{sech} s = \infty$ near such points. We seek an asymptotic expansion near to one of the points closest to the real axis, i.e. $s = \frac{i\pi}{2}$. In particular, if we let

$$s = \frac{i\pi}{2} + \sigma \ ,$$

then

$$\theta_0 = -\frac{\pi}{2} + 2i\tanh^{-1}(e^\sigma) \ ,$$

and

$$\theta_0 \sim i\ln\left(-\frac{2}{\sigma}\right) - \frac{\pi}{2} + \dots \qquad \text{as } \sigma \to 0.$$

Further, from HOT (i.e. higher order terms),

$$\theta \sim -\frac{\pi}{2} + i\left[\ln\left(-\frac{2}{\sigma}\right) - 2\left(\frac{\varepsilon}{\sigma}\right)^2 + \frac{50}{3}\left(\frac{\varepsilon}{\sigma}\right)^4 + \dots\right] \quad \text{as} \quad \sigma \to 0 \ .$$

This expansion becomes disordered for $\sigma = \mathcal{O}(\varepsilon)$. Hence when σ is this small we rescale:

$$\begin{aligned} s &= \frac{i\pi}{2} + \varepsilon z \ , \\ \theta &= i\ln\left(\frac{2}{\varepsilon}\right) - \frac{\pi}{2} + i\varphi(z,\varepsilon) \ . \end{aligned}$$

Then

$$\varphi''' + \varphi' = e^\varphi - \left(\frac{\varepsilon}{2}\right)^2 e^{-\varphi} \ , \tag{6.29}$$

and from matching we require that

$$\varphi \to -\ln(-z) - \frac{2}{z^2} + \dots \quad \text{as} \quad \mathrm{Re}\,(z) \to -\infty.$$

We seek an asymptotic solution to (6.29):

$$\varphi = \varphi_0 + \varepsilon^2 \varphi_1 + \dots \,,$$

then

$$\varphi_0''' + \varphi_0' = e^{\varphi_0} \,, \tag{6.30}$$

and

$$\varphi_0 \to -\ln(-z) - \frac{2}{z^2} \quad \text{as} \quad \mathrm{Re}\,(z) \to -\infty. \tag{6.31}$$

It is possible to prove that $\exists$ a unique solution for φ_0 in $\mathrm{Re}\,(z) \leqslant 0$. The strategy is therefore to:

(a) integrate (6.30) from $\mathrm{Re}\,(z) = -\infty$ to $\mathrm{Re}\,(z) = 0$ along a line on which $\mathrm{Im}\,(z) = \text{constant} < 0$;

(b) continue this solution down $\mathrm{Re}\,(z) = 0$ to $s = 0$ and compute $\theta''(0, \varepsilon)$.

Write

$$\varphi_0 = -\ln(-z) - \frac{2}{z^2} + \dots + \tilde{\varphi} \,, \tag{6.32}$$

and linearise (6.30) for large $|z|$. We find that

$$\tilde{\varphi} = \alpha \tilde{\varphi}_1 + \beta \tilde{\varphi}_2 + \gamma \tilde{\varphi}_3 \,,$$

where

$$\tilde{\varphi}_1 \sim -\frac{1}{z} + \frac{4}{z^3} + \dots \,,$$
$$\tilde{\varphi}_2 \sim z^{\frac{1}{2}} e^{iz} \left(1 + \frac{3}{8} \frac{i}{z} + \dots \right) \,,$$
$$\tilde{\varphi}_3 \sim z^{\frac{1}{2}} e^{-iz} \left(1 - \frac{3}{8} \frac{i}{z} + \dots \right) \,.$$

The matching condition (6.31) implies that if we let $\operatorname{Re}(z) \to -\infty$ along $\operatorname{Im}(z) = $ constant, then we deduce that in this 'direction'

$$\alpha = \beta = \gamma = 0 \; .$$

This does **not** mean that $\alpha = \beta = \gamma = 0$ in the direction specified by $\operatorname{Im}(z) \to -\infty$ with $\operatorname{Re}(z) = 0$ because $\varphi_3(z)$ is exponentially small in that direction. Hence, while we might expect that

$$\alpha = \beta = 0 \quad \text{for} \quad \operatorname{Im}(z) \to -\infty, \quad \operatorname{Re}(z) = 0 \, ,$$

it is possible that $\gamma \neq 0$ in that direction.

In order to get a handle on these terms, we note that when $\operatorname{Re}(z) = 0$, the algebraic terms in (6.32) are real valued, hence as $\operatorname{Im}(z) \to -\infty$ with $\operatorname{Re}(z) = 0$

$$\operatorname{Im}(\varphi(z)) \sim -\frac{\pi}{2} + \Gamma |z|^{\frac{1}{2}} e^{-|z|} \left(1 + \mathcal{O}\left(|z|^{-1}\right)\right) \; ,$$

where $\Gamma = \operatorname{Im}\left(\gamma e^{-i\pi/4}\right)$. Moreover, numerical solutions to (6.30) subject to (6.31) show that

$$\Gamma \approx 2.11 \; ;$$

a result that can also be obtained analytically using Borel summation. Hence

$$\operatorname{Re}(\theta(s, \varepsilon)) \sim -\Gamma \, |z|^{\frac{1}{2}} \, e^{-|z|} \left(1 + \mathcal{O}\left(|z|^{-1}\right)\right)$$

as $\operatorname{Im}(z) \to -\infty$ with $\operatorname{Re}(z) = 0 = \operatorname{Re}(s)$. With a little more effort one can conclude, by integrating along $\operatorname{Re}(s) = 0$ back to $s = 0$, that

$$\theta''(0, \varepsilon) \sim 2\Gamma \varepsilon^{-\frac{5}{2}} \exp(-\pi/2\varepsilon) \; ,$$

which is **exponentially small**. This term is non-zero because of a **Stokes-line effect.**

11/04

19/07 Often exponentially small terms do not matter, but they do here. We conclude that $\theta(s, \varepsilon)$ is not antisym-
19/08 metric, and hence the well-posed problem does not represent a needle crystal. Indeed, no needle crystal
19/14 solutions exist for small ε.

13/18
13/19
13/20
13/22

7 Method of Multiple Scales

11/95

Multiple scales is a useful technique for a number of problems. For instance, it underlies much of the theory of '*ray-tracing*'.

One of the simpler, if important, uses of multiple scales is to describe the evolution of linear waves through slowly varying media (e.g. sound waves through the atmosphere). For such examples, the different scales are often immediately apparent (e.g. the wavelength of sound, and the depth of the troposphere).

We will concentrate on nonlinear problems where the need for two (or more) scales is necessary, but may not be immediately apparent.

MAE: Two or more processes with different scales; processes act separately in different regions.
13/15 MS: Two or more processes each with own scale; processes act simultaneously.

7.1 Van der Pol oscillator

The Van der Pol oscillator is described by the equation

$$\ddot{x} + \underbrace{\varepsilon\dot{x}(x^2-1)}_{\substack{\text{nonlinear friction}\\ -\text{ve}: \quad |x|<1 \\ +\text{ve}: \quad |x|>1}} + x = 0\ , \qquad t \geqslant 0\ , \tag{7.1}$$

where $0 < \varepsilon \ll 1$. Typical initial conditions might be $x = 1,\ \dot{x} = 0$ at $t = 0$ (although the precise initial conditions are not crucial for what follows).

Solutions are found to tend to a finite amplitude oscillation, during which energy losses when $|x| > 1$ are balanced by energy gains when $|x| < 1$.

7.1.1 Regular perturbation

Try

$$x = x_0 + \varepsilon x_1 + \dots\ . \tag{7.2}$$

Then at leading order

$$\ddot{x}_0 + x_0 = 0 \quad \Rightarrow \quad x_0 = \cos t\ . \tag{7.3}$$

At the next order

$$\begin{aligned} \ddot{x}_1 + x_1 = \dot{x}_0(1-x_0^2) &= -\sin^3 t \\ &= -\tfrac{3}{4}\sin t + \tfrac{1}{4}\sin 3t\ , \end{aligned} \tag{7.4a}$$

and

$$x_1 = \tfrac{3}{8}\left(t\cos t - \sin t\right) - \tfrac{1}{32}\left(\sin 3t - 3\sin t\right)\ . \tag{7.4b}$$

Note that the expansion loses its asymptoticness when

$$\varepsilon x_1 = \operatorname{ord}(x_0) \quad \text{i.e. when} \quad t = \operatorname{ord}\left(\tfrac{1}{\varepsilon}\right)\ . \tag{7.5}$$

The 'problem' is that the ε-damping term slowly changes the oscillation amplitude on a time scale of $\operatorname{ord}(\varepsilon^{-1})$ by the slow accumulation of small effects.

7.1.2 Multiple scales expansion

The oscillator has two processes:

Harmonic oscillation on time scale of ord(1).	Slow drift in amplitude (and possible phase) on time scale of ord(ε^{-1}).
$\tau = t$	$T = \varepsilon t$
The 'fast' time scale.	The 'slow' time scale.

We treat τ and T as independent variables:

- the rapidly changing features are modelled by τ,
- the slowly changing features are modelled by T.

Hence we seek a solution with the form

$$x(t;\varepsilon) = x(\tau, T; \varepsilon) \ , \tag{7.6a}$$

where the two variables are introduced as an artifice in order to remove secular effects. We use the chain rule to compute derivatives:

$$\frac{d}{dt}\, x(t;\varepsilon) = \frac{\partial x}{\partial \tau}(\tau, T; \varepsilon) + \varepsilon \frac{\partial x}{\partial T}(\tau, T; \varepsilon) \ , \tag{7.6b}$$

$$\ddot{x} = x_{\tau\tau} + 2\varepsilon x_{\tau T} + \varepsilon^2 x_{TT} \ . \tag{7.6c}$$

We now seek an asymptotic expansion of the form

$$x = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \dots \ , \tag{7.7}$$

and require the expansion to be valid for $T = \mathrm{ord}(1)$, i.e. $t = \mathrm{ord}(\varepsilon^{-1})$. Then at leading order

ε^0 :

$$x_{0\tau\tau} + x_0 = 0 \ , \quad t \geqslant 0 \ , \tag{7.8a}$$

$$x_0 = 1 \quad , \quad x_{0\tau} = 0 \ , \quad \text{at} \quad t = 0 \ . \tag{7.8b}$$

This has solution in terms of trigonometric functions (we could alternatively use complex notation, as we shall see below),

$$x_0 = R_0(T) \, \cos\left(\tau + \theta_0(T)\right) \ , \tag{7.8c}$$

where, in order to satisfy the initial conditions,

$$R_0(0) = 1 \quad , \quad \theta_0(0) = 0 \ . \tag{7.8d}$$

The functions R_0 and θ_0 are not fixed at this stage — we need equations for them. At next order we have that

ε^1 :

$$\begin{aligned} x_{1\tau\tau} + x_1 &= -x_{0\tau}\left(x_0^2 - 1\right) - 2x_{0\tau T} \\ &= 2R_0\theta_{0T} \cos\left(\tau + \theta_0\right) + \left(2R_{0T} + \tfrac{1}{4}R_0^3 - R_0\right) \sin(\tau + \theta_0) + \tfrac{1}{4}R_0^3 \sin 3(\tau + \theta_0) \, , \end{aligned} \tag{7.9a}$$

together with the initial conditions

$$x_1 = 0 \quad , \quad x_{1\tau} = -x_{0T} = -R_{0T} \quad \text{at} \quad t = 0 \ . \tag{7.9b}$$

The solution is

$$\begin{aligned} x_1 = {} & R_0\theta_{0T}\, \tau \sin(\tau + \theta_0(T)) - \tfrac{1}{2}\left(2R_{0T} + \tfrac{1}{4}R_0^3 - R_0\right) \tau \cos(\tau + \theta_0(T)) \\ & - \tfrac{1}{32}R_0^3 \sin 3(\tau + \theta_0(T)) + R_1 \sin\left(\tau + \theta_1(T)\right) \ . \end{aligned} \tag{7.9c}$$

However, the asymptotic expansion will not be valid for $\tau = \mathrm{ord}(\varepsilon^{-1})$ unless

$$R_0\theta_{0T} = 0 \quad , \quad 2R_{0T} + \tfrac{1}{4}R_0^3 - R_0 = 0 \ . \tag{7.10a}$$

This is the 'secularity' or 'integrability' condition of Poincaré. Using the initial conditions we deduce that

$$\theta_0 = 0 \quad , \quad R_0 = \frac{2}{(1+3e^{-T})^{\frac{1}{2}}} \; . \tag{7.10b}$$

In particular note that $R_0 \to 2$ as $T \to \infty$. It follows that the solution for x_1 becomes

$$x_1 = R_1 \, \sin\left(\tau + \theta_1(T)\right) - \tfrac{1}{32} R_0^3 \, \sin 3\left(\tau + \theta_0(T)\right) \; , \tag{7.11a}$$

while the initial conditions for R_1 and θ_1 become

$$\begin{gathered} R_1(0) \, \sin\theta_1(0) = 0 \; , \\ R_1(0) \, \cos\theta_1(0) - \tfrac{3}{32} R_0^3(0) \, \cos 3\theta_0(0) = -R_{0T}(0) \, , \end{gathered}$$

i.e.

$$\theta_1(0) = 0 \quad , \quad R_1(0) = -\tfrac{9}{32} \; . \tag{7.11b}$$

The equations governing R_1 and θ_1 are determined by the secularity condition for the x_2 problem. However, we then find that there is insufficient freedom in R_1 and θ_1 to avoid breaking the asymptoticness when $T = \operatorname{ord}(1)$. This problem can be avoided by introducing a super slow time scale, $T_2 = \varepsilon^2 t$.

Alternative approach to deriving (7.10a). Instead of solving explicitly for x_1, we could use a condition based on requiring x_1 to be periodic over the time scale τ. For instance, we could require that (cf. inner products and Sturm-Liouville operators and integrating by parts twice)

$$\int_0^{2\pi} \left(x_{1\tau\tau} + x_1\right) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \left(\tau + \theta_0\right) \, d\tau = 0 \; , \tag{7.12a}$$

i.e.

$$\int_0^{2\pi} \left(x_{0\tau}\left(x_0^2 - 1\right) + 2x_{0\tau T}\right) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \left(\tau + \theta_0\right) \, d\tau = 0 \; . \tag{7.12b}$$

On performing the integrals, (7.10a) is again recovered. This is known as the Fredholm alternative.

7.1.3 A simple example of super slow time scale

Consider the exact solution to the equation

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0 \, ,$$

i.e.

$$x = e^{-\varepsilon t} \, \cos\!\left(\left(1 - \varepsilon^2\right)^{\frac{1}{2}} t\right) \; .$$

This has:

(a) an oscillation on the time scale $t = \operatorname{ord}(1)$,

(b) an amplitude drift on the time scale $t = \operatorname{ord}(\varepsilon^{-1})$, and

(c) a phase drift on the time scale $t = \operatorname{ord}(\varepsilon^{-2})$.

In general, when working to $\operatorname{ord}\left(\varepsilon^k\right)$ on a time scale $\operatorname{ord}\left(\varepsilon^{k-n}\right)$, one must expect to have a hierarchy of n slow time scales.

7.2 Mathieu Equation

As a further example of multiple-scales consider solutions to the Mathieu equation:

$$\ddot{y} + \left(\omega^2 + \varepsilon \cos t\right) y = 0 \,. \tag{7.13}$$

The coefficients are 2π-periodic. This equation describes the small amplitude oscillations of a pendulum whose length changes slightly in time. If the natural oscillation frequency is near a multiple of *half* the forcing frequency, then the amplitude of the pendulum will increase in time. This is an example of
12/95 **parametric excitation**.

7.2.1 Floquet Theory (for second order ODEs)

First note that, since the coefficients of the Mathieu equation are 2π periodic, if $y(t)$ is a solution, then $y(t+2\pi)$ is also a solution. Further since the equation is second order, we can write the general solution as

$$y(t) = Ay_1(t) + By_2(t) \,. \tag{7.14}$$

Combining these results we see that we can write

$$y_j(t+2\pi) = \alpha_j y_1(t) + \beta_j y_2(t) \,, \tag{7.15a}$$

and hence

$$\begin{aligned} y(t+2\pi) &= Ay_1(t+2\pi) + By_2(t+2\pi) && (7.15b)\\ &= \left(A\alpha_1 + B\alpha_2\right) y_1(t) + \left(A\beta_1 + B\beta_2\right) y_2(t) \\ &= A'y_1(t) + B'y_2(t)\,, && (7.15c) \end{aligned}$$

where, in matrix notation,

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}}_{\mathsf{P}} \begin{pmatrix} A \\ B \end{pmatrix} \,. \tag{7.15d}$$

Suppose (A, B) is an eigenvector of P with eigenvalue λ; then

$$A' = \lambda A\,, \quad B' = \lambda B\,, \tag{7.16a}$$

and

$$y(t+2\pi) = \lambda y(t) \quad \text{for all } t\,. \tag{7.16b}$$

Let $\mu = \ln \lambda / 2\pi$ and define

$$\varphi(t) = e^{-\mu t} y(t) \,. \tag{7.17a}$$

Then from (7.16b)

$$\varphi(t+2\pi) = e^{-\mu(t+2\pi)} y(t+2\pi) = e^{-\mu t} y(t) = \varphi(t) \quad \text{for all } t \,, \tag{7.17b}$$

and hence

$$y(t) = e^{\mu t} \varphi(t) \,, \tag{7.17c}$$

14/17 where $\varphi(t)$ is a 2π-*periodic* function.

Since the Mathieu equation is second order, there will be two eigenvalues λ, or equivalently two constants μ, and two eigenvectors (we sidestep the degenerate case of one eigenvector). Then the system is said to be

$$\begin{aligned} \text{unstable if, for either eigenvalue,} \quad & \mathrm{Re}\,(\mu) > 0\,, \\ \text{stable if, for both eigenvalues,} \quad & \mathrm{Re}\,(\mu) \leqslant 0\,. \end{aligned}$$

In the case of the Mathieu equation, if $y(t)$ is a solution, so is $y(-t)$. Thus for stability we must have $Re(\mu) = 0$ for both eigenvalues.

It is possible to show that there are regions of the (ω^2, ε) plane where solutions are stable, and other
14/15 regions where solutions are unstable. We will attempt to find the 'stability boundaries' when $|\varepsilon| \ll 1$ by
14/18 seeking small amplitude periodic solutions, and identifying regions of parameter space where they do not
14/19 exist.
14/22

7.2.2 $\omega \neq \pm n/2$

Try the Poincaré expansion

$$y = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots \; . \tag{7.18}$$

From substitution into the Mathieu equation we obtain:

$$O(\varepsilon^0): \qquad \ddot{y}_0 + \omega^2 y_0 = 0\,, \tag{7.19a}$$

$$O(\varepsilon^1): \qquad \ddot{y}_1 + \omega^2 y_1 = -y_0 \cos t\,. \tag{7.19b}$$

If we seek a real solution, then

$$y_0 = A_0 \exp\big(\imath\omega t\big) + A_0^* \exp\big(-\imath\omega t\big)\;, \tag{7.20a}$$

$$= A_0 \exp\big(\imath\omega t\big) + \text{c.c.}\;, \tag{7.20b}$$

and

$$\ddot{y}_1 + \omega^2 y_1 = -\tfrac{1}{2}A_0 \exp\big(\imath(\omega+1)t\big) - \tfrac{1}{2}A_0 \exp\big(\imath(\omega-1)t\big) + \text{c.c}\;. \tag{7.20c}$$

It follows that there are 'secular' terms if $\omega \pm 1 = -\omega$, i.e. if $\omega = \mp\frac{1}{2}$. Further, it is possible to show that higher-order terms are secular only if $\omega = \pm n/2$. Thus if $\omega \neq \pm n/2$, we can solve at all orders to show that

$$y(t) = \exp\big(\imath\omega t\big)\,\varphi(t) + \text{c.c.}\;,$$

14/20 where φ is 2π-periodic. We conclude that for $\varepsilon \ll 1$ and $\omega \neq \pm n/2$, the solution is stable.

7.2.3 $\left|\omega^2 - \frac{1}{4}\right| \ll 1$

See Example Sheet 3.

7.2.4 $\left|\omega^2 - 1\right| \ll 1$

Suppose that $|\omega^2 - 1| \ll 1$, and seek a solution of the form

$$\omega^2 = 1 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots$$

From §7.2.2 we anticipate that resonance will only occur at second order. Hence if $a_1 \neq 0$, we expect there to be no instability; thus we set $a_1 = 0$.

ε^0 : 1^{st} harmonic
ε^1 : 0^{th} & 2^{nd} harmonics
ε^2 : 1^{st} & 3^{rd} harmonics
$\uparrow$can force resonance

This suggests that we should consider an ord(ε^{-2}) slow time scale. Try

$$\tau = t\,, \quad T = \varepsilon^2 t\,, \tag{7.21a}$$

$$y = y_0(\tau, T) + \varepsilon y_1(\tau, T) + \varepsilon^2 y_2(\tau, T) + \dots\,. \tag{7.21b}$$

At leading order the governing equations is

$$\varepsilon^0: \qquad y_{0\tau\tau} + y_0 = 0\;, \tag{7.22a}$$

with solution

$$y_0 = A_0(T)e^{\imath\tau} + \text{c.c.}\;. \tag{7.22b}$$

At next order

$$\varepsilon^1: \qquad y_{1\tau\tau} + y_1 = -y_0 \cos\tau$$
$$= -\tfrac{1}{2} A_0 (e^{2\imath\tau} + 1) + \text{c.c.}\,, \tag{7.22c}$$

with solution

$$y_1 = -\tfrac{1}{2}(A_0 + A_0^*) + \tfrac{1}{6}\left(A_0 e^{2\imath\tau} + A_0^* e^{-2\imath\tau}\right)\,, \tag{7.22d}$$

where any homogeneous component can [usually] be absorbed by a suitable redefinition of A_0. At next order

$$\varepsilon^2: \qquad y_{2\tau\tau} + y_2 = -2y_{0\tau T} - a_2 y_0 - \tfrac{1}{2} y_1 \left(e^{\imath\tau} + e^{-\imath\tau}\right)$$
$$= \left(-2\imath A_{0T} + (\tfrac{1}{6} - a_2)A_0 + \tfrac{1}{4}A_0^*\right) e^{\imath\tau} - \tfrac{1}{12} A_0 e^{3\imath\tau} + \text{c.c.}\,, \tag{7.23a}$$

where * denotes a complex conjugate. For asymptoticness not to be lost when $T = \text{ord}(1)$, it follows from the secularity condition that

$$2\beta_T + \left(\tfrac{5}{12} - a_2\right)\alpha = 0 \quad , \quad 2\alpha_T + \left(\tfrac{1}{12} + a_2\right)\beta = 0\,, \tag{7.23b}$$

where $A_0 = \alpha + \imath\beta$. Hence the oscillation is unstable on the slow time scale T if

$$\left(\tfrac{5}{12} - a_2\right)\left(\tfrac{1}{12} + a_2\right) > 0\,, \tag{7.23c}$$

i.e. if

$$-\tfrac{1}{12} < a_2 < \tfrac{5}{12}\,. \tag{7.23d}$$

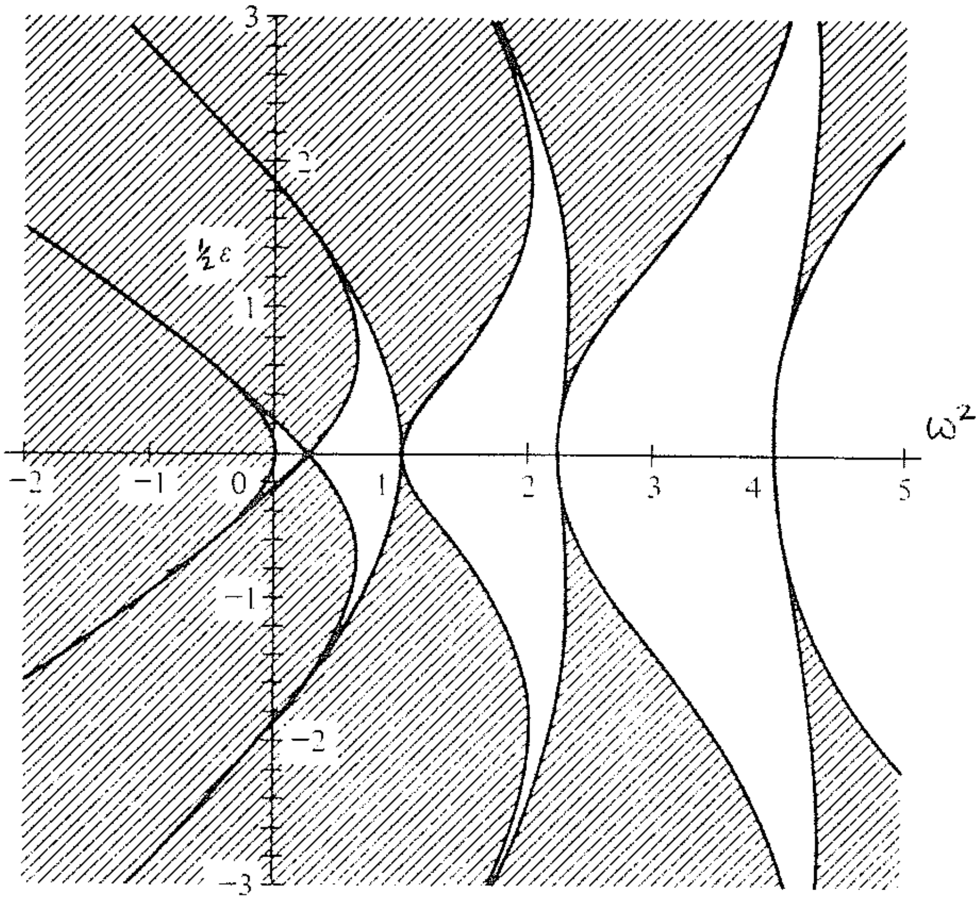


Figure 7.5: Plot of the stability boundaries of solutions to the Mathieu equation. In the white regions of the (ω^2, ε) plane, all solutions of the Mathieu equation are stable, while in the cross-hatched regions there is an unstable solution. When $\varepsilon = 0$, the cross-hatched regions meet the ω^2 axis at $\omega = n/2$, $n = 0, 1, 2 \ldots$. Source: *Advanced Mathematical Methods for Scientists and Engineers*, by C.M. Bender and S.A. Orszag.

7.3 WKBJLG Theory

Terminology: omit the J if not in Cambridge, and omit the LG if a physicist.

This theory is concerned with asymptotic solutions to equations with slowly varying coefficients, e.g.

$$\ddot{x} + f(\varepsilon t)\, x = 0\ . \tag{7.24}$$

It has both linear and nonlinear variants. A generalisation to two or more independent variables is called *ray theory*.

7.3.1 Leading-order solution

Initially assume that $f = \omega^2 > 0$, and seek a multiple scales solution with

$$\tau = t\,, \quad T = \varepsilon t\ , \tag{7.25a}$$
$$x \equiv x(\tau, T) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \dots\ . \tag{7.25b}$$

Then at leading order

$$x_{0\tau\tau} + \omega^2(T)\, x_0 = 0\ , \tag{7.26a}$$

with solution

$$x_0 = R_0(T) \cos\left(\omega(T)\, \tau + \theta_0(T)\right)\ . \tag{7.26b}$$

At next order

$$\begin{aligned} x_{1\tau\tau} + \omega^2 x_1 &= -2x_{0\tau T} \\ &= 2(\omega R_0)_T \sin(\omega\tau + \theta_0) + 2\omega R_0\ (\omega_T\, \tau + \theta_{0T}) \cos(\omega\tau + \theta_0)\ . \end{aligned} \tag{7.27a}$$

The secularity condition implies that

$$\theta_T(T) = -\tau\omega_T(T)\ , \tag{7.27b}$$

but this is 'impossible', because the fast variable appears in the 'drift' equation for the slow dependence. In some sense we want 'θ to be larger'. Instead replace the solutions for x_0 with

$$x_0(\tau, T) = R_0(T) \cos(\theta(T))\ , \tag{7.28a}$$

where

$$\theta = \tfrac{1}{\varepsilon}\Theta_0(T) + \Theta_1(T) + \dots\ , \tag{7.28b}$$

so that small variations in Θ_0 on the T timescale produce $\mathcal{O}(1)$ changes in θ.

Since

$$\theta_t = \Theta_{0T} + \varepsilon\Theta_{1T} + \dots\ , \tag{7.29a}$$

it follows that

$$\begin{aligned} \dot{x}_0 &= -R_0\theta_{0T} \sin\theta + \varepsilon\big(R_{0T} \cos\theta - R_0\Theta_{1T} \sin\theta\big) + \dots\ , \\ \ddot{x}_0 &= -R_0\theta_{0T}^2 \cos\theta - \varepsilon\big((2R_{0T}\Theta_{0T} + R_0\Theta_{0TT}) \sin\theta + 2R_0\Theta_{1T}\Theta_{0T} \cos\theta\big) + \dots\ . \end{aligned}$$

On substituting these expansions into (7.24) we find that at leading order

$$\theta_{0T}^2 = \omega^2\ , \quad \text{i.e.} \quad \theta_{0T} = \omega\ , \tag{7.29b}$$

where $\omega > 0$ wlog. On applying the secularity condition to the equation for x_1 we obtain

$$\left.\begin{aligned} 2R_0\Theta_{1T}\Theta_{0T} &= 0 \\ 2R_{0T}\Theta_{0T} + R_0\Theta_{0TT} &= 0 \end{aligned}\right\} \qquad \begin{aligned} \Theta_1 &= \text{const.} \\ R_0^2\omega &= \text{const.} \end{aligned} \tag{7.29c}$$

Remark. While the local 'energy' $E = \frac{1}{2}R_0^2\omega^2$ is *not* conserved, the 'action' E/ω is conserved (recall that for a standard harmonic oscillator $E = \frac{1}{2}(\dot{x}^2 + \omega^2 x^2)$)

Hence the multiple scales solution has the form

$$x \sim \tfrac{1}{[f(\varepsilon t)]^{1/4}} \left(a\cos\theta + b\sin\theta\right) \;, \tag{7.30a}$$

where a and b are constants, and

$$\theta = \int_0^t \big[f(\varepsilon q)\big]^{\frac{1}{2}}\, dq \;. \tag{7.30b}$$

A similar analysis is possible if $f < 0$, except that exponentially growing/decaying solutions are found rather than harmonically oscillating ones. In particular

$$x \sim \tfrac{1}{[-f(\varepsilon t)]^{1/4}} \left(Ae^{-\varphi} + Be^{\varphi}\right) \;, \tag{7.30c}$$

where and A and B are constants, and

$$\varphi = \int_0^t \big[-f(\varepsilon q)\big]^{\frac{1}{2}}\, dq \;. \tag{7.30d}$$

Remark. In order to obtain higher order approximations, at first sight it might appear that super slow time scales, $T_n = \varepsilon^n t$, are needed. However, with care, this is not necessary (see the last example
13/95 sheet).
15/17

7.3.2 Turning points

What if $f = 0$ at some point? The solutions (7.30a) and (7.30c) are then singular. In order to investigate this case, we assume without loss of generality that $f(0) = 0$ and $f'(0) < 0$.

We recall that when $\varepsilon t = \mathrm{ord}(1)$, we have (7.30a) as solution for $t < 0$ (since $f > 0$),
(7.30c) as solution for $t > 0$ (since $f < 0$).

In order to have a complete solution we need the relationship between (a, b) and (A, B). To this end we observe that when $|\varepsilon t| \ll 1$,

$$\ddot{x} + \varepsilon t f'(0)\, x \approx 0 \;. \tag{7.31a}$$

Therefore, all times are of a comparable scale when

$$\frac{x}{t^2} \sim \varepsilon f'(0)\, tx \quad \Rightarrow \quad t \sim \big|\varepsilon f'(0)\big|^{-\frac{1}{3}} \;. \tag{7.31b}$$

Thus we introduce 'medium time', s, defined by

$$s = t\big(-\varepsilon f'(0)\big)^{\frac{1}{3}} \;. \tag{7.31c}$$

Based on the magnitudes of (7.30a) and (7.30c) when $t = \mathrm{ord}(\varepsilon^{-\frac{1}{3}})$, i.e. $s = \mathrm{ord}(1)$, we scale x by

$$x = \frac{1}{\varepsilon^{\frac{1}{6}}} X_0 + \dots \;. \tag{7.31d}$$

15/15

The leading-order governing equation is then Airy's equation,

$$X_{0ss} - sX_0 = 0 \;, \tag{7.32a}$$

with solution

$$X_0 = \alpha\, \mathrm{Ai}(s) + \beta\, \mathrm{Bi}(s) \;, \tag{7.32b}$$

15/18 where α and β are constants.

This solution must match with those valid when $\varepsilon t = \mathrm{ord}(1)$. First we match (7.32b) as $s \to \infty$ to (7.30c) as $\varepsilon t \to 0+$. From the asymptotic expansions for the Airy function, etc.

$$\text{(7.32b)}: \qquad X_0 \sim \frac{1}{s^{1/4}\sqrt{\pi}} \left(\tfrac{1}{2}\alpha \exp\big(-\tfrac{2}{3}s^{\frac{3}{2}}\big) + \beta \exp\big(\tfrac{2}{3}s^{\frac{3}{2}}\big)\right) \;, \tag{7.33a}$$

$$\text{(7.30c)}: \qquad x_0 \sim \frac{1}{\big[-\varepsilon t f'(0)\big]^{\frac{1}{4}}} \left(A\exp(-\varphi) + B\exp(\varphi)\right) \;, \tag{7.33b}$$

where

$$\varphi \sim \tfrac{2}{3}\big[-\varepsilon f'(0)\big]^{\frac{1}{2}} t^{\frac{3}{2}} = \tfrac{2}{3} s^{\frac{3}{2}} . \tag{7.33c}$$

Hence from matching

$$\frac{\alpha}{2\sqrt{\pi}} = \frac{A}{\big[-f'(0)\big]^{1/6}} , \qquad \frac{\beta}{\sqrt{\pi}} = \frac{B}{\big[-f'(0)\big]^{1/6}} . \tag{7.33d}$$

Note that the determination of A this way is 'dangerous' since that part of the solution is exponentially small in (7.30c).

We can similarly match (7.32b) as $s \to -\infty$ to (7.30a) as $\varepsilon t \to 0-$. From above

$$\text{(7.32b)}: \qquad X_0 \sim \frac{1}{\sqrt{\pi}(-s)^{1/4}} \left(\alpha \sin\Theta + \beta\cos\Theta\right) \ , \quad \Theta = \tfrac{2}{3}(-s)^{\frac{3}{2}} + \tfrac{1}{4}\pi \ , \tag{7.34a}$$

$$\text{(7.30a)}: \qquad x_0 \sim \frac{1}{\big[\varepsilon t f'(0)\big]^{1/4}} \left(a\cos\theta + b\sin\theta\right) \ , \quad \theta \sim -\tfrac{2}{3}\big[-\varepsilon f'(0)\big]^{\frac{1}{2}}(-t)^{\frac{3}{2}} = -\tfrac{2}{3}(-s)^{\frac{3}{2}} \ . \tag{7.34b}$$

These two expansions match if:

$$\frac{a}{\big[-f'(0)\big]^{1/6}} = \frac{\beta+\alpha}{(2\pi)^{1/2}} , \qquad \frac{b}{\big[-f'(0)\big]^{1/6}} = \frac{\beta-\alpha}{(2\pi)^{1/2}} . \tag{7.34c}$$

We therefore have the *connection formulae*

15/19
15/20
15/22
(long)

$$A = \frac{a-b}{2\sqrt{2}} \quad , \quad B = \frac{a+b}{\sqrt{2}} . \tag{7.35}$$

7.4 Ray Theory

Consider waves propagating through a slowly varying medium. Assume that they are governed by

$$\mathcal{L}(\partial_t, \partial_x; \varepsilon x, \varepsilon t)\varphi = \varepsilon\mathcal{N}(\partial_t, \partial_x, \varphi; \varepsilon x, \varepsilon t, \varepsilon) \ , \tag{7.36a}$$

where $\mathcal{L}$ is a linear operator,
$\mathcal{N}$ is a nonlinear operator,

and $X = \varepsilon x$ and $T = \varepsilon t$ represent the slowly varying nature of the medium. For instance

$$\mathcal{L}\varphi \equiv \left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x}\left(c^2(X,T)\frac{\partial}{\partial x}\right)\right)\varphi = 0 \ . \tag{7.36b}$$

Seek a solution of the form

$$\varphi = [A_0(X,T) + \varepsilon A_1(X,T) + \dots]\exp\left(\frac{\imath}{\varepsilon}\,\theta(X,T)\right) + c.c. \ . \tag{7.37a}$$

Then

$$\varphi_t = i\theta_T[A_0 + \varepsilon A_1 + \dots]e^{i\theta/\varepsilon} + \varepsilon[A_{0T} + \varepsilon A_{1T} + \dots]e^{i\theta/\varepsilon} + c.c. \ , \tag{7.37b}$$

and the leading order approximation to (7.36a) becomes

$$\mathcal{L}(i\theta_T, i\theta_X; X, T) = 0 \ , \tag{7.38a}$$

i.e. the *dispersion relation*

$$\mathcal{L}(-i\omega, ik; X, T) = 0 \ , \tag{7.38b}$$

where $\omega = -\theta_T$ is defined to be the [real] frequency, and $k = \theta_X$ is defined to be the [real] wave number. (7.38b) is often rewritten in the form

$$\omega = \Omega(k; X, T) \ . \tag{7.38c}$$

Consider small variations about the values X_0 and T_0 by writing $X = X_0 + \varepsilon\delta x$ and $T = T_0 + \varepsilon\delta t$ where $|\varepsilon\delta x|, |\varepsilon\delta t| \ll 1$. Then

$$\begin{aligned}\exp\left(\frac{i}{\varepsilon}\,\theta(X_0+\delta X, T_0+\delta T)\right) &\approx \exp\left(\frac{i\theta(X_0,T_0)}{\varepsilon}\right)\exp\left(\frac{i\theta_X}{\varepsilon}\,\varepsilon\delta x + \ldots + \frac{i\theta_T}{\varepsilon}\,\varepsilon\delta t\right)\\ &\approx \exp\left(\frac{i\theta(X_0,T_0)}{\varepsilon}\right)\exp\left(ik\delta x - i\omega\delta t + \ldots\right) .\end{aligned}$$

Hence the definitions of ω and k are consistent with convention. Further, because

$$\theta_{XT} - \theta_{TX} = 0 \, , \tag{7.39a}$$

it follows that

$$k_T + \omega_X = 0 \, , \tag{7.39b}$$

and hence from (7.38c) that

$$k_T + c_g k_X = -\frac{\partial\Omega}{\partial X} \, , \tag{7.39c}$$

where $c_g = \frac{\partial\Omega}{\partial k}$ is the group velocity. In characteristic form

$$\frac{dk}{dT} = -\frac{\partial\Omega}{\partial X} \qquad \text{on} \qquad \frac{dX}{dT} = c_g \, . \tag{7.40a}$$

A **ray** is a path along the characteristic traversed with speed c_g. In general rays are curved.

Exercise. Show that

$$\frac{d\omega}{dT} = \frac{\partial\Omega}{\partial T} \qquad \text{on} \qquad \frac{dX}{dT} = c_g \, . \tag{7.40b}$$

Hamilton's Equations. Consider the transformations:

$$\begin{array}{rcl} X & \to & q \\ k(X,T) & \to & p \\ \Omega(k;X,T) & \to & H(q,p,T) \, , \end{array} \tag{7.41a}$$

then (7.40a) becomes

$$\frac{dp}{dT} = -\frac{\partial H}{\partial q} \, , \quad \frac{dq}{dT} = \frac{\partial H}{\partial p} \, . \tag{7.41b}$$

These are just Hamilton's equations; hence waves move like particles with speed c_g. Further, from (7.38c)

$$\frac{\partial\theta}{\partial T} + H\left(q, \frac{\partial\theta}{\partial q}, T\right) = 0 \, . \tag{7.41c}$$

This is the Hamilton-Jacobi equation with the phase, $\theta(q,T)$, as the action.

7.4.1 Model example

Consider the equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x}\left(c^2(X,T)\frac{\partial}{\partial x}\right)\right)\varphi = 0 \, . \tag{7.42}$$

Substitute

$$\varphi = (A_0(X,T) + \varepsilon A_1(X,T) + \ldots)\exp\left(\frac{i\theta(X,T}{\varepsilon}\right) + c.c. \, , \tag{7.43a}$$

then

$$\varepsilon^0 : \qquad -\omega^2 A_0 + c^2 k^2 A_0 = 0 \, , \tag{7.43b}$$

$$\varepsilon^1 : \qquad -\omega^2 A_1 + c^2 k^2 A_1 = i(\omega_T A_0 + 2\omega A_{0T}) + 2cc_X ikA_0 + ic^2(k_X A_0 + 2kA_{0X}) \, . \tag{7.43c}$$

Hence at leading order we deduce the *dispersion relation*

$$\omega = \pm ck \,, \tag{7.44a}$$

and at first order it follows that

$$(\omega A_0^2)_T + 2cc_X kA_0^2 + c^2(kA_0^2)_X = 0 \;, \tag{7.44b}$$

or equivalently

$$(\omega A_0^2)_T + (c_g\omega A_0^2)_X = 0 \;, \tag{7.44c}$$

where $c_g = \pm c = \frac{\omega}{k}$. In this case no further information comes from the complex conjugate equation. Write

$$A_0 = r_0 e^{i\psi_0} \;, \tag{7.45a}$$

then, on taking real and imaginary parts,

$$\psi_{0T} + c_g\psi_{0X} = 0 \;, \tag{7.45b}$$

and

$$(\omega r_0^2)_T + (c_g\omega r_0^2)_X = 0 \;. \tag{7.45c}$$

Wave action. The local time and spatial averaged energy density of a wave satisfying (7.42) is given by

$$\begin{aligned} E &= \frac{\omega k}{4\pi^2}\int_0^{2\pi/\omega}\int_0^{2\pi/k} \tfrac{1}{2}\left(\varphi_t^2 + c^2\varphi_x^2\right)\,dx\,dt \\ &= \tfrac{1}{2}\omega^2 r_0^2 + O(\varepsilon)\,. \end{aligned} \tag{7.46}$$

14/95 Hence (7.45c) represents conservation of **wave action** E/ω.

7.4.2 Conservation of wave action for sound waves *(only outlined in lectures)*

For 1D sound waves the governing equations are

$$\begin{aligned} \rho(u_t + uu_z) &= -\,p_z \,, \\ \rho_t + (\rho u)_z &= 0 \,, \\ S_t + uS_z &= 0 \,, \\ p &\equiv p(\rho, S)\,. \end{aligned}$$

Consider small perturbation from a basic, slowly varying, state of the form

$$\begin{aligned} \rho &= \rho_0(Z) + \tilde{\rho}\,, \\ p &= p_0(\rho_0, S_0) + \tilde{p}\,, \\ S &= S_0(Z) + \tilde{S}\,, \end{aligned}$$

where

$$z = \varepsilon Z\,.$$

Assume that the basic state is at constant pressure so that

$$\left.\frac{\partial p}{\partial \rho}\right|_S \rho_{0Z} + \left.\frac{\partial p}{\partial S}\right|_\rho S_{0Z} = 0\,,$$

or equivalently

$$c_0^2(Z)\rho_{0Z} + p_{0S}(Z)S_{0Z} = 0\,.$$

Linearised perturbations satisfy the equations

$$\begin{aligned}\rho_0 u_t &= -\tilde{p}_z\,,\\ \tilde{\rho}_t + (\rho_0 u)_z &= 0\,,\\ \tilde{S}_t + \varepsilon u S_{0Z} &= 0\,,\\ \tilde{p} &= c_0^2(z)\tilde{\rho} + p_{0S}\tilde{S}\,.\end{aligned}$$

Hence

$$\begin{aligned}\tilde{p}_t &= c_0^2\tilde{\rho}_t + p_{0S}\tilde{S}_t\\ &= -c_0^2(\rho_0 u)_z - \varepsilon p_{0S}\, u S_{0Z}\\ &= -c_0^2(\rho_0 u)_z + \varepsilon c_0^2 \rho_{0Z} u\,,\end{aligned}$$

and

$$\begin{aligned}\tilde{p}_{tt} &= -c_0^2\rho_0 u_{zt}\\ &= c_0^2\rho_0\left(\frac{\tilde{p}}{\rho_0}\right)_z\,.\end{aligned}$$

The governing wave equation for pressure perturbations is thus (cf. (7.42))

$$\tilde{p}_{tt} - c_0^2(Z)\rho_0(Z)\left(\frac{\tilde{p}_z}{\rho_0(Z)}\right)_z = 0\,. \tag{7.47}$$

Multiple-scales analysis. On the basis of earlier, seek a solution of the form

$$\begin{aligned}\tilde{p} =& [a_0(Z,T) + \varepsilon a_1(Z,T) + \dots]\exp\left(\frac{i\theta(Z,T)}{\varepsilon}\right) + \text{c.c.}\,,\\ \tilde{p}_z =& ika_0 e^{i\frac{\theta}{\varepsilon}} + \varepsilon(ika_1 + a_{0Z})e^{i\frac{\theta}{\varepsilon}} + \text{c.c.}\,,\\ \tilde{p}_{zz} =& -k^2 a_0 e^{i\frac{\theta}{\varepsilon}} + \varepsilon(-k^2 a_1 + ik_Z a_0 + 2ika_{0Z})e^{i\frac{\theta}{\varepsilon}} + \text{c.c.}\,,\\ \tilde{p}_t =& -i\omega a_0 e^{i\frac{\theta}{\varepsilon}} + \varepsilon(-i\omega a_1 + a_{0T})e^{i\frac{\theta}{\varepsilon}} + \text{c.c.}\,,\\ \tilde{p}_{tt} =& -\omega^2 a_0 e^{i\frac{\theta}{\varepsilon}} + \varepsilon(-\omega^2 a_1 - i\omega_T a_0 - 2i\omega a_{0T})e^{i\frac{\theta}{\varepsilon}} + \text{c.c.}\,.\end{aligned}$$

Substitute into the governing equation

$$\tilde{p}_{tt} - c_0^2\tilde{p}_{zz} + \frac{\varepsilon c_0^2\rho_{0Z}}{\rho_0}\tilde{p}_z = 0\,,$$

then on collecting terms of the same power of ε one obtains

$$\begin{aligned}\varepsilon^0:&\qquad -\omega^2 a_0 + c_0^2k^2a_0 = 0\,,\\ \varepsilon^1:&\qquad -\omega^2 a_1 - i\omega_T a_0 - 2i\omega a_{0T} + k^2c_0^2a_1 - ik_Zc_0^2a_0 - 2ikc_0^2a_{0Z} + \frac{c_0^2\rho_{0Z}}{\rho_0}ika_0 = 0\,.\end{aligned}$$

These yield the dispersion relation

$$\omega^2 = k^2c_0^2\,,$$

and the amplitude equation

$$(\omega a_0^2)_T + c_0^2(ka_0^2)_Z - \frac{\rho_{0Z}}{\rho_0}c_0^2ka_0^2 = 0\,.$$

From the governing equations

$$\begin{aligned}u =& \frac{ka_0}{\omega\rho_0}e^{i\frac{\theta}{\varepsilon}} + \dots + \text{c.c.}\,,\\ \tilde{S} =& -\varepsilon\frac{ika_0}{\omega^2\rho_0}S_{0Z}e^{i\frac{\theta}{\varepsilon}} + \dots + \text{c.c.}\,,\\ \tilde{\rho} =& \frac{a_0}{c_0^2}e^{i\frac{\theta}{\varepsilon}} + \dots + \text{c.c.}\,.\end{aligned}$$

The mean local energy density is given by

$$E = \langle \tfrac{1}{2}\rho_0 u^2 \rangle + \langle \tfrac{1}{2} c_0^2 \tilde{\rho}^2/\rho_0 \rangle$$
$$= \frac{|a_0|^2}{2\rho_0 c_0^2}\,,$$

and hence

$$\frac{E}{\omega} = \frac{|a_0|^2}{2\rho_0 c_0^2 \omega}\,,$$

where $\omega = \pm k c_0$ is the dispersion relation, and the [local] group velocity is given by $c_g = \pm c_0$. Thus

$$\frac{c_g E}{\omega} = \frac{k|a_0|^2}{2\rho_0 \omega^2}\,.$$

Henceforth assume a_0 is real for simplicity. Then

$$\begin{aligned}
\left(\frac{E}{\omega}\right)_T + \left(\frac{c_g E}{\omega}\right)_Z &= \frac{a_0 a_{0T}}{\rho_0 c_0^2 \omega} - \frac{\omega_T a_0^2}{2\rho_0 c_0^2 \omega^2} + \frac{k a_0 a_{0Z}}{\rho_0 \omega^2} - \frac{\rho_{0Z} k a_0^2}{2\rho_0^2 \omega^2} - \frac{k a_0^2 \omega_Z}{\rho_0 \omega^3} + \frac{k_Z a_0^2}{2\rho_0 \omega^2} \\
&= \frac{a_0}{2\rho_0 c_0^2 \omega^2}\left[2\omega a_{0T} - \omega_T a_0 + 2k c_0^2 a_{0Z} + k_Z c_0^2 a_0 - \frac{\rho_{0Z} k c_0^2 a_0}{\rho_0} - \frac{2k a_0 c_0^2 \omega_Z}{\omega}\right], \\
&= -\frac{a_0^2}{\rho_0 c_0^2 \omega^2}\left[\omega_T + \frac{k c_0^2 \omega_Z}{\omega}\right],
\end{aligned}$$

from making use of the amplitude equation. Further, because the dispersion relation is independent of time, from (7.40b)

$$\omega_T + c_g \omega_Z = 0\,,$$

i.e.

$$\omega_T + \frac{k c_0^2}{\omega}\omega_Z = 0\,.$$

16/15
16/17
16/18
16/19
16/20
16/22

Hence wave action is conserved:

$$\left(\frac{E}{\omega}\right)_T + \left(\frac{c_g E}{\omega}\right)_Z = 0\,. \tag{7.48}$$