Mathematical Tripos: IA Vectors & Matrices

Contents

-1 Vectors & Matrices: Introduction

-0.7 Schedule

This is a copy from the booklet of schedules.[1] Schedules are minimal for lecturing and maximal for examining; that is to say, all the material in the schedules will be lectured and only material in the schedules will be examined. The numbers in square brackets at the end of paragraphs of the schedules indicate *roughly* the number of lectures that will be devoted to the material in the paragraph.

VECTORS AND MATRICES **24 lectures, Michaelmas term**

Complex numbers
Review of complex numbers, including complex conjugate, inverse, modulus, argument and Argand diagram. Informal treatment of complex logarithm, *n-th* roots and complex powers. de Moivre's theorem. [2]

Vectors
Review of elementary algebra of vectors in $\mathbb{R}^3$, including scalar product. Brief discussion of vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$; scalar product and the Cauchy–Schwarz inequality. Concepts of linear span, linear independence, subspaces, basis and dimension.

Suffix notation: including summation convention, δ_{ij} and ϵ_{ijk}. Vector product and triple product: definition and geometrical interpretation. Solution of linear vector equations. Applications of vectors to geometry, including equations of lines, planes and spheres. [5]

Matrices
Elementary algebra of 3×3 matrices, including determinants. Extension to $n \times n$ complex matrices. Trace, determinant, non-singular matrices and inverses. Matrices as linear transformations; examples of geometrical actions including rotations, reflections, dilations, shears; kernel and image. [4]

Simultaneous linear equations: matrix formulation; existence and uniqueness of solutions, geometric interpretation; Gaussian elimination. [3]

Symmetric, anti-symmetric, orthogonal, hermitian and unitary matrices. Decomposition of a general matrix into isotropic, symmetric trace-free and antisymmetric parts. [1]

Eigenvalues and Eigenvectors
Eigenvalues and eigenvectors; geometric significance. [2]

Proof that eigenvalues of hermitian matrix are real, and that distinct eigenvalues give an orthogonal basis of eigenvectors. The effect of a general change of basis (similarity transformations). Diagonalization of general matrices: sufficient conditions; examples of matrices that cannot be diagonalized. Canonical forms for 2×2 matrices. [5]

Discussion of quadratic forms, including change of basis. Classification of conics, cartesian and polar forms. [1]

Rotation matrices and Lorentz transformations as transformation groups. [1]

Appropriate books

Alan F Beardon *Algebra and Geometry.* CUP 2005 (£21.99 paperback, £48.00 hardback).

D.E. Bourne and P.C. Kendall *Vector Analysis and Cartesian Tensors.* Nelson Thornes 1992 (£30.75 paperback).

James J. Callahan *The Geometry of Spacetime: An Introduction to Special and General Relativity.* Springer 2000 (£51.00).

[1] See the link from http://www.maths.cam.ac.uk/undergrad/course/.

John W. Dettman *Mathematical Methods in Physics and Engineering.* Dover, 1988 (Not in schedules, out of print).
Richard Kaye and Robert Wilson *Linear Algebra.* Oxford science publications, 1998 (£23.00).
E. Sernesi *Linear Algebra: A Geometric Approach.* CRC Press 1993 (£38.99 paperback).
Gilbert Strang *Linear Algebra and Its Applications.* Thomson Brooks/Cole, 2006 (£42.81 paperback).

-0.6 Lectures

- Lectures will start at 10:05 promptly with a summary of the last lecture. Please be on time since it is distracting to have people walking in late.
- I will endeavour to have a 2 minute break in the middle of the lecture for a rest and/or jokes and/or politics and/or paper aeroplanes[2]; students seem to find that the break makes it easier to concentrate throughout the lecture.[3]
- I will aim to finish by 10:55, but am not going to stop dead in the middle of a long proof/explanation.
- I will stay around for a few minutes at the front after lectures in order to answer questions.
- By all means chat to each other *quietly* if I am unclear, but please do not discuss, say, last night's football results, or who did (or did not) get drunk and/or laid. Such chatting is a distraction, as are mobile phones ringing in the middle of lectures: **please turn your mobile phones off**.
- I want you to learn. I will do my best to be clear but you *must* read through and understand your notes before the next lecture …otherwise there is a high probability that you will get hopelessly lost. An understanding of your notes will not diffuse into you just because you have carried your notes around for a week …or put them under your pillow.
- I welcome *constructive* heckling. If I am inaudible, illegible, unclear or just plain wrong then please shout out.
- I aim to avoid the words *trivial*, *easy*, *obvious* and *yes*[4]. Let me know if I fail. I will occasionally use *straightforward* or *similarly to last time*; if it is not, email me (`S.J.Cowley@damtp.cam.ac.uk`) or catch me at the end of the next lecture.
- Sometimes I may confuse both you and *myself* (I am not infallible), and may not be able to extract myself in the middle of a lecture. Under such circumstances I will have to plough on as a result of time constraints; however I will clear up any problems at the beginning of the next lecture.
- The course is on the pureish side of *applied* mathematics, but is applied mathematics. Hence do not always expect pure mathematical levels of rigour; having said that all the outline/sketch 'proofs' could in principle be tightened up given sufficient time.
- If anyone is colour blind please come and tell me which colour pens you cannot read.
- Finally, I was in your position 36 years ago and nearly gave up the Tripos. If you feel that the course is going over your head, or you are spending more than 10 or so hours (including lectures) a week on it, come and chat.

-0.5 Printed Notes

- Printed notes will be handed out for the course …so that you can listen to me rather than having to scribble things down. If it is not in the notes or on the example sheets it should not be in the exam.
- Any notes will only be available in lectures and only *once* for each set of notes.

[2] If you throw paper aeroplanes please pick them up. I will pick up the first one to stay in the air for 10 seconds.
[3] Having said that, research suggests that within the first 20 minutes I will, at some point, have lost the attention of all of you.
[4] But I will fail miserably in the case of *yes*.

- I do not keep back-copies (otherwise my office would be an even worse mess) ... from which you may conclude that I will not have copies of last time's notes (so please do not ask).

- There will only be approximately as many copies of the notes as there were students at the lecture on the previous Saturday.[5] We are going to fell a forest as it is, and I have no desire to be even more environmentally unsound.

- Please do not take copies for your absent friends unless they are *ill*, but if they are ill then please take copies.[6]

- The notes are deliberately *not* available on the WWW; they are an adjunct to lectures and are not meant to be used independently.

- If you do not want to attend lectures then there are a number of excellent textbooks that you can use in place of my notes.

- With one or two exceptions, figures/diagrams are deliberately omitted from the notes. I was taught to do this at my teaching course on *How To Lecture* ... the aim being that it might help you to stay awake if you have to write something down from time to time.

- There are a number of unlectured worked examples in the notes. In the past I have been tempted to not include these because I was worried that students would be unhappy with material in the notes that was not lectured. However, a vote in one of my previous lecture courses was overwhelming in favour of including unlectured worked examples.

- Please email me corrections to the notes and example sheets (`S.J.Cowley@damtp.cam.ac.uk`).

-0.4 Example Sheets

- There will be four main example sheets. They will be available on the WWW at about the same time as I hand them out (see `http://damtp.cam.ac.uk/user/examples/`). There will also be at least one supplementary 'study' sheet and a preliminary sheet 0.

- You should be able to do example sheets 1/2/3/4 after lectures 6/12/18/24 respectively, or thereabouts. Please bear this in mind when arranging supervisions. Personally I suggest that you do not have your first supervision before the middle of week 3 of lectures.

- There is some repetition on the sheets by design; pianists do scales, athletes do press-ups, mathematicians do algebra/manipulation.

- Your supervisors might like to know (a) that the example sheets will be heavily based on the sheets from last year, and (b) that if they send a nice email to me they can get copies of my printed notes.

-0.3 Computer Software

Vectors & Matrices is a theoretical course. However, in the Easter term there will be lectures and practicals on computing as a preparation for the Part IB *Computational Projects* course.[7] In order to complete the investigations that are part of the *Computational Projects* course you will need to use a computer language. To this end, in the Easter term the Faculty will provide you with a free copy of MATLAB (short for 'MATrix LABoratory') for your desktop or laptop.[8] As a preliminary to the Easter

[5] With the exception of the first two lectures for the pedants.
[6] If you really have been ill and cannot find a copy of the notes, then come and see me, but bring your sick-note.
[7] The Computational Projects course is an introduction to the techniques of solving problems in mathematics using computational methods. The projects are intended to be exercises in independent investigation somewhat like those a mathematician might be asked to undertake in the real world.
[8] The Faculty recommends (especially if you have not programmed before), and supports, use of MATLAB for the *Computational Projects* course. However, you are not required to use MATLAB, and are free to write your programs in any computing language whatsoever, e.g. Mathematica, Maple, R, C, Python, Visual Basic (although, other than MATLAB, these languages are not supported by the Faculty).

term material, when we get to the 'matrix' part of this course I may use MATLAB to illustrate one or more larger calculations that are impracticable by hand. At this stage, and if some of you are keen, I *may* be able to provide a limited number of copies of MATLAB early.

Some of you may also have heard of the *Mathematica* software package. While we do not teach this package to undergraduates, some of you might like to explore its capabilities and/or 'play' with it. Under an agreement between the Faculty and the suppliers of Mathematica (Wolfram Research), mathematics students can download versions of Mathematica for the Linux, MacOS and Windows operating systems from

`http://www.damtp.cam.ac.uk/computing/software/mathematica/`

Please note that the agreement expires in June 2013, and there is no guarantee that the agreement will be extended.

-0.2 Acknowledgements

The following notes were adapted (i.e. stolen) from those of Peter Haynes (my esteemed Head of Department), Tom Körner (who writes excellent general interest mathematics books, and who has a web page full of useful resources: see http://www.dpmms.cam.ac.uk/~twk/) and Robert Hunt.

-0.1 Some History and Culture

Most of *Vectors & Matrices* used to be the first 24 lectures of a *Algebra & Geometry* course.[9] A key aspect of the latter course was to show how the same mathematical entity could be understood by either an algebraic or a geometric approach. To some extent we adopt the same culture; hence for the same mathematical entity we will swap between algebraic and geometric descriptions (having checked that they are equivalent). Our aim will be to use the *easiest* way of looking at the same thing.

[9] *Algebra & Geometry* was 48 lectures long, and was lectured on consecutive days (bar Sunday). Sanity eventually prevailed, and the course was split into *Vectors & Matrices* and *Groups*.

0 Revision

You should check that you recall the following.

0.1 The Greek Alphabet

A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ϵ	epsilon	P	ρ	rho
Z	ζ	zeta	Σ	σ	sigma
H	η	eta	T	τ	tau
Θ	θ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega

There are also typographic variations of epsilon (i.e. ε), phi (i.e. φ), and rho (i.e. ϱ).

0.2 Sums and Elementary Transcendental Functions

0.2.1 The sum of a geometric progression

$$\sum_{k=0}^{n-1} \omega^k = \frac{1-\omega^n}{1-\omega}\,. \tag{0.1}$$

0.2.2 The binomial theorem

The binomial theorem for the expansion of powers of sums states that for a non-negative integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\,, \tag{0.2a}$$

where the *binomial coefficients* are given by

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}\,. \tag{0.2b}$$

0.2.3 The exponential function

One way to define the exponential function, $\exp(x)$, is by the series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,. \tag{0.3a}$$

From this definition one can deduce (after a little bit of work) that the exponential function has the following properties

$$\exp(0) = 1\,, \tag{0.3b}$$
$$\exp(1) = e \approx 2.71828183\,, \tag{0.3c}$$
$$\exp(x+y) = \exp(x)\exp(y)\,, \tag{0.3d}$$
$$\exp(-x) = \frac{1}{\exp(x)}\,. \tag{0.3e}$$

Exercise. Show that if x is integer or rational then

$$e^x = \exp(x)\,. \tag{0.4a}$$

If x is irrational we define e^x to be $\exp(x)$, i.e.

$$e^x \equiv \exp(x)\,. \tag{0.4b}$$

0.2.4 The logarithm

For a real number $x > 0$, the logarithm of x, i.e. $\log x$ (or $\ln x$ if you really want), is defined as the unique solution y of the equation

$$\exp(y) = x\,. \tag{0.5a}$$

It has the following properties

$$\log(1) = 0\,, \tag{0.5b}$$
$$\log(e) = 1\,, \tag{0.5c}$$
$$\log(\exp(x)) = x\,, \tag{0.5d}$$
$$\log(xy) = \log(x) + \log(y)\,, \tag{0.5e}$$
$$\log(y) = -\log\left(\frac{1}{y}\right)\,. \tag{0.5f}$$

Exercise. Show that if x is integer or rational then

$$\log(y^x) = x\log(y)\,. \tag{0.6a}$$

If x is irrational we define $\log(y^x)$ to be $x\log(y)$, i.e.

$$y^x \equiv \exp(x\log(y))\,. \tag{0.6b}$$

0.2.5 The cosine and sine functions

The cosine and sine functions are defined by the series

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-)^n x^{2n}}{2n!}\,, \tag{0.7a}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-)^n x^{2n+1}}{(2n+1)!}\,. \tag{0.7b}$$

0.2.6 Certain trigonometric identities

You should recall the following

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)\,, \tag{0.8a}$$
$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)\,, \tag{0.8b}$$
$$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}\,, \tag{0.8c}$$
$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\,, \tag{0.8d}$$
$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\,, \tag{0.8e}$$
$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\,, \tag{0.8f}$$
$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\,. \tag{0.8g}$$

0.2.7 The cosine rule

Let ABC be a triangle. Let the lengths of the sides opposite vertices A, B and C be a, b and c respectively. Further suppose that the angles subtended at A, B and C are α, β and γ respectively. Then the cosine rule (also known as the cosine formula or law of cosines) states that

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos\alpha\,, && (0.9a)\\
b^2 &= a^2 + c^2 - 2ac\cos\beta\,, && (0.9b)\\
c^2 &= a^2 + b^2 - 2ab\cos\gamma\,. && (0.9c)
\end{aligned}
$$

Exercise: draw the figure (if it's not there).

0.3 Elemetary Geometry

0.3.1 The equation of a line

In 2D Cartesian co-ordinates, (x, y), the equation of a line with slope m which passes through (x_0, y_0) is given by

$$y - y_0 = m(x - x_0)\,. \qquad (0.10a)$$

In parametric form the equation of this line is given by

$$x = x_0 + a\,t\,, \quad y = y_0 + a\,m\,t\,, \qquad (0.10b)$$

where t is the parametric variable and a is an arbitrary real number.

0.3.2 The equation of a circle

In 2D Cartesian co-ordinates, (x, y), the equation of a circle of radius r and centre (p, q) is given by

$$(x - p)^2 + (y - q)^2 = r^2\,. \qquad (0.11)$$

0.3.3 Plane polar co-ordinates (r, θ)

In plane polar co-ordinates the co-ordinates of a point are given in terms of a radial distance, r, from the origin and a polar angle, θ, where $0 \leqslant r < \infty$ and $0 \leqslant \theta < 2\pi$. In terms of 2D Cartesian co-ordinates, (x, y),

$$x = r\cos\theta\,, \quad y = r\sin\theta\,. \qquad (0.12a)$$

From inverting (0.12a) it follows that

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2}\,, && (0.12b)\\
\theta &= \arctan\left(\frac{y}{x}\right)\,, && (0.12c)
\end{aligned}
$$

where the choice of arctan should be such that $0 < \theta < \pi$ if $y > 0$, $\pi < \theta < 2\pi$ if $y < 0$, $\theta = 0$ if $x > 0$ and $y = 0$, and $\theta = \pi$ if $x < 0$ and $y = 0$.

Exercise: draw the figure (if it's not there).

Remark: sometimes ρ and/or ϕ are used in place of r and/or θ respectively.

0.4 Complex Numbers

All of you should have the equivalent of a *Further Mathematics* AS-level, and hence should have encountered complex numbers before. The following is 'revision', just in case you have not!

0.4.1 Real numbers

The real numbers are denoted by $\mathbb{R}$ and consist of:

integers,	denoted by $\mathbb{Z}$,	$\ldots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots$
rationals,	denoted by $\mathbb{Q}$,	p/q where $p,\ q$ are integers $(q \neq 0)$
irrationals,		the rest of the reals, e.g. $\sqrt{2},\ e,\ \pi,\ \pi^2$.

We sometimes visualise real numbers as lying on a line (e.g. between any two distinct points on a line there is another point, and between any two distinct real numbers there is always another real number).

0.4.2 i and the general solution of a quadratic equation

Consider the quadratic equation

$$\alpha z^2 + \beta z + \gamma = 0 \quad : \quad \alpha,\, \beta,\, \gamma \in \mathbb{R} \quad , \alpha \neq 0\,,$$

where $\in$ means 'belongs to'. This has two roots

$$z_1 = -\frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \quad \text{and} \quad z_2 = -\frac{\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}\,. \tag{0.13}$$

If $\beta^2 \geqslant 4\alpha\gamma$ then the roots are real (there is a repeated root if $\beta^2 = 4\alpha\gamma$). If $\beta^2 < 4\alpha\gamma$ then the square root is not equal to any real number. In order that we can *always* solve a quadratic equation, we introduce

$$i = \sqrt{-1}\,. \tag{0.14}$$

Remark: note that i is sometimes denoted by j by engineers (and MATLAB).

If $\beta^2 < 4\alpha\gamma$, (0.13) can now be rewritten

$$z_1 = -\frac{\beta}{2\alpha} + i\frac{\sqrt{4\alpha\gamma - \beta^2}}{2\alpha} \quad \text{and} \quad z_2 = -\frac{\beta}{2\alpha} - i\frac{\sqrt{4\alpha\gamma - \beta^2}}{2\alpha}\,, \tag{0.15}$$

where the square roots are now real [numbers]. Subject to us being happy with the introduction and existence of i, we can now *always* solve a quadratic equation.

0.4.3 Complex numbers (by algebra)

Complex numbers are denoted by $\mathbb{C}$. We define a complex number, say z, to be a number with the form

$$z = a + ib, \quad \text{where} \quad a,\, b \in \mathbb{R}, \tag{0.16}$$

where $i = \sqrt{-1}$ (see (0.14)). We say that $z \in \mathbb{C}$.

For $z = a + ib$, we sometimes write

$$\begin{aligned} a = \mathrm{Re}\,(z) &\quad : \quad \text{the real part of } z, \\ b = \mathrm{Im}\,(z) &\quad : \quad \text{the imaginary part of } z. \end{aligned}$$

Remarks.

(i) $\mathbb{C}$ contains all real numbers since if $a \in \mathbb{R}$ then $a + i.0 \in \mathbb{C}$.

(ii) A complex number $0 + i.b$ is said to be *pure imaginary*.

(iii) Extending the number system from real ($\mathbb{R}$) to complex ($\mathbb{C}$) allows a number of important generalisations, e.g. it is now possible to always to solve a quadratic equation (see §0.4.2), and it makes solving certain differential equations much easier.

(iv) Complex numbers were first used by Tartaglia (1500-1557) and Cardano (1501-1576). The terms *real* and *imaginary* were first introduced by Descartes (1596-1650).

Theorem 0.1. *The representation of a complex number z in terms of its real and imaginary parts is unique.*

Proof. Assume $\exists\ a, b, c, d \in \mathbb{R}$ such that

$$z = a + ib = c + id.$$

Then $a - c = i\,(d - b)$, and so $(a - c)^2 = -\,(d - b)^2$. But the only number greater than or equal to zero that is equal to a number that is less than or equal to zero, is zero. Hence $a = c$ and $b = d$. $\square$

Corollary 0.2. *If $z_1 = z_2$ where $z_1, z_2 \in \mathbb{C}$, then* $\mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2)$ *and* $\mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)$.

0.4.4 Algebraic manipulation of complex numbers

In order to manipulate complex numbers simply follow the rules for reals, but adding the rule $i^2 = -1$. Hence for $z_1 = a + ib$ and $z_2 = c + id$, where $a, b, c, d \in \mathbb{R}$, we have that

$$\text{addition/subtraction}: \quad z_1 + z_2 = (a + ib) \pm (c + id) = (a \pm c) + i\,(b \pm d)\ ; \tag{0.17a}$$

$$\begin{aligned}\text{multiplication}: \quad z_1\, z_2 &= (a + ib)\,(c + id) = ac + ibc + ida + (ib)\,(id)\\ &= (ac - bd) + i\,(bc + ad)\ ;\end{aligned} \tag{0.17b}$$

$$\text{inverse}: \quad z_1^{-1} = \frac{1}{z} = \frac{1}{a + ib}\,\frac{a - ib}{a - ib} = \frac{a}{a^2 + b^2} - \frac{ib}{a^2 + b^2}. \tag{0.17c}$$

Remark. All the above operations on elements of $\mathbb{C}$ result in new elements of $\mathbb{C}$. This is described as *closure*: $\mathbb{C}$ is closed under addition and multiplication.

Exercises.

(i) For z_1^{-1} as defined in (0.17c), check that $z_1\, z_1^{-1} = 1 + i.0$.

(ii) Show that addition is *commutative* and *associative*, i.e.

$$z_1 + z_2 = z_2 + z_1 \quad \text{and} \quad z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3\,. \tag{0.18a}$$

(iii) Show that multiplication is *commutative* and *associative*, i.e.

$$z_1 z_2 = z_2 z_1 \quad \text{and} \quad z_1(z_2 z_3) = (z_1 z_2) z_3\,. \tag{0.18b}$$

(iv) Show that multiplication is *distributive* over addition, i.e.

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3\,. \tag{0.18c}$$

1 Complex Numbers

1.0 Why Study This?

For the same reason as we study real numbers, $\mathbb{R}$: because they are useful and occur throughout mathematics.

1.1 Functions of Complex Numbers

We may extend the idea of functions to complex numbers. A complex-valued function f is one that takes a complex number as 'input' and defines a new complex number $f(z)$ as 'output'.

1.1.1 Complex conjugate

The *complex conjugate* of $z = a + ib$, which is usually written as $\overline{z}$, but sometimes as z^*, is defined as $a - ib$, i.e.

$$\text{if} \quad z = a + ib \quad \text{then} \quad \overline{z} \equiv z^* = a - ib. \tag{1.1}$$

Exercises. Show that

(i)
$$\overline{\overline{z}} = z\,; \tag{1.2a}$$

(ii)
$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}\,; \tag{1.2b}$$

(iii)
$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}\,; \tag{1.2c}$$

(iv)
$$\overline{(z^{-1})} = (\overline{z})^{-1}\,. \tag{1.2d}$$

Definition. Given a complex-valued function f, the complex conjugate function $\overline{f}$ is defined by

$$\overline{f}\,(\overline{z}) = \overline{f\,(z)}, \quad \text{and hence from (1.2a)} \quad \overline{f}\,(z) = \overline{f\,(\overline{z})}. \tag{1.3}$$

Example. Let $f\,(z) = pz^2 + qz + r$ with $p, q, r \in \mathbb{C}$ then by using (1.2b) and (1.2c)

$$\overline{f}\,(\overline{z}) \equiv \overline{f\,(z)} = \overline{pz^2 + qz + r} = \overline{p}\,\overline{z}^2 + \overline{q}\,\overline{z} + \overline{r}.$$

Hence $\overline{f}\,(z) = \overline{p}\,z^2 + \overline{q}\,z + \overline{r}$.

1.1.2 Modulus

The *modulus* of $z = a + ib$, which is written as $|z|$, is defined as

$$|z| = \left(a^2 + b^2\right)^{1/2}\,. \tag{1.4}$$

Exercises. Show that

(i)
$$|z|^2 = z\,\overline{z}\,; \tag{1.5a}$$

(ii)
$$z^{-1} = \frac{\overline{z}}{|z|^2}\,. \tag{1.5b}$$

1/03

1.2 The Argand Diagram: Complex Numbers by Geometry

Consider the set of points in two dimensional (2D) space referred to Cartesian axes. Then we can represent each $z = x+iy \in \mathbb{C}$ by the point (x, y), i.e. the real and imaginary parts of z are viewed as co-ordinates in an xy plot. We label the 2D *vector* between the origin and (x, y), say $\vec{OP}$, by the complex number z. Such a plot is called an *Argand diagram* (cf. the *number line* for real numbers).

Remarks.

(i) The xy plane is referred to as the *complex plane.* We refer to the x-axis as the *real axis*, and the y-axis as the *imaginary axis.*

(ii) The Argand diagram was invented by Caspar Wessel (1797), and re-invented by Jean-Robert Argand (1806).

Modulus. The modulus of z corresponds to the magnitude of the vector $\vec{OP}$ since

$$|z| = \left(x^2 + y^2\right)^{1/2} .$$

Complex conjugate. If $\vec{OP}$ represents z, then $\vec{OP'}$ represents $\bar{z}$, where P' is the point $(x, -y)$; i.e. P' is P reflected in the x-axis.

Addition. Let $z_1 = x_1+iy_1$ be associated with P_1, and $z_2 = x_2 + iy_2$ be associated with P_2. Then

$$z_3 = z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2) \ ,$$

is associated with the point P_3 that is obtained by completing the parallelogram $P_1\,O\,P_2P_3$. In terms of vector addition

$$\begin{aligned} \vec{OP_3} &= \vec{OP_1} + \vec{OP_2} \\ &= \vec{OP_2} + \vec{OP_1}\,, \end{aligned}$$

which is sometimes called the *triangle law.*

1/02

Theorem 1.1. *If $z_1, z_2 \in \mathbb{C}$ then*

$$|z_1 + z_2| \leqslant |z_1| + |z_2|\,, \tag{1.6a}$$
$$|z_1 - z_2| \geqslant \big|\,|z_1| - |z_2|\,\big|\,. \tag{1.6b}$$

Remark. Result (1.6a) is known as the *triangle inequality* (and is in fact one of many triangle inequalities).

Proof. Self-evident by geometry. Alternatively, by the cosine rule (0.9a)

$$\begin{aligned} |z_1 + z_2|^2 &= |z_1|^2 + |z_2|^2 - 2|z_1|\,|z_2|\cos\psi \\ &\leqslant |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| \\ &= (|z_1| + |z_2|)^2 \ . \end{aligned}$$

(1.6b) follows from (1.6a). Let $z_1' = z_1 + z_2$ and $z_2' = z_2$, so that $z_1 = z_1' - z_2'$ and $z_2 = z_2'$. Then (1.6a) implies that

$$|z_1'| \leqslant |z_1' - z_2'| + |z_2'|\,,$$

and hence that

$$|z_1' - z_2'| \geqslant |z_1'| - |z_2'|\,.$$

Interchanging z_1' and z_2' we also have that

$$|z_2' - z_1'| = |z_1' - z_2'| \geqslant |z_2'| - |z_1'|\,.$$

(1.6b) follows. □

1.3 Polar (Modulus/Argument) Representation

Another helpful representation of complex numbers is obtained by using plane polar co-ordinates to represent position in Argand diagram. Let $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\begin{aligned} z = x + iy &= r\cos\theta + ir\sin\theta \\ &= r\left(\cos\theta + i\sin\theta\right)\,. \end{aligned} \tag{1.7}$$

Note that

$$|z| = \left(x^2 + y^2\right)^{1/2} = r\,. \tag{1.8}$$

- Hence r is the modulus of z (mod(z) for short).
- θ is called the *argument* of z (arg (z) for short).
- The expression for z in terms of r and θ is called
1/06 the *modulus/argument form*.

The pair (r, θ) specifies z uniquely. However, z does *not* specify (r, θ) uniquely, since adding $2n\pi$ to θ ($n \in \mathbb{Z}$, i.e. the integers) does not change z. For each z there is a unique value of the argument θ such that $-\pi < \theta \leqslant \pi$, sometimes called the *principal value* of the argument.

Remark. In order to get a unique value of the argument it is sometimes more convenient to restrict θ to $0 \leqslant \theta < 2\pi$ (or to restrict θ to $-\pi < \theta \leqslant \pi$ or to . . .).

1.3.1 Geometric interpretation of multiplication

Consider z_1, z_2 written in modulus argument form:

$$\begin{aligned} z_1 &= r_1\left(\cos\theta_1 + i\sin\theta_1\right)\,, \\ z_2 &= r_2\left(\cos\theta_2 + i\sin\theta_2\right)\,. \end{aligned}$$

Then, using (0.8a) and (0.8b),

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \left(\cos\theta_1 . \cos\theta_2 - \sin\theta_1 . \sin\theta_2 \right. \\ &\qquad \left. + i\left(\sin\theta_1 . \cos\theta_2 + \sin\theta_2 . \cos\theta_1\right)\right) \\ &= r_1 r_2 \left(\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right)\,. \end{aligned} \tag{1.9}$$

Hence

$$|z_1 z_2| = |z_1|\,|z_2|\,, \tag{1.10a}$$
$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \quad (+2n\pi \text{ with } n \text{ an arbitrary integer}). \tag{1.10b}$$

In words: multiplication of z_1 by z_2 scales z_1 by $|\,z_2\,|$ and rotates z_1 by $\arg(z_2)$.

1/07 *Exercise.* Find an equivalent result for z_1/z_2.

1.4 The Exponential Function

1.4.1 The real exponential function

The real exponential function, $\exp(x)$, is defined by the power series

$$\exp(x) = \exp x = 1 + x + \frac{x^2}{2!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,. \tag{1.11}$$

This series converges for all $x \in \mathbb{R}$ (see the *Analysis I* course).

Worked exercise. Show for $x, y \in \mathbb{R}$ that

$$\exp(x)\,\exp(y) = \exp(x+y)\,. \tag{1.12a}$$

Solution (well nearly a solution).

$$\begin{aligned}
\exp(x)\,\exp(y) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!} \\
&= \sum_{r=0}^{\infty} \sum_{m=0}^{r} \frac{x^{r-m}}{(r-m)!} \frac{y^m}{m!} \quad \text{for } n = r-m \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{m=0}^{r} \frac{r!}{(r-m)!m!}\, x^{r-m} y^m \\
&= \sum_{r=0}^{\infty} \frac{(x+y)^r}{r!} \quad \text{by the binomial theorem} \\
&= \exp(x+y)\,.
\end{aligned}$$

1/08

Definition. We write

$$\exp(1) = e\,. \tag{1.12b}$$

Worked exercise. Show for $n, p, q \in \mathbb{Z}$, where without loss of generality (wlog) $q > 0$, that:

$$e^n = \exp(n) \qquad \text{and} \qquad e^{\frac{p}{q}} = \exp\left(\frac{p}{q}\right).$$

Solution. For $n = 1$ there is nothing to prove. For $n \geqslant 2$, and using (1.12a),

$$\exp(n) = \exp(1)\exp(n-1) = e\,\exp(n-1)\,, \quad \text{and thence by induction} \quad \exp(n) = e^n\,.$$

From the power series definition (1.11) with $n = 0$:

$$\exp(0) = 1 = e^0\,.$$

Also from (1.12a) we have that

$$\exp(-1)\exp(1) = \exp(0)\,, \quad \text{and thence} \quad \exp(-1) = \frac{1}{e} = e^{-1}\,.$$

For $n \leqslant -2$ proceed by induction as above.

Next note from applying (1.12a) q times that

$$\left(\exp\left(\frac{p}{q}\right)\right)^q = \exp\left(p\right) = e^p\,.$$

Thence on taking the positive qth root

$$\exp\left(\frac{p}{q}\right) = e^{\frac{p}{q}}\,.$$

Definition. For irrational x, define

$$e^x = \exp(x)\,. \tag{1.12c}$$

From the above it follows that if $y \in \mathbb{R}$, then it is consistent to write $\exp(y) = e^y$.

1.4.2 The complex exponential function

Definition. For $z \in \mathbb{C}$, the *complex exponential* is defined by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}\,. \tag{1.13a}$$

This series converges for all finite $|z|$ (again see the *Analysis I* course).

Definition. For $z \in \mathbb{C}$ and $z \notin \mathbb{R}$ we define

$$e^z = \exp(z)\,, \tag{1.13b}$$

Remarks.

(i) When $z \in \mathbb{R}$ these definitions are consistent with (1.11) and (1.12c).

(ii) For $z_1, z_2 \in \mathbb{C}$,

$$\exp\left(z_1\right)\,\exp\left(z_2\right) = \exp\left(z_1 + z_2\right)\,, \tag{1.13c}$$

with the proof essentially as for (1.12a).

(iii) From (1.13b) and (1.13c)

$$e^{z_1}e^{z_2} = \exp(z_1)\exp(z_2) = \exp(z_1 + z_2) = e^{z_1+z_2}\,. \tag{1.13d}$$

1/09

1.4.3 The complex trigonometric functions

Definition.

$$\cos\,w = \sum_{n=0}^{\infty} (-1)^n\,\frac{w^{2n}}{(2n)!} \quad \text{and} \quad \sin\,w = \sum_{n=0}^{\infty} (-1)^n\,\frac{w^{2n+1}}{(2n+1)!}\,. \tag{1.14}$$

Remark. For $w \in \mathbb{R}$ these definitions of cosine and sine are consistent with (0.7a) and (0.7b).

Theorem 1.2. *For $w \in \mathbb{C}$*

$$\exp(iw) \equiv e^{iw} = \cos w + i \sin w\,. \tag{1.15}$$

Unlectured Proof. From (1.13a) and (1.14) we have that

$$\begin{aligned}
\exp(iw) &= \sum_{n=0}^{\infty} \frac{(iw)^n}{n!} = 1 + iw - \frac{w^2}{2} - i\frac{w^3}{3!}\dots \\
&= \left(1 - \frac{w^2}{2!} + \frac{w^4}{4!}\dots\right) + i\left(w - \frac{w^3}{3!} + \frac{w^5}{5!}\dots\right) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n+1}}{(2n+1)!} \\
&= \cos w + i \sin w\,,
\end{aligned}$$

which is as required (as long as we do not mind living dangerously and re-ordering infinite series). □

Remarks.

(i) From taking the complex conjugate of (1.15) and then exchanging w for $\bar{w}$, or otherwise,

$$\exp(-iw) \equiv e^{-iw} = \cos w - i \sin w\,. \tag{1.16}$$

(ii) From (1.15) and (1.16) it follows that

$$\cos w = \tfrac{1}{2}\left(e^{iw} + e^{-iw}\right)\,, \quad \text{and} \quad \sin w = \frac{1}{2i}\left(e^{iw} - e^{-iw}\right)\,. \tag{1.17}$$

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2/02

1.4.4 Relation to modulus/argument form

Let $w = \theta$ where $\theta \in \mathbb{R}$. Then from (1.15)

$$e^{i\theta} = \cos\theta + i \sin\theta\,. \tag{1.18}$$

It follows from the polar representation (1.7) that

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}\,, \tag{1.19}$$

with (again) $r = |z|$ and $\theta = \arg z$. In this representation the multiplication of two complex numbers is rather elegant:

$$z_1 z_2 = \left(r_1 e^{i\,\theta_1}\right)\left(r_2 e^{i\,\theta_2}\right) = r_1 r_2\, e^{i(\theta_1+\theta_2)}\,,$$

confirming (1.10a) and (1.10b).

1.4.5 Modulus/argument expression for 1

Consider solutions of

$$z = re^{i\,\theta} = 1\,. \tag{1.20a}$$

Since by definition $r, \theta \in \mathbb{R}$, it follows that $r = 1$ and

$$e^{i\,\theta} = \cos\theta + i\,\sin\theta = 1\,,$$

and thence that $\cos\theta = 1$ and $\sin\theta = 0$. We deduce that

$$r = 1 \quad \text{and} \quad \theta = 2k\pi \quad \text{with} \quad k \in \mathbb{Z}. \tag{1.20b}$$

2/03

1.5 Roots of Unity

A root of unity is a solution of $z^n = 1$, with $z \in \mathbb{C}$ and n a positive integer.

Theorem 1.3. *There are n solutions of $z^n = 1$ (i.e. there are n 'nth roots of unity')*

Proof. One solution is $z = 1$. Seek more general solutions of the form $z = r\,e^{i\,\theta}$, with the restriction that $0 \leqslant \theta < 2\pi$ so that θ is not multi-valued. Then, from (0.18b) and (1.13d),

$$\left(r\,e^{i\,\theta}\right)^n = r^n \left(e^{i\,\theta}\right)^n = r^n e^{in\theta} = 1\,; \tag{1.21a}$$

hence from (1.20a) and (1.20b), $r^n = 1$ and $n\,\theta = 2k\pi$ with $k \in \mathbb{Z}$. We conclude that within the requirement that $0 \leqslant \theta < 2\pi$, there are n distinct roots given by

$$r = 1 \quad \text{and} \quad \theta = \frac{2k\pi}{n} \quad \text{with} \quad k = 0, 1, \ldots, n-1. \tag{1.21b}$$

□

Remark. If we write $\omega = e^{2\pi\,i/n}$, then the roots of $z^n = 1$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. Further, for $n \geqslant 2$ it follows from the sum of a geometric progression, (0.1), that

$$1 + \omega + \cdots + \omega^{n-1} = \sum_{k=0}^{n-1} \omega^k = \frac{1-\omega^n}{1-\omega} = 0\,, \tag{1.22}$$

because $\omega^n = 1$.

Geometric example. Solve $z^5 = 1$.

Solution. Put $z = e^{i\,\theta}$, then we require that

$$e^{5i\,\theta} = e^{2\pi k i} \quad \text{for} \quad k \in \mathbb{Z}.$$

There are thus five distinct roots given by

$$\theta = 2\pi k/5 \quad \text{with} \quad k = 0, 1, 2, 3, 4.$$

Larger (or smaller) values of k yield no new roots. If we write $\omega = e^{2\pi\,i/5}$, then the roots are 1, ω, ω^2, ω^3 and ω^4, and from (1.22)

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0\,.$$

Each root corresponds to a vertex of a pentagon. 2/06

1.6 Logarithms and Complex Powers

We know already that if $x \in \mathbb{R}$ and $x > 0$, the equation $e^y = x$ has a unique real solution, namely $y = \log x$ (or $\ln x$ if you prefer).

Definition. For $z \in \mathbb{C}$ define $\log z$ as 'the' solution w of

$$e^w = z\,. \tag{1.23a}$$

Remarks.

(i) By definition

$$\exp(\log(z)) = z\,. \tag{1.23b}$$

(ii) Let $y = \log(z)$ and take the logarithm of both sides of (1.23b) to conclude that

$$\begin{aligned} \log(\exp(y)) &= \log(\exp(\log(z))) \\ &= \log(z) \\ &= y\,. \end{aligned} \tag{1.23c}$$

To understand the nature of the complex logarithm let $w = u + iv$ with $u, v \in \mathbb{R}$. Then from (1.23a) $e^{u+iv} = e^w = z = re^{i\theta}$, and hence

$$\begin{aligned} e^u &= |z| = r\,, \\ v &= \arg z = \theta + 2k\pi \quad \text{for any } k \in \mathbb{Z}\,. \end{aligned}$$

Thus

$$\log z = w = u + iv = \log |z| + i \arg z\,. \tag{1.24a}$$

Remark. Since $\arg z$ is a multi-valued function, so is $\log z$.

Definition. The *principal value* of $\log z$ is such that

$$-\pi < \arg z = \operatorname{Im}(\log z) \leqslant \pi. \tag{1.24b}$$

Example. If $z = -x$ with $x \in \mathbb{R}$ and $x > 0$, then

$$\begin{aligned} \log z &= \log |-x| + i \arg(-x) \\ &= \log |x| + (2k+1)i\pi \quad \text{for any } k \in \mathbb{Z}\,. \end{aligned}$$

The principal value of $\log(-x)$ is $\log |x| + i\pi$.

1.6.1 Complex powers

Recall the definition of x^a, for $x, a \in \mathbb{R}$, $x > 0$ and a irrational, namely

$$x^a = e^{a \log x} = \exp(a \log x)\,.$$

Definition. For $z \neq 0$, $z, w \in \mathbb{C}$, define z^w by

$$z^w = e^{w \log z}. \tag{1.25}$$

Remark. Since $\log z$ is multi-valued so is z^w, i.e. z^w is only defined up to an arbitrary multiple of $e^{2ki\pi w}$, for any $k \in \mathbb{Z}$.[10]

[10] If $z, w \in \mathbb{R}$, $z > 0$, then whether z^w is single-valued or multi-valued depends on whether we are working in the *field* of real numbers or the *field* of complex numbers. You can safely ignore this footnote if you do not know what a *field* is.

Examples.

(i) For $a, b \in \mathbb{C}$ it follows from (1.25) that

$$z^{ab} = \exp(ab \log z) = \exp(b(a \log z)) = y^b \,,$$

where

$$\log y = a \log z \,.$$

But from the definition of the logarithm (1.23b) we have that $\exp(\log(z)) = z$. Hence

$$y = \exp(a \log z) \equiv z^a \,,$$

and thus (after a little thought for the the second equality)

$$z^{ab} = \left(z^a\right)^b = \left(z^b\right)^a \,. \tag{1.26}$$

(ii) *Unlectured.* The value of i^i is given by

$$\begin{aligned} i^i &= e^{i \log i} \\ &= e^{i(\log|i| + i \arg i)} \\ &= e^{i(\log 1 + 2ki\,\pi + i\pi/2)} \\ &= e^{-\left(2k+\frac{1}{2}\right)\pi} \quad \text{for any } k \in \mathbb{Z} \quad \text{(which is real).} \end{aligned}$$

2/07

1.7 De Moivre's Theorem

Theorem 1.4. *De Moivre's theorem states that for $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$*

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n \,. \tag{1.27}$$

Proof. From (1.18) and (1.26)

$$\begin{aligned} \cos n\theta + i \sin n\theta &= e^{i\,(n\theta)} \\ &= \left(e^{i\,\theta}\right)^n \\ &= (\cos\,\theta + i \sin\,\theta)^n \,. \end{aligned}$$

□

Remark. Although de Moivre's theorem requires $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$, equality (1.27) also holds for $\theta, n \in \mathbb{C}$ in the sense that when $(\cos\theta + i\sin\theta)^n$ as defined by (1.25) is multi-valued, the single-valued $(\cos n\theta + i \sin n\theta)$ is equal to *one* of the values of $(\cos\theta + i\sin\theta)^n$.

Unlectured Alternative Proof. (1.27) is true for $n = 0$. Now argue by induction.

Assume true for $n = p > 0$, i.e. assume that $(\cos\theta + i\sin\theta)^p = \cos p\theta + i \sin p\theta$. Then

$$\begin{aligned} (\cos\theta + i\sin\theta)^{p+1} &= (\cos\theta + i\sin\theta)\,(\cos\theta + i\sin\theta)^p \\ &= (\cos\theta + i\sin\theta)\,(\cos p\theta + i\sin p\theta) \\ &= \cos\theta.\cos p\theta - \sin\theta.\sin p\theta + i\,(\sin\theta.\cos p\theta + \cos\theta.\sin p\theta) \\ &= \cos\,(p+1)\,\theta + i \sin\,(p+1)\,\theta \,. \end{aligned}$$

Hence the result is true for $n = p + 1$, and so holds for all $n \geqslant 0$. Now consider $n < 0$, say $n = -p$. Then, using the proved result for $p > 0$,

$$\begin{aligned} (\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-p} \\ &= \frac{1}{(\cos\theta + i\sin\theta)^p} \\ &= \frac{1}{\cos p\,\theta + i\sin p\theta} \\ &= \cos p\theta - i\sin p\,\theta \\ &= \cos n\,\theta + i\sin n\theta \end{aligned}$$

Hence de Moivre's theorem is true $\forall\; n \in \mathbb{Z}$.

1.8 Lines and Circles in the Complex Plane

1.8.1 Lines

For fixed $z_0, w \in \mathbb{C}$ with $w \neq 0$, and varying $\lambda \in \mathbb{R}$, the equation

$$z = z_0 + \lambda w \qquad (1.28a)$$

represents in the Argand diagram (complex plane) points on straight line through z_0 and parallel to w.

Remark. Since $\lambda \in \mathbb{R}$, it follows that $\lambda = \bar{\lambda}$, and hence, since $\lambda = (z - z_0)/w$, that

$$\frac{z - z_0}{w} = \frac{\bar{z} - \bar{z}_0}{\bar{w}}\,.$$

Thus

$$z\bar{w} - \bar{z}w = z_0\bar{w} - \bar{z}_0 w \qquad (1.28b)$$

is an alternative representation of the line.

Worked exercise. Show that $z_0\bar{w} - \bar{z}_0 w = 0$ if and only if (iff) the line (1.28a) passes through the origin.

Solution. If the line passes through the origin then put $z = 0$ in (1.28b), and the result follows. If $z_0\bar{w} - \bar{z}_0 w = 0$, then the equation of the line is $z\bar{w} - \bar{z}w = 0$. This is satisfied by $z = 0$, and hence the line passes through the origin.

Exercise. For non-zero $w, z \in \mathbb{C}$ show that if $z\bar{w} - \bar{z}w = 0$, then $z = \gamma w$ for some $\gamma \in \mathbb{R}$.

1.8.2 Circles

In the Argand diagram, a circle of radius $r \neq 0$ and centre v ($r \in \mathbb{R}$, $v \in \mathbb{C}$) is given by

$$S = \{z \in \mathbb{C} \; : \mid z - v \mid = r\}\,, \qquad (1.29a)$$

i.e. the set of complex numbers z such that $|z - v| = r$.

Remarks.

- If $z = x + iy$ and $v = p + iq$ then
$$|z - v|^2 = (x - p)^2 + (y - q)^2 = r^2\,,$$
which is the equation for a circle with centre (p, q) and radius r in Cartesian coordinates (see (0.11)).
- Since $|z - v|^2 = (\bar{z} - \bar{v})(z - v)$, an alternative equation for the circle is
$$|z|^2 - \bar{v}z - v\bar{z} + |v|^2 = r^2\,. \qquad (1.29b)$$

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2/08

1.9 Möbius Transformations

Möbius transformations used to be part of the schedule. If you would like a preview see

http://www.youtube.com/watch?v=0z1fIsUNhO4 or
http://www.youtube.com/watch?v=JX3VmDgiFnY.

2 Vector Algebra

2.0 Why Study This?

Many scientific quantities just have a magnitude, e.g. time, temperature, density, concentration. Such quantities can be completely specified by a single number. We refer to such numbers as *scalars*. You have learnt how to manipulate such scalars (e.g. by addition, subtraction, multiplication, differentiation) since your first day in school (or possibly before that). A scalar, e.g. temperature T, that is a function of position (x, y, z) is referred to as a *scalar field*; in the case of our example we write $T \equiv T(x, y, z)$.

However other quantities have both a magnitude and a direction, e.g. the position of a particle, the velocity of a particle, the direction of propagation of a wave, a force, an electric field, a magnetic field. You need to know how to manipulate these quantities (e.g. by addition, subtraction, multiplication and, next term, differentiation) if you are to be able to describe them mathematically.

2.1 Vectors

Geometric definition. A quantity that is specified by a [positive] magnitude and a direction in space is called a *vector*.

Geometric representation. We will represent a vector $\mathbf{v}$ as a line segment, say $\overrightarrow{AB}$, with length $|\mathbf{v}|$ and with direction/sense from A to B.

Remarks.

- For the purpose of this course the notes will represent vectors in bold, e.g. $\mathbf{v}$. On the overhead/blackboard I will put a squiggle under the $\underset{\sim}{v}$.[11]
- The magnitude of a vector $\mathbf{v}$ is written $|\mathbf{v}|$.
- Two vectors $\mathbf{u}$ and $\mathbf{v}$ are equal if they have the same magnitude, i.e. $|\mathbf{u}| = |\mathbf{v}|$, and they are in the same direction, i.e. $\mathbf{u}$ is parallel to $\mathbf{v}$ and in both vectors are in the same direction/sense.
- A vector, e.g. force $\mathbf{F}$, that is a function of position (x, y, z) is referred to as a *vector field*; in the case of our example we write $\mathbf{F} \equiv \mathbf{F}(x, y, z)$.

2.1.1 Examples

(i) Every point P in 3D (or 2D) space has a *position vector*, $\mathbf{r}$, from some chosen origin O, with $\mathbf{r} = \overrightarrow{OP}$ and $r = OP = |\mathbf{r}|$.

Remarks.

- Often the position vector is represented by $\mathbf{x}$ rather than $\mathbf{r}$, but even then the length (i.e. magnitude) is usually represented by r.
- The position vector is an example of a vector field.

(ii) Every complex number corresponds to a unique point in the complex plane, and hence to the position vector of that point. 4/03 2/09

[11] Sophisticated mathematicians use neither **bold** nor *squiggles*; absent-minded lecturers like myself sometimes forget the *squiggles* on the overhead, but sophisticated students allow for that.

2.2 Properties of Vectors

2.2.1 Addition

Vectors add according to the geometric *parallelogram rule*:

$$\mathbf{a}+\mathbf{b} = \mathbf{c}\,, \tag{2.1a}$$

or equivalently

$$\vec{OA}+\vec{OB} = \vec{OC}\,, \tag{2.1b}$$

where $OACB$ is a parallelogram.

Remarks.

(i) Since a vector is defined by its magnitude and direction it follows that $\vec{OB}=\vec{AC}$ and $\vec{OA}=\vec{BC}$. Hence from parallelogram rule it is also true that

$$\vec{OC}=\vec{OA}+\vec{AC}=\vec{OB}+\vec{BC}\ . \tag{2.2}$$

We deduce that vector addition is *commutative*, i.e.

VA(i) $$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}\,. \tag{2.3}$$

2/10

(ii) Similarly we can deduce geometrically that vector addition is *associative*, i.e.

VA(ii) $$\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}\,. \tag{2.4}$$

(iii) If $|\mathbf{a}|=0$, write $\mathbf{a}=\mathbf{0}$, where $\mathbf{0}$ is the null vector or zero vector.[12] For all vectors $\mathbf{b}$

VA(iii) $$\mathbf{b}+\mathbf{0}=\mathbf{b}\,,\quad \text{and from (2.3)}\quad \mathbf{0}+\mathbf{b}=\mathbf{b}\,. \tag{2.5}$$

(iv) Define the vector $-\mathbf{a}$ to be parallel to $\mathbf{a}$, to have the same magnitude as $\mathbf{a}$, but to have the opposite direction/sense (so that it is *anti-parallel*). This is called the *negative* or *inverse* of $\mathbf{a}$ and is such that

VA(iv) $$(-\mathbf{a})+\mathbf{a}=\mathbf{0}\,. \tag{2.6a}$$

Define subtraction of vectors by

$$\mathbf{b}-\mathbf{a}\equiv\mathbf{b}+(-\mathbf{a})\,. \tag{2.6b}$$

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2.2.2 Multiplication by a scalar

If $\lambda\in\mathbb{R}$ then $\lambda\mathbf{a}$ has magnitude $|\lambda||\mathbf{a}|$, is parallel to $\mathbf{a}$, and it has the same direction/sense as $\mathbf{a}$ if $\lambda>0$, has the opposite direction/sense as $\mathbf{a}$ if $\lambda<0$, and is the zero vector, $\mathbf{0}$, if $\lambda=0$ (see also (2.9b) below).

A number of geometric properties follow from the above definition. In what follows $\lambda,\mu\in\mathbb{R}$.

Distributive law:

SM(i) $$(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}\,, \tag{2.7a}$$

SM(ii) $$\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}\,. \tag{2.7b}$$

[12] I have attempted to always write $\mathbf{0}$ in bold in the notes; if I have got it wrong somewhere then please email me. However, on the overhead/blackboard you will need to be more 'sophisticated'. I will try and get it right, but I will not always. Depending on context 0 will sometimes mean 0 and sometimes $\mathbf{0}$.

Associative law:

SM(iii) $$\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}\,. \tag{2.8}$$

Multiplication by 1, 0 and -1:

SM(iv)
$$\begin{aligned} 1\,\mathbf{a} &= \mathbf{a}\,, && && (2.9a)\\ 0\,\mathbf{a} &= \mathbf{0} && \text{since } 0\,|\mathbf{a}| = 0, && (2.9b)\\ (-1)\,\mathbf{a} &= -\mathbf{a} && \text{since } |-1|\,|\mathbf{a}| = |\mathbf{a}| \text{ and } -1 < 0. && (2.9c) \end{aligned}$$

Definition. The vector $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ is described as a *linear combination* of $\mathbf{a}$ and $\mathbf{b}$.

Unit vectors. Suppose $\mathbf{a} \neq \mathbf{0}$, then define
$$\widehat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}\,. \tag{2.10}$$

$\widehat{\mathbf{a}}$ is termed a *unit vector* since
$$|\widehat{\mathbf{a}}| = \left|\frac{1}{|\mathbf{a}|}\right| |\mathbf{a}| = \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1\,.$$

A $\widehat{\ }$ is often used to indicate a unit vector, but note that this is a convention that is often broken (e.g. see §2.8.1).

2.2.3 Example: the midpoints of the sides of any quadrilateral form a parallelogram

This an example of the fact that the rules of vector manipulation and their geometric interpretation can be used to prove geometric theorems.

Denote the vertices of the quadrilateral by A, B, C and D. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ represent the sides $\overrightarrow{DA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$, and let P, Q, R and S denote the respective midpoints. Then since the quadrilateral is closed
$$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0\,. \tag{2.11}$$

Further
$$\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ} = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}\,.$$

Similarly, and by using (2.11),
$$\begin{aligned} \overrightarrow{RS} &= \tfrac{1}{2}(\mathbf{c} + \mathbf{d})\\ &= -\tfrac{1}{2}(\mathbf{a} + \mathbf{b})\\ &= -\overrightarrow{PQ}\;. \end{aligned}$$

Thus $\overrightarrow{PQ} = \overrightarrow{SR}$, i.e. PQ and SR have equal magnitude and are parallel; similarly $\overrightarrow{QR} = \overrightarrow{PS}$. Hence $PQSR$
04/06 is a parallelogram.

2.3 Vector Spaces

2.3.1 Algebraic definition

So far we have taken a geometric approach; now we take an algebraic approach. A *vector space over the real numbers* is a set V of elements, or 'vectors', together with two *binary operations*

- *vector addition* denoted for $\mathbf{a}, \mathbf{b} \in V$ by $\mathbf{a} + \mathbf{b}$, where $\mathbf{a} + \mathbf{b} \in V$ so that there is closure under vector addition;

- *scalar multiplication* denoted for $\lambda \in \mathbb{R}$ and $\mathbf{a} \in V$ by $\lambda\mathbf{a}$, where $\lambda\mathbf{a} \in V$ so that there is closure under scalar multiplication;

satisfying the following eight axioms or rules:[13]

VA(i) addition is *commutative*, i.e. for all $\mathbf{a}, \mathbf{b} \in V$

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}\,; \tag{2.12a}$$

VA(ii) addition is *associative*, i.e. for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}\,; \tag{2.12b}$$

VA(iii) there exists an element $\mathbf{0} \in V$, called the *null or zero vector*, such that for all $\mathbf{a} \in V$

$$\mathbf{a} + \mathbf{0} = \mathbf{a}\,, \tag{2.12c}$$

i.e. vector addition has an identity element;

VA(iv) for all $\mathbf{a} \in V$ there exists an additive *negative or inverse vector* $\mathbf{a}' \in V$ such that

$$\mathbf{a} + \mathbf{a}' = \mathbf{0}\,; \tag{2.12d}$$

SM(i) scalar multiplication is distributive over scalar addition, i.e. for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{a} \in V$

$$(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}\,; \tag{2.12e}$$

SM(ii) scalar multiplication is distributive over vector addition, i.e. for all $\lambda \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in V$

$$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}\,; \tag{2.12f}$$

SM(iii) scalar multiplication of vectors is 'associative',[14] i.e. for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{a} \in V$

$$\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}\,, \tag{2.12g}$$

SM(iv) scalar multiplication has an identity element, i.e. for all $\mathbf{a} \in V$

$$1\,\mathbf{a} = \mathbf{a}\,, \tag{2.12h}$$

where 1 is the multiplicative identity in $\mathbb{R}$.

2.3.2 Examples

(i) For the set of vectors in 3D space, vector addition and scalar multiplication of vectors (as defined in §2.2.1 and §2.2.2 respectively) satisfy the eight axioms or rules VA(i)-(iv) and SM(i)-(iv): see (2.3), (2.4), (2.5), (2.6a), (2.7a), (2.7b), (2.8) and (2.9a). Hence the set of vectors in 3D space form a *vector space over the real numbers*.

(ii) Let $\mathbb{R}^n$ be the set of all $n-$tuples $\{\mathbf{x} = (x_1, x_2, \dots, x_n) : x_j \in \mathbb{R} \text{ with } j = 1, 2, \dots, n\}$, where n is any strictly positive integer. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with $\mathbf{x}$ as above and $\mathbf{y} = (y_1, y_2, \dots, y_n)$, define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)\,, && (2.13a)\\ \lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \dots, \lambda x_n)\,, && (2.13b)\\ \mathbf{0} &= (0, 0, \dots, 0)\,, && (2.13c)\\ \mathbf{x}' &= (-x_1, -x_2, \dots, -x_n)\,. && (2.13d) \end{aligned}$$

It is straightforward to check that VA(i)-(iv) and SM(i)-(iv) are satisfied. Hence $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

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[13] The first four mean that V is an *abelian group* under addition.

[14] Strictly we are not asserting the associativity of an operation, since there are two different operations in question, namely scalar multiplication of vectors (e.g. $\mu\mathbf{a}$) and multiplication of real numbers (e.g. $\lambda\mu$).

2.3.3 Properties of Vector Spaces (Unlectured)

(i) *The zero vector* $\mathbf{0}$ *is unique.* For suppose that $\mathbf{0}$ and $\mathbf{0}'$ are both zero vectors in V then from (2.12a) and (2.12c) $\mathbf{a} = \mathbf{0} + \mathbf{a}$ and $\mathbf{a} + \mathbf{0}' = \mathbf{a}$ for all $\mathbf{a} \in V$, and hence

$$\mathbf{0}' = \mathbf{0} + \mathbf{0}' = \mathbf{0}\,.$$

(ii) *The additive inverse of a vector* $\mathbf{a}$ *is unique.* For suppose that both $\mathbf{b}$ and $\mathbf{c}$ are additive inverses of $\mathbf{a}$ then

$$\begin{aligned}\mathbf{b} &= \mathbf{b} + \mathbf{0}\\ &= \mathbf{b} + (\mathbf{a} + \mathbf{c})\\ &= (\mathbf{b} + \mathbf{a}) + \mathbf{c}\\ &= \mathbf{0} + \mathbf{c}\\ &= \mathbf{c}\,.\end{aligned}$$

Definition. We denote the unique inverse of $\mathbf{a}$ by $-\mathbf{a}$.

(iii) *Definition.* The existence of a unique negative/inverse vector allows us to *subtract* as well as add vectors, by defining

$$\mathbf{b} - \mathbf{a} \equiv \mathbf{b} + (-\mathbf{a})\,. \tag{2.14}$$

(iv) *Scalar multiplication by* 0 *yields the zero vector.* For all $\mathbf{a} \in V$

$$0\mathbf{a} = \mathbf{0}\,, \tag{2.15}$$

since

$$\begin{aligned}0\mathbf{a} &= 0\mathbf{a} + \mathbf{0}\\ &= 0\mathbf{a} + (\mathbf{a} + (-\mathbf{a}))\\ &= (0\mathbf{a} + \mathbf{a}) + (-\mathbf{a})\\ &= (0\mathbf{a} + 1\mathbf{a}) + (-\mathbf{a})\\ &= (0 + 1)\mathbf{a} + (-\mathbf{a})\\ &= \mathbf{a} + (-\mathbf{a})\\ &= \mathbf{0}\,.\end{aligned}$$

Hence (2.9b) is a *consequential* property.

(v) *Scalar multiplication by* -1 *yields the additive inverse of the vector.* For all $\mathbf{a} \in V$,

$$(-1)\mathbf{a} = -\mathbf{a}\,, \tag{2.16}$$

since

$$\begin{aligned}(-1)\mathbf{a} &= (-1)\mathbf{a} + \mathbf{0}\\ &= (-1)\mathbf{a} + (\mathbf{a} + (-\mathbf{a}))\\ &= ((-1)\mathbf{a} + \mathbf{a}) + (-\mathbf{a})\\ &= (-1 + 1)\mathbf{a} + (-\mathbf{a})\\ &= 0\mathbf{a} + (-\mathbf{a})\\ &= \mathbf{0} + (-\mathbf{a})\\ &= -\mathbf{a}\,.\end{aligned}$$

Hence (2.9c) is a *consequential* property.

(vi) *Scalar multiplication with the zero vector yields the zero vector.* For all $\lambda \in \mathbb{R}$, $\lambda\mathbf{0} = \mathbf{0}$. To see this we first observe that $\lambda\mathbf{0}$ is a zero vector since

$$\begin{aligned} \lambda\mathbf{0} + \lambda\mathbf{a} &= \lambda(\mathbf{0} + \mathbf{a}) \\ &= \lambda\mathbf{a}\,. \end{aligned}$$

Next we appeal to the fact that the zero vector is unique to conclude that

$$\lambda\mathbf{0} = \mathbf{0}\,. \tag{2.17}$$

(vii) *If* $\lambda\mathbf{a} = \mathbf{0}$ *then either* $\mathbf{a} = \mathbf{0}$ *and/or* $\lambda = 0$. First suppose that $\lambda \neq 0$, in which case there exists λ^{-1} such that $\lambda^{-1}\lambda = 1$. Then we conclude that

$$\begin{aligned} \mathbf{a} &= 1\,\mathbf{a} \\ &= (\lambda^{-1}\lambda)\mathbf{a} \\ &= \lambda^{-1}(\lambda\mathbf{a}) \\ &= \lambda^{-1}\mathbf{0} \\ &= \mathbf{0}\,. \end{aligned}$$

If $\lambda\mathbf{a} = \mathbf{0}$, where $\lambda \in \mathbb{R}$ and $\mathbf{a} \in V$, then one possibility from (2.15) is that $\lambda = 0$. So if $\lambda\mathbf{a} = \mathbf{0}$ then either $\lambda = 0$ or $\mathbf{a} = \mathbf{0}$.

(viii) *Negation commutes freely.* This is because for all $\lambda \in \mathbb{R}$ and $\mathbf{a} \in V$

$$\begin{aligned} (-\lambda)\mathbf{a} &= (\lambda(-1))\mathbf{a} \\ &= \lambda((-1)\mathbf{a}) \\ &= \lambda(-\mathbf{a})\,, \end{aligned} \tag{2.18a}$$

and

$$\begin{aligned} (-\lambda)\mathbf{a} &= ((-1)\lambda)\mathbf{a} \\ &= (-1)(\lambda\mathbf{a}) \\ &= -(\lambda\mathbf{a})\,. \end{aligned} \tag{2.18b}$$

2.4 Scalar Product

Definition. The *scalar product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined geometrically to be the real (scalar) number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta\,, \tag{2.19}$$

where $0 \leqslant \theta \leqslant \pi$ is the non-reflex angle between $\mathbf{a}$ and $\mathbf{b}$ once they have been placed 'tail to tail' or 'head to head'.

Remark. The *scalar product* is also referred to as the *dot product.*

2.4.1 Properties of the scalar product

(i) The scalar product is *commutative*:

SP(i)
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}\,. \tag{2.20}$$

(ii) The scalar product of a vector with itself is the square of its modulus, and is thus positive:

SP(iii)
$$\mathbf{a}^2 \equiv \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \geqslant 0\,. \tag{2.21a}$$

Further, the scalar product of a vector with itself is zero [if and] only if the vector is the zero vector, i.e.

SP(iv)
$$\mathbf{a} \cdot \mathbf{a} = 0 \quad \Rightarrow \quad \mathbf{a} = \mathbf{0}\,. \tag{2.21b}$$

(iii) If $0 \leqslant \theta < \frac{1}{2}\pi$, then $\mathbf{a}\cdot\mathbf{b} > 0$, while if $\frac{1}{2}\pi < \theta \leqslant \pi$, then $\mathbf{a}\cdot\mathbf{b} < 0$.

(iv) If $\mathbf{a}\cdot\mathbf{b} = 0$ and $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{a}$ and $\mathbf{b}$ must be orthogonal, i.e. $\theta = \frac{1}{2}\pi$.

(v) Suppose $\lambda \in \mathbb{R}$. If $\lambda > 0$ then

$$\begin{aligned} \mathbf{a}\cdot(\lambda\mathbf{b}) &= |\mathbf{a}||\lambda\mathbf{b}|\cos\theta \\ &= |\lambda||\mathbf{a}||\mathbf{b}|\cos\theta \\ &= |\lambda|\,\mathbf{a}\cdot\mathbf{b} \\ &= \lambda\,\mathbf{a}\cdot\mathbf{b}\,. \end{aligned}$$

If instead $\lambda < 0$ then

$$\begin{aligned} \mathbf{a}\cdot(\lambda\mathbf{b}) &= |\mathbf{a}||\lambda\mathbf{b}|\cos(\pi-\theta) \\ &= -|\lambda||\mathbf{a}||\mathbf{b}|\cos\theta \\ &= -|\lambda|\,\mathbf{a}\cdot\mathbf{b} \\ &= \lambda\,\mathbf{a}\cdot\mathbf{b}\,. \end{aligned}$$

Similarly, or by using (2.20), $(\lambda\mathbf{a})\cdot\mathbf{b} = \lambda\,\mathbf{a}\cdot\mathbf{b}$. In summary

$$\mathbf{a}\cdot(\lambda\mathbf{b}) = (\lambda\mathbf{a})\cdot\mathbf{b} = \lambda\,\mathbf{a}\cdot\mathbf{b}\,. \tag{2.22}$$

2.4.2 Projections

The *projection* of $\mathbf{a}$ onto $\mathbf{b}$ is that part of $\mathbf{a}$ that is parallel to $\mathbf{b}$ (which here we will denote by $\mathbf{a}^{\perp}$).

From geometry, $|\mathbf{a}^{\perp}| = |\mathbf{a}|\,\cos\theta$ (assume for the time being that $\cos\theta \geqslant 0$). Thus since $\mathbf{a}^{\perp}$ is parallel to $\mathbf{b}$, and hence $\widehat{\mathbf{b}}$ the unit vector in the direction of $\mathbf{b}$:

$$\mathbf{a}^{\perp} = |\mathbf{a}^{\perp}|\,\widehat{\mathbf{b}} = |\mathbf{a}|\,\cos\theta\,\widehat{\mathbf{b}}\,. \tag{2.23a}$$

Exercise. Show that (2.23a) remains true if $\cos\theta < 0$.

Hence from (2.19)

$$\mathbf{a}^{\perp} = |\mathbf{a}|\,\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\,\widehat{\mathbf{b}} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\,\mathbf{b} = (\mathbf{a}\cdot\widehat{\mathbf{b}})\,\widehat{\mathbf{b}}\,. \tag{2.23b}$$

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2.4.3 The scalar product is distributive over vector addition

We wish to show that

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}\,. \tag{2.24a}$$

The result is [clearly] true if $\mathbf{a} = \mathbf{0}$, so henceforth assume $\mathbf{a} \neq \mathbf{0}$. Then from (2.23b) (after exchanging $\mathbf{a}$ for $\mathbf{b}$, and $\mathbf{b}$ or $\mathbf{c}$ or $(\mathbf{b}+\mathbf{c})$ for $\mathbf{a}$, etc.)

$$\begin{aligned} \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a} + \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|^2}\,\mathbf{a} &= \{\text{projection of } \mathbf{b} \text{ onto } \mathbf{a}\} + \{\text{projection of } \mathbf{c} \text{ onto } \mathbf{a}\} \\ &= \{\text{projection of } (\mathbf{b}+\mathbf{c}) \text{ onto } \mathbf{a}\} \qquad \text{(by geometry)} \\ &= \frac{\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})}{|\mathbf{a}|^2}\,\mathbf{a}\,, \end{aligned}$$

which is of the form, for some $\alpha, \beta, \gamma \in \mathbb{R}$,

$$\alpha\mathbf{a} + \beta\mathbf{a} = \gamma\mathbf{a}\,.$$

From SM(i), i.e. (2.7a), first note $\alpha\mathbf{a}+\beta\mathbf{a}=(\alpha+\beta)\mathbf{a}$; then 'dot' both sides with $\mathbf{a}$ to deduce that

$$(\alpha+\beta)\,\mathbf{a}\cdot\mathbf{a}=\gamma\,\mathbf{a}\cdot\mathbf{a}\,,\quad\text{and hence (since } \mathbf{a}\neq\mathbf{0}\text{) that}\quad \alpha+\beta=\gamma\,.$$

The result (2.24a) follows.

Remark. Similarly, from using (2.22) it follows for $\lambda,\mu\in\mathbb{R}$ that

SP(ii)
$$\mathbf{a}\cdot(\lambda\mathbf{b}+\mu\mathbf{c})=\lambda\,\mathbf{a}\cdot\mathbf{b}+\mu\,\mathbf{a}\cdot\mathbf{c}\,. \tag{2.24b}$$

2.4.4 Example: the cosine rule

$$\begin{aligned}
BC^2 &\equiv |\,\vec{BC}\,|^2 \;=\; |\,\vec{BA}+\vec{AC}\,|^2\\
&= \left(\vec{BA}+\vec{AC}\right)\cdot\left(\vec{BA}+\vec{AC}\right)\\
&= \vec{BA}\cdot\vec{BA}+\vec{BA}\cdot\vec{AC}+\vec{AC}\cdot\vec{BA}+\vec{AC}\cdot\vec{AC}\\
&= BA^2+2\,\vec{BA}\cdot\vec{AC}+AC^2\\
&= BA^2+2\,BA\;AC\;\cos\theta+AC^2\\
&= BA^2-2\,BA\;AC\;\cos\alpha+AC^2\,.
\end{aligned}$$

2.4.5 Algebraic definition of a scalar product

For an n-dimensional vector space V over the real numbers assign for every pair of vectors $\mathbf{a},\,\mathbf{b}\in V$ a *scalar product*, or *inner product*, $\mathbf{a}\cdot\mathbf{b}\in\mathbb{R}$ with the following properties.

SP(i) *Symmetry*, i.e.

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}\,. \tag{2.25a}$$

SP(ii) *Linearity* in the 'second' argument, i.e. for $\mathbf{a},\mathbf{b},\mathbf{c}\in V$ and $\lambda,\mu\in\mathbb{R}$

$$\mathbf{a}\cdot(\lambda\mathbf{b}+\mu\mathbf{c})=\lambda\,\mathbf{a}\cdot\mathbf{b}+\mu\,\mathbf{a}\cdot\mathbf{c}\,. \tag{2.25b}$$

Remark. From properties (2.25a) and (2.25b) we deduce that there is linearity in the 'first' argument, i.e. for $\mathbf{a},\mathbf{b},\mathbf{c}\in V$ and $\lambda,\mu\in\mathbb{R}$

$$\begin{aligned}
(\lambda\mathbf{a}+\mu\mathbf{c})\cdot\mathbf{b} &= \mathbf{b}\cdot(\lambda\mathbf{a}+\mu\mathbf{c})\\
&= \lambda\mathbf{b}\cdot\mathbf{a}+\mu\mathbf{b}\cdot\mathbf{c}\\
&= \lambda\mathbf{a}\cdot\mathbf{b}+\mu\mathbf{c}\cdot\mathbf{b}\,.
\end{aligned} \tag{2.25c}$$

Corollary. For $\lambda\in\mathbb{R}$ it follows from (2.17) and (2.25b) that

$$\mathbf{a}\cdot\mathbf{0}=\mathbf{a}\cdot(\lambda\mathbf{0})=\lambda\,\mathbf{a}\cdot\mathbf{0}\,,\quad\text{and hence that}\quad(\lambda-1)\,\mathbf{a}\cdot\mathbf{0}=0\,.$$

Choose λ with $\lambda\neq1$ to deduce that

$$\mathbf{a}\cdot\mathbf{0}=0\,. \tag{2.25d}$$

SP(iii) *Non-negativity*, i.e. a scalar product of a vector with itself should be positive, i.e.

$$\mathbf{a}\cdot\mathbf{a}\geqslant 0\,. \tag{2.25e}$$

This allows us to define

$$|\mathbf{a}|^2=\mathbf{a}\cdot\mathbf{a}\,, \tag{2.25f}$$

where the real positive number $|\mathbf{a}|$ is a *norm* (cf. length) of the vector $\mathbf{a}$. It follows from (2.25d) with $\mathbf{a}=\mathbf{0}$ that

$$|\mathbf{0}|^2=\mathbf{0}\cdot\mathbf{0}=0\,. \tag{2.25g}$$

SP(iv) *Non-degeneracy*, i.e. the only vector of zero norm should be the zero vector, i.e.

$$|\mathbf{a}|=0\quad\Rightarrow\quad\mathbf{a}=\mathbf{0}\,. \tag{2.25h}$$

Alternative notation. An alternative notation for scalar products and norms is

$$\langle\, \mathbf{a} \mid \mathbf{b}\, \rangle \equiv \mathbf{a}\cdot\mathbf{b}\,, \tag{2.26a}$$

$$\|\mathbf{a}\| \equiv |\mathbf{a}| = (\mathbf{a}\cdot\mathbf{a})^{\frac{1}{2}}\,. \tag{2.26b}$$

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2.4.6 The Schwarz inequality (a.k.a. the Cauchy-Schwarz inequality)

Schwarz's inequality states that

$$|\mathbf{a}\cdot\mathbf{b}| \leqslant |\mathbf{a}|\,|\mathbf{b}|\,, \tag{2.27}$$

with equality only when $\mathbf{a}=\mathbf{0}$ and/or $\mathbf{b}=\mathbf{0}$, or when $\mathbf{a}$ is a scalar multiple of $\mathbf{b}$.

Proof. We present two proofs. First a geometric argument using (2.19):

$$|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}||\mathbf{b}||\cos\theta| \leqslant |\mathbf{a}||\mathbf{b}|$$

with equality when $\mathbf{a}=\mathbf{0}$ and/or $\mathbf{b}=\mathbf{0}$, or if $|\cos\theta|=1$, i.e. if $\mathbf{a}$ and $\mathbf{b}$ are parallel.

Second an algebraic argument that can be generalised. To start this proof consider, for $\lambda\in\mathbb{R}$,

$$\begin{aligned} 0 \leqslant |\mathbf{a}+\lambda\mathbf{b}|^2 &= (\mathbf{a}+\lambda\mathbf{b})\cdot(\mathbf{a}+\lambda\mathbf{b}) && \text{from (2.25e) and (2.25e) (or (2.21a))}\\ &= \mathbf{a}\cdot\mathbf{a}+\lambda\mathbf{a}\cdot\mathbf{b}+\lambda\mathbf{b}\cdot\mathbf{a}+\lambda^2\mathbf{b}\cdot\mathbf{b} && \text{from (2.25b) and (2.25c) (or (2.20) and (2.24a))}\\ &= |\mathbf{a}|^2+(2\mathbf{a}\cdot\mathbf{b})\,\lambda+|\mathbf{b}|^2\,\lambda^2 && \text{from (2.25a) and (2.25f) (or (2.20)).} \end{aligned}$$

We have two cases to consider: $\mathbf{b}=\mathbf{0}$ and $\mathbf{b}\neq\mathbf{0}$. First, suppose that $\mathbf{b}=\mathbf{0}$, so that $|\mathbf{b}|=0$. Then from (2.25d) it follows that (2.27) is satisfied as an equality.

If $|\mathbf{b}|\neq 0$ the right-hand-side is a quadratic in λ that, since it is not negative, has at most one real root. Hence '$b^2\leqslant 4ac$', i.e.

$$(2\mathbf{a}\cdot\mathbf{b})^2 \leqslant 4|\mathbf{a}|^2|\mathbf{b}|^2\,.$$

Schwarz's inequality follows on taking the positive square root, with equality only if $\mathbf{a}=-\lambda\mathbf{b}$ for some λ.

□

2.4.7 Triangle inequality

This triangle inequality is a generalisation of (1.6a) and states that

$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}|+|\mathbf{b}|\,. \tag{2.28}$$

Proof. Again there are a number of proofs. Geometrically (2.28) must be true since the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. More formally we could proceed in an analogous manner to (1.6a) using the cosine rule.

An algebraic proof that generalises is to take the positive square root of the following inequality:

$$\begin{aligned} |\mathbf{a}+\mathbf{b}|^2 &= |\mathbf{a}|^2+2\,\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2 && \text{from above with } \lambda=1\\ &\leqslant |\mathbf{a}|^2+2|\mathbf{a}\cdot\mathbf{b}|+|\mathbf{b}|^2 &&\\ &\leqslant |\mathbf{a}|^2+2|\mathbf{a}|\,|\mathbf{b}|+|\mathbf{b}|^2 && \text{from (2.27)}\\ &\leqslant (|\mathbf{a}|+|\mathbf{b}|)^2\,. && \end{aligned}$$

□

Remark. In the same way that (1.6a) can be extended to (1.6b) we can similarly deduce that

$$|\mathbf{a}-\mathbf{b}| \geqslant \big|\,|\mathbf{a}|-|\mathbf{b}|\,\big|\,. \tag{2.29}$$

2.5 Vector Product

Definition. From a geometric standpoint, the *vector product* $\mathbf{a}\times\mathbf{b}$ of an ordered pair $\mathbf{a}$, $\mathbf{b}$ is a vector such that

(i)
$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta\,, \tag{2.30}$$
with $0\leqslant\theta\leqslant\pi$ defined as before;

(ii) $\mathbf{a}\times\mathbf{b}$ is perpendicular/orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ (if $\mathbf{a}\times\mathbf{b}\neq\mathbf{0}$);

(iii) $\mathbf{a}\times\mathbf{b}$ has the sense/direction defined by the 'right-hand rule', i.e. take a right hand, point the index finger in the direction of $\mathbf{a}$, the second finger in the direction of $\mathbf{b}$, and then $\mathbf{a}\times\mathbf{b}$ is in the direction of the thumb.

Remarks.

(i) The vector product is also referred to as the *cross product*.

(ii) An alternative notation (that is falling out of favour except on my overhead/blackboard) is $\mathbf{a}\wedge\mathbf{b}$.

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2.5.1 Properties of the vector product

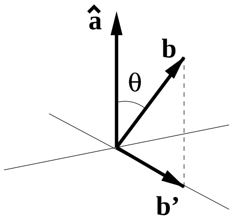
(i) The vector product is *anti-commutative* (from the right-hand rule):
$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}\,. \tag{2.31a}$$

(ii) The vector product of a vector with itself is zero:
$$\mathbf{a}\times\mathbf{a} = \mathbf{0}\,. \tag{2.31b}$$

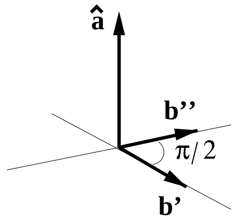
(iii) If $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ and $\mathbf{a}\neq\mathbf{0}$ and $\mathbf{b}\neq\mathbf{0}$, then $\theta=0$ or $\theta=\pi$, i.e. $\mathbf{a}$ and $\mathbf{b}$ are parallel (or equivalently there exists $\lambda\in\mathbb{R}$ such that $\mathbf{a}=\lambda\mathbf{b}$).

(iv) It follows from the definition of the vector product
$$\mathbf{a}\times(\lambda\mathbf{b}) = \lambda\,(\mathbf{a}\times\mathbf{b})\ . \tag{2.31c}$$

(v) For given $\widehat{\mathbf{a}}$ and $\mathbf{b}$, suppose that the vector $\mathbf{b}''(\widehat{\mathbf{a}},\mathbf{b})$ is constructed by two operations. First project $\mathbf{b}$ onto a plane orthogonal to $\widehat{\mathbf{a}}$ to generate the vector $\mathbf{b}'$. Next obtain $\mathbf{b}''$ by rotating $\mathbf{b}'$ about $\widehat{\mathbf{a}}$ by $\frac{\pi}{2}$ in an 'anti-clockwise' direction ('anti-clockwise' when looking in the opposite direction to $\widehat{\mathbf{a}}$).



b' is the projection of **b** onto the plane perpendicular to **â**



b'' is the result of rotating the vector **b'** through an angle π/2 anti−clockwise about **â** (looking in the opposite direction to **â**)

By geometry $|\mathbf{b}'| = |\mathbf{b}|\,\sin\theta = |\widehat{\mathbf{a}}\times\mathbf{b}|$, and $|\mathbf{b}''|=|\mathbf{b}'|$. Further, by construction $\widehat{\mathbf{a}}$, $\mathbf{b}'$ and $\mathbf{b}''$ form an orthogonal right-handed triple. It follows that
$$\mathbf{b}''(\widehat{\mathbf{a}},\mathbf{b}) \equiv \widehat{\mathbf{a}}\times\mathbf{b}\,. \tag{2.31d}$$

(vi) We can use this geometric construction to show that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}\,. \tag{2.31e}$$

To this end first note from geometry that

$$\begin{aligned} \{\text{projection of } \mathbf{b} \text{ onto plane} \perp \text{to } \mathbf{a}\} \quad &+ \quad \{\text{projection of } \mathbf{c} \text{ onto plane} \perp \text{to } \mathbf{a}\} \\ &= \quad \{\text{projection of } (\mathbf{b}+\mathbf{c}) \text{ onto plane} \perp \text{to } \mathbf{a}\}\,, \end{aligned}$$

i.e.

$$\mathbf{b}' + \mathbf{c}' \quad = \quad (\mathbf{b}+\mathbf{c})'\,.$$

Next note that by rotating $\mathbf{b}'$, $\mathbf{c}'$ and $(\mathbf{b}+\mathbf{c})'$ by $\frac{\pi}{2}$ 'anti-clockwise' about $\mathbf{a}$ it follows that

$$\mathbf{b}'' + \mathbf{c}'' = (\mathbf{b}+\mathbf{c})''\,.$$

Hence from property (2.31d)

$$\widehat{\mathbf{a}} \times \mathbf{b} + \widehat{\mathbf{a}} \times \mathbf{c} = \widehat{\mathbf{a}} \times (\mathbf{b} + \mathbf{c})\,.$$

The result (2.31e) follows from multiplying by $|\mathbf{a}|$.

2.5.2 Vector area of a triangle/parallelogram

Let O, A and B denote the vertices of a triangle, and let NB be the *altitude* through B. Denote $\vec{OA}$ and $\vec{OB}$ by $\mathbf{a}$ and $\mathbf{b}$ respectively. Then

$$\text{area of triangle} = \tfrac{1}{2}\,OA\,.\,NB = \tfrac{1}{2}\,OA\,.\,OB\,\sin\theta = \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}|\,.$$

The quantity $\frac{1}{2}\mathbf{a}\times\mathbf{b}$ is referred to as the *vector area* of the triangle. It has the same magnitude as the area of the triangle, and is normal to OAB, i.e. normal to the plane containing $\mathbf{a}$ and $\mathbf{b}$.

Let O, A, C and B denote the vertices of a parallelogram, with $\vec{OA}$ and $\vec{OB}$ as before. Then

$$\text{area of parallelogram} = |\mathbf{a}\times\mathbf{b}|\,,$$

and the *vector area* of the parallelogram is $\mathbf{a}\times\mathbf{b}$.

2.6 Triple Products

Given the scalar ('dot') product and the vector ('cross') product, we can form two triple products.

Scalar triple product:

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = -\,(\mathbf{b}\times\mathbf{a})\cdot\mathbf{c}\,, \tag{2.32}$$

from using (2.20) and (2.31a).

Vector triple product:

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = -\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = -\,(\mathbf{b}\times\mathbf{a})\times\mathbf{c} = \mathbf{c}\times(\mathbf{b}\times\mathbf{a})\,, \tag{2.33}$$

5/06 from using (2.31a).
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2.6.1 Properties of the scalar triple product

Volume of parallelepipeds. The volume of a parallelepiped (or parallelipiped or parallelopiped or parallelopipede or parallelopipedon) with edges $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is

$$\begin{aligned} \text{Volume} &= \text{Base Area} \times \text{Height} \\ &= |\mathbf{a}\times\mathbf{b}|\,|\mathbf{c}|\,\cos\phi \\ &= |\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| && (2.34a) \\ &= (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \geqslant 0\,, && (2.34b) \end{aligned}$$

if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ have the sense of the right-hand rule.

Identities. Assume that the ordered triple $(\mathbf{a},\mathbf{b},\mathbf{c})$ has the sense of the right-hand rule. Then so do the ordered triples $(\mathbf{b},\mathbf{c},\mathbf{a})$, and $(\mathbf{c},\mathbf{a},\mathbf{b})$. Since the ordered scalar triple products will all equal the volume of the same parallelepiped it follows that

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}\,. \tag{2.35a}$$

Further the ordered triples $(\mathbf{a},\mathbf{c},\mathbf{b})$, $(\mathbf{b},\mathbf{a},\mathbf{c})$ and $(\mathbf{c},\mathbf{b},\mathbf{a})$ all have the sense of the left-hand rule, and so their scalar triple products must all equal the 'negative volume' of the parallelepiped; thus

$$(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b} = (\mathbf{b}\times\mathbf{a})\cdot\mathbf{c} = (\mathbf{c}\times\mathbf{b})\cdot\mathbf{a} = -\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\,. \tag{2.35b}$$

It also follows from the first two expression in (2.35a), and from the commutative property (2.20), that

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\,, \tag{2.35c}$$

and hence the order of the 'cross' and 'dot' is inconsequential.[15] For this reason we sometimes use the notation

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\,. \tag{2.35d}$$

Coplanar vectors. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar then

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = 0\,,$$

since the volume of the parallelepiped is zero. Conversely if non-zero $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $[\mathbf{a},\mathbf{b},\mathbf{c}]=0$, then $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.

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2.7 Spanning Sets, Linear Independence, Bases and Components

There are a large number of vectors in 2D or 3D space. Is there a way of expressing these vectors as a combination of [a small number of] other vectors?

2.7.1 2D Space

First consider 2D space, an origin O, and two non-zero and *non-parallel* vectors $\mathbf{a}$ and $\mathbf{b}$. Then the position vector $\mathbf{r}$ of any point P in the plane can be expressed as

$$\mathbf{r} = \vec{OP} = \lambda\mathbf{a} + \mu\mathbf{b}\,, \tag{2.36}$$

for suitable and unique real scalars λ and μ.

[15] What is important is the order of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Geometric construction. Draw a line through P parallel to $\vec{OA}= \mathbf{a}$ to intersect $\vec{OB}= \mathbf{b}$ (or its extension) at N (all non-parallel lines intersect). Then there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\vec{ON}= \mu\mathbf{b} \quad \text{and} \quad \vec{NP}= \lambda\mathbf{a}\,,$$

and hence

$$\mathbf{r} = \vec{OP}= \lambda\mathbf{a} + \mu\mathbf{b}\,.$$

Definition. We say that the set $\{\mathbf{a}, \mathbf{b}\}$ *spans* the set of vectors lying in the plane.

Uniqueness. For given $\mathbf{a}$ and $\mathbf{b}$, λ and μ are unique. For suppose that λ and μ are not-unique, and that there exists $\lambda, \lambda', \mu, \mu' \in \mathbb{R}$ such that

$$\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b} = \lambda'\mathbf{a} + \mu'\mathbf{b}\,.$$

Rearranging the above it follows that

$$(\lambda - \lambda')\mathbf{a} = (\mu' - \mu)\mathbf{b}\,. \tag{2.37}$$

Hence, since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, $\lambda - \lambda' = \mu - \mu' = 0$. [16]

Definition. We refer to (λ, μ) as the *components* of $\mathbf{r}$ with respect to the ordered pair of vectors $\mathbf{a}$ and $\mathbf{b}$.

Definition. If for two vectors $\mathbf{a}$ and $\mathbf{b}$ and $\alpha, \beta \in \mathbb{R}$,

$$\alpha\,\mathbf{a} + \beta\,\mathbf{b} = \mathbf{0} \quad \Rightarrow \quad \alpha = \beta = 0\,, \tag{2.38}$$

then we say that $\mathbf{a}$ and $\mathbf{b}$ are *linearly independent*.

Definition. We say that the set $\{\mathbf{a}, \mathbf{b}\}$ is a *basis* for the set of vectors lying the in plane if it is a spanning set and $\mathbf{a}$ and $\mathbf{b}$ are linearly independent.

Remarks.

(i) Any two non-parallel vectors are linearly independent.

(ii) The set $\{\mathbf{a}, \mathbf{b}\}$ does not have to be orthogonal to be a basis.

(iii) In 2D space a basis always consists of two vectors.[17]

2.7.2 3D Space

Next consider 3D space, an origin O, and three non-zero and non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (i.e. $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$). Then the position vector $\mathbf{r}$ of any point P in space can be expressed as

$$\mathbf{r} = \vec{OP}= \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}\,, \tag{2.39}$$

for suitable and unique real scalars λ, μ and ν.

[16] Alternatively, we may conclude this by 'crossing' (2.37) first with $\mathbf{a}$, and then 'crossing' (2.37) with $\mathbf{b}$.
[17] As you will see below, and in *Linear Algebra*, this is a tautology.

Geometric construction. Let $\Pi_{\mathbf{ab}}$ be the plane containing $\mathbf{a}$ and $\mathbf{b}$. Draw a line through P parallel to $\overrightarrow{OC}= \mathbf{c}$. This line cannot be parallel to $\Pi_{\mathbf{ab}}$ because $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not coplanar. Hence it will intersect $\Pi_{\mathbf{ab}}$, say at N, and there will exist $\nu \in \mathbb{R}$ such that $\overrightarrow{NP}= \nu\mathbf{c}$. Further, since $\overrightarrow{ON}$ lies in the plane $\Pi_{\mathbf{ab}}$, from §2.7.1 there exists $\lambda, \mu \in \mathbb{R}$ such that $\overrightarrow{ON}= \lambda\mathbf{a} + \mu\mathbf{b}$. It follows that

$$\begin{aligned} \mathbf{r} = \overrightarrow{OP} &= \overrightarrow{ON} + \overrightarrow{NP} \\ &= \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}\,, \end{aligned}$$

for some $\lambda, \mu, \nu \in \mathbb{R}$.

We conclude that if $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$ then the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *spans* 3D space. 6/02

Uniqueness. We can show that λ, μ and ν are unique by construction. Suppose that $\mathbf{r}$ is given by (2.39) and consider

$$\begin{aligned} \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) &= (\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c}) \\ &= \lambda\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mu\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) + \nu\mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) \\ &= \lambda\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\,, \end{aligned}$$

since $\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{c}) = 0$. Hence, and similarly or by permutation,

$$\lambda = \frac{[\mathbf{r}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\,, \quad \mu = \frac{[\mathbf{r}, \mathbf{c}, \mathbf{a}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\,, \quad \nu = \frac{[\mathbf{r}, \mathbf{a}, \mathbf{b}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}\,. \tag{2.40}$$

Definition. We refer to (λ, μ, ν) as the *components* of $\mathbf{r}$ with respect to the ordered triple of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Definition. If for three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and $\alpha, \beta, \gamma \in \mathbb{R}$,

$$\alpha\,\mathbf{a} + \beta\,\mathbf{b} + \gamma\,\mathbf{c} = \mathbf{0} \quad \Rightarrow \quad \alpha = \beta = \gamma = 0\,, \tag{2.41}$$

then we say that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are *linearly independent*.

Remarks.

(i) If $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$ the uniqueness of λ, μ and ν means that since $(0, 0, 0)$ is *a* solution to

$$\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = \mathbf{0}\,, \tag{2.42}$$

it is also *the unique* solution, and hence the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is *linearly independent*.

(ii) If $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$ the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ both spans 3D space and is linearly independent, it is hence a *basis* for 3D space.

(iii) $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ do not have to be mutually orthogonal to be a basis.

5/08 (iv) In 3D space a basis always consists of three vectors.[18]

[18] Another tautology.

2.8 Orthogonal Bases

2.8.1 The Cartesian or standard basis in 3D

We have noted that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ do not have to be mutually orthogonal (or right-handed) to be a basis. However, matters are simplified if the basis vectors are mutually orthogonal and have unit magnitude, in which case they are said to define a *orthonormal basis*. It is also conventional to order them so that they are right-handed.

Let OX, OY, OZ be a right-handed set of Cartesian axes. Let

$\mathbf{i}$ be the unit vector along OX,
$\mathbf{j}$ be the unit vector along OY,
$\mathbf{k}$ be the unit vector along OZ,

where it is not conventional to add a $\widehat{\ }$. Then the ordered set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ forms a basis for 3D space satisfying

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1\,, \tag{2.43a}$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0\,, \tag{2.43b}$$

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}\,, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}\,, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}\,, \tag{2.43c}$$

$$[\mathbf{i}, \mathbf{j}, \mathbf{k}] = 1\,. \tag{2.43d}$$

Definition. If for a vector $\mathbf{v}$ and a Cartesian basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$,

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\,, \tag{2.44}$$

where $v_x, v_y, v_z \in \mathbb{R}$, we define (v_x, v_y, v_z) to be the *Cartesian components* of $\mathbf{v}$ with respect to the ordered basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

By 'dotting' (2.44) with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively, we deduce from (2.43a) and (2.43b) that

$$v_x = \mathbf{v}\cdot\mathbf{i}\,, \quad v_y = \mathbf{v}\cdot\mathbf{j}\,, \quad v_z = \mathbf{v}\cdot\mathbf{k}\,. \tag{2.45}$$

Hence for all 3D vectors $\mathbf{v}$

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{i})\,\mathbf{i} + (\mathbf{v}\cdot\mathbf{j})\,\mathbf{j} + (\mathbf{v}\cdot\mathbf{k})\,\mathbf{k}\,. \tag{2.46}$$

Remarks.

(i) Assuming that we know the basis vectors (and remember that there are an uncountably infinite number of Cartesian axes), we often write

$$\mathbf{v} = (v_x, v_y, v_z)\ . \tag{2.47}$$

In terms of this notation

$$\mathbf{i} = (1, 0, 0)\,, \quad \mathbf{j} = (0, 1, 0)\,, \quad \mathbf{k} = (0, 0, 1)\,. \tag{2.48}$$

(ii) If the point P has the Cartesian co-ordinates (x, y, z), then the position vector

$$\overrightarrow{OP} = \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,, \quad \text{i.e.} \quad \mathbf{r} = (x, y, z)\,. \tag{2.49}$$

(iii) Every vector in 2D/3D space may be uniquely represented by two/three real numbers, so we often write $\mathbb{R}^2/\mathbb{R}^3$ for 2D/3D space.

2.8.2 Direction cosines

If $\mathbf{t}$ is a unit vector with components (t_x, t_y, t_z) with respect to the ordered basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, then

$$t_x = \mathbf{t}\cdot\mathbf{i} = |\mathbf{t}|\,|\mathbf{i}|\cos\alpha\,, \tag{2.50a}$$

6/06 where α is the angle between $\mathbf{t}$ and $\mathbf{i}$. Hence if β and γ are the angles between $\mathbf{t}$ and $\mathbf{j}$, and $\mathbf{t}$ and $\mathbf{k}$, respectively, the *direction cosines* of $\mathbf{t}$ are defined by

$$\mathbf{t} = (\cos\alpha, \cos\beta, \cos\gamma)\ . \tag{2.50b}$$

2.9 Vector Component Identities

Suppose that

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}\,, \quad \mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k} \quad \text{and} \quad \mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}\,. \tag{2.51}$$

Then we can deduce a number of vector identities for components (and one true vector identity).

Addition. From repeated application of (2.7a), (2.7b) and (2.8)

$$\lambda\mathbf{a} + \mu\mathbf{b} = (\lambda a_x + \mu b_x)\mathbf{i} + (\lambda a_y + \mu b_y)\mathbf{j} + (\lambda a_z + \mu b_z)\mathbf{k}\,. \tag{2.52}$$

Scalar product. From repeated application of (2.8), (2.20), (2.22), (2.24a), (2.43a) and (2.43b)

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\ &= a_x\mathbf{i}\cdot b_x\mathbf{i} + a_x\mathbf{i}\cdot b_y\mathbf{j} + a_x\mathbf{i}\cdot b_z\mathbf{k} + \dots \\ &= a_x\,b_x\,\mathbf{i}\cdot\mathbf{i} + a_x\,b_y\,\mathbf{i}\cdot\mathbf{j} + a_x\,b_z\,\mathbf{i}\cdot\mathbf{k} + \dots \\ &= a_x\,b_x + a_y\,b_y + a_z\,b_z\,. \end{aligned} \tag{2.53}$$

Vector product. From repeated application of (2.8), (2.31a), (2.31b), (2.31c), (2.31e), (2.43c)

$$\begin{aligned} \mathbf{a}\times\mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\times(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) \\ &= a_x\mathbf{i}\times b_x\mathbf{i} + a_x\mathbf{i}\times b_y\mathbf{j} + a_x\mathbf{i}\times b_z\mathbf{k} + \dots \\ &= a_x\,b_x\,\mathbf{i}\times\mathbf{i} + a_x\,b_y\,\mathbf{i}\times\mathbf{j} + a_x\,b_z\,\mathbf{i}\times\mathbf{k} + \dots \\ &= (a_y\,b_z - a_z\,b_y)\mathbf{i} + (a_z\,b_x - a_x\,b_z)\mathbf{j} + (a_x\,b_y - a_y\,b_x)\mathbf{k}\,. \end{aligned} \tag{2.54}$$

Scalar triple product. From (2.53) and (2.54)

$$\begin{aligned} (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= ((a_y\,b_z - a_z\,b_y)\mathbf{i} + (a_z\,b_x - a_x\,b_z)\mathbf{j} + (a_x\,b_y - a_y\,b_x)\mathbf{k})\cdot(c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}) \\ &= a_x\,b_y\,c_z + a_y\,b_z\,c_x + a_z\,b_x\,c_y - a_x\,b_z\,c_y - a_y\,b_x\,c_z - a_z\,b_y\,c_x\,. \end{aligned} \tag{2.55}$$

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Vector triple product. We wish to show that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\,. \tag{2.56}$$

Remark. Identity (2.56) has no component in the direction $\mathbf{a}$, i.e. no component in the direction of the vector outside the parentheses.

To prove this, begin with the x-component of the left-hand side of (2.56). Then from (2.54)

$$\begin{aligned} \big(\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\big)_x \equiv \big(\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\big)\cdot\mathbf{i} &= a_y\,(\mathbf{b}\times\mathbf{c})_z - a_z\,(\mathbf{b}\times\mathbf{c})_y \\ &= a_y\,(b_x\,c_y - b_y\,c_x) - a_z\,(b_z\,c_x - b_x\,c_z) \\ &= (a_yc_y + a_zc_z)b_x + a_xb_xc_x - a_xb_xc_x - (a_yb_y + a_zb_z)c_x \\ &= (\mathbf{a}\cdot\mathbf{c})b_x - (\mathbf{a}\cdot\mathbf{b})c_x \\ &= \big((\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\big)\cdot\mathbf{i} \equiv \big((\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\big)_x\,. \end{aligned}$$

Now proceed similarly for the y and z components, or note that if its true for one component it must be
7/03 true for all components because of the arbitrary choice of axes.

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2.10 Higher Dimensional Spaces

We can 'boot-strap' to higher dimensional spaces.

2.10.1 $\mathbb{R}^n$

Recall that for fixed positive integer n, we defined $\mathbb{R}^n$ to be the set of all $n-$tuples

$$\{\mathbf{x} = (x_1, x_2, \ldots, x_n) : x_j \in \mathbb{R} \text{ with } j = 1, 2, \ldots, n\}, \tag{2.57}$$

with vector addition, scalar multiplication, the zero vector and the inverse vector defined by (2.13a), (2.13b), (2.13c) and (2.13d) respectively.

2.10.2 Linear independence, spanning sets and bases in $\mathbb{R}^n$

Definition. A set of m vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\ \mathbf{v}_m\}$, $\mathbf{v}_j \in \mathbb{R}^n$, $j = 1, 2, \ldots, m$, is *linearly independent* if for all scalars $\lambda_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$,

$$\sum_{i=1}^{m} \lambda_i \mathbf{v}_i = \mathbf{0} \quad \Rightarrow \quad \lambda_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, m. \tag{2.58}$$

Otherwise, the vectors are said to be *linearly dependent* since there exist scalars $\lambda_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$, not all of which are zero, such that

$$\sum_{i=1}^{m} \lambda_i \mathbf{v}_i = \mathbf{0}.$$

Definition. A subset $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots\ \mathbf{u}_m\}$ of vectors in $\mathbb{R}^n$ is a *spanning set* for $\mathbb{R}^n$ if for every vector $\mathbf{v} \in \mathbb{R}^n$, there exist scalars $\lambda_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$, such that

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \ldots + \lambda_m \mathbf{u}_m. \tag{2.59}$$

Remark. We state (but do not prove) that $m \geqslant n$, and that for a given $\mathbf{v}$ the λ_j are not necessarily unique if $m > n$.

Definition. A linearly independent subset of vectors that spans $\mathbb{R}^n$ is a *basis* of $\mathbb{R}^n$.

Remark. Every basis of $\mathbb{R}^n$ has n elements (this statement will be proved in *Linear Algebra*).

Property (unproven). If the set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots\ \mathbf{u}_n\}$ is a basis of $\mathbb{R}^n$, then for every vector $\mathbf{v} \in \mathbb{R}^n$ there exist *unique* scalars $\lambda_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$, such that

$$\mathbf{v} = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \ldots + \lambda_n \mathbf{u}_n. \tag{2.60}$$

Definition. We refer to the λ_j, $j = 1, 2, \ldots, n$, as the *components* of $\mathbf{v}$ with respect to the ordered basis S.

Remarks.

(i) The x_j, $j = 1, 2, \ldots, n$, in the n-tuple (2.57) might be viewed as the components of a vector with respect to the *standard basis*

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \ \ldots\ , \mathbf{e}_n = (0, 0, \ldots, 1). \tag{2.61a}$$

As such the definitions of vector addition and scalar multiplication in $\mathbb{R}^n$, i.e. (2.13a) and (2.13b) respectively, are consistent with our component notation in 2D and 3D.

(ii) In 3D we identify $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively, i.e.

$$\mathbf{e}_1 = \mathbf{i}, \quad \mathbf{e}_2 = \mathbf{j}, \quad \mathbf{e}_3 = \mathbf{k}. \tag{2.61b}$$

2.10.3 Dimension

At school you used two axes to describe 2D space, and three axes to describe 3D space. We noted above that 2D/3D space [always] has two/three vectors in a basis (corresponding to two/three axes). We now turn this view of the world on its head.

Definition. We define the *dimension* of a space as the number of vectors in a basis of the space.

Remarks.

(i) This definition depends on the proof (given in *Linear Algebra*) that every basis of a *vector space* has the same number of elements/vectors.

(ii) Hence $\mathbb{R}^n$ has dimension n.

2.10.4 The scalar product for $\mathbb{R}^n$

We define the *scalar product* on $\mathbb{R}^n$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as (cf. (2.53))

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \,. \tag{2.62a}$$

Exercise. Show that for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$,

SP(i) $$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x} \,, \tag{2.62b}$$

SP(ii) $$\mathbf{x} \cdot (\lambda \mathbf{y} + \mu \mathbf{z}) = \lambda\, \mathbf{x} \cdot \mathbf{y} + \mu\, \mathbf{x} \cdot \mathbf{z} \,, \tag{2.62c}$$

SP(iii) $$\mathbf{x} \cdot \mathbf{x} \geqslant 0 \,, \tag{2.62d}$$

SP(iv) $$\mathbf{x} \cdot \mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{x} = \mathbf{0} \,. \tag{2.62e}$$

Remarks.

(i) The *length*, or *Euclidean norm*, of a vector $\mathbf{x} \in \mathbb{R}^n$ is *defined* to be

$$|\mathbf{x}| \equiv (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} ,$$

while the *interior angle* θ between two vectors $\mathbf{x}$ and $\mathbf{y}$ is *defined* to be

$$\theta = \arccos \left(\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|} \right) .$$

(ii) The Cauchy-Schwarz inequality (2.27) holds (use the 'long' algebraic proof).

(iii) Non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are *defined* to be *orthogonal* if $\mathbf{x} \cdot \mathbf{y} = 0$.

(iv) We need to be a little careful with the definition (2.62a). It is important to appreciate that the scalar product for $\mathbb{R}^n$ as defined by (2.62a) is consistent with the scalar product for $\mathbb{R}^3$ defined in (2.19) only when the x_i and y_i are components with respect to an orthonormal basis. To this end it is helpful to view (2.62a) as the scalar product with respect to the *standard basis*

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \; \mathbf{e}_2 = (0, 1, \ldots, 0), \; \ldots, \; \mathbf{e}_n = (0, 0, \ldots, 1).$$

For the case when the x_i and y_i are components with respect to a non-orthonormal basis (e.g. a non-orthogonal basis), the scalar product in n-dimensional space equivalent to (2.19) has a more complicated form (for which we need matrices).

2.10.5 $\mathbb{C}^n$

Definition. For fixed positive integer n, define $\mathbb{C}^n$ to be the set of n-tuples $(z_1, z_2, \ldots, z_n)$ of complex numbers $z_i \in \mathbb{C}$, $i = 1, \ldots, n$. For $\lambda \in \mathbb{C}$ and complex vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$, where

$$\mathbf{u} = (u_1, \ldots, u_n) \quad \text{and} \quad \mathbf{v} = (v_1, \ldots, v_n),$$

define vector addition and scalar multiplication by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n) \in \mathbb{C}^n\,, \tag{2.63a}$$
$$\lambda\mathbf{u} = (\lambda u_1, \ldots, \lambda u_n) \in \mathbb{C}^n\,. \tag{2.63b}$$

Remark. The standard basis for $\mathbb{R}^n$, i.e.

$$\mathbf{e}_1 = (1, 0, \ldots, 0),\ \ldots\ , \mathbf{e}_n = (0, 0, \ldots, 1), \tag{2.64a}$$

also serves as a standard basis for $\mathbb{C}^n$ since

(i) the $\mathbf{e}_i$ $(i = 1, \ldots, n)$ are still linearly independent, i.e. for all scalars $\lambda_i \in \mathbb{C}$

$$\sum_{i=1}^{n} \lambda_i \mathbf{e}_i = (\lambda_1, \ldots, \lambda_n) = \mathbf{0} \quad \Rightarrow \quad \lambda_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, n\,,$$

(ii) and we can express any $\mathbf{z} \in \mathbb{C}^n$ in terms of components as

$$\mathbf{z} = (z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i \mathbf{e}_i\,. \tag{2.64b}$$

Hence $\mathbb{C}^n$ has dimension n when viewed as a vector space over $\mathbb{C}$.

2.10.6 The scalar product for $\mathbb{C}^n$

We define the *scalar product* on $\mathbb{C}^n$ for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i^* v_i = u_1^* v_1 + u_2^* v_2 + \ldots + u_n^* v_n\,, \tag{2.65}$$

where * denotes a complex conjugate (do **not** forget the complex conjugate).

Exercise. Show that for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ and $\lambda, \mu \in \mathbb{C}$,

SP(i)* $$\mathbf{u} \cdot \mathbf{v} = (\mathbf{v} \cdot \mathbf{u})^*\,, \tag{2.66a}$$
SP(ii) $$\mathbf{u} \cdot (\lambda\mathbf{v} + \mu\mathbf{w}) = \lambda\, \mathbf{u} \cdot \mathbf{v} + \mu\, \mathbf{u} \cdot \mathbf{w}\,, \tag{2.66b}$$
SP(iii) $$|\mathbf{u}|^2 \equiv \mathbf{u} \cdot \mathbf{u} \geqslant 0\,, \tag{2.66c}$$
SP(iv) $$|\mathbf{u}| = 0 \quad \Rightarrow \quad \mathbf{u} = \mathbf{0}\,. \tag{2.66d}$$

Remark. After minor modifications, the long proof of the Cauchy-Schwarz inequality (2.27) again holds.

6/08 *Definition.* Non-zero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ are said to be *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

2.11 Suffix Notation

So far we have used *dyadic notation* for vectors. *Suffix notation* is an alternative means of expressing vectors (and tensors). Once familiar with suffix notation, it is generally easier to manipulate vectors using suffix notation.[19]

[19] Although there are dissenters to that view.

In (2.44) and (2.47) we introduced the notation

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = (v_x, v_y, v_z)\,.$$

An alternative is to let $v_x = v_1$, $v_y = v_2$, and $v_z = v_3$, and use (2.61b) to write

$$\begin{aligned} \mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 &= (v_1, v_2, v_3) &\quad (2.67a)\\ &= \{v_i\} \quad \text{for} \quad i = 1, 2, 3\,. &\quad (2.67b)\end{aligned}$$

Suffix notation. We will refer to $\mathbf{v}$ as $\{v_i\}$, with the $i = 1, 2, 3$ understood; i is then termed a *free suffix.*

Remark. Sometimes we will denote the i^{th} component of the vector $\mathbf{v}$ by $(\mathbf{v})_i$, i.e. $(\mathbf{v})_i = v_i$.

Example: the position vector. The position vector $\mathbf{r}$ can be written as

$$\mathbf{r} = (x, y, z) = (x_1, x_2, x_3) = \{x_i\}\,. \tag{2.68}$$

Remark. The use of $\mathbf{x}$, rather than $\mathbf{r}$, for the position vector in dyadic notation possibly seems more understandable given the above expression for the position vector in suffix notation. Henceforth we will use $\mathbf{x}$ and $\mathbf{r}$ interchangeably.

2.11.1 Dyadic and suffix equivalents

If two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal, we write

$$\mathbf{a} = \mathbf{b}\,, \tag{2.69a}$$

or equivalently in component form

$$\begin{aligned} a_1 &= b_1\,, &\quad (2.69b)\\ a_2 &= b_2\,, &\quad (2.69c)\\ a_3 &= b_3\,. &\quad (2.69d)\end{aligned}$$

In suffix notation we express this equality as

$$a_i = b_i \quad \text{for} \quad i = 1, 2, 3\,. \tag{2.69e}$$

This is a vector equation; when we omit the 'for $i = 1, 2, 3$', it is understood that the one free suffix i ranges through $1, 2, 3$ so as to give three component equations. Similarly

$$\begin{aligned} \mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b} \quad &\Leftrightarrow \quad c_i = \lambda a_i + \mu b_i\\ &\Leftrightarrow \quad c_j = \lambda a_j + \mu b_j\\ &\Leftrightarrow \quad c_\alpha = \lambda a_\alpha + \mu b_\alpha\\ &\Leftrightarrow \quad c_{\yen} = \lambda a_{\yen} + \mu b_{\yen}\,,\end{aligned}$$

where is is assumed that i, j, α and ¥, respectively, range through $(1, 2, 3)$.[20]

Remark. It does not matter what letter, or symbol, is chosen for the free suffix, but it must be the same in each term.

Dummy suffices. In suffix notation the scalar product becomes

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= a_1b_1 + a_2b_2 + a_3b_3\\ &= \sum_{i=1}^{3} a_ib_i\\ &= \sum_{k=1}^{3} a_kb_k\,, \quad \text{etc.,}\end{aligned}$$

[20] In higher dimensions the suffices would be assumed to range through the number of dimensions.

where the i, k, etc. are referred to as *dummy suffices* since they are 'summed out' of the equation. Similarly

$$\mathbf{a}\cdot\mathbf{b}=\lambda \quad\Leftrightarrow\quad \sum_{\alpha=1}^{3} a_\alpha b_\alpha=\lambda\,,$$

where we note that the equivalent equation on the right hand side has no free suffices since the dummy suffix (in this case α) has again been summed out.

Further examples.

(i) As another example consider the equation $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}=\mathbf{d}$. In suffix notation this becomes

$$\sum_{k=1}^{3}(a_k b_k)\,c_i=\sum_{k=1}^{3}a_k b_k c_i=d_i\,, \tag{2.70}$$

where k is the dummy suffix, and i is the free suffix that is assumed to range through $(1,2,3)$. It is essential that we used different symbols for both the dummy and free suffices!

(ii) In suffix notation the expression $(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d})$ becomes

$$\begin{aligned}(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) &= \left(\sum_{i=1}^{3}a_i b_i\right)\left(\sum_{j=1}^{3}c_j d_j\right)\\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}a_i b_i c_j d_j\,,\end{aligned}$$

6/09 6/10 where, especially after the rearrangement, it is essential that the dummy suffices are different.

2.11.2 Summation convention

In the case of free suffices we are assuming that they range through $(1,2,3)$ without the need to explicitly say so. Under Einstein's summation convention the explicit sum, $\sum$, can be omitted for dummy suffices.[21] In particular

- if a suffix appears once it is taken to be a free suffix and ranged through,
- if a suffix appears twice it is taken to be a dummy suffix and summed over,
- if a suffix appears more than twice *in one term* of an equation, ***something has gone wrong*** (unless there is an explicit sum).

Remark. This notation is powerful because it is highly abbreviated (and so aids calculation, especially in examinations), but the above rules *must* be followed, and remember to check your answers (e.g. the free suffices should be identical on each side of an equation).

Examples. Under suffix notation and the summation convention

$$\begin{array}{rcl}\mathbf{a}+\mathbf{b}=\mathbf{c} & \text{can be written as} & a_i+b_i=c_i\,,\\ (\mathbf{a}\cdot\mathbf{b})\mathbf{c}=\mathbf{d} & \text{can be written as} & a_i\,b_i\,c_j=d_j\,,\\ ((\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b})_j & \text{can be written as} & a_i\,b_i\,c_j-a_k\,c_k\,b_j\,,\\ & \text{or can be written as} & a_i\,b_i\,c_j-a_i\,c_i\,b_j\,,\\ & \text{or can be written as} & a_i(b_i\,c_j-c_i\,b_j)\,.\end{array}$$

[21] Learning to omit the explicit sum is a bit like learning to change gear when starting to drive. At first you have to remind yourself that the sum is there, in the same way that you have to think consciously where to move gear knob. **With practice** you will learn to note the existence of the sum unconsciously, in the same way that an experienced driver changes gear unconsciously; however you will crash a few gears on the way!

Under suffix notation the following equations make no sense

$$a_k = b_j \qquad \text{because the free suffices are different,}$$
$$((\mathbf{a}\cdot\mathbf{b})\mathbf{c})_i = a_i\, b_i\, c_i \qquad \text{because } i \text{ is repeated more than twice in one term on the left-hand side.}$$

Under suffix notation the following equation is problematical (and probably best avoided unless you will always remember to double count the i on the right-hand side)

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$$n_i\, n_i = n_i^2 \quad \text{because } i \text{ occurs twice on the left-hand side and only once on the right-hand side.}$$

2.11.3 Kronecker delta

The Kronecker delta, δ_{ij}, $i, j = 1, 2, 3$, is a set of nine numbers defined by

$$\delta_{11} = 1\,, \quad \delta_{22} = 1\,, \quad \delta_{33} = 1\,, \tag{2.71a}$$
$$\delta_{ij} = 0 \quad \text{if} \quad i \neq j\,. \tag{2.71b}$$

This can be written as a *matrix* equation:

$$\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.71c}$$

Properties.

(i) δ_{ij} is symmetric, i.e.

$$\delta_{ij} = \delta_{ji}\,.$$

(ii) Using the definition of the delta function:

$$\begin{aligned} a_i\delta_{i1} &= \sum_{i=1}^{3} a_i\delta_{i1} \\ &= a_1\delta_{11} + a_2\delta_{21} + a_3\delta_{31} \\ &= a_1\,. \end{aligned} \tag{2.72a}$$

Similarly

$$a_i\delta_{ij} = a_j \qquad \text{and} \qquad a_j\delta_{ij} = a_i\,. \tag{2.72b}$$

(iii)

$$\delta_{ij}\delta_{jk} = \sum_{j=1}^{3} \delta_{ij}\delta_{jk} = \delta_{ik}\,. \tag{2.72c}$$

(iv)

$$\delta_{ii} = \sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3\,. \tag{2.72d}$$

(v)

$$a_p\,\delta_{pq}\,b_q = a_p\,b_p = a_q\,b_q = \mathbf{a}\cdot\mathbf{b}\,. \tag{2.72e}$$

2.11.4 More on basis vectors

Now that we have introduced suffix notation, it is more convenient to write $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ for the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ (see also (2.61b)). An alternative notation is $\mathbf{e}^{(1)}$, $\mathbf{e}^{(2)}$ and $\mathbf{e}^{(3)}$, where the use of superscripts may help emphasise that the 1, 2 and 3 are labels rather than components.

Then in terms of the superscript notation

$$\mathbf{e}^{(i)} \cdot \mathbf{e}^{(j)} = \delta_{ij}\,, \tag{2.73a}$$

$$\mathbf{a} \cdot \mathbf{e}^{(i)} = a_i\,. \tag{2.73b}$$

Thus the ith component of $\mathbf{e}^{(j)}$ is given by

$$\begin{aligned}\left(\mathbf{e}^{(j)}\right)_i &= \mathbf{e}^{(j)} \cdot \mathbf{e}^{(i)} \\ &= \delta_{ij}\,.\end{aligned} \tag{2.73c}$$

Similarly

$$\left(\mathbf{e}^{(i)}\right)_j = \delta_{ij}\,, \tag{2.73d}$$

and equivalently

8/06 $$\left(\mathbf{e}_j\right)_i = \left(\mathbf{e}_i\right)_j = \delta_{ij}\,. \tag{2.73e}$$

2.11.5 Part one of a dummy's guide to permutations[22]

A permutation of degree n is a[n invertible] function, or map, that rearranges n distinct objects amongst themselves. We will consider permutations of the set of the first n strictly positive integers $\{1, 2, \ldots, n\}$.

If $n = 3$ there are 6 permutations (including the identity permutation) that re-arrange $\{1, 2, 3\}$ to

$$\{1,2,3\},\ \{2,3,1\},\ \{3,1,2\}, \tag{2.74a}$$

$$\{1,3,2\},\ \{2,1,3\},\ \{3,2,1\}. \tag{2.74b}$$

(2.74a) and (2.74b) are known, respectively, as even and odd permutations of $\{1, 2, 3\}$.

Remark. Slightly more precisely: an ordered sequence is an even/odd permutation if the number of pairwise swaps (or exchanges or transpositions) necessary to recover the original ordering, in this case $\{1\,2\,3\}$, is even/odd.[23]

2.11.6 The Levi-Civita symbol or alternating tensor

Definition. We define ε_{ijk} $(i, j, k = 1, 2, 3)$ to be the set of 27 quantities such that

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i\,j\,k \text{ is an even permutation of } 1\,2\,3; \\ -1 & \text{if } i\,j\,k \text{ is an odd permutation of } 1\,2\,3; \\ 0 & \text{otherwise} \end{cases} \tag{2.75}$$

The non-zero components of ε_{ijk} are therefore

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \tag{2.76a}$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1 \tag{2.76b}$$

Further

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{ikj} = -\varepsilon_{kji} = -\varepsilon_{jik}\,. \tag{2.76c}$$ 8/02

Worked exercise. For a symmetric tensor s_{ij}, $i, j = 1, 2, 3$, such that $s_{ij} = s_{ji}$ evaluate $\varepsilon_{ijk} s_{ij}$.

[22] Which is not the same thing as *Permutations for Dummies*.
[23] This definition requires a proof of the fact that the number of pairwise swaps is always even or odd: see *Groups*.

Solution. By relabelling the dummy suffices we have from (2.76c) and the symmetry of s_{ij} that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon_{ijk}s_{ij} = \sum_{a=1}^{3}\sum_{b=1}^{3}\varepsilon_{abk}s_{ab} = \sum_{j=1}^{3}\sum_{i=1}^{3}\varepsilon_{jik}s_{ji} = \sum_{i=1}^{3}\sum_{j=1}^{3} -\varepsilon_{ijk}s_{ij}\,, \tag{2.77a}$$

or equivalently

$$\varepsilon_{ijk}s_{ij} = \varepsilon_{abk}s_{ab} = \varepsilon_{jik}s_{ji} = -\varepsilon_{ijk}s_{ij}\,. \tag{2.77b}$$

Hence we conclude that

$$\varepsilon_{ijk}s_{ij} = 0\,. \tag{2.77c}$$

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2.11.7 The vector product in suffix notation

We claim that

$$(\mathbf{a}\times\mathbf{b})_i = \sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\,a_j\,b_k = \varepsilon_{ijk}\,a_j\,b_k\,, \tag{2.78}$$

where we note that there is one free suffix and two dummy suffices.

Check.

$$(\mathbf{a}\times\mathbf{b})_1 = \sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{1jk}\,a_j\,b_k = \varepsilon_{123}\,a_2\,b_3 + \varepsilon_{132}\,a_3\,b_2 = a_2\,b_3 - a_3\,b_2\,,$$

as required from (2.54). Do we need to do more?

Example. From (2.72b), (2.73e) and (2.78)

$$\begin{aligned}\left(\mathbf{e}^{(j)}\times\mathbf{e}^{(k)}\right)_i &= \varepsilon_{ilm}\left(\mathbf{e}^{(j)}\right)_l\left(\mathbf{e}^{(k)}\right)_m\\ &= \varepsilon_{ilm}\,\delta_{jl}\,\delta_{km}\\ &= \varepsilon_{ijk}\,.\end{aligned} \tag{2.79}$$

2.11.8 An identity

Theorem 2.1.

$$\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\,\delta_{kq} - \delta_{jq}\,\delta_{kp}\,. \tag{2.80}$$

Remark. There are four free suffices/indices on each side, with i as a dummy suffix on the left-hand side. Hence (2.80) represents 3^4 equations.

Proof. First suppose, say, $j = k = 1$; then

$$\begin{aligned}\text{LHS} &= \varepsilon_{i11}\varepsilon_{ipq} = 0\,,\\ \text{RHS} &= \delta_{1p}\,\delta_{1q} - \delta_{1q}\,\delta_{1p} = 0\,.\end{aligned}$$

Similarly whenever $j = k$ (or $p = q$). Next suppose, say, $j = 1$ and $k = 2$; then

$$\begin{aligned}\text{LHS} &= \varepsilon_{i12}\,\varepsilon_{ipq}\\ &= \varepsilon_{312}\,\varepsilon_{3pq}\\ &= \begin{cases} 1 & \text{if } p=1, q=2\\ -1 & \text{if } p=2, q=1\\ 0 & \text{otherwise}\end{cases}\,,\end{aligned}$$

while

$$\begin{aligned} \text{RHS} &= \delta_{1p}\,\delta_{2q} - \delta_{1q}\,\delta_{2p} \\ &= \begin{cases} 1 & \text{if } p=1, q=2 \\ -1 & \text{if } p=2, q=1 \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

Similarly whenever $j \neq k$. □

Example. Take $j = p$ in (2.80) as an example of a repeated suffix; then

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$$\begin{aligned} \varepsilon_{ipk}\varepsilon_{ipq} &= \delta_{pp}\,\delta_{kq} - \delta_{pq}\,\delta_{kp} \\ &= 3\delta_{kq} - \delta_{kq} \;=\; 2\delta_{kq} . \end{aligned} \tag{2.81}$$

2.11.9 Scalar triple product

In suffix notation the scalar triple product is given by

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$$\begin{aligned} \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= a_i(\mathbf{b}\times\mathbf{c})_i \\ &= \varepsilon_{ijk}\,a_i\,b_j\,c_k . \end{aligned} \tag{2.82}$$

2.11.10 Vector triple product

Using suffix notation for the vector triple product we recover

$$\begin{aligned} \big(\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\big)_i &= \varepsilon_{ijk}\,a_j\,(\mathbf{b}\times\mathbf{c})_k \\ &= \varepsilon_{ijk}\,a_j\,\varepsilon_{klm}\,b_l\,c_m \\ &= -\varepsilon_{kji}\,\varepsilon_{klm}\,a_j\,b_l\,c_m \\ &= -(\delta_{jl}\delta_{im} - \delta_{jm}\delta_{il})\,a_j\,b_l\,c_m \\ &= a_j\,b_i\,c_j - a_j\,b_j\,c_i \\ &= \big((\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\big)_i , \end{aligned}$$

in agreement with (2.56).

2.11.11 Yet another proof of Schwarz's inequality (Unlectured)

This time using the summation convention:

$$\begin{aligned} \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - |\mathbf{x}\cdot\mathbf{y}|^2 &= x_ix_iy_jy_j - x_iy_ix_jy_j \\ &= \tfrac{1}{2}x_ix_iy_jy_j + \tfrac{1}{2}x_jx_jy_iy_i - x_iy_ix_jy_j \\ &= \tfrac{1}{2}(x_iy_j - x_jy_i)(x_iy_j - x_jy_i) \\ &\geqslant 0 . \end{aligned}$$

2.12 Vector Equations

When presented with a vector equation one approach might be to write out the equation in components, e.g. $(\mathbf{x}-\mathbf{a})\cdot\mathbf{n} = 0$ would become

$$x_1\,n_1 + x_2\,n_2 + x_3\,n_3 = a_1\,n_1 + a_2\,n_2 + a_3\,n_3 . \tag{2.83}$$

For given $\mathbf{a}, \mathbf{n} \in \mathbb{R}^3$ this is a single equation for three unknowns $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, and hence we
EJH might expect two arbitrary parameters in the solution (as we shall see is the case in (2.89) below). An
7/07 alternative, and often better, way forward is to use vector manipulation to make progress.

Worked Exercise. For given $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ find solutions $\mathbf{x} \in \mathbb{R}^3$ to

$$\mathbf{x} - (\mathbf{x}\times\mathbf{a})\times\mathbf{b} = \mathbf{c} . \tag{2.84}$$

Solution. First expand the vector triple product using (2.56):

$$\mathbf{x} - \mathbf{a}(\mathbf{b}\cdot\mathbf{x}) + \mathbf{x}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{c}\;;$$

then dot this with $\mathbf{b}$:

$$\mathbf{b}\cdot\mathbf{x} = \mathbf{b}\cdot\mathbf{c}\;;$$

then substitute this result into the previous equation to obtain:

$$\mathbf{x}(1+\mathbf{a}\cdot\mathbf{b}) = \mathbf{c} + \mathbf{a}(\mathbf{b}\cdot\mathbf{c})\;;$$

now rearrange to deduce that

$$\mathbf{x} = \frac{\mathbf{c} + \mathbf{a}(\mathbf{b}\cdot\mathbf{c})}{(1+\mathbf{a}\cdot\mathbf{b})}\,.$$

Remark. For the case when $\mathbf{a}$ and $\mathbf{c}$ are not parallel we could have alternatively sought a solution using $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{a}\times\mathbf{c}$ as a basis.

2.13 Lines, Planes and Spheres

Certain geometrical objects can be described by vector equations.

2.13.1 Lines

Consider the line through a point A parallel to a vector $\mathbf{t}$, and let P be a point on the line. Then the vector equation for a point on the line is given by

$$\vec{OP} \;=\; \vec{OA} + \vec{AP}$$

or equivalently, for some $\lambda \in \mathbb{R}$,

$$\mathbf{x} \;=\; \mathbf{a} + \lambda\mathbf{t}\,. \tag{2.85a}$$

We may eliminate λ from the equation by noting that $\mathbf{x}-\mathbf{a} = \lambda\mathbf{t}$, and hence

$$(\mathbf{x}-\mathbf{a})\times\mathbf{t} = \mathbf{0}\,. \tag{2.85b}$$

This is an equivalent equation for the line since the solutions to (2.85b) for $\mathbf{t}\neq\mathbf{0}$ are either $\mathbf{x}=\mathbf{a}$ or
9/03 $(\mathbf{x}-\mathbf{a})$ parallel to $\mathbf{t}$.

Remark. Equation (2.85b) has many solutions; the multiplicity of the solutions is represented by a single arbitrary scalar.

Worked Exercise. For given $\mathbf{u},\mathbf{t}\in\mathbb{R}^3$ find solutions $\mathbf{x}\in\mathbb{R}^3$ to

$$\mathbf{u} = \mathbf{x}\times\mathbf{t}\,. \tag{2.86}$$

Solution. First 'dot' (2.86) with $\mathbf{t}$ to obtain

$$\mathbf{t}\cdot\mathbf{u} = \mathbf{t}\cdot(\mathbf{x}\times\mathbf{t}) = 0\,.$$

Thus there are no solutions unless $\mathbf{t}\cdot\mathbf{u}=0$. Next 'cross' (2.86) with $\mathbf{t}$ to obtain

$$\mathbf{t}\times\mathbf{u} = \mathbf{t}\times(\mathbf{x}\times\mathbf{t}) = (\mathbf{t}\cdot\mathbf{t})\mathbf{x} - (\mathbf{t}\cdot\mathbf{x})\mathbf{t}\,.$$

Hence

$$\mathbf{x} = \frac{\mathbf{t}\times\mathbf{u}}{|\mathbf{t}|^2} + \frac{(\mathbf{t}\cdot\mathbf{x})\mathbf{t}}{|\mathbf{t}|^2}\,.$$

Finally observe that if $\mathbf{x}$ is a solution to (2.86) so is $\mathbf{x}+\mu\mathbf{t}$ for any $\mu\in\mathbb{R}$, i.e. solutions of (2.86) can only be found up to an arbitrary multiple of $\mathbf{t}$. Hence the general solution to (2.86), assuming that $\mathbf{t}\cdot\mathbf{u}=0$, is

$$\mathbf{x} = \frac{\mathbf{t}\times\mathbf{u}}{|\mathbf{t}|^2} + \mu\mathbf{t}\,, \tag{2.87}$$

9/02 i.e. a straight line in direction $\mathbf{t}$ through $(\mathbf{t}\times\mathbf{u})/|\mathbf{t}|^2$.

2.13.2 Planes

Consider a plane that goes through a point A and that is orthogonal to a unit vector $\mathbf{n}$; $\mathbf{n}$ is the *normal* to the plane. Let P be any point in the plane. Then (cf. (2.83))

$$\begin{aligned} \overrightarrow{AP} \cdot \mathbf{n} &= 0\,, \\ (\,\overrightarrow{AO} + \overrightarrow{OP}\,) \cdot \mathbf{n} &= 0\,, \\ (\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} &= 0\,. \end{aligned} \tag{2.88a}$$

Let Q be the point in the plane such that $\overrightarrow{OQ}$ is parallel to $\mathbf{n}$. Suppose that $\overrightarrow{OQ} = d\mathbf{n}$, then d is the distance of the plane to O. Further, since Q is in the plane, it follows from (2.88a) that

$$(d\mathbf{n} - \mathbf{a}) \cdot \mathbf{n} = 0\,, \quad \text{and hence} \quad \mathbf{a} \cdot \mathbf{n} = d\mathbf{n}^2 = d\,.$$

The equation of the plane is thus

$$\mathbf{x} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = d\,. \tag{2.88b}$$

Remarks.

(i) If $\mathbf{l}$ and $\mathbf{m}$ are two linearly independent vectors such that $\mathbf{l} \cdot \mathbf{n} = 0$ and $\mathbf{m} \cdot \mathbf{n} = 0$ (so that both vectors lie in the plane), then any point $\mathbf{x}$ in the plane may be written as

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{l} + \mu \mathbf{m}\,, \tag{2.89}$$

where $\lambda, \mu \in \mathbb{R}$.

(ii) (2.89) is a solution to equation (2.88b). The arbitrariness in the *two* independent arbitrary scalars λ and μ means that the equation has [uncountably] many solutions.

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Worked exercise. Under what conditions do the two lines $L_1 : (\mathbf{x} - \mathbf{a}) \times \mathbf{t} = 0$ and $L_2 : (\mathbf{x} - \mathbf{b}) \times \mathbf{u} = 0$ intersect?

Solution. If the lines are to intersect they cannot be parallel, hence $\mathbf{t}$ and $\mathbf{u}$ must be linearly independent. L_1 passes through $\mathbf{a}$; let L_2' be the line passing through $\mathbf{a}$ parallel to $\mathbf{u}$. Let Π be the plane containing L_1 and L_2', with normal $\mathbf{t} \times \mathbf{u}$. Hence from (2.88a) the equation specifying points on the plane Π is

$$\Pi : \qquad (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{t} \times \mathbf{u}) = 0\,. \tag{2.90}$$

Because L_2 is parallel to L_2' and thence Π, either L_2 intersects Π nowhere (in which case L_1 does not intersect L_2), or L_2 lies in Π (in which case L_1 intersects L_2). If the latter case, then $\mathbf{b}$ lies in Π and we deduce that a *necessary* condition for the lines to intersect is that

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{t} \times \mathbf{u}) = 0\,. \tag{2.91}$$

Further, we can show that (2.91) is also a *sufficient* condition for the lines to intersect (assuming that they are not parallel). For if (2.91) holds, then $(\mathbf{b} - \mathbf{a})$ must lie in the plane through the origin that is normal to $(\mathbf{t} \times \mathbf{u})$. This plane is spanned by $\mathbf{t}$ and $\mathbf{u}$, and hence there must exist $\alpha, \beta \in \mathbb{R}$ such that

$$\mathbf{b} - \mathbf{a} = \alpha \mathbf{t} + \beta \mathbf{u}\,.$$

Let $\mathbf{x}$ be the point specified as follows:

$$\mathbf{x} = \mathbf{a} + \alpha\mathbf{t} = \mathbf{b} - \beta\mathbf{u} \;;$$

then from the equation of a line, (2.85a), we deduce that $\mathbf{x}$ is a point on both L_1 and L_2 (as required).

Hyper-plane in $\mathbb{R}^n$*.* The (hyper-)plane in $\mathbb{R}^n$ that passes through $\mathbf{b} \in \mathbb{R}^n$ and has normal $\mathbf{n} \in \mathbb{R}^n$, is given by

$$\Pi = \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x} - \mathbf{b}) \cdot \mathbf{n} = 0,\ \mathbf{b}, \mathbf{n} \in \mathbb{R}^n \text{ with } |\mathbf{n}| = 1\}. \tag{2.92}$$

2.13.3 Spheres

Sphere in $\mathbb{R}^3$*.* The sphere in $\mathbb{R}^3$ with centre O and radius $r \in \mathbb{R}$ is given by

$$\Sigma = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = r > 0,\ r \in \mathbb{R}\}. \tag{2.93a}$$

Hyper-sphere in $\mathbb{R}^n$*.* The (hyper-)sphere in $\mathbb{R}^n$ with centre $\mathbf{a} \in \mathbb{R}^n$ and radius $r \in \mathbb{R}$ is given by

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$$\Sigma = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{a}| = r > 0,\ r \in \mathbb{R},\ \mathbf{a} \in \mathbb{R}^n\}. \tag{2.93b}$$

2.14 Subspaces

2.14.1 Subspaces: informal discussion

Suppose the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis for 3D space. Form a set consisting of any two linearly independent combinations of these vectors, e.g. $\{\mathbf{a}, \mathbf{b}\}$, or $\{\mathbf{a} + \mathbf{c}, \mathbf{a} + \mathbf{b} - \mathbf{c}\}$. This new set spans a 2D plane that we view as a *subspace* of 3D space.

Remarks.

(i) There are an infinite number of 2D subspaces of 3D space.

(ii) Similarly, we can define [an infinite number of] 1D subspaces of 3D space (or 2D space).

2.14.2 Subspaces: formal definition

Definition. A non-empty subset U of the elements of a vector space V is called a *subspace* of V if U is a vector space under the same operations (i.e. vector addition and scalar multiplication) as are used to define V.

Proper subspaces. Strictly V and $\{\mathbf{0}\}$ (i.e. the set containing the zero vector only) are subspaces of V. A *proper subspace* is a subspace of V that is not V or $\{\mathbf{0}\}$.

Theorem 2.2. *A subset U of a vector space V is a subspace of V if and only if under operations defined on V*

(i) *for each* $\mathbf{x}, \mathbf{y} \in U$, $\mathbf{x} + \mathbf{y} \in U$,

(ii) *for each* $\mathbf{x} \in U$ *and* $\lambda \in \mathbb{R}$, $\lambda\mathbf{x} \in U$,

i.e. if and only if U is closed under vector addition and scalar multiplication.

Remark. We can combine (i) and (ii) as the single condition

$$\text{for each } \mathbf{x}, \mathbf{y} \in U \text{ and } \lambda, \mu \in \mathbb{R},\ \lambda\mathbf{x} + \mu\mathbf{y} \in U\,. \tag{2.94}$$

Proof.

Only if. If U is a subspace then it is a vector space, and hence (i) and (ii) hold from the definition of a vector space..

If. It is straightforward to show that VA(i), VA(ii), SM(i), SM(ii), SM(iii) and SM(iv) hold, since the elements of U are also elements of V. We need demonstrate that VA(iii) (i.e. $\mathbf{0}$ is an element of U), and VA(iv) (i.e. the every element has an inverse in U) hold.

VA(iii). For each $\mathbf{x} \in U$, it follows from (ii) that $0\mathbf{x} \in U$; but since also $\mathbf{x} \in V$ it follows from (2.9b) or (2.15) that $0\mathbf{x} = \mathbf{0}$. Hence $\mathbf{0} \in U$.

VA(iv). For each $\mathbf{x} \in U$, it follows from (ii) that $(-1)\mathbf{x} \in U$; but since also $\mathbf{x} \in V$ it follows from (2.9c) or (2.16) that $(-1)\mathbf{x} = -\mathbf{x}$. Hence $-\mathbf{x} \in U$.

□

2.14.3 Examples

(i) For $n \geqslant 2$ let $U = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = (x_1, x_2, \dots, x_{n-1}, 0) \text{ with } x_j \in \mathbb{R} \text{ and } j = 1, 2, \dots, n-1\}$. Then U is a subspace of $\mathbb{R}^n$ since, for $\mathbf{x}, \mathbf{y} \in U$ and $\lambda, \mu \in \mathbb{R}$,

$$\lambda(x_1, x_2, \dots, x_{n-1}, 0) + \mu(y_1, y_2, \dots, y_{n-1}, 0) = (\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, \dots, \lambda x_{n-1} + \mu y_{n-1}, 0)\,.$$

Thus U is closed under vector addition and scalar multiplication, and is hence a subspace.

(ii) Consider the set $W = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n \alpha_i x_i = 0\}$ for given scalars $\alpha_j \in \mathbb{R}$ $(j = 1, 2, \dots, n)$. W is a a hyper-plane through the origin (see (2.92)), and is a subspace of V since, for $\mathbf{x}, \mathbf{y} \in W$ and $\lambda, \mu \in \mathbb{R}$,

$$\lambda\mathbf{x} + \mu\mathbf{y} = (\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, \dots, \lambda x_n + \mu y_n)\,,$$

and

$$\sum_{i=1}^n \alpha_i(\lambda x_i + \mu y_i) = \lambda \sum_{i=1}^n \alpha_i x_i + \mu \sum_{i=1}^n \alpha_i y_i = 0\,.$$

Thus W is closed under vector addition and scalar multiplication, and is hence a subspace.

(iii) For $n \geqslant 2$ consider the set $\widetilde{W} = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n \alpha_i x_i = 1\}$ for given scalars $\alpha_j \in \mathbb{R}$ $(j = 1, 2, \dots, n)$ not all of which are zero (wlog $\alpha_1 \neq 0$, if not reorder the numbering of the axes). $\widetilde{W}$ is another hyper-plane, but in this case it does not pass through the origin. It is *not* a subspace of $\mathbb{R}^n$. To see this *either* note that $\mathbf{0} \notin \widetilde{W}$, *or* consider $\mathbf{x} \in \widetilde{W}$ such that

$$\mathbf{x} = (\alpha_1^{-1}, 0, \dots, 0)\,.$$

Then $\mathbf{x} \in \widetilde{W}$ but $\mathbf{x}+\mathbf{x} \notin \widetilde{W}$ since $\sum_{i=1}^n \alpha_i(x_i + x_i) = 2$. Thus $\widetilde{W}$ is not closed under vector addition, and so $\widetilde{W}$ cannot be a subspace of $\mathbb{R}^n$.

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3 Matrices and Linear Maps

3.0 Why Study This?

A matrix is a rectangular table of elements. Matrices are used for a variety of purposes, e.g. describing both *linear* equations and *linear* maps. They are important since, *inter alia*, many problems in the real world are linear, e.g. electromagnetic waves satisfy linear equations, and almost all sounds you hear are 'linear' (exceptions being sonic booms). Moreover, many computational approaches to solving nonlinear problems involve 'linearisations' at some point in the process (since computers are good at solving linear problems). The aim of this section is to familiarise you with matrices.

3.1 An Example of a Linear Map

Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be an orthonormal basis in 2D, and let $\mathbf{u}$ be vector with components (u_1, u_2). Suppose that $\mathbf{u}$ is rotated by an angle θ to become the vector $\mathbf{u}'$ with components (u_1', u_2'). How are the (u_1', u_2') related to the (u_1, u_2)?

Let $u = |\mathbf{u}| = |\mathbf{u}'|$, $\mathbf{u}\cdot\mathbf{e}_1 = u\cos\psi$ and $\mathbf{u}\cdot\mathbf{e}_2 = u\sin\psi$, then by geometry

$$u_1' = u\cos(\psi+\theta) = u(\cos\psi\cos\theta - \sin\psi\sin\theta) = u_1\cos\theta - u_2\sin\theta \tag{3.1a}$$
$$u_2' = u\sin(\psi+\theta) = u(\cos\psi\sin\theta + \sin\psi\cos\theta) = u_1\sin\theta + u_2\cos\theta \tag{3.1b}$$

We can express this in suffix notation as (after temporarily suppressing the summation convention)

$$u_i' = \sum_{j=1}^{2} R_{ij} u_j\,, \tag{3.2}$$

where

$$R_{11} = \cos\theta\,, \qquad R_{12} = -\sin\theta\,, \tag{3.3a}$$
$$R_{21} = \sin\theta\,, \qquad R_{22} = \cos\theta\,, \tag{3.3b}$$

3.1.1 Matrix notation

The above equations can be written in a more convenient form by using matrix notation. Let u and u' be the *column matrices*, or *column vectors*,

$$\mathsf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad \mathsf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} \tag{3.4a}$$

respectively, and let R be the 2×2 *square matrix*

$$\mathsf{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}. \tag{3.4b}$$

Remarks.

- We call the R_{ij}, $i, j = 1, 2$ the *elements* of the matrix R.
- Sometimes we write either $\mathsf{R} = \{R_{ij}\}$ or $R_{ij} = (\mathsf{R})_{ij}$.
- The first suffix i is the *row* number, while the second suffix j is the *column* number.

- We now have **bold u** denoting a vector, *italic* u_i denoting a component of a vector, and sans serif u denoting a column matrix of components.
- To try and avoid confusion we have introduced for a short while a specific notation for a column matrix, i.e. u. However, in the case of a column matrix of vector components, i.e. a column vector, an accepted convention is to use the standard notation for a vector, i.e. $\mathbf{u}$. Hence we now have

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (u_1, u_2)\,, \tag{3.5}$$

where we draw attention to the commas on the RHS.

Equation (3.2) can now be expressed in matrix notation as

$$\mathsf{u}' = \mathsf{R}\mathsf{u} \qquad \text{or equivalently} \qquad \mathbf{u}' = \mathsf{R}\mathbf{u}\,, \tag{3.6a}$$

where a *matrix multiplication rule* has been defined in terms of matrix elements as

$$u'_i = \sum_{j=1}^{2} R_{ij} u_j\,, \tag{3.6b}$$

and the [2D] *rotation matrix*, $\mathsf{R}(\theta)$, is given by

$$\mathsf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.6c}$$

3.2 Linear Maps

3.2.1 Notation

Definition. Let A, B be sets. A *map* $\mathcal{T}$ of A into B is a rule that assigns to each $x \in A$ a unique $x' \in B$. We write

$$\mathcal{T} : A \to B \quad \text{and/or} \quad x \mapsto x' = \mathcal{T}(x)$$

Definition. A is the *domain* of $\mathcal{T}$.

Definition. B is the *range*, or *codomain*, of $\mathcal{T}$.

Definition. $\mathcal{T}(x) = x'$ is the *image of* x *under* $\mathcal{T}$.

Definition. $\mathcal{T}(A)$ is the *image of* A *under* $\mathcal{T}$, i.e. the set of all image points $x' \in B$ of $x \in A$.

Remark. $\mathcal{T}(A) \subseteq B$, but there may be elements of B that are not images of any $x \in A$.

3.2.2 Definition

We shall consider linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, or $\mathbb{C}^n$ to $\mathbb{C}^m$, for $m, n \in \mathbb{Z}^+$. For definiteness we will work with real linear maps, but the extension to complex linear maps is straightforward (except where noted otherwise).

Definition. Let V, W be real vector spaces, e.g. $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$. The map $\mathcal{T} : V \to W$ is a *linear map* or *linear transformation* if for all $\mathbf{a}, \mathbf{b} \in V$ and $\lambda, \mu \in \mathbb{R}$,

(i) $$\mathcal{T}(\mathbf{a} + \mathbf{b}) = \mathcal{T}(\mathbf{a}) + \mathcal{T}(\mathbf{b})\,, \tag{3.7a}$$

(ii) $$\mathcal{T}(\lambda\mathbf{a}) = \lambda\mathcal{T}(\mathbf{a})\,, \tag{3.7b}$$

or equivalently if

$$\mathcal{T}(\lambda\mathbf{a} + \mu\mathbf{b}) = \lambda\mathcal{T}(\mathbf{a}) + \mu\mathcal{T}(\mathbf{b})\,. \tag{3.8}$$

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Property. $T(V)$ is a subspace of W, since for $T(\mathbf{a}), T(\mathbf{b}) \in T(V)$

$$\lambda T(\mathbf{a}) + \mu T(\mathbf{b}) = T(\lambda\mathbf{a} + \mu\mathbf{b}) \in T(V) \quad \text{for all} \quad \lambda, \mu \in \mathbb{R}. \tag{3.9}$$

Now apply Theorem 2.2 on page 38.

The zero element. Since $T(V)$ is a subspace, it follows that $\mathbf{0} \in T(V)$. However, we can say more than that. Set $\mathbf{b} = \mathbf{0} \in V$ in (3.7a), to deduce that

$$T(\mathbf{a}) = T(\mathbf{a} + \mathbf{0}) = T(\mathbf{a}) + T(\mathbf{0}) \quad \text{for all} \quad \mathbf{a} \in V\,.$$

Thus from the uniqueness of the zero element it follows that $T(\mathbf{0}) = \mathbf{0} \in W$.

15/03 *Remark.* $T(\mathbf{b}) = \mathbf{0} \in W$ does **not** imply $\mathbf{b} = \mathbf{0} \in V$.

3.2.3 Examples

(i) Consider translation in $\mathbb{R}^3$, i.e. consider

$$\mathbf{x} \mapsto \mathbf{x}' = \mathcal{T}(\mathbf{x}) = \mathbf{x} + \mathbf{a}\,, \quad \text{where} \quad \mathbf{a} \in \mathbb{R}^3 \quad \text{and} \quad \mathbf{a} \neq \mathbf{0}\,. \tag{3.10a}$$

This is *not* a linear map by the *strict* definition of a linear map since

$$\mathcal{T}(\mathbf{x}) + \mathcal{T}(\mathbf{y}) = \mathbf{x} + \mathbf{a} + \mathbf{y} + \mathbf{a} = \mathcal{T}(\mathbf{x} + \mathbf{y}) + \mathbf{a} \neq \mathcal{T}(\mathbf{x} + \mathbf{y})\,.$$

(ii) Consider the projection, $\mathcal{P}_{\mathbf{n}} : \mathbb{R}^3 \to \mathbb{R}^3$, onto a line with direction $\mathbf{n} \in \mathbb{R}^3$ as defined by (cf. (2.23b))

$$\mathbf{x} \mapsto \mathbf{x}' = \mathcal{P}_{\mathbf{n}}(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{n})\mathbf{n} \quad \text{where} \quad \mathbf{n} \cdot \mathbf{n} = 1\,. \tag{3.10b}$$

From the observation that

$$\begin{aligned} \mathcal{P}_{\mathbf{n}}(\lambda\mathbf{x}_1 + \mu\mathbf{x}_2) &= ((\lambda\mathbf{x}_1 + \mu\mathbf{x}_2) \cdot \mathbf{n})\mathbf{n} \\ &= \lambda(\mathbf{x}_1 \cdot \mathbf{n})\mathbf{n} + \mu(\mathbf{x}_2 \cdot \mathbf{n})\mathbf{n} \\ &= \lambda\mathcal{P}_{\mathbf{n}}(\mathbf{x}_1) + \mu\mathcal{P}_{\mathbf{n}}(\mathbf{x}_2) \end{aligned}$$

we conclude that this is a linear map.

Remark. The image of $\mathcal{P}_{\mathbf{n}}$ is given by $\mathcal{P}_{\mathbf{n}}(\mathbb{R}^3) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda\mathbf{n} \text{ for } \lambda \in \mathbb{R}\}$, which is a 1-dimensional subspace of $\mathbb{R}^3$.

(iii) As an example of a map where the domain has a higher dimension than the range consider $\mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$(x, y, z) \mapsto (x', y') = \mathcal{S}(\mathbf{x}) = (x + y,\ 2x - z)\,. \tag{3.10c}$$

$\mathcal{S}$ is a linear map since

$$\begin{aligned} \mathcal{S}(\lambda\mathbf{x}_1 + \mu\mathbf{x}_2) &= ((\lambda x_1 + \mu x_2) + (\lambda y_1 + \mu y_2),\ 2(\lambda x_1 + \mu x_2) - (\lambda z_1 + \mu z_2)) \\ &= \lambda(x_1 + y_1,\ 2x_1 - z_1) + \mu(x_2 + y_2,\ 2x_2 - z_2) \\ &= \lambda\mathcal{S}(\mathbf{x}_1) + \mu\mathcal{S}(\mathbf{x}_2)\,. \end{aligned}$$

Remarks.

(a) The orthonormal basis vectors for $\mathbb{R}^3$, i.e. $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$ and $\mathbf{e}_3 = (0, 0, 1)$ are mapped to the vectors

$$\left.\begin{aligned} \mathcal{S}(\mathbf{e}_1) &= (1, 2) \\ \mathcal{S}(\mathbf{e}_2) &= (1, 0) \\ \mathcal{S}(\mathbf{e}_3) &= (0, -1) \end{aligned}\right\} \quad \text{which are linearly dependent and span } \mathbb{R}^2.$$

Hence $\mathcal{S}(\mathbb{R}^3) = \mathbb{R}^2$.

(b) We observe that

$$\mathbb{R}^3 = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \quad \text{and} \quad \mathcal{S}(\mathbb{R}^3) = \mathbb{R}^2 = \operatorname{span}\{\mathcal{S}(\mathbf{e}_1), \mathcal{S}(\mathbf{e}_2), \mathcal{S}(\mathbf{e}_3)\}.$$

(c) We also observe that

$$\mathcal{S}(\mathbf{e}_1) - \mathcal{S}(\mathbf{e}_2) + 2\mathcal{S}(\mathbf{e}_3) = \mathbf{0} \in \mathbb{R}^2 \quad \text{which means that} \quad \mathcal{S}(\lambda(\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3)) = \mathbf{0} \in \mathbb{R}^2\,,$$

for all $\lambda \in \mathbb{R}$. Thus the whole of the subspace of $\mathbb{R}^3$ spanned by $\{\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3\}$, i.e. the 1-dimensional line specified by $\lambda(\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3)$, is mapped onto $\mathbf{0} \in \mathbb{R}^2$.

(iv) As an example of a map where the domain has a lower dimension than the range, let $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^4$ where

$$(x, y) \mapsto \mathcal{T}(x, y) = (x + y, x, y - 3x, y)\,. \tag{3.10d}$$

$\mathcal{T}$ is a linear map since

$$\begin{aligned}
\mathcal{T}(\lambda\mathbf{x}_1 + \mu\mathbf{x}_2) &= ((\lambda x_1 + \mu x_2) + (\lambda y_1 + \mu y_2), \lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2 - 3(\lambda x_1 + \mu x_2), \lambda y_1 + \mu y_2) \\
&= \lambda(x_1 + y_1, x_1, y_1 - 3x_1, y_1) + \mu(x_2 + y_2, x_2, y_2 - 3x_2, y_2) \\
&= \lambda\mathcal{T}(\mathbf{x}_1) + \mu\mathcal{T}(\mathbf{x}_2)\,.
\end{aligned}$$

Remarks.

(a) In this case we observe that the orthonormal basis vectors of $\mathbb{R}^2$ are mapped to the vectors

$$\left.\begin{array}{lcll}
\mathcal{T}(\mathbf{e}_1) &=& \mathcal{T}((1,0)) &= (1,1,-3,0) \\
\mathcal{T}(\mathbf{e}_2) &=& \mathcal{T}((0,1)) &= (1,0,1,1)
\end{array}\right\} \quad \text{which are linearly independent,}$$

and which form a basis for $\mathcal{T}(\mathbb{R}^2)$. Thus the subspace $\mathcal{T}(\mathbb{R}^2) = \operatorname{span}\{(1,1,-3,0), (1,0,1,1)\}$ of $\mathbb{R}^4$ is two-dimensional.

(b) The only solution to $\mathcal{T}(\mathbf{x}) = \mathbf{0}$ is $\mathbf{x} = 0$.

3.3 Rank, Kernel and Nullity

Let $\mathcal{T} : V \to W$ be a linear map (say with $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$). Recall that $\mathcal{T}(V)$ is the image of V under $\mathcal{T}$, and that $\mathcal{T}(V)$ is a subspace of W.

Definition. The *rank* of $\mathcal{T}$ is the dimension of the image, i.e.

$$r(\mathcal{T}) = \dim \mathcal{T}(V)\,. \tag{3.11}$$

Examples. For $\mathcal{S}$ defined in (3.10c) and for $\mathcal{T}$ defined in (3.10d) we have seen that $r(\mathcal{S}) = 2$ and $r(\mathcal{T}) = 2$.

Definition. The subset of V that maps to the zero element in W is called the *kernel*, or *null space*, of $\mathcal{T}$, i.e.

$$K(\mathcal{T}) = \{\mathbf{v} \in V : \mathcal{T}(\mathbf{v}) = \mathbf{0} \in W\}\,. \tag{3.12}$$

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Theorem 3.1. *$K(\mathcal{T})$ is a subspace of V.*

Proof. We will use Theorem 2.2 on page 38. To do so we need to show that if $\mathbf{u}, \mathbf{v} \in K(\mathcal{T})$ and $\lambda, \mu \in \mathbb{R}$ then $\lambda\mathbf{u} + \mu\mathbf{v} \in K(\mathcal{T})$. However from (3.8)

$$\begin{aligned} \mathcal{T}(\lambda\mathbf{u} + \mu\mathbf{v}) &= \lambda\mathcal{T}(\mathbf{u}) + \mu\mathcal{T}(\mathbf{v}) \\ &= \lambda\mathbf{0} + \mu\mathbf{0} \\ &= \mathbf{0}, \end{aligned}$$

and hence $\lambda\mathbf{u} + \mu\mathbf{v} \in K(\mathcal{T})$. □

Remark. Since $\mathcal{T}(\mathbf{0}) = \mathbf{0} \in W$, $\mathbf{0} \in K(\mathcal{T}) \subseteq V$, so $K(\mathcal{T})$ contains at least $\mathbf{0}$.

Definition. The *nullity of* $\mathcal{T}$ is defined to be the dimension of the kernel, i.e.

$$n(\mathcal{T}) = \dim K(\mathcal{T}). \tag{3.13}$$

Examples. For $\mathcal{S}$ defined in (3.10c) and for $\mathcal{T}$ defined in (3.10d) we have seen that $n(\mathcal{S}) = 1$ and $n(\mathcal{T}) = 0$.

Theorem 3.2 (The Rank-Nullity Theorem)**.** *Let $\mathcal{T} : V \to W$ be a linear map, then*

$$r(\mathcal{T}) + n(\mathcal{T}) = \dim V = \text{dimension of domain}. \tag{3.14}$$

Proof. See *Linear Algebra*. □

Examples. For $\mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^2$ defined in (3.10c), and for $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^4$ defined in (3.10d), we have that

$$r(\mathcal{S}) + n(\mathcal{S}) = 2 + 1 = \dim \mathbb{R}^3 = \text{dimension of domain},$$
$$r(\mathcal{T}) + n(\mathcal{T}) = 2 + 0 = \dim \mathbb{R}^2 = \text{dimension of domain}.$$

3.3.1 Unlectured further examples

(i) Consider projection onto a line, $\mathcal{P}_{\mathbf{n}} : \mathbb{R}^3 \to \mathbb{R}^3$, where as in (2.23b) and (3.10b)

$$\mathbf{x} \mapsto \mathbf{x}' = \mathcal{P}_{\mathbf{n}}(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{n})\mathbf{n},$$

and $\mathbf{n}$ is a fixed unit vector. Then

$$\mathcal{P}_{\mathbf{n}}(\mathbb{R}^3) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda\mathbf{n},\ \lambda \in \mathbb{R}\},$$

which is a line in $\mathbb{R}^3$; thus $r(\mathcal{P}_{\mathbf{n}}) = 1$. Further, the kernel is given by

$$K(\mathcal{P}_{\mathbf{n}}) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{n} = 0\},$$

which is a plane in $\mathbb{R}^3$; thus $n(\mathcal{P}_{\mathbf{n}}) = 2$. We conclude that in accordance with (3.14)

$$r(\mathcal{P}_{\mathbf{n}}) + n(\mathcal{P}_{\mathbf{n}}) = 3 = \dim \mathbb{R}^3 = \text{dimension of domain}.$$

(ii) Consider the map $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$(x, y) \mapsto \mathcal{T}(x, y) = (2x + 3y, 4x + 6y, -2x - 3y) = (2x + 3y)(1, 2, -1).$$

$\mathcal{T}(\mathbb{R}^2)$ is the line $\mathbf{x} = \lambda(1, 2, -1) \in \mathbb{R}^3$, and so the rank of the map is given by

$$r(\mathcal{T}) = \dim \mathcal{T}(\mathbb{R}^2) = 1.$$

Further, $\mathbf{x} = (x, y) \in K(\mathcal{T})$ if $2x + 3y = 0$, so

$$K(\mathcal{T}) = \{\mathbf{x} = (-3s, 2s) : s \in \mathbb{R}\},$$

which is a line in $\mathbb{R}^2$. Thus

$$n(\mathcal{T}) = \dim K(\mathcal{T}) = 1.$$

We conclude that in accordance with (3.14)

$$r(\mathcal{T}) + n(\mathcal{T}) = 2 = \dim \mathbb{R}^2 = \text{dimension of domain}.$$

3.4 Composition of Maps

Suppose that $\mathcal{S} : U \to V$ and $\mathcal{T} : V \to W$ are linear maps (say with $U = \mathbb{R}^\ell$, $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$) such that

$$\mathbf{u} \mapsto \mathbf{v} = \mathcal{S}(\mathbf{u})\,, \quad \mathbf{v} \mapsto \mathbf{w} = \mathcal{T}(\mathbf{v})\,. \tag{3.15}$$

Definition. The *composite* or *product* map $\mathcal{TS}$ is the map $\mathcal{TS} : U \to W$ such that

$$\mathbf{u} \mapsto \mathbf{w} = \mathcal{T}(\mathcal{S}(\mathbf{u}))\,,$$

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where we note that $\mathcal{S}$ acts first, then $\mathcal{T}$.

Remark. For the map to be well-defined the domain of $\mathcal{T}$ must include the image of $\mathcal{S}$ (as assumed above).

3.4.1 Examples

(i) Let $\mathcal{P}_\mathbf{n}$ be a projection onto a line (see (3.10b)):

$$\mathcal{P}_\mathbf{n} : \mathbb{R}^3 \to \mathbb{R}^3\,.$$

Since the image $\subseteq$ domain we may apply the map twice, and show (by geometry or algebra) that

$$\mathcal{P}_\mathbf{n}\,\mathcal{P}_\mathbf{n} = \mathcal{P}_\mathbf{n}^2 = \mathcal{P}_\mathbf{n}\,. \tag{3.16}$$

(ii) For the maps $\mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^2$ and $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}^4$ as defined in (3.10c) and (3.10d), the composite map $\mathcal{TS} : \mathbb{R}^3 \to \mathbb{R}^4$ is such that

$$\begin{aligned}\mathcal{TS}(x,y,z) &= \mathcal{T}(\mathcal{S}(x,y,z))\\ &= \mathcal{T}(x+y, 2x-z)\\ &= (3x+y-z, x+y, -x-3y-z, 2x-z)\,.\end{aligned}$$

16/02 9/09 *Remark.* $\mathcal{ST}$ not well defined because the range of $\mathcal{T}$ is not the domain of $\mathcal{S}$.[24]

3.5 Bases and the Matrix Description of Maps

Let $\{\mathbf{e}_j\}$ $(j = 1, \dots, n)$ be a basis for $\mathbb{R}^n$ (not necessarily orthonormal or even orthogonal, although it may help to think of it as an orthonormal basis). Any $\mathbf{x} \in \mathbb{R}^n$ has a unique expansion in terms of this basis, namely

$$\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j\,,$$

where the x_j are the components of $\mathbf{x}$ with respect to the given basis.

Consider a linear map $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$, where $m, n \in \mathbb{Z}^+$, i.e. a map $\mathbf{x} \mapsto \mathbf{x}' = \mathcal{A}(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x}' \in \mathbb{R}^m$. From the definition of a linear map, (3.8), it follows that

$$\mathcal{A}(\mathbf{x}) = \mathcal{A}\left(\sum_{j=1}^{n} x_j \mathbf{e}_j\right) = \sum_{j=1}^{n} x_j \mathcal{A}(\mathbf{e}_j)\,, \tag{3.17a}$$

where $\mathcal{A}(\mathbf{e}_j)$ is the image of $\mathbf{e}_j$. In keeping with the above notation

$$\mathbf{e}'_j = \mathcal{A}(\mathbf{e}_j) \quad \text{where} \quad \mathbf{e}'_j \in \mathbb{R}^m. \tag{3.17b}$$

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[24] Or, to be pedantic, a subspace of the domain of $\mathcal{S}$.

Let $\{\mathbf{f}_i\}$ $(i = 1, \dots, m)$ be a basis of $\mathbb{R}^m$, then since any vector in $\mathbb{R}^m$ can be expressed in terms of the $\{\mathbf{f}_i\}$, there exist $A_{ij} \in \mathbb{R}$ $(i = 1, \dots, m,\ j = 1, \dots, n)$ such that

$$\mathbf{e}'_j = \mathcal{A}(\mathbf{e}_j) = \sum_{i=1}^{m} A_{ij}\, \mathbf{f}_i\,. \tag{3.18a}$$

A_{ij} is the i^{th} component of $\mathbf{e}'_j = \mathcal{A}(\mathbf{e}_j)$ with respect to the basis $\{\mathbf{f}_i\}$ $(i = 1, \dots, m)$, i.e.

$$A_{ij} = (\mathbf{e}'_j)_i = (\mathcal{A}(\mathbf{e}_j))_i\,. \tag{3.18b}$$

Hence, from (3.17a) and (3.18a), it follows that for general $\mathbf{x} \in \mathbb{R}^n$

$$\begin{aligned} \mathbf{x}' \;=\; \mathcal{A}(\mathbf{x}) \;&=\; \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} A_{ij}\, \mathbf{f}_i \right) \\ &=\; \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij}\, x_j \right) \mathbf{f}_i\,. \end{aligned} \tag{3.19a}$$

Thus, in component form

$$\mathbf{x}' = \sum_{i=1}^{m} x'_i\, \mathbf{f}_i \quad \text{where} \quad x'_i \;=\; (\mathcal{A}(\mathbf{x}))_i = \sum_{j=1}^{n} A_{ij}\, x_j\,. \tag{3.19b}$$

Alternatively, in long hand

$$\begin{array}{ccccccccc} x'_1 & = & A_{11}x_1 & + & A_{12}x_2 & + & \dots & + & A_{1n}x_n\,, \\ x'_2 & = & A_{21}x_1 & + & A_{22}x_2 & + & \dots & + & A_{2n}x_n\,, \\ \vdots & = & \vdots & + & \vdots & + & \ddots & + & \vdots\quad, \\ x'_m & = & A_{m1}x_1 & + & A_{m2}x_2 & + & \dots & + & A_{mn}x_n\,, \end{array} \tag{3.19c}$$

or in terms of the suffix notation and summation convention introduced earlier

$$x'_i = A_{ij}x_j\,. \tag{3.19d}$$

Since $\mathbf{x}$ was an arbitrary vector, what this means is that once we know the A_{ij} we can calculate the results of the mapping $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ for all elements of $\mathbb{R}^n$. In other words, the mapping $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ is, once bases in $\mathbb{R}^n$ and $\mathbb{R}^m$ have been chosen, completely specified by the $m \times n$ quantities A_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$.

Remark. Comparing the definition of A_{ij} from (3.18a) with the relationship (3.19d) between the components of $\mathbf{x}' = \mathcal{A}(\mathbf{x})$ and $\mathbf{x}$, we note that in some sense the relations 'go opposite ways'.

$$\mathcal{A}(\mathbf{e}_j) = \sum_{i=1}^{m} \mathbf{f}_i\, A_{ij} \quad (j = 1, \dots, n)\,,$$

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$$x'_i = (\mathcal{A}(\mathbf{x}))_i = \sum_{j=1}^{n} A_{ij}x_j \quad (i = 1, \dots, n)\,.$$

3.5.1 Matrix notation

As in (3.6a) and (3.6c) of §3.1.1, the above equations can be written in a more convenient form by using matrix notation. Let x and x' now be the *column matrices*, or *column vectors*,

$$\mathsf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathsf{x}' = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_m \end{pmatrix} \tag{3.20a}$$

respectively, and let A be the $m \times n$ *rectangular matrix*

$$\mathsf{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix}. \tag{3.20b}$$

Remarks.

(i) As before the first suffix i is the *row* number, the second suffix j is the *column* number, and we call the A_{ij} the *elements* of the matrix A.

(ii) In the case of maps from $\mathbb{C}^n$ to $\mathbb{C}^m$, the A_{ij} are in general complex numbers (i.e. A is in general a complex matrix).

Using the same rules of multiplication as before, equation (3.19b), or equivalently (3.19c), or equivalently (3.19d), can now be expressed in matrix notation as

$$\underbrace{\begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_m \end{pmatrix}}_{\substack{m\times 1 \text{ matrix} \\ \text{(column vector} \\ \text{with } m \text{ rows)}}} = \underbrace{\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix}}_{\substack{m\times n \text{ matrix} \\ (m \text{ rows, } n \text{ columns})}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\substack{n\times 1 \text{ matrix} \\ \text{(column vector} \\ \text{with } n \text{ rows)}}} \tag{3.21a}$$

i.e.

$$\mathsf{x}' = \mathsf{A}\mathsf{x}\,, \quad \text{or equivalently} \quad \mathbf{x}' = \mathsf{A}\mathbf{x}\,, \quad \text{where} \quad \mathsf{A} = \{A_{ij}\}\,. \tag{3.21b}$$

Remarks.

(i) Since $A_{ij} = (\mathbf{e}'_j)_i$ from (3.18b), it follows that

$$\mathsf{A} = \begin{pmatrix} (\mathbf{e}'_1)_1 & (\mathbf{e}'_2)_1 & \dots & (\mathbf{e}'_n)_1 \\ (\mathbf{e}'_1)_2 & (\mathbf{e}'_2)_2 & \dots & (\mathbf{e}'_n)_2 \\ \vdots & \vdots & \ddots & \vdots \\ (\mathbf{e}'_1)_m & (\mathbf{e}'_2)_m & \dots & (\mathbf{e}'_n)_m \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \dots & \mathbf{e}'_n \end{pmatrix}, \tag{3.22}$$

where the $\mathbf{e}'_i$ on the RHS are to be interpreted as column vectors.

(ii) The elements of A depend on the choice of bases. Hence when specifying a matrix A associated with a map $\mathcal{A}$, it is necessary to give the bases with respect to which it has been constructed.

(iii) For $i = 1, \dots, m$ let $\mathbf{r}_i$ be the vector with components equal to the elements of the ith row of A, i.e.

$$\mathbf{r}_i = (A_{i1}, A_{i2}, \dots, A_{in})\,, \quad \text{for } i = 1, 2, \dots, m.$$

Then for *real* linear maps we see that in terms of the scalar product for vectors[25]

$$i^{\text{th}} \text{ row of } \mathsf{x}' = x'_i = \mathbf{r}_i \cdot \mathbf{x}\,, \quad \text{for } i = 1, 2, \dots, m.$$

[25] For complex linear maps this is not the case since the scalar product (2.65) involves a complex conjugate.

3.5.2 Examples (including some important definitions of maps)

We consider maps $\mathbb{R}^3 \mapsto \mathbb{R}^3$; then since the domain and range are the same we choose $\mathbf{f}_j = \mathbf{e}_j$ $(j = 1, 2, 3)$. Further we take the $\{\mathbf{e}_j\}$, $j = 1, 2, 3$, to be an *orthonormal* basis.

(i) *Rotation.* Consider rotation by an angle θ about the x_3 axis. Under such a rotation

$$\begin{aligned}
\mathbf{e}_1 \mapsto \mathbf{e}_1' &= \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta\,,\\
\mathbf{e}_2 \mapsto \mathbf{e}_2' &= -\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta\,,\\
\mathbf{e}_3 \mapsto \mathbf{e}_3' &= \mathbf{e}_3\,.
\end{aligned}$$

Thus from (3.22) the rotation matrix, $\mathsf{R}(\theta)$, is given by (cf.(3.6c))

$$\mathsf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.23}$$

(ii) *Reflection.* Consider reflection in the plane $\Pi = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x}\cdot\mathbf{n} = 0 \text{ and } |\mathbf{n}| = 1\}$; here $\mathbf{n}$ is a constant unit vector.

For a point P, let N be the foot of the perpendicular from P to the plane. Suppose also that

$$\vec{OP} = \mathbf{x} \;\mapsto\; \mathcal{H}_\Pi(\mathbf{x}) = \mathbf{x}' = \vec{OP'}\,.$$

Then

$$\vec{NP'} = \vec{PN} = -\vec{NP}\,,$$

and so

$$\vec{OP'} = \vec{OP} + \vec{PN} + \vec{NP'} = \vec{OP} - 2\,\vec{NP}\,.$$

But $|NP| = |\mathbf{x}\cdot\mathbf{n}|$ and

$$\vec{NP} = \begin{cases} |NP|\mathbf{n} & \text{if } \vec{NP} \text{ has the same sense as } \mathbf{n}, \text{ i.e. } \mathbf{x}\cdot\mathbf{n} > 0 \\ -|NP|\mathbf{n} & \text{if } \vec{NP} \text{ has the opposite sense as } \mathbf{n}, \text{ i.e. } \mathbf{x}\cdot\mathbf{n} < 0\,. \end{cases}$$

Hence $\vec{NP} = (\mathbf{x}\cdot\mathbf{n})\mathbf{n}$, and

$$\vec{OP'} = \mathbf{x}' = \mathcal{H}_\Pi(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}\cdot\mathbf{n})\mathbf{n}\,. \tag{3.24}$$

We wish to construct the matrix H that represents $\mathcal{H}_\Pi$ (with respect to an orthonormal basis). To this end consider the action of $\mathcal{H}_\Pi$ on each member of an orthonormal basis. Recalling that for an orthonormal basis $\mathbf{e}_j\cdot\mathbf{n} = n_j$, it follows that

$$\mathcal{H}_\Pi(\mathbf{e}_1) = \mathbf{e}_1' = \mathbf{e}_1 - 2n_1\mathbf{n} = \begin{pmatrix}1\\0\\0\end{pmatrix} - 2n_1\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix} = \begin{pmatrix}1-2n_1^2\\-2n_1n_2\\-2n_1n_3\end{pmatrix}.$$

This is the first column of H. Similarly we obtain

$$\mathsf{H} = \begin{pmatrix} 1-2n_1^2 & -2n_1n_2 & -2n_1n_3 \\ -2n_1n_2 & 1-2n_2^2 & -2n_2n_3 \\ -2n_1n_3 & -2n_2n_3 & 1-2n_3^2 \end{pmatrix}. \tag{3.25a}$$

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An easier derivation? Alternatively the same result can be obtained using suffix notation since from (3.24)

$$\begin{aligned}
x_i' &= x_i - 2x_jn_jn_i \\
&= \delta_{ij}x_j - 2x_jn_jn_i \\
&= (\delta_{ij} - 2n_in_j)x_j \\
&\equiv H_{ij}x_j\,.
\end{aligned}$$

Hence

$$(\mathsf{H})_{ij} = H_{ij} = \delta_{ij} - 2n_i n_j\,, \quad i,j = 1,2,3. \tag{3.25b}$$ 10/08

Unlectured worked exercise. Show that the reflection mapping is *isometric*, i.e. show that distances are preserved by the mapping.

Answer. Suppose for $j = 1, 2$ that

$$\mathbf{x}_j \mapsto \mathbf{x}'_j = \mathbf{x}_j - 2(\mathbf{x}_j \cdot \mathbf{n})\mathbf{n}\,.$$

Let $\mathbf{x}_{12} = \mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{x}'_{12} = \mathbf{x}'_1 - \mathbf{x}'_2$, then

$$\begin{aligned} |\mathbf{x}'_{12}|^2 &= |\mathbf{x}_1 - 2(\mathbf{x}_1 \cdot \mathbf{n})\mathbf{n} - \mathbf{x}_2 + 2(\mathbf{x}_2 \cdot \mathbf{n})\mathbf{n}|^2 \\ &= |\mathbf{x}_{12} - 2(\mathbf{x}_{12} \cdot \mathbf{n})\mathbf{n}|^2 \\ &= \mathbf{x}_{12}\mathbf{x}_{12} - 4(\mathbf{x}_{12} \cdot \mathbf{n})(\mathbf{x}_{12} \cdot \mathbf{n}) + 4(\mathbf{x}_{12} \cdot \mathbf{n})^2\mathbf{n}^2 \\ &= |\mathbf{x}_{12}|^2\,, \end{aligned}$$

since $\mathbf{n}^2 = 1$, and as required for isometry.

(iii) *Unlectured.* Consider the map $\mathcal{Q}_\mathbf{b} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{x} \mapsto \mathbf{x}' = \mathcal{Q}_\mathbf{b}(\mathbf{x}) = \mathbf{b} \times \mathbf{x}\,. \tag{3.25z}$$

In order to construct the map's matrix Q with respect to an orthonormal basis, first note that

$$\mathcal{Q}_\mathbf{b}(\mathbf{e}_1) = (b_1, b_2, b_3) \times (1,0,0) = (0, b_3, -b_2)\,.$$

Now use formula (3.22) and similar expressions for $\mathcal{Q}_\mathbf{b}(\mathbf{e}_2)$ and $\mathcal{Q}_\mathbf{b}(\mathbf{e}_3)$ to deduce that

$$\mathsf{Q}_\mathbf{b} = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}. \tag{3.27a}$$

The elements $\{Q_{ij}\}$ of $\mathsf{Q}_\mathbf{b}$ could also be derived as follows. From (3.19d) and (3.25z)

$$Q_{ij}x_j = x'_i = \varepsilon_{ijk} b_j x_k = (\varepsilon_{ikj} b_k) x_j\,.$$

Hence, in agreement with (3.27a),

$$Q_{ij} = \varepsilon_{ikj} b_k = -\varepsilon_{ijk} b_k\,. \tag{3.27b}$$

(iv) *Dilatation.* Consider the mapping $\mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{x} \mapsto \mathbf{x}'$ where

$$x'_1 = \lambda x_1\,, \quad x'_2 = \mu x_2\,, \quad x'_3 = \nu x_3 \quad \text{where } \lambda, \mu, \nu \in \mathbb{R} \text{ and } \lambda, \mu, \nu > 0.$$

Then

$$\mathbf{e}'_1 = \lambda \mathbf{e}_1\,, \quad \mathbf{e}'_2 = \mu \mathbf{e}_2\,, \quad \mathbf{e}'_3 = \nu \mathbf{e}_3\,,$$

and so the map's matrix with respect to an orthonormal basis, say D, is given by

$$\mathsf{D} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \tag{3.28}$$

where D is a *dilatation matrix*. The effect on the unit cube $0 \leqslant x_1 \leqslant 1$, $0 \leqslant x_2 \leqslant 1$, $0 \leqslant x_3 \leqslant 1$, of this map is to send it to $0 \leqslant x'_1 \leqslant \lambda$, $0 \leqslant x'_2 \leqslant \mu$, $0 \leqslant x'_3 \leqslant \nu$, i.e. to a cuboid that has been stretched or contracted by different factors along the different Cartesian axes. If $\lambda = \mu = \nu$ then the transformation is called a *pure dilatation*. 17/03

(v) *Shear.* A simple shear is a transformation in the plane, e.g. the x_1x_2-plane, that displaces points in one direction, e.g. the x_1 direction, by an amount proportional to the distance in that plane from, say, the x_1-axis. Under this transformation

$$\mathbf{e}_1 \mapsto \mathbf{e}_1' = \mathbf{e}_1\,, \quad \mathbf{e}_2 \mapsto \mathbf{e}_2' = \mathbf{e}_2 + \lambda\mathbf{e}_1\,, \quad \mathbf{e}_3 \mapsto \mathbf{e}_3' = \mathbf{e}_3\,, \quad \text{where } \lambda \in \mathbb{R}. \tag{3.29}$$

For this example the map's *shear matrix* (with respect to an orthonormal basis), say S_λ, is given by

$$\mathsf{S}_\lambda = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.30}$$

Check.

$$\mathsf{S}_\lambda \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} a + \lambda b \\ b \\ 0 \end{pmatrix}.$$

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3.6 Algebra of Matrices

3.6.1 Addition

Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathcal{B} : \mathbb{R}^n \to \mathbb{R}^m$ be linear maps. Define the linear map $(\mathcal{A} + \mathcal{B}) : \mathbb{R}^n \to \mathbb{R}^m$ by

$$(\mathcal{A} + \mathcal{B})(\mathbf{x}) = \mathcal{A}(\mathbf{x}) + \mathcal{B}(\mathbf{x})\,. \tag{3.31a}$$

Suppose that $\mathsf{A} = \{A_{ij}\}$, $\mathsf{B} = \{B_{ij}\}$ and $(\mathsf{A} + \mathsf{B}) = \{(A+B)_{ij}\}$ are the $m \times n$ matrices associated with the maps, then (using the summation convention)

$$\begin{aligned} (\mathsf{A} + \mathsf{B})_{ij} x_j &= ((\mathcal{A} + \mathcal{B})(\mathbf{x}))_i \\ &= \mathcal{A}(\mathbf{x})_i + (\mathcal{B})(\mathbf{x})_i \\ &= (\mathsf{A}_{ij} + \mathsf{B}_{ij}) x_j\,. \end{aligned}$$

Hence, for consistency, matrix addition must be defined by

$$\mathsf{A} + \mathsf{B} = \{A_{ij} + B_{ij}\}\,. \tag{3.31b}$$

3.6.2 Multiplication by a scalar

Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then for given $\lambda \in \mathbb{R}$ define the linear map $(\lambda\mathcal{A})$ such that

$$(\lambda\mathcal{A})(\mathbf{x}) = \lambda(\mathcal{A}(\mathbf{x}))\,. \tag{3.32a}$$

Let $\mathsf{A} = \{A_{ij}\}$ be the matrix of $\mathcal{A}$, then from (3.19a)

$$\begin{aligned} ((\lambda\mathcal{A})(\mathbf{x}))_i &= (\lambda\mathcal{A}(\mathbf{x}))_i \\ &= \lambda(A_{ij} x_j) \\ &= (\lambda A_{ij}) x_j\,. \end{aligned}$$

Hence, for consistency the matrix of $\lambda\mathcal{A}$ must be

$$\lambda\mathsf{A} = \{\lambda A_{ij}\}\,, \tag{3.32b}$$

which we use as the definition of a matrix multiplied by a scalar.

3.6.3 Matrix multiplication

Let $\mathcal{S} : \mathbb{R}^\ell \to \mathbb{R}^n$ and $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ be linear maps. For given bases for $\mathbb{R}^\ell$, $\mathbb{R}^m$ and $\mathbb{R}^n$, let

$$\begin{aligned} \mathsf{S} = \{S_{ij}\} &\quad \text{be the } n \times \ell \text{ matrix of } \mathcal{S}, \\ \mathsf{T} = \{T_{ij}\} &\quad \text{be the } m \times n \text{ matrix of } \mathcal{T}. \end{aligned}$$

Now consider the composite map $\mathcal{W} = \mathcal{T}\mathcal{S} : \mathbb{R}^\ell \to \mathbb{R}^m$, with associated $m \times \ell$ matrix $\mathsf{W} = \{W_{ij}\}$. If

$$\mathbf{x}' = \mathcal{S}(\mathbf{x}) \quad \text{and} \quad \mathbf{x}'' = \mathcal{T}(\mathbf{x}')\,, \tag{3.33a}$$

then from (3.19d),

$$x'_j = S_{jk}x_k \quad \text{and} \quad x''_i = T_{ij}x'_j \quad \text{(s.c.)}, \tag{3.33b}$$

and thus

$$x''_i = T_{ij}(S_{jk}x_k) = (T_{ij}S_{jk})x_k \quad \text{(s.c.)}. \tag{3.34a}$$

However,

$$\mathbf{x}'' = \mathcal{T}\mathcal{S}(\mathbf{x}) = \mathcal{W}(\mathbf{x}) \quad \text{and} \quad x''_i = W_{ik}x_k \quad \text{(s.c.)}\,. \tag{3.34b}$$

Hence because (3.34a) and (3.34b) must identical for arbitrary $\mathbf{x}$, it follows that

$$W_{ik} = T_{ij}S_{jk} \quad \text{(s.c.)}\,. \tag{3.35}$$

We interpret (3.35) as defining the elements of the matrix product TS. In words, for *real* matrices

the ik^{th} element of TS equals the scalar product of the i^{th} row of T with the k^{th} column of S.

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Remarks.

(i) The above definition of matrix multiplication is consistent with the special case when S is a column matrix (or column vector), i.e. the $n = 1$ special case considered in (3.21b).

(ii) For matrix multiplication to be well defined, the number of columns of T must equal the number of rows of S; this is the case above since T is a $m \times n$ matrix, while S is a $n \times \ell$ matrix.

(iii) If A is a $p \times q$ matrix, and B is a $r \times s$ matrix, then

$$\begin{aligned} &\mathsf{AB} \text{ exists only if } q = r, \text{ and is then a } p \times s \text{ matrix;} \\ &\mathsf{BA} \text{ exists only if } s = p, \text{ and is then a } r \times q \text{ matrix.} \end{aligned}$$

For instance

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} g & h & i \\ j & k & \ell \end{pmatrix} = \begin{pmatrix} ag+bj & ah+bk & ai+b\ell \\ cg+dj & ch+dk & ci+d\ell \\ eg+fj & eh+fk & ei+f\ell \end{pmatrix},$$

while

$$\begin{pmatrix} g & h & i \\ j & k & \ell \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} ga+hc+ie & gb+hd+if \\ ja+kc+\ell e & jb+kd+\ell f \end{pmatrix}.$$

(iv) Even if $p = q = r = s$, so that both AB and BA exist and have the same number of rows and columns,

$$\mathsf{AB} \neq \mathsf{BA} \quad \text{in general,} \tag{3.36}$$

i.e. matrix multiplication is not commutative. For instance

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad = \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

while

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Property. The multiplication of matrices is associative, i.e. if $\mathsf{A} = \{A_{ij}\}$, $\mathsf{B} = \{B_{ij}\}$ and $\mathsf{C} = \{C_{ij}\}$ are matrices such that AB and BC exist, then

$$\mathsf{A}(\mathsf{BC}) = (\mathsf{AB})\mathsf{C} . \tag{3.37}$$

Proof. In terms of suffix notation (and the summation convention)

$$\begin{aligned} (\mathsf{A}(\mathsf{BC}))_{ij} &= A_{ik}(\mathsf{BC})_{kj} = A_{ik}B_{k\ell}C_{\ell j} = A_{i\pounds}B_{\pounds\yen}C_{\yen j} , \\ ((\mathsf{AB})\mathsf{C})_{ij} &= (\mathsf{AB})_{ik}C_{kj} = A_{i\ell}B_{\ell k}C_{kj} = A_{i\pounds}B_{\pounds\yen}C_{\yen j} . \end{aligned}$$ □

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3.6.4 Transpose

Definition. If $\mathsf{A} = \{A_{ij}\}$ is a $m \times n$ matrix, then its *transpose* A^{T} is defined to be a $n \times m$ matrix with elements

$$(\mathsf{A}^{\mathrm{T}})_{ij} = (\mathsf{A})_{ji} = A_{ji} . \tag{3.38a}$$

Property.

$$(\mathsf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathsf{A} . \tag{3.38b}$$

Examples.

(i)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} .$$

(ii)

If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a column vector, $\mathbf{x}^{\mathrm{T}} = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}$ is a row vector.

Remark. Recall that commas are sometimes important:

$$\begin{aligned} \mathbf{x} \quad &= \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad = \quad (x_1, x_2, \ldots, x_n) , \\ \mathbf{x}^{\mathrm{T}} \quad &= \quad \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix} . \end{aligned}$$

Property. If $\mathsf{A} = \{A_{ij}\}$ and $\mathsf{B} = \{B_{ij}\}$ are matrices such that AB exists, then

$$(\mathsf{AB})^{\mathrm{T}} = \mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} . \tag{3.39}$$

Proof.

$$
\begin{aligned}
((\mathsf{AB})^{\mathrm{T}})_{ij} &= (\mathsf{AB})_{ji} \\
&= A_{jk} B_{ki} \\
&= (\mathsf{B})_{ki} (\mathsf{A})_{jk} \\
&= (\mathsf{B}^{\mathrm{T}})_{ik} (\mathsf{A}^{\mathrm{T}})_{kj} \\
&= (\mathsf{B}^{\mathrm{T}} \mathsf{A}^{\mathrm{T}})_{ij} \,.
\end{aligned}
$$

□

Example. Let $\mathbf{x}$ and $\mathbf{y}$ be 3×1 column vectors, and let $\mathsf{A} = \{A_{ij}\}$ be a 3×3 matrix. Then

$$\mathbf{x}^{\mathrm{T}} \mathsf{A} \mathbf{y} = x_i A_{ij} y_j = x_{\pounds} A_{\pounds \yen} y_{\yen} \,.$$

is a 1×1 matrix, i.e. a scalar. Further we can confirm that a scalar is its own transpose:

$$(\mathbf{x}^{\mathrm{T}} \mathsf{A} \mathbf{y})^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}} \mathsf{A}^{\mathrm{T}} \mathbf{x} = y_i A_{ji} x_j = x_j A_{ji} y_i = x_{\pounds} A_{\pounds \yen} y_{\yen} = \mathbf{x}^{\mathrm{T}} \mathsf{A} \mathbf{y} \,.$$

Definition. The *Hermitian conjugate* or *conjugate transpose* or *adjoint* of a matrix $\mathsf{A} = \{A_{ij}\}$, where $A_{ij} \in \mathbb{C}$, is defined to be

$$\mathsf{A}^{\dagger} = (\mathsf{A}^{\mathrm{T}})^* = (\mathsf{A}^*)^{\mathrm{T}} \,. \tag{3.40}$$

Property. Similarly to transposes

$$A^{\dagger\dagger} = A \,. \tag{3.41a}$$

Property. If $\mathsf{A} = \{A_{ij}\}$ and $\mathsf{B} = \{B_{ij}\}$ are matrices such that AB exists, then

$$(\mathsf{AB})^{\dagger} = \mathsf{B}^{\dagger} \mathsf{A}^{\dagger} \,. \tag{3.41b}$$

Proof. Add a few complex conjugates to the proof above.

3.6.5 Symmetric and Hermitian Matrices

Definition. A square $n \times n$ matrix $\mathsf{A} = \{A_{ij}\}$ is *symmetric* if

$$\mathsf{A} = \mathsf{A}^{\mathrm{T}} \,, \quad \text{i.e.} \quad A_{ij} = A_{ji} \,. \tag{3.42a}$$

Definition. A square $n \times n$ [complex] matrix $\mathsf{A} = \{A_{ij}\}$ is *Hermitian* if

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$$\mathsf{A} = \mathsf{A}^{\dagger} \,, \quad \text{i.e.} \quad A_{ij} = A^*_{ji} \,. \tag{3.42b}$$

Definition. A square $n \times n$ matrix $\mathsf{A} = \{A_{ij}\}$ is *antisymmetric* if

$$\mathsf{A} = -\mathsf{A}^{\mathrm{T}} \,, \quad \text{i.e.} \quad A_{ij} = -A_{ji} \,. \tag{3.43a}$$

Remark. For an antisymmetric matrix, $A_{11} = -A_{11}$, i.e. $A_{11} = 0$. Similarly we deduce that all the *diagonal* elements of antisymmetric matrices are zero, i.e.

$$A_{11} = A_{22} = \ldots = A_{nn} = 0 \,. \tag{3.43b}$$

Definition. A square $n \times n$ [complex] matrix $\mathsf{A} = \{A_{ij}\}$ is *skew-Hermitian* if

$$\mathsf{A} = -\mathsf{A}^{\dagger} \,, \quad \text{i.e.} \quad A_{ij} = -A^*_{ji} \,. \tag{3.43c}$$

Exercise. Show that all the *diagonal* elements of Hermitian and skew-Hermitian matrices are real and pure imaginary respectively.

Examples.

(i) A symmetric 3×3 matrix S has the form

$$\mathsf{S} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix},$$

i.e. it has six independent elements.

(ii) An antisymmetric 3×3 matrix A has the form

$$\mathsf{A} = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix},$$

i.e. it has three independent elements.

Remark. Let $a = v_3$, $b = v_2$ and $c = v_1$, then (cf. (3.27a) and (3.27b))

$$\mathsf{A} = \{A_{ij}\} = \{\varepsilon_{ijk} v_k\}. \tag{3.44}$$

Thus each antisymmetric 3×3 matrix corresponds to a unique vector $\mathbf{v}$ in $\mathbb{R}^3$.

3.6.6 Trace

Definition. The *trace* of a square $n \times n$ matrix $\mathsf{A} = \{A_{ij}\}$ is equal to the sum of the diagonal elements, i.e.

$$Tr(\mathsf{A}) = A_{ii} \quad \text{(s.c.)}. \tag{3.45}$$

Remark. Let $\mathsf{B} = \{B_{ij}\}$ be a $m \times n$ matrix and $\mathsf{C} = \{C_{ij}\}$ be a $n \times m$ matrix, then BC and CB both exist, but are not usually equal (even if $m = n$). However

$$\begin{aligned} Tr(\mathsf{BC}) &= (\mathsf{BC})_{ii} = B_{ij} C_{ji}, \\ Tr(\mathsf{CB}) &= (\mathsf{CB})_{ii} = C_{ij} B_{ji} = B_{ij} C_{ji}, \end{aligned}$$

11/08 and hence $Tr(\mathsf{BC}) = Tr(\mathsf{CB})$ (even if $m \neq n$ so that the matrices are of different sizes).

3.6.7 The unit or identity matrix

Definition. The *unit* or *identity* $n \times n$ matrix is defined to be

$$\mathsf{I} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \tag{3.46}$$

i.e. all the elements are 0 except for the diagonal elements that are 1.

Example. The 3×3 identity matrix is given by

$$\mathsf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \{\delta_{ij}\}. \tag{3.47}$$

Property. Define the Kronecker delta in $\mathbb{R}^n$ such that

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{for } i, j = 1, 2, \ldots, n. \tag{3.48}$$

Let $\mathsf{A} = \{A_{ij}\}$ be a $n \times n$ matrix, then

$$\begin{aligned} (\mathsf{IA})_{ij} &= \delta_{ik} A_{kj} = A_{ij}\,, \\ (\mathsf{AI})_{ij} &= A_{ik} \delta_{kj} = A_{ij}\,, \end{aligned}$$

i.e.

$$\mathsf{IA} = \mathsf{AI} = \mathsf{A}\,. \tag{3.49}$$

3.6.8 Decomposition of a Square Matrix into Isotropic, Symmetric Trace-Free and Antisymmetric Parts

Suppose that B be a $n \times n$ square matrix. Construct the matrices A and S from B as follows:

$$\mathsf{A} = \tfrac{1}{2}\left(\mathsf{B} - \mathsf{B}^{\mathrm{T}}\right)\,, \tag{3.50a}$$

$$\mathsf{S} = \tfrac{1}{2}\left(\mathsf{B} + \mathsf{B}^{\mathrm{T}}\right)\,. \tag{3.50b}$$

Then A and S are antisymmetric and symmetric matrices respectively, and B is the sum of A and S, since

$$\mathsf{A}^{\mathrm{T}} = \tfrac{1}{2}\left(\mathsf{B}^{\mathrm{T}} - \mathsf{B}\right) = -\mathsf{A}\,, \tag{3.51a}$$

$$\mathsf{S}^{\mathrm{T}} = \tfrac{1}{2}\left(\mathsf{B}^{\mathrm{T}} + \mathsf{B}\right) = \mathsf{S}\,, \tag{3.51b}$$

$$\mathsf{A} + \mathsf{S} = \tfrac{1}{2}\left(\mathsf{B} - \mathsf{B}^{\mathrm{T}} + \mathsf{B} + \mathsf{B}^{\mathrm{T}}\right) = \mathsf{B}\,. \tag{3.51c}$$

Let $n\,\sigma$ be the trace of S, i.e. $\sigma = \frac{1}{n}\,S_{ii}$, and write

$$\mathsf{E} = \mathsf{S} - \sigma \mathsf{I}\,. \tag{3.52}$$

Then E is a trace-free symmetric tensor, and

$$\mathsf{B} = \sigma \mathsf{I} + \mathsf{E} + \mathsf{A}\,, \tag{3.53}$$

which represents the decomposition of a square matrix into isotropic, symmetric trace-free and antisymmetric parts.

Remark. For current purposes we define an *isotropic matrix* to be a scalar multiple of the identity matrix. Strictly we are interested in *isotropic tensors*, which you will encounter in the *Vector Calculus* course.

Application. Consider *small* deformations of a solid body. If $\mathbf{x}$ is the position vector of a point in the body, suppose that it is deformed to $\mathsf{B}\mathbf{x}$. Then

(i) σ represents the average [uniform] dilatation (i.e. expansion or contraction) of the deformation,

(ii) E is a measure of the *strain* of the deformation,

(iii) $\mathsf{A}\mathbf{x}$ is that part of the displacement representing the average rotation of the deformation (A is sometimes referred to as the *spin* matrix).

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3.6.9 The inverse of a matrix

Definition. Let A be a $m \times n$ matrix. A $n \times m$ matrix B is a *left inverse* of A if $\mathsf{BA} = \mathsf{I}$. A $n \times m$ matrix C is a *right inverse* of A if $\mathsf{AC} = \mathsf{I}$.

Property. If B is a left inverse of A and C is a right inverse of A then $\mathsf{B} = \mathsf{C}$ and we write $\mathsf{B} = \mathsf{C} = \mathsf{A}^{-1}$.

Proof. From (3.37), (3.49), $\mathsf{BA} = \mathsf{I}$ and $\mathsf{AC} = \mathsf{I}$ it follows that

$$\mathsf{B} = \mathsf{BI} = \mathsf{B}(\mathsf{AC}) = (\mathsf{BA})\mathsf{C} = \mathsf{IC} = \mathsf{C}\,. \qquad \square$$

Remark. This property is based on the premise that both a left inverse and right inverse exist. In general, the existence of a left inverse does not necessarily guarantee the existence of a right inverse, or vice versa. However, in the case of a *square* matrix, the existence of a left inverse does imply the existence of a right inverse, and vice versa (see *Linear Algebra* for a general proof). The above property then implies that they are the same matrix.

Definition. Let A be a $n \times n$ matrix. A is said to be *invertible* if there exists a $n \times n$ matrix B such that

$$\mathsf{BA} = \mathsf{AB} = \mathsf{I}\,. \tag{3.54}$$

The matrix B is called the *inverse* of A, is unique (see above) and is denoted by A^{-1} (see above).

Property. From (3.54) it follows that $\mathsf{A} = \mathsf{B}^{-1}$ (in addition to $\mathsf{B} = \mathsf{A}^{-1}$). Hence

$$\mathsf{A} = (\mathsf{A}^{-1})^{-1}\,. \tag{3.55}$$

Property. Suppose that A and B are both invertible $n \times n$ matrices. Then

$$(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1}\,. \tag{3.56}$$

Proof. From using (3.37), (3.49) and (3.54) it follows that

$$\begin{aligned} \mathsf{B}^{-1}\mathsf{A}^{-1}(\mathsf{AB}) &= \mathsf{B}^{-1}(\mathsf{A}^{-1}\mathsf{A})\mathsf{B} = \mathsf{B}^{-1}\mathsf{IB} = \mathsf{B}^{-1}\mathsf{B} = \mathsf{I}\,,\\ (\mathsf{AB})\mathsf{B}^{-1}\mathsf{A}^{-1} &= \mathsf{A}(\mathsf{BB}^{-1})\mathsf{A}^{-1} = \mathsf{AIA}^{-1} = \mathsf{AA}^{-1} = \mathsf{I}\,. \qquad \square \end{aligned}$$

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3.6.10 Orthogonal and unitary matrices

Definition. An $n \times n$ real matrix $\mathsf{A} = \{A_{ij}\}$ is *orthogonal* if

$$\mathsf{AA}^{\mathrm{T}} = \mathsf{I} = \mathsf{A}^{\mathrm{T}}\mathsf{A}\,, \tag{3.57a}$$

i.e. if A is invertible and $\mathsf{A}^{-1} = \mathsf{A}^{\mathrm{T}}$.

Property: orthogonal rows and columns. In components (3.57a) becomes

$$(\mathsf{A})_{ik}(\mathsf{A}^{\mathrm{T}})_{kj} = A_{ik}A_{jk} = \delta_{ij}\,. \tag{3.57b}$$

Thus the real scalar product of the i^{th} and j^{th} rows of A is zero unless $i = j$ in which case it is 1. This implies that the rows of A form an orthonormal set. Similarly, since $\mathsf{A}^{\mathrm{T}}\mathsf{A} = \mathsf{I}$,

$$(\mathsf{A}^{\mathrm{T}})_{ik}(\mathsf{A})_{kj} = A_{ki}A_{kj} = \delta_{ij}\,, \tag{3.57c}$$

and so the columns of A also form an orthonormal set.

Property: map of an orthonormal basis. Suppose that the map $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ has a matrix A with respect to an orthonormal basis. Then from (3.22) we recall that

$$\begin{array}{lcll} \mathbf{e}_1 & \mapsto & \mathsf{A}\mathbf{e}_1 & \text{the first column of } \mathsf{A}, \\ \mathbf{e}_2 & \mapsto & \mathsf{A}\mathbf{e}_2 & \text{the second column of } \mathsf{A}, \\ \vdots & \mapsto & \vdots & \\ \mathbf{e}_n & \mapsto & \mathsf{A}\mathbf{e}_n & \text{the } n^{\text{th}} \text{ column of } \mathsf{A}. \end{array}$$

Thus if A is an orthogonal matrix the $\{\mathbf{e}_i\}$ transform to an orthonormal set (which may be right-handed or left-handed depending on the sign of $\det \mathsf{A}$, where we define $\det \mathsf{A}$ below).

Examples.

(i) With respect to an orthonormal basis, rotation by an angle θ about the x_3 axis has the matrix (see (3.23))

$$\mathsf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.58a}$$

R is orthogonal since both the rows and the columns are orthogonal vectors, and thus

$$\mathsf{R}\mathsf{R}^{\mathrm{T}} = \mathsf{R}^{\mathrm{T}}\mathsf{R} = \mathsf{I}\,. \tag{3.58b}$$

(ii) By geometry (or by algebra by invoking formula (3.24) twice), the application of a reflection map $\mathcal{H}_\Pi$ twice results in the identity map, i.e.

$$\mathcal{H}_\Pi^2 = I\,. \tag{3.59a}$$

Further, from (3.25b) the matrix of $\mathcal{H}_\Pi$ with respect to an orthonormal basis is specified by

$$(\mathsf{H})_{ij} = \{\delta_{ij} - 2n_i n_j\}\,.$$

It follows from (3.59a), or a little manipulation, that

$$\mathsf{H}^2 = \mathsf{I}\,. \tag{3.59b}$$

Moreover H is symmetric, hence

$$\mathsf{H} = \mathsf{H}^{\mathrm{T}}\,, \quad \text{and so} \quad \mathsf{H}^2 = \mathsf{H}\mathsf{H}^{\mathrm{T}} = \mathsf{H}^{\mathrm{T}}\mathsf{H} = \mathsf{I}\,. \tag{3.60}$$

19/02 Thus H is orthogonal.

Preservation of the real scalar product. Under a map represented by an orthogonal matrix with respect to an orthonormal basis, a real scalar product is preserved. For suppose that, in component form, $\mathsf{x} \mapsto \mathsf{x}' = \mathsf{A}\mathsf{x}$ and $\mathsf{y} \mapsto \mathsf{y}' = \mathsf{A}\mathsf{y}$ (note the use of $\mathsf{sans\ serif}$), then

$$\begin{aligned} \mathbf{x}' \cdot \mathbf{y}' &= \mathsf{x}'^{\mathrm{T}}\mathsf{y}' && \text{(since the basis is orthonormal)} \\ &= (\mathsf{x}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}})(\mathsf{A}\mathsf{y}) \\ &= \mathsf{x}^{\mathrm{T}}\,\mathsf{I}\,\mathsf{y} \\ &= \mathsf{x}^{\mathrm{T}}\mathsf{y} \\ &= \mathbf{x} \cdot \mathbf{y} && \text{(since the basis is orthonormal)}\,. \end{aligned}$$

Isometric maps. If a linear map is represented by an orthogonal matrix A with respect to an orthonormal basis, then the map is an isometry (i.e. distances are preserved by the mapping) since

$$\begin{aligned} |\mathbf{x}' - \mathbf{y}'|^2 &= (\mathbf{x}' - \mathbf{y}') \cdot (\mathbf{x}' - \mathbf{y}') \\ &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ &= |\mathbf{x} - \mathbf{y}|^2\,. \end{aligned}$$

Hence $|\mathbf{x}' - \mathbf{y}'| = |\mathbf{x} - \mathbf{y}|$, i.e. lengths are preserved.

Remark. The only length preserving maps of $\mathbb{R}^3$ are translations (which are not linear maps) and reflections and rotations (which we have already seen are associated with orthogonal matrices).

Definition. A complex square matrix U is said to be *unitary* if its Hermitian conjugate is equal to its inverse, i.e. if

$$\mathsf{U}^\dagger = \mathsf{U}^{-1}\,. \tag{3.61}$$

Remark. Unitary matrices are to complex matrices what orthogonal matrices are to real matrices. Similar properties to those above for orthogonal matrices also hold for unitary matrices when references to *real* scalar products are replaced by references to *complex* scalar products (as defined by (2.65)).

3.7 Determinants

3.7.1 Determinants for 3×3 matrices

Recall that the signed volume of the $\mathbb{R}^3$ parallelepiped defined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ (positive if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are right-handed, negative if left-handed).

Consider the effect of a linear map, $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$, on the volume of the unit cube defined by orthonormal basis vectors $\mathbf{e}_i$. Let $\mathsf{A} = \{A_{ij}\}$ be the matrix associated with $\mathcal{A}$, then the volume of the mapped cube is, with the aid of (2.73e) and (3.19d), given by

$$\begin{aligned}
\mathbf{e}'_1 \cdot \mathbf{e}'_2 \times \mathbf{e}'_3 &= \varepsilon_{ijk} (\mathbf{e}'_1)_i (\mathbf{e}'_2)_j (\mathbf{e}'_3)_k \\
&= \varepsilon_{ijk} A_{i\ell} (\mathbf{e}_1)_\ell \, A_{jm} (\mathbf{e}_2)_m \, A_{kn} (\mathbf{e}_3)_n \\
&= \varepsilon_{ijk} A_{i\ell} \delta_{1\ell} \, A_{jm} \delta_{2m} \, A_{kn} \delta_{3n} \\
&= \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}\,.
\end{aligned} \tag{3.62}$$

Definition. The *determinant* of a 3×3 matrix A is given by

$$\begin{aligned}
\det \mathsf{A} &= \varepsilon_{ijk} A_{i1} A_{j2} A_{k3} && (3.63\text{a})\\
&= A_{11}(A_{22}A_{33} - A_{32}A_{23}) + A_{21}(A_{32}A_{13} - A_{12}A_{33}) + A_{31}(A_{12}A_{23} - A_{22}A_{13}) && (3.63\text{b})\\
&= A_{11}(A_{22}A_{33} - A_{23}A_{32}) + A_{12}(A_{23}A_{31} - A_{21}A_{33}) + A_{13}(A_{21}A_{32} - A_{22}A_{31}) && (3.63\text{c})\\
&= \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} && (3.63\text{d})\\
&= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31}\,. && (3.63\text{e})
\end{aligned}$$

Alternative notation. Alternative notations for the determinant of the matrix A include

$$\det \mathsf{A} \equiv |\,\mathsf{A}\,| \equiv \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{vmatrix}\,. \tag{3.64}$$

Remarks.

(i) A linear map $\mathbb{R}^3 \to \mathbb{R}^3$ is *volume preserving* if and only if the determinant of its matrix with respect to an orthonormal basis is ± 1 (strictly 'an' should be replaced by 'any', but we need some extra machinery before we can prove that; see also (5.25a)).

(ii) If $\{\mathbf{e}_i\}$ is a right-handed orthonormal basis then the set $\{\mathbf{e}'_i\}$ is right-handed (but not necessarily orthonormal or even orthogonal) if $\begin{vmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{vmatrix} > 0$, and left-handed if $\begin{vmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{vmatrix} < 0$.

(iii) If we like subscripts of subscripts

$$\det \mathsf{A} = \varepsilon_{i_1 i_2 i_3} A_{i_1 1} A_{i_2 2} A_{i_3 3} = \varepsilon_{j_1 j_2 j_3} A_{1 j_1} A_{2 j_2} A_{3 j_3}\,.$$

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Exercises.

(i) Show that the determinant of the rotation matrix R defined in (3.23) is $+1$.

(ii) Show that the determinant of the reflection matrix H defined in (3.25a), or equivalently (3.25b), is -1 (since reflection sends a right-handed set of vectors to a left-handed set).

Triple-scalar product representation. Let $A_{i1} = \alpha_i$, $A_{j2} = \beta_j$, $A_{k3} = \gamma_k$, then from (3.63a)

$$\det \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \varepsilon_{ijk}\alpha_i\beta_j\gamma_k = \boldsymbol{\alpha}\cdot(\boldsymbol{\beta}\times\boldsymbol{\gamma})\,. \tag{3.65}$$

An abuse of notation. This is not for the faint-hearted. If you are into the abuse of notation show that

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\,, \tag{3.66}$$

19/03 where we are treating the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, as 'components'.

3.7.2 Determinants for 2×2 matrices

A map $\mathbb{R}^3 \to \mathbb{R}^3$ is effectively two-dimensional if $A_{13} = A_{31} = A_{23} = A_{32} = 0$ and $A_{33} = 1$ (cf. (3.23)). Hence for a 2×2 matrix A we define the determinant to be given by (see (3.63b) or (3.63c))

$$\det \mathsf{A} \equiv |\,\mathsf{A}\,| \equiv \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}\,. \tag{3.67}$$

Remark. A map $\mathbb{R}^2 \to \mathbb{R}^2$ is area preserving if $\det \mathsf{A} = \pm 1$.

Observation. We note that the expressions (3.63b) and (3.63c) for the determinant of a 3×3 matrix can be re-written as

$$\det \mathsf{A} = A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{21}(A_{12}A_{33} - A_{13}A_{32}) + A_{31}(A_{12}A_{23} - A_{22}A_{13}) \tag{3.68a}$$
$$= A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})\,, \tag{3.68b}$$

which in turn can be re-written as

$$\det \mathsf{A} = A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{21}\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} + A_{31}\begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} \tag{3.69a}$$
$$= A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13}\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}\,. \tag{3.69b}$$

(3.69a) and (3.69b) are expansions of $\det \mathsf{A}$ in terms of elements of the first column and row, respectively, of A and determinants of 2×2 sub-matrices.

Remark. Note the sign pattern in (3.69a) and (3.69b). 20/02 12/08

3.7.3 Part two of a dummy's guide to permutations

As noted in part one, a permutation of degree n is a[n invertible] map that rearranges n distinct objects amongst themselves. We will consider permutations of the set of the first n strictly positive integers $\{1, 2, \ldots, n\}$, and will state properties that will be proved at some point in the *Groups* course.

Remark. There are $n!$ permutations in the set S_n of all permutations of $\{1, 2, \ldots, n\}$.[26]

[26] S_n is actually a group, referred to as the *symmetric group*.

Notation. If ρ is a permutation, we write the map ρ in the form

$$\rho = \begin{pmatrix} 1 & 2 & \dots & n \\ \rho(1) & \rho(2) & \dots & \rho(n) \end{pmatrix} . \tag{3.70a}$$

It is not necessary to write the columns in order, e.g.

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 4 & 2 & 6 & 3 \\ 5 & 4 & 1 & 6 & 2 & 3 \end{pmatrix} . \tag{3.70b}$$

Inverse. If

$$\rho = \begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix} , \quad \text{then} \quad \rho^{-1} = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 2 & \dots & n \end{pmatrix} . \tag{3.70c}$$

Fixed Points. k is a *fixed point* of ρ if $\rho(k) = k$. By convention fixed points can be omitted from the expression for ρ, e.g. '3' is a *fixed point* in (3.70b) so

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 4 & 2 & 6 \\ 5 & 4 & 1 & 6 & 2 \end{pmatrix} . \tag{3.70d}$$

Disjoint Permutations. Two permutations ρ_1 and ρ_2 are disjoint if for every k in $\{1, 2, \dots, n\}$ either $\rho_1(k) = k$ or $\rho_2(k) = k$. For instance

$$\rho_1 = \begin{pmatrix} 1 & 5 & 4 \\ 5 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \rho_2 = \begin{pmatrix} 2 & 6 \\ 6 & 2 \end{pmatrix} . \tag{3.70e}$$

are disjoint permutations.

Remark. In general permutations do *not* commute. However, the composition maps formed by disjoint permutations do commute, i.e. if ρ_1 and ρ_2 are disjoint permutations then $\rho_2\rho_1 = \rho_1\rho_2$.

Cycles. A *cycle* $(a_1, a_2, \dots, a_q)$ of length q (or q-cycle) is the permutation

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{q-1} & a_q \\ a_2 & a_3 & \dots & a_q & a_1 \end{pmatrix} . \tag{3.70f}$$

For instance ρ_1 and ρ_2 in (3.70e) are a 3-cycle and a 2-cycle respectively.

The Standard Representation. Any permutation ρ can be expressed as a product of disjoint (and hence commuting) cycles, say

$$\rho = \rho_m \dots \rho_2\rho_1 . \tag{3.70g}$$

For instance from (3.70b), (3.70e) and (3.70j)

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 4 \\ 5 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 6 & 2 \end{pmatrix} = (1, 5, 4)\,(2, 6) . \tag{3.70h}$$

This description can be shown to be unique up to the ordering of the factors, and is called the *standard representation* of ρ.

Transpositions. A 2-cycle, e.g. (a_1, a_2), is called a *transposition*. Since $(a_1, a_2) = (a_2, a_1)$ a transposition is its own inverse. A q-cycle can be expressed as a product of transpositions, viz:

$$(a_1, a_2, \dots, a_q) = (a_1, a_q) \dots (a_1, a_3)\,(a_1, a_2) . \tag{3.70i}$$

For instance

$$\rho_1 = \begin{pmatrix} 1 & 5 & 4 \\ 5 & 4 & 1 \end{pmatrix} = (1, 5, 4) = (1, 4)\,(1, 5) . \tag{3.70j}$$

Product of Transpositions. It follows from (3.70i) and (3.70g) that any permutation can be represented as a product of transpositions (this representation is not unique). For instance

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 4 & 2 \end{pmatrix} = (1, 4)\,(1, 5)\,(2, 6) = (5, 1)\,(5, 4)\,(2, 6) = (4, 5)\,(4, 1)\,(2, 6) . \tag{3.70k}$$

The Sign of a Permutation. If a permutation ρ is expressed as a product of transpositions, then it can be *proved* that the number of transpositions is always either even or odd. Further, suppose that ρ can be expressed as a product of r transpositions, then it is consistent to define the *sign*, $\epsilon(\rho)$, of ρ to be $(-)^r$. For instance, $\epsilon(\rho) = -1$ for the permutation in (3.70k).

Further, it follows that if ρ and σ are permutations then

$$\epsilon(\rho\sigma) = \epsilon(\rho)\,\epsilon(\sigma)\,, \tag{3.70l}$$

and thence by choosing $\sigma = \rho^{-1}$ that

$$\epsilon(\rho^{-1}) = \epsilon(\rho)\,. \tag{3.70m}$$

The Levi-Civita Symbol. We can now generalise the Levi-Civita symbol to higher dimensions:

$$\varepsilon_{j_1 j_2 \dots j_n} = \begin{cases} +1 & \text{if } (j_1, j_2, \dots, j_n) \text{ is an even permutation of } (1, 2, \dots, n) \\ -1 & \text{if } (j_1, j_2, \dots, j_n) \text{ is an odd permutation of } (1, 2, \dots, n) \\ 0 & \text{if any two labels are the same} \end{cases}\,. \tag{3.70n}$$

Thus if $(j_1, j_2, \dots, j_n)$ is a permutation, $\varepsilon_{j_1 j_2 \dots j_n}$ is the sign of the permutation, otherwise $\varepsilon_{j_1 j_2 \dots j_n}$
12/09 is zero.

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3.7.4 Determinants for $n \times n$ matrices

We define the determinant of a $n \times n$ matrix A by

$$\det \mathsf{A} = \sum_{i_1 i_2 \dots i_n} \varepsilon_{i_1 i_2 \dots i_n} A_{i_1 1} A_{i_2 2} \dots A_{i_n n}\,. \tag{3.71a}$$

Remarks.

(a) This definition is consistent with (3.63a) for 3×3 matrices. It is also consistent with our definition of the determinant of 2×2 matrices.

(b) The only non-zero contributions to (3.71a) come from terms in which the factors $A_{i_k k}$ are drawn once and only once from each row and column.

(c) An equivalent definition to (3.71a) is

$$\det \mathsf{A} = \sum_{\sigma \in S_n} \epsilon(\sigma) A_{\sigma(1)1} A_{\sigma(2)2} \dots A_{\sigma(n)n}\,, \tag{3.71b}$$

where in essence we have expressed $(i_1, i_2, \dots, i_n)$ in terms of a permutation σ.

12/07 *Exercise.* Show that $\det \mathsf{I} = 1$.

3.7.5 Properties of determinants

(i) For any square matrix A

$$\det \mathsf{A} = \det \mathsf{A}^{\mathrm{T}}\,. \tag{3.72}$$

Proof. Consider a single term in (3.71b) and let ρ be a permutation, then (writing $\sigma(\rho(1)) = \sigma\rho(1)$)

$$A_{\sigma(1)1} A_{\sigma(2)2} \dots A_{\sigma(n)n} = A_{\sigma\rho(1)\rho(1)} A_{\sigma\rho(2)\rho(2)} \dots A_{\sigma\rho(n)\rho(n)}\,, \tag{3.73}$$

because the product on the right is simply a re-ordering. Now choose $\rho = \sigma^{-1}$, and use the fact that $\epsilon(\sigma) = \epsilon(\sigma^{-1})$ from (3.70m), to conclude that

$$\det \mathsf{A} = \sum_{\sigma \in S_n} \epsilon(\sigma^{-1}) A_{1\sigma^{-1}(1)} A_{2\sigma^{-1}(2)} \dots A_{n\sigma^{-1}(n)}\,.$$

Every permutation has an inverse, so summing over σ is equivalent to summing over σ^{-1}; hence

$$\begin{aligned}\det \mathsf{A} &= \sum_{\sigma\in S_n} \epsilon(\sigma) A_{1\sigma(1)}A_{2\sigma(2)}\dots A_{n\sigma(n)} &(3.74a)\\ &= \det \mathsf{A}^{\mathrm{T}}\,. & \square\end{aligned}$$

Remarks.

(a) For 3×3 matrices (3.72) follows immediately from (3.63a) and (3.63d).

(b) From (3.74a) it follows that (cf. (3.71a))

$$\det \mathsf{A} = \sum_{j_1 j_2\dots j_n} \varepsilon_{j_1 j_2\dots j_n} A_{1j_1}A_{2j_2}\dots A_{nj_n}\,. \tag{3.74b}$$

(ii) If a matrix B is obtained from A by multiplying any *single* row or column of A by λ then

$$\det \mathsf{B} = \lambda \det \mathsf{A}\,, \tag{3.75}$$

Proof. From (3.72) we only need to prove the result for rows. Suppose that row r is multiplied by λ then for $j = 1,2,\dots,n$

$$B_{ij} = A_{ij} \text{ if } i\neq r, \quad B_{rj} = \lambda A_{rj}\,.$$

Hence from (3.74b)

$$\begin{aligned}\det \mathsf{B} &= \varepsilon_{j_1\dots j_r\dots j_n} B_{1j_1}\dots B_{rj_r}\dots B_{nj_n}\\ &= \varepsilon_{j_1\dots j_r\dots j_n} A_{1j_1}\dots \lambda A_{rj_r}\dots A_{nj_n}\\ &= \lambda\varepsilon_{j_1\dots j_r\dots j_n} A_{1j_1}\dots A_{rj_r}\dots A_{nj_n}\\ &= \lambda \det \mathsf{A}\,.\end{aligned}$$

(iii) The determinant of the $n\times n$ matrix $\lambda\mathsf{A}$ is given by

$$\det(\lambda\mathsf{A}) = \lambda^n \det \mathsf{A}\,. \tag{3.76}$$

Proof. From (3.74b)

$$\begin{aligned}\det(\lambda\mathsf{A}) &= \varepsilon_{j_1\dots j_r\dots j_n} \lambda A_{1j_1}\lambda A_{2j_2}\dots \lambda A_{nj_n}\\ &= \lambda^n\varepsilon_{j_1\dots j_r\dots j_n} A_{1j_1}A_{2j_2}\dots A_{nj_n}\\ &= \lambda^n \det \mathsf{A}\,.\end{aligned}$$

(iv) If a matrix B is obtained from A by interchanging two rows or two columns, then $\det \mathsf{B} = -\det \mathsf{A}$.

Proof. From (3.72) we only need to prove the result for rows. Suppose that rows r and s are interchanged then for $j = 1,2,\dots,n$

$$B_{ij} = A_{ij} \text{ if } i\neq r,s, \quad B_{rj} = A_{sj}\,, \quad \text{and} \quad B_{sj} = A_{rj}\,.$$

Then from (3.74b)

$$\begin{aligned}\det \mathsf{B} &= \varepsilon_{j_1\dots j_r\dots j_s\dots j_n} B_{1j_1}\dots B_{rj_r}\dots B_{sj_s}\dots B_{nj_n}\\ &= \varepsilon_{j_1\dots j_r\dots j_s\dots j_n} A_{1j_1}\dots A_{sj_r}\dots A_{rj_s}\dots A_{nj_n}\\ &= \varepsilon_{j_1\dots j_s\dots j_r\dots j_n} A_{1j_1}\dots A_{sj_s}\dots A_{rj_r}\dots A_{nj_n} &&\text{relabel } j_r\leftrightarrow j_s\\ &= -\varepsilon_{j_1\dots j_r\dots j_s\dots j_n} A_{1j_1}\dots A_{rj_r}\dots A_{sj_s}\dots A_{nj_n} &&\text{permutate } j_r \text{ and } j_s\\ &= -\det \mathsf{A}\,. && \square\end{aligned}$$

Remark. If two rows or two columns of A are identical then $\det \mathsf{A} = 0$.

(v) If a matrix B is obtained from A by adding to a given row/column of A a multiple of another row/column, then

$$\det \mathsf{B} = \det \mathsf{A}\,. \tag{3.77}$$

Proof. From (3.72) we only need to prove the result for rows. Suppose that to row r is added row $s \neq r$ multiplied by λ; then for $j = 1, 2, \ldots, n$

$$B_{ij} = A_{ij} \text{ if } i \neq r, \quad B_{rj} = A_{rj} + \lambda A_{sj}\,.$$

Hence from (3.74b), and the fact that a determinant is zero if two rows are equal:

$$\begin{aligned}\det \mathsf{B} &= \varepsilon_{j_1 \ldots j_r \ldots j_n} B_{1j_1} \ldots B_{rj_r} \ldots B_{nj_n} \\ &= \varepsilon_{j_1 \ldots j_r \ldots j_n} A_{1j_1} \ldots A_{rj_r} \ldots A_{nj_n} + \varepsilon_{j_1 \ldots j_r \ldots j_n} A_{1j_1} \ldots \lambda A_{sj_r} \ldots A_{nj_n} \\ &= \varepsilon_{j_1 \ldots j_r \ldots j_n} A_{1j_1} \ldots A_{rj_r} \ldots A_{nj_n} + \lambda\, 0 \\ &= \det \mathsf{A}\,.\end{aligned}$$

(vi) If the rows or columns of a matrix A are linearly dependent then $\det \mathsf{A} = 0$.

Proof. From (3.72) we only need to prove the result for rows. Suppose that the r^{th} row is linearly dependent on the other rows. Express this row as a linear combination of the other rows. The value of $\det \mathsf{A}$ is unchanged by subtracting this linear combination from the r^{th} row, and the result is a row of zeros. If follows from (3.74a) or (3.74b) that $\det \mathsf{A} = 0$.

Contrapositive. The contrapositive of this result is that if $\det \mathsf{A} \neq 0$ the rows (and columns) of A cannot be linearly dependent, i.e. they must be linearly independent. The converse, i.e. that if $\det \mathsf{A} = 0$ the rows (and columns) must be linearly dependent, is also true (and proved on page 74 or thereabouts).

3×3 *matrices.* For 3×3 matrices both original statement and the converse can be obtained from (3.65). In particular, since $\boldsymbol{\alpha} \cdot (\boldsymbol{\beta} \times \boldsymbol{\gamma}) = 0$ if and only if $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are coplanar (i.e. linearly dependent), $\det \mathsf{A} = 0$ if and only if the columns of A are linearly dependent (or from (3.72) if and only if the rows of A are linearly dependent).

3.7.6 The determinant of a product

We first need a preliminary result. Let ρ be a permutation, then

$$\epsilon(\rho) \det \mathsf{A} = \sum_{\sigma \in S_n} \epsilon(\sigma) A_{\sigma(1)\rho(1)} A_{\sigma(2)\rho(2)} \ldots A_{\sigma(n)\rho(n)}\,. \tag{3.78a}$$

Proof.

$$\begin{aligned}\epsilon(\rho) \det \mathsf{A} &= \epsilon(\rho) \sum_{\tau \in S_n} \epsilon(\tau) A_{\tau(1)1} A_{\tau(2)2} \ldots A_{\tau(n)n} && \text{from (3.71b) with } \sigma = \tau \\ &= \sum_{\tau \in S_n} \epsilon(\tau\rho) A_{\tau\rho(1)\rho(1)} A_{\tau\rho(2)\rho(2)} \ldots A_{\tau\rho(n)\rho(n)} && \text{from (3.70l) and (3.73)} \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) A_{\sigma(1)\rho(1)} A_{\sigma(2)\rho(2)} \ldots A_{\sigma(n)\rho(n)}\,,\end{aligned}$$

since summing over $\sigma = \tau\rho$ is equivalent to summing over τ. □

Similarly

$$\epsilon(\rho) \det \mathsf{A} = \sum_{\sigma \in S_n} \epsilon(\sigma) A_{\rho(1)\sigma(1)} A_{\rho(2)\sigma(2)} \ldots A_{\rho(n)\sigma(n)}\,, \tag{3.78b}$$

or equivalently

$$\varepsilon_{p_1 p_2 \ldots p_n} \det \mathsf{A} = \sum_{i_1 i_2 \ldots i_n} \varepsilon_{i_1 i_2 \ldots i_n} A_{i_1 p_1} A_{i_2 p_2} \ldots A_{i_n p_n}\,, \tag{3.78c}$$

$$= \sum_{j_1 j_2 \ldots j_n} \varepsilon_{j_1 j_2 \ldots j_n} A_{p_1 j_1} A_{p_2 j_2} \ldots A_{p_n j_n}\,. \tag{3.78d}$$

Theorem 3.3. *If* A *and* B *are both square matrices, then*

$$\det \mathsf{AB} = (\det \mathsf{A})(\det \mathsf{B})\,. \tag{3.79}$$

Proof.

$$\begin{aligned}
\det \mathsf{AB} &= \varepsilon_{i_1 i_2 \dots i_n} (\mathsf{AB})_{i_1 1} (\mathsf{AB})_{i_2 2} \dots (\mathsf{AB})_{i_n n} && \text{from (3.71a)}\\
&= \varepsilon_{i_1 i_2 \dots i_n} A_{i_1 k_1} B_{k_1 1} A_{i_2 k_2} B_{k_2 2} \dots A_{i_n k_n} B_{k_n n} && \text{from (3.35)}\\
&= \varepsilon_{i_1 i_2 \dots i_n} A_{i_1 k_1} A_{i_2 k_2} \dots A_{i_n k_n} B_{k_1 1} B_{k_2 2} \dots B_{k_n n} &&\\
&= \varepsilon_{k_1 k_2 \dots k_n} \det \mathsf{A}\, B_{k_1 1} B_{k_2 2} \dots B_{k_n n} && \text{from (3.78c)}\\
&= \det \mathsf{A} \det \mathsf{B}\,. && \text{from (3.71a)} \quad \square
\end{aligned}$$

Theorem 3.4. *If* A *is orthogonal then*

$$\det \mathsf{A} = \pm 1\,. \tag{3.80}$$

Proof. If A is orthogonal then $\mathsf{AA}^{\mathrm{T}} = \mathsf{I}$. It follows from (3.72) and (3.79) that

$$(\det \mathsf{A})^2 = (\det \mathsf{A})(\det \mathsf{A}^{\mathrm{T}}) = \det(\mathsf{AA}^{\mathrm{T}}) = \det \mathsf{I} = 1\,.$$

Hence $\det \mathsf{A} = \pm 1$. $\square$

Remark. This has already been verified for some reflection and rotation matrices.

3.7.7 Alternative proof of (3.78c) and (3.78d) for 3×3 matrices (unlectured)

If $\mathsf{A} = \{A_{ij}\}$ is a 3×3 matrix, then the expressions (3.78c) and (3.78d) are equivalent to

$$\varepsilon_{pqr} \det \mathsf{A} = \varepsilon_{ijk} A_{ip} A_{jq} A_{kr}\,, \tag{3.81a}$$

$$\varepsilon_{pqr} \det \mathsf{A} = \varepsilon_{ijk} A_{pi} A_{qj} A_{rk}\,. \tag{3.81b}$$

Proof. Start with (3.81b), and suppose that $p = 1$, $q = 2$, $r = 3$. Then (3.81b) is just (3.63d). Next suppose that p and q are swapped. Then the sign of the left-hand side of (3.81b) reverses, while the right-hand side becomes

$$\varepsilon_{ijk} A_{qi} A_{pj} A_{rk} = \varepsilon_{jik} A_{qj} A_{pi} A_{rk} = -\varepsilon_{ijk} A_{pi} A_{qj} A_{rk}\,,$$

so the sign of right-hand side also reverses. Similarly for swaps of p and r, or q and r. It follows that (3.81b) holds for any $\{p, q, r\}$ that is a permutation of $\{1, 2, 3\}$.

Suppose now that two (or more) of p, q and r in (3.81b) are equal. Wlog take $p = q = 1$, say. Then the left-hand side is zero, while the right-hand side is

$$\varepsilon_{ijk} A_{1i} A_{1j} A_{rk} = \varepsilon_{jik} A_{1j} A_{1i} A_{rk} = -\varepsilon_{ijk} A_{1i} A_{1j} A_{rk}\,,$$

which is also zero. Having covered all cases we conclude that (3.81b) is true.

Similarly for (3.81a) starting from (3.63a) $\square$

3.7.8 Minors and cofactors

For a square $n \times n$ matrix $\mathsf{A} = \{A_{ij}\}$, define A^{ij} to be the $(n-1) \times (n-1)$ square matrix obtained by eliminating the i^{th} row and the j^{th} column of A. Hence

$$\mathsf{A}^{ij} = \begin{pmatrix}
A_{11} & \dots & A_{1(j-1)} & A_{1(j+1)} & \dots & A_{1n}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
A_{(i-1)1} & \dots & A_{(i-1)(j-1)} & A_{(i-1)(j+1)} & \dots & A_{(i-1)n}\\
A_{(i+1)1} & \dots & A_{(i+1)(j-1)} & A_{(i+1)(j+1)} & \dots & A_{(i+1)n}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
A_{n1} & \dots & A_{n(j-1)} & A_{n(j+1)} & \dots & A_{nn}
\end{pmatrix}.$$

Definition. Define the *minor*, M_{ij}, of the ij^{th} element of square matrix A to be the determinant of the square matrix obtained by eliminating the i^{th} row and the j^{th} column of A, i.e.

$$M_{ij} = \det \mathsf{A}^{ij}\,. \tag{3.82a}$$

Definition. Define the *cofactor* Δ_{ij} of the ij^{th} element of square matrix A as

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$$\Delta_{ij} = (-)^{i-j} M_{ij} = (-)^{i-j} \det \mathsf{A}^{ij}\,. \tag{3.82b}$$

3.7.9 An alternative expression for determinants

Notation. First some notation. In what follows we will use a $\bar{}$ to denote that a symbol that is missing from an otherwise natural sequence. For instance for some I we might re-order (3.74b) to obtain

$$\det \mathsf{A} = \sum_{j_I=1}^{n} A_{Ij_I} \sum_{j_1 j_2 \ldots \overline{j_I} \ldots j_n}^{n} \varepsilon_{j_1 j_2 \ldots j_I \ldots j_n} A_{1j_1} A_{2j_2} \ldots \overline{A_{Ij_I}} \ldots A_{nj_n}\,. \tag{3.83a}$$

Further, let σ be the permutation that re-orders $(1, \ldots, j_I, \ldots, n)$ so that j_I is moved to the I^{th} position, with the rest of the numbers in their natural order. Hence

$$\sigma = \begin{cases} \begin{pmatrix} 1 & \ldots & I & I+1 & \ldots & j_I - 1 & j_I & \ldots & n \\ 1 & \ldots & j_I & I & \ldots & j_I - 2 & j_I - 1 & \ldots & n \end{pmatrix} = (I, j_I), \ldots, (j_I - 2, j_I), (j_I - 1, j_I) \\ \hfill \text{if } j_I > I \\ \\ \begin{pmatrix} 1 & \ldots & j_I & j_I + 1 & \ldots & I - 1 & I & \ldots & n \\ 1 & \ldots & j_I + 1 & j_I + 2 & \ldots & I & j_I & \ldots & n \end{pmatrix} = (I, j_I), \ldots, (j_I + 2, j_I), (j_I + 1, j_I) \\ \hfill \text{if } j_I < I \end{cases},$$

with σ being the identity if $j_I = I$. It follows that $\epsilon(\sigma) = (-)^{I - j_I}$. Next let ρ be the permutation that maps $(1, \ldots, \overline{j_I}, \ldots, n)$ to $(j_1, \ldots, \overline{j_I}, \ldots, j_n)$, i.e.

$$\rho = \begin{pmatrix} 1 & \ldots & \ldots & \overline{j_I} & \ldots & n \\ j_1 & \ldots & \overline{j_I} & \ldots & \ldots & j_n \end{pmatrix}.$$

Then $\epsilon(\rho) = \varepsilon_{j_1 j_2 \ldots \overline{j_I} \ldots j_n}$, and the permutation $\rho\sigma$ reorders $(1, \ldots, j_I, \ldots, n)$ to $(j_1, \ldots, j_I, \ldots, j_n)$. It follows that

$$\varepsilon_{j_1 j_2 \ldots j_I \ldots j_n} = (-)^{I - j_I} \varepsilon_{j_1 j_2 \ldots \overline{j_I} \ldots j_n}\,. \tag{3.83b}$$

Unlectured example. If $n = 4$, $j_1 = 4$, $j_2 = 3$, $j_3 = 1$, $j_4 = 2$ and $I = 2$, then

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = (2,3)\,, \qquad \rho = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 2 \end{pmatrix} = (1,2)(1,4)\,.$$

and

$$\rho\sigma = (1,2)(1,4)(2,3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ j_1 & j_2 & j_3 & j_4 \end{pmatrix}.$$

Alternative Expression. We now observe from (3.82a), (3.82b), (3.83a) and (3.83b) that

$$\begin{aligned} \det \mathsf{A} &= \sum_{j_I=1}^{n} A_{Ij_I} (-)^{I - j_I} \left(\sum_{j_1 j_2 \ldots \overline{j_I} \ldots j_n}^{n} \varepsilon_{j_1 j_2 \ldots \overline{j_I} \ldots j_n} A_{1j_1} A_{2j_2} \ldots \overline{A_{Ij_I}} \ldots A_{nj_n} \right) \\ &= \sum_{j_I=1}^{n} A_{Ij_I} (-)^{I - j_I} M_{Ij_I} \qquad (3.84\text{a}) \\ &= \sum_{k=1}^{n} A_{Ik} \Delta_{Ik} \quad \text{for any } 1 \leqslant I \leqslant n \text{ (beware: no s.c. over } I\text{).} \qquad (3.84\text{b}) \end{aligned}$$

Similarly starting from (3.71a)

$$\det \mathsf{A} = \sum_{i_J=1}^{n} A_{i_J J}(-)^{J-i_J} M_{i_J J} \tag{3.84c}$$

$$= \sum_{k=1}^{n} A_{kJ}\Delta_{kJ} \quad \text{for any } 1 \leqslant J \leqslant n \text{ (beware: no s.c. over } J). \tag{3.84d}$$

Equations (3.84b) and (3.84d), which are known as the *Laplace expansion formulae*, express $\det \mathsf{A}$ as a sum of n determinants each with one less row and column then the original. In principle we
13/09 could now recurse until the matrices are reduced in size to, say, 1×1 matrices.
13/10

Examples.

(a) For 3×3 matrices see (3.68a) for (3.84d) with $J = 1$ and (3.68b) for (3.84b) with $I = 1$.

(b) Let us evaluate the following matrix using (3.84d) with $J = 2$, then

$$\mathsf{A} = \begin{vmatrix} 2 & 4 & 2 \\ 3 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -4\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}$$
$$= -4(1) + 2(-2) - 0(-4) = -8\,, \tag{3.85a}$$

or using (3.84b) with $I = 3$, then

$$\mathsf{A} = \begin{vmatrix} 2 & 4 & 2 \\ 3 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = +2\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$$
$$= 2(0) - 0(-4) + 1(-8) = -8\,. \tag{3.85b}$$

3.7.10 Practical evaluation of determinants

We can now combine (3.84b) and (3.84d) with earlier properties of determinants to obtain a practical method of evaluating determinants.

First we note from (3.85a) and (3.85b) that it helps calculation to multiply the cofactors by zero; the ideal row I or column J to choose would therefore have only one non-zero entry. However, we can arrange this. Suppose for instance that $A_{11} \neq 0$, then subtract A_{21}/A_{11} times the first from the second row to obtain a new matrix B. From (3.77) of property (v) on page 62 the determinant is unchanged, i.e. $\det \mathsf{B} = \det \mathsf{A}$, but now $B_{21} = 0$. Rename B as A and recurse until $A_{i1} = 0$ for $i = 2, \dots, n$. Then from (3.84d)

$$\det \mathsf{A} = A_{11}\Delta_{11}\,.$$

Now recurse starting from the $(n-1) \times (n-1)$ matrix A^{11}. If $(\mathsf{A}^{11})_{11} = 0$ either swap rows and/or columns so that $(\mathsf{A}^{11})_{11} \neq 0$, or alternatively choose another row/column to subtract multiples of.

Example. For A in (3.85a) or (3.85b), note that $A_{32} = 0$ so first subtract twice column 3 from column 1, then at the 2×2 matrix stage add twice column 1 to column 2, ...

$$\begin{vmatrix} 2 & 4 & 2 \\ 3 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 4 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -2 & 4 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 1 & 4 \end{vmatrix} = -2\,\begin{vmatrix}4\end{vmatrix} = 4\,\begin{vmatrix}-2\end{vmatrix} = -8\,.$$

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4 Matrix Inverses and Linear Equations

4.0 Why Study This?

This section continues our study of linear mathematics. The 'real' world can often be described by equations of various sorts. Some models result in linear equations of the type studied here. However, even when the real world results in more complicated models, the solution of these more complicated models often involves the solution of linear equations.

4.1 Solution of Two Linear Equations in Two Unknowns

Consider two linear equations in two unknowns:

$$A_{11}x_1 + A_{12}x_2 = d_1 \tag{4.1a}$$
$$A_{21}x_1 + A_{22}x_2 = d_2 \tag{4.1b}$$

or equivalently

$$\mathsf{A}\mathbf{x} = \mathbf{d} \tag{4.2a}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad \text{and} \quad \mathsf{A} = \{A_{ij}\} \quad (\text{a } 2 \times 2 \text{ matrix}). \tag{4.2b}$$

Now solve by forming suitable linear combinations of the two equations (e.g. $A_{22} \times (4.1\text{a}) - A_{12} \times (4.1\text{b})$)

$$\begin{aligned} (A_{11}A_{22} - A_{21}A_{12})x_1 &= A_{22}d_1 - A_{12}d_2\,, \\ (A_{21}A_{12} - A_{22}A_{11})x_2 &= A_{21}d_1 - A_{11}d_2\,. \end{aligned}$$

From (3.67) we have that

$$(A_{11}A_{22} - A_{21}A_{12}) = \det \mathsf{A} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}.$$

Thus, if $\det \mathsf{A} \neq 0$, the equations have a *unique* solution

$$\begin{aligned} x_1 &= (A_{22}d_1 - A_{12}d_2)/\det \mathsf{A}\,, \\ x_2 &= (-A_{21}d_1 + A_{11}d_2)/\det \mathsf{A}\,, \end{aligned}$$

i.e.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\det \mathsf{A}} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}. \tag{4.3a}$$

However, from left multiplication of (4.2a) by A^{-1} (if it exists) we have that

$$\mathbf{x} = \mathsf{A}^{-1}\mathbf{d}\,. \tag{4.3b}$$

We therefore conclude that

$$\mathsf{A}^{-1} = \frac{1}{\det \mathsf{A}} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}. \tag{4.4}$$

21/03 *Exercise.* Check that $\mathsf{A}\mathsf{A}^{-1} = \mathsf{A}^{-1}\mathsf{A} = \mathsf{I}$.

4.2 The Inverse of a $n \times n$ Matrix

Recall from (3.84b) and (3.84d) that the determinant f a $n \times n$ matrix A can be expressed in terms of the *Laplace expansion formulae*

$$\det \mathsf{A} = \sum_{k=1}^{n} A_{Ik}\Delta_{Ik} \quad \text{for any } 1 \leqslant I \leqslant n \tag{4.5a}$$
$$= \sum_{k=1}^{n} A_{kJ}\Delta_{kJ} \quad \text{for any } 1 \leqslant J \leqslant n. \tag{4.5b}$$

Let B be the matrix obtained by replacing the I^{th} row of A with one of the other rows, say the i^{th}. Since B has two identical rows $\det \mathsf{B} = 0$. However, replacing the I^{th} row by something else does not change the cofactors Δ_{Ij} of the elements in the I^{th} row. Hence the cofactors Δ_{Ij} of the I^{th} row of A are also the cofactors of the I^{th} row of B. Thus applying (4.5a) to the matrix B we conclude that

$$\begin{aligned} 0 &= \det \mathsf{B} \\ &= \sum_{k=1}^{n} B_{Ik}\Delta_{Ik} \\ &= \sum_{k=1}^{n} A_{ik}\Delta_{Ik} \quad \text{for any } i \neq I,\ 1 \leqslant i, I \leqslant n. \end{aligned} \tag{4.6a}$$

Similarly by replacing the J^{th} column and using (4.5b)

$$0 = \sum_{k=1}^{n} A_{kj}\Delta_{kJ} \quad \text{for any } j \neq J,\ 1 \leqslant j, J \leqslant n. \tag{4.6b}$$

We can combine (4.5a), (4.5b), (4.6a) and (4.6b) to obtain the formulae, using the summation convention,

$$A_{ik}\Delta_{jk} = \delta_{ij} \det \mathsf{A}\,, \tag{4.7a}$$

$$A_{ki}\Delta_{kj} = \delta_{ij} \det \mathsf{A}\,. \tag{4.7b}$$

Theorem 4.1. *Given a $n \times n$ matrix A with $\det \mathsf{A} \neq 0$, let B be the $n \times n$ with elements*

$$B_{ij} = \frac{1}{\det \mathsf{A}} \Delta_{ji}\,, \tag{4.8a}$$

then

$$\mathsf{AB} = \mathsf{BA} = \mathsf{I}\,. \tag{4.8b}$$

Proof. From (4.7a) and (4.8a),

$$\begin{aligned} (\mathsf{AB})_{ij} &= A_{ik}B_{kj} \\ &= \frac{A_{ik}\Delta_{jk}}{\det \mathsf{A}} \\ &= \frac{\delta_{ij}\det \mathsf{A}}{\det \mathsf{A}} \\ &= \delta_{ij}\,. \end{aligned}$$

Hence $\mathsf{AB} = \mathsf{I}$. Similarly from (4.7b) and (4.8a), $\mathsf{BA} = \mathsf{I}$. It follows that $\mathsf{B} = \mathsf{A}^{-1}$ and A is invertible. $\square$

We conclude that

$$(\mathsf{A}^{-1})_{ij} = \frac{1}{\det \mathsf{A}} \Delta_{ji}\,. \tag{4.9}$$

Example. Consider the simple shear matrix (see (3.30))

$$\mathsf{S}_\gamma = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\det \mathsf{S}_\gamma = 1$, and after a little manipulation

$$\begin{array}{lll} \Delta_{11} = 1\,, & \Delta_{12} = 0\,, & \Delta_{13} = 0\,, \\ \Delta_{21} = -\gamma\,, & \Delta_{22} = 1\,, & \Delta_{23} = 0\,, \\ \Delta_{31} = 0\,, & \Delta_{32} = 0\,, & \Delta_{33} = 1\,. \end{array}$$

Hence

$$\mathsf{S}_\gamma^{-1} = \begin{pmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & -\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathsf{S}_{-\gamma}\,.$$

21/02 This makes physical sense in that the effect of a shear γ is reversed by changing the sign of γ.

4.3 Solving Linear Equations

4.3.1 Inhomogeneous and homogeneous problems

Suppose that we wish to solve the system of equations

$$\mathsf{A}\mathbf{x} = \mathbf{d}\,, \tag{4.10}$$

where A is a given $m \times n$ matrix, $\mathbf{x}$ is a $n \times 1$ column vector of unknowns, and $\mathbf{d}$ is a given $m \times 1$ column vector.

Definition. If $\mathbf{d} \neq \mathbf{0}$ then the system of equations (4.10) is said to be a system of *inhomogeneous* equations.

Definition. The system of equations

$$\mathsf{A}\mathbf{x} = \mathbf{0} \tag{4.11}$$

is said to be a system of *homogeneous* equations.

4.3.2 Solving linear equations: the slow way

Start by assuming that $m = n$ so that A is a square matrix, and that that $\det \mathsf{A} \neq 0$. Then (4.10) has the *unique* solution

$$\mathbf{x} = \mathsf{A}^{-1}\mathbf{d}\,. \tag{4.12}$$

If we wished to solve (4.10) numerically, one method would be to calculate A^{-1} using (4.9), and then form $\mathsf{A}^{-1}\mathbf{d}$. However, before proceeding, let us estimate how many mathematical operations are required to calculate the inverse using formula (4.9).

- The most expensive *single* step is calculating the determinant. Suppose we use one of the Laplace expansion formulae (4.5a) or (4.5b) to do this (noting, in passing, that as a by-product of this approach we will calculate one of the required rows/columns of cofactors).
- Let f_n be the number of operations needed to calculate a $n \times n$ determinant. Since each $n \times n$ determinant requires us to calculate n smaller $(n-1) \times (n-1)$ determinants, plus perform n multiplications and $(n-1)$ additions,

  $$f_n = n f_{n-1} + 2n - 1\,.$$

  Similarly, it follows that each $(n-1) \times (n-1)$ determinant requires the calculation of $(n-1)$ smaller $(n-2) \times (n-2)$ determinants; etc. We conclude that the calculation of a determinant using (4.5a) or (4.5b) requires $f_n = O(n!)$ operations; note that this is marginally better than the $(n-1)n! = O((n+1)!)$ operations that would result from using (3.71a) or (3.71b).

  Exercise: As $n \to \infty$ show that $f_n \to n!\,e + k$, where k is a constant (calculate k for a mars bar, or see Appendix B on page D).
- To calculate the other $n(n-1)$ cofactors requires $O(n(n-1)(n-1)!)$ operations.
- Once the inverse has been obtained, the final multiplication $\mathsf{A}^{-1}\mathbf{d}$ only requires $O(n^2)$ operations, so for large n it follows that the calculation of the cofactors for the inverse dominate.
- The solution of (4.12) by this method therefore takes $O(n.n!)$ operations, or equivalently $O((n+1)!)$ operations ... which is rather large.

4.3.3 Equivalent systems of equations: an example

Suppose that we wish to solve

$$\begin{aligned} x+y &= 1 && (4.13a)\\ x-y &= 0\,, && (4.13b)\end{aligned}$$

then we might write

$$\mathsf{x}=\begin{pmatrix}x\\y\end{pmatrix}\,,\quad \mathsf{A}=\begin{pmatrix}1&1\\1&-1\end{pmatrix}\quad\text{and}\quad \mathsf{d}=\begin{pmatrix}1\\0\end{pmatrix}\,. \tag{4.13c}$$

However, if we swap the order of the rows we are still solving the same system but now

$$\begin{aligned} x-y &= 0 && (4.14a)\\ x+y &= 1\,, && (4.14b)\end{aligned}$$

while we might write

$$\mathsf{x}=\begin{pmatrix}x\\y\end{pmatrix}\,,\quad \mathsf{A}=\begin{pmatrix}1&-1\\1&1\end{pmatrix}\quad\text{and}\quad \mathsf{d}=\begin{pmatrix}0\\1\end{pmatrix}\,. \tag{4.14c}$$

Similarly if we effectively swap the order of the columns

$$\begin{aligned} y+x &= 1 && (4.15a)\\ -y+x &= 0\,, && (4.15b)\end{aligned}$$

we might write

$$\mathsf{x}=\begin{pmatrix}y\\x\end{pmatrix}\,,\quad \mathsf{A}=\begin{pmatrix}1&1\\-1&1\end{pmatrix}\quad\text{and}\quad \mathsf{d}=\begin{pmatrix}1\\0\end{pmatrix}\,. \tag{4.15c}$$

Whatever our [consistent] choice of x, A and d, the solution for x and y is the same.

4.3.4 Solving linear equations: a faster way by Gaussian elimination

A better method, than using the Laplace expansion formulae, is to use *Gaussian elimination*. Suppose we wish to solve

$$\begin{aligned} A_{11}\,x_1 + A_{12}\,x_2 \;+\ldots+ A_{1n}\,x_n &= d_1\,, && (4.16a)\\ A_{21}\,x_1 + A_{22}\,x_2 \;+\ldots+ A_{2n}\,x_n &= d_2\,, && (4.16b)\\ \vdots\quad + \quad\vdots\quad + \ddots + \quad \vdots\quad &= \vdots\;, \\ A_{m1}\,x_1 + A_{m2}\,x_2 + \ldots + A_{mn}\,x_n &= d_m\,. && (4.16c)\end{aligned}$$

Assume $A_{11}\neq 0$, otherwise re-order the equations (i.e. rows) so that $A_{11}\neq 0$. If that is not possible then the first column is illusory (and x_1 can be equal to anything), so relabel $x_2\to x_1$, ..., $x_n\to x_{n-1}$ and $x_1\to x_n$, and start again.

Now use (4.16a) to eliminate x_1 from (4.16b) to (4.16c) by forming

$$(4.16b) - \frac{A_{21}}{A_{11}}\times(4.16a)\quad\ldots\quad(4.16c)-\frac{A_{m1}}{A_{11}}\times(4.16a)\,,$$

so as to obtain (4.16a) plus

$$\begin{aligned} \left(A_{22}-\frac{A_{21}}{A_{11}}A_{12}\right)x_2+\ldots+\left(A_{2n}-\frac{A_{21}}{A_{11}}A_{1n}\right)x_n &= d_2-\frac{A_{21}}{A_{11}}d_1\,, && (4.17a)\\ \vdots\qquad\qquad + \ddots + \qquad\qquad \vdots\qquad\qquad &= \qquad\vdots\qquad, \\ \left(A_{m2}-\frac{A_{m1}}{A_{11}}A_{12}\right)x_2+\ldots+\left(A_{mn}-\frac{A_{m1}}{A_{11}}A_{1n}\right)x_n &= d_m-\frac{A_{m1}}{A_{11}}d_1\,. && (4.17b)\end{aligned}$$

In order to simplify notation let

$$\begin{array}{ccc} A_{22}^{(2)} = \left(A_{22} - \frac{A_{21}}{A_{11}} A_{12}\right) , & A_{2n}^{(2)} = \left(A_{2n} - \frac{A_{21}}{A_{11}} A_{1n}\right) , & d_2^{(2)} = d_2 - \frac{A_{21}}{A_{11}} d_1 , \\ \vdots & \vdots & \vdots \\ A_{m2}^{(2)} = \left(A_{m2} - \frac{A_{m1}}{A_{11}} A_{12}\right) , & A_{mn}^{(2)} = \left(A_{mn} - \frac{A_{m1}}{A_{11}} A_{1n}\right) , & d_m^{(2)} = d_m - \frac{A_{m1}}{A_{11}} d_1 , \end{array}$$

so that (4.16a) and (4.17a), ..., (4.17b) become

$$\begin{aligned} A_{11}\, x_1 + A_{12}\, x_2 + \ldots + A_{1n}\, x_n &= d_1 , && (4.18a) \\ A_{22}^{(2)} x_2 + \ldots + A_{2n}^{(2)} x_n &= d_2^{(2)} , && (4.18b) \\ \vdots \quad + \ddots + \quad \vdots \quad &= \vdots \\ A_{m2}^{(2)} x_2 + \ldots + A_{mn}^{(2)} x_n &= d_m^{(2)} . && (4.18c) \end{aligned}$$

In essence we have reduced a $m \times n$ system of equations for $x_1, x_2, \ldots, x_n$ to a $(m-1) \times (n-1)$ system of equations for $x_2, \ldots, x_n$. We now recurse. Assume $A_{22}^{(2)} \neq 0$, otherwise re-order the equations (i.e. rows) so that $A_{22}^{(2)} \neq 0$. If that is not possible then the [new] first column is absent (and x_2 can be equal to anything), so relabel $x_3 \to x_2$, ..., $x_n \to x_{n-1}$ and $x_2 \to x_n$ (also remembering to relabel the first equation (4.18a)), and continue.

At the end of the calculation the system will have the form

$$\begin{aligned} A_{11}\, x_1 + A_{12}\, x_2 + \ldots + A_{1r}\, x_r + \ldots + A_{1n}\, x_n &= d_1 , \\ A_{22}^{(2)}\, x_2 + \ldots + A_{2r}^{(2)}\, x_r + \ldots + A_{2n}^{(2)} x_n &= d_2^{(2)} , \\ \ddots + \quad \vdots \quad + \ldots + \quad \vdots \quad &= \vdots \;, \\ A_{rr}^{(r)}\, x_r + \ldots + A_{rn}^{(r)} x_n &= d_r^{(r)} , \\ 0 &= d_{r+1}^{(r)} , \\ \vdots &= \vdots , \\ 0 &= d_m^{(r)} , \end{aligned}$$

where $r \leqslant m$ and $A_{ii}^{(i)} \neq 0$. There are now three possibilities. 14/08

(i) If $r < m$ and at least one of the numbers $d_{r+1}^{(r)}, \ldots, d_m^{(r)}$ is non-zero, then the equations are inconsistent and there is no solution. Such a system of equations is said to be *overdetermined*.

Example. Suppose

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 , \\ 6x_1 + 3x_2 + 3x_3 &= 0 , \\ 6x_1 + 2x_2 + 4x_3 &= 6 . \end{aligned}$$

Eliminate the first column below the leading diagonal to obtain

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 , \\ -x_2 + x_3 &= -6 , \\ -2x_2 + 2x_3 &= 0 . \end{aligned}$$

Eliminate the second column below the leading diagonal to obtain

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 , \\ -x_2 + x_3 &= -6 , \\ 0 &= 12 . \end{aligned}$$

The last equation is inconsistent.

(ii) If $r = n \leqslant m$ (and $d_i^{(r)} = 0$ for $i = r+1, \dots, m$ if $r < m$), there is a unique solution for x_n (the solution to the n^{th} equation), and thence by *back substitution* for x_{n-1} (by solving the $(n-1)^{\text{th}}$ equation), $x_{n-2}, \dots, x_1$. Such a system of equations is said to be *determined*.

Example. Suppose

$$\begin{aligned} 2x_1 + 5x_2 &= 2\,, \\ 4x_1 + 3x_2 &= 11\,. \end{aligned}$$

Then from multiplying the first equation by 2 and subtracting that from the second equation we obtain

$$\begin{aligned} 2x_1 + 5x_2 &= 2\,, \\ -7x_2 &= 7\,. \end{aligned}$$

14/09 So $x_2 = -1$, and by back substitution into the first equation we deduce that $x_1 = 7/2$.

14/10

(iii) If $r < n$ (and $d_i^{(r)} = 0$ for $i = r+1, \dots, m$ if $r < m$), there are infinitely many solutions. Any of these solutions is obtained by choosing values for the unknowns $x_{r+1}, \dots, x_n$, solving the r^{th} equation for x_r, then the $(r-1)^{\text{th}}$ equation for x_{r-1}, and so on by back substitution until a solution for x_1 is obtained. Such a system of equations is said to be *underdetermined*.

Example. Suppose

$$\begin{aligned} x_1 + \ x_2 &= 1\,, \\ 2x_1 + 2x_2 &= 2\,. \end{aligned}$$

Eliminate the first column below the leading diagonal to obtain

$$\begin{aligned} x_1 + x_2 &= 1\,, \\ 0 &= 0\,. \end{aligned}$$

14/07 The general solution is therefore $x_1 = 1 - x_2$, for any value of x_2.

Remarks.

(a) In order to avoid rounding error it is good practice to *partial pivot*, i.e. to reorder rows so that in the leading column the coefficient with the largest modulus is on the leading diagonal. For instance, at the first stage reorder the equations so that

$$|A_{11}| = \max(\,|A_{11}|, |A_{21}|, \dots, |A_{m1}|\,)\,.$$

(b) The operation count at stage r is $O((m-r)(n-r))$. If $m = n$ the total operation count will therefore be $O(n^3)$ ... which is significantly less than $O((n+1)!)$ if $n \gg 1$.

(c) The inverse and the determinant of a matrix can be calculated similarly in $O(n^3)$ operations (and it is possible to do even better).

(d) A linear system is said to be *consistent* if it has at least one solution, but *inconsistent* if it has no solutions at all.

(e) The following are referred to as elementary row operations:

   (i) interchange of two rows,
   (ii) addition of a constant multiple of one row to another,
   (iii) multiplication of a row by a non-zero constant.

   We refer to a linear system as being row-equivalent to another linear system if it can be obtained from that linear system by elementary row operations. Row equivalent linear systems have the same set of solutions.

   We also note that if we accompany the interchange of two columns with an appropriate relabelling of the x_i, we again obtain an equivalent linear system.

(f) In the case when $m = n$, so that the determinant of A is defined, the first two elementary row operations, i.e.

(i) interchange of two rows,

(ii) addition of a constant multiple of one row to another,

together with

(iii) interchange of two columns,

change the value of the determinant by at most a sign. We conclude that if there are k row and column swaps then $O(n^3)$ operations yields

$$\det \mathsf{A} = (-)^k \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1r} & A_{1\,r+1} & \dots & A_{1n} \\ 0 & A_{22}^{(2)} & \dots & A_{2r}^{(2)} & A_{2\,r+1}^{(2)} & \dots & A_{2n}^{(2)} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & A_{rr}^{(r)} & A_{r\,r+1}^{(r)} & \dots & A_{rn}^{(r)} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{vmatrix}, \tag{4.19a}$$

where $A_{11} \neq 0$ and $A_{jj}^{(j)} \neq 0$ $(j = 2, \dots, r)$. It follows that if $r < n$, then $\det \mathsf{A} = 0$, while if $r = n$

$$\det \mathsf{A} = (-)^k A_{11}\, A_{22}^{(2)} \dots A_{nn}^{(n)} \neq 0\,. \tag{4.19b}$$

4.4 The Rank of a Matrix

4.4.1 Definition

In (3.11) we defined the rank of a linear map $\mathcal{A}$, $\mathbb{R}^n \mapsto \mathbb{R}^m$ to be the dimension of the image, i.e.

$$r(\mathcal{A}) = \dim \mathcal{A}(\mathbb{R}^n)\,. \tag{4.20}$$

Let $\{\mathbf{e}_j\}$ $(j = 1, \dots, n)$ be a standard basis of $\mathbb{R}^n$. Then, since by definition a basis spans the domain, $\{\mathcal{A}(\mathbf{e}_j)\}$ $(j = 1, \dots, n)$ must span the image of $\mathcal{A}$. Further, the number of linearly independent vectors in this set must equal $r(\mathcal{A})$.

We also recall from (3.17b) and (3.22) that $\mathcal{A}(\mathbf{e}_j)$ $(j = 1, \dots, n)$ are the column vectors of the matrix A associated with the map $\mathcal{A}$. It follows that

$$\begin{aligned} r(\mathcal{A}) &= \dim \operatorname{span}\{\mathsf{A}\mathbf{e}_1, \mathsf{A}\mathbf{e}_2, \dots, \mathsf{A}\mathbf{e}_n\} \\ &= \text{number of linearly independent columns of } \mathsf{A}. \end{aligned}$$

Definition. The *column rank* of a matrix A is defined to be the maximum number of linearly independent columns of A. The *row rank* of a matrix A is defined to be the maximum number of linearly independent rows of A.

4.4.2 There is only one rank

Theorem 4.2. *The row rank of a matrix is equal to its column rank. We denote the rank of a matrix* A *by* $\operatorname{rank}\mathsf{A}$*, or* $r(\mathcal{A})$ *if there is an associated map* $\mathcal{A}$*.*

Proof. Let r be the row rank of the matrix A, i.e. suppose that A has a linearly independent set of r row vectors. Denote these row vectors by $\mathsf{v}_k^{\mathrm{T}}$ $(k = 1, \dots, r)$ where, in terms of matrix coefficients,

$$\mathsf{v}_k^{\mathrm{T}} = \begin{pmatrix} v_{k1} & v_{k2} & \dots & v_{kn} \end{pmatrix} \quad \text{for } k = 1, \dots, r\,.$$

Denote the i^{th} row of A, by $\mathsf{r}_i^{\mathrm{T}}$, i.e.

$$\mathsf{r}_i^{\mathrm{T}} = \begin{pmatrix} A_{i1} & A_{i2} & \dots & A_{in} \end{pmatrix} .$$

Then since every row of A can be written as a linear combination of the $\{\mathsf{v}_k^{\mathrm{T}}\}$ we have that

$$\mathsf{r}_i^{\mathrm{T}} = \sum_{k=1}^{r} c_{ik} \mathsf{v}_k^{\mathrm{T}} \quad (1 \leqslant i \leqslant m) ,$$

for some coefficients c_{ik} $(1 \leqslant i \leqslant m,\ 1 \leqslant k \leqslant r)$. In terms of matrix coefficients

$$A_{ij} = \sum_{k=1}^{r} c_{ik} v_{kj} \quad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n) .$$

Alternatively, this expression can be written as

$$\begin{pmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{pmatrix} = \sum_{k=1}^{r} v_{kj} \begin{pmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{mk} \end{pmatrix} .$$

We conclude that any column of A can be expressed as a linear combination of the r column vectors c_k $(1 \leqslant k \leqslant r)$, where

$$\mathsf{c}_k = \begin{pmatrix} c_{1k} \\ c_{2k} \\ \vdots \\ c_{mk} \end{pmatrix} \quad (1 \leqslant k \leqslant r).$$

Therefore the column rank of A must be less than or equal to r, i.e.

$$r(\mathcal{A}) \leqslant r .$$

We now apply the same argument to A^{T} to deduce that

$$r \leqslant r(\mathcal{A}) ,$$

and hence that $r = r(\mathcal{A}) \equiv \operatorname{rank} \mathsf{A}$. □

4.4.3 Calculation of rank

The number of linearly independent row vectors does not change under elementary row operations. Nor does the number of linearly independent row vectors change if we reorder the basis vectors $\{\mathbf{e}_j\}$ $(j = 1, \dots, n)$, i.e. reorder the columns. Hence we can use the technique of Gaussian elimination to calculate the rank. In particular, from (4.19a) we see that the row rank (and column rank) of the matrix A is r.

Remark. For the case when $m = n$ suppose that $\det \mathsf{A} = 0$. Then from (4.19b) it follows $r < n$, and thence from (4.19a) that the rows and columns of A are linearly dependent.

4.5 Solving Linear Equations: Homogeneous Problems

Henceforth we will restrict attention to the case $m = n$.

If $\det \mathsf{A} \neq 0$ then from Theorem 4.1 on page 68, A is invertible and A^{-1} exists. In such circumstances it follows that the system of equations $\mathsf{A}\mathbf{x} = \mathbf{d}$ has the *unique* solution $\mathbf{x} = \mathsf{A}^{-1}\mathbf{d}$.

If the system is homogeneous, i.e. if $\mathbf{d} = \mathbf{0}$, then if $\det \mathsf{A} \neq 0$ the unique solution is $\mathbf{x} = \mathsf{A}^{-1}\mathbf{0} = \mathbf{0}$. The contrapositive to this is that if $\mathsf{A}\mathbf{x} = \mathbf{0}$ has a solution with $\mathbf{x} \neq \mathbf{0}$, then $\det \mathsf{A} = 0$.

21/06 This section is concerned with understanding more fully what happens if $\det \mathsf{A} = 0$.

4.5.1 Geometrical view of $\mathsf{A}\mathbf{x} = \mathbf{0}$

We start with a geometrical view of the solutions to $\mathsf{A}\mathbf{x} = \mathbf{0}$ for the special case when A is a *real* 3×3 matrix.

As before let $\mathsf{r}_i^{\mathrm{T}}$ $(i = 1, 2, 3)$ be the row matrix with components equal to the elements of the ith row of A; hence

$$\mathsf{A} = \begin{pmatrix} \mathsf{r}_1^{\mathrm{T}} \\ \mathsf{r}_2^{\mathrm{T}} \\ \mathsf{r}_3^{\mathrm{T}} \end{pmatrix} . \tag{4.21}$$

The equations $\mathsf{A}\mathbf{x} = \mathbf{d}$ and $\mathsf{A}\mathbf{x} = \mathbf{0}$ may then be expressed as

$$\mathsf{r}_i^{\mathrm{T}} \mathsf{x} \;\equiv\; \mathbf{r}_i \cdot \mathbf{x} \;=\; d_i \qquad (i = 1, 2, 3)\,, \tag{4.22a}$$
$$\mathsf{r}_i^{\mathrm{T}} \mathsf{x} \;\equiv\; \mathbf{r}_i \cdot \mathbf{x} \;=\; 0 \qquad (i = 1, 2, 3)\,, \tag{4.22b}$$

respectively. Since each of these individual equations represents a plane in $\mathbb{R}^3$, the solution of each set of 3 equations is the intersection of 3 planes.

For the homogeneous equations (4.22b) the three planes each pass through the origin, O. There are three possibilities:

(i) the three planes intersect only at O;

(ii) the three planes have a common line (including O);

(iii) the three planes coincide.

22/03
15/08 We show that which of these cases occurs depends on $\operatorname{rank} \mathsf{A}$.

(i) If $\det \mathsf{A} \neq 0$ then $(\mathbf{r}_1 \times \mathbf{r}_2) \cdot \mathbf{r}_3 \neq 0$, and the set $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ consists of three linearly independent vectors; hence $\operatorname{span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\} = \mathbb{R}^3$ and $\operatorname{rank} \mathsf{A} = 3$. The first two equations of (4.22b) imply that $\mathbf{x}$ must lie on the intersection of the planes $\mathbf{r}_1 \cdot \mathbf{x} = 0$ and $\mathbf{r}_2 \cdot \mathbf{x} = 0$, i.e. $\mathbf{x}$ must lie on the line

$$\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda \mathbf{t}, \lambda \in \mathbb{R}, \mathbf{t} = \mathbf{r}_1 \times \mathbf{r}_2\}.$$

The final condition $\mathbf{r}_3 \cdot \mathbf{x} = 0$ then implies that $\lambda = 0$ (since we have assumed that $(\mathbf{r}_1 \times \mathbf{r}_2) \cdot \mathbf{r}_3 \neq 0$), and hence that $\mathbf{x} = \mathbf{0}$, i.e. the three planes intersect only at the origin. The *solution space*, i.e. the *kernel* of the associated map $\mathcal{A}$, thus has zero dimension when $\operatorname{rank} \mathsf{A} = 3$, i.e. $n(\mathcal{A}) = 0$.

(ii) Next suppose that $\det \mathsf{A} = 0$. In this case the set $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is linearly dependent with the dimension of $\operatorname{span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ being equal to 2 or 1. First we consider the case when it is 2, i.e. the case when $\operatorname{rank} \mathsf{A} = 2$. Assume wlog that $\mathbf{r}_1$ and $\mathbf{r}_2$ are two linearly independent vectors. Then as above the first two equations of (4.22b) again imply that $\mathbf{x}$ must lie on the line

$$\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda \mathbf{t}, \lambda \in \mathbb{R}, \mathbf{t} = \mathbf{r}_1 \times \mathbf{r}_2\}.$$

Since $(\mathbf{r}_1 \times \mathbf{r}_2) \cdot \mathbf{r}_3 = 0$, all points in this line satisfy $\mathbf{r}_3 \cdot \mathbf{x} = 0$. Hence the intersection of the three planes is a line, i.e. the solution for $\mathbf{x}$ is a line. The *solution space* thus has dimension one when $\operatorname{rank} \mathsf{A} = 2$, i.e. $n(\mathcal{A}) = 1$.

(iii) Finally we need to consider the case when the dimension of $\operatorname{span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is 1, i.e. $\operatorname{rank} \mathsf{A} = 1$. The three row vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ must then all be parallel. This means that each of $\mathbf{r}_1 \cdot \mathbf{x} = 0$, $\mathbf{r}_2 \cdot \mathbf{x} = 0$ and $\mathbf{r}_3 \cdot \mathbf{x} = 0$ implies the others. Thus the intersection of the three planes is a plane, i.e. solutions to (4.22b) lie on a plane. If $\mathbf{a}$ and $\mathbf{b}$ are any two linearly independent vectors such that $\mathbf{a} \cdot \mathbf{r}_1 = \mathbf{b} \cdot \mathbf{r}_1 = 0$, then we may specify the plane, and thus the solution space, by (cf. (2.89))

$$\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \lambda \mathbf{a} + \mu \mathbf{b} \text{ where } \lambda, \mu \in \mathbb{R}\}\,.$$

The *solution space* thus has dimension two when $\operatorname{rank} \mathsf{A} = 1$, i.e. $n(\mathcal{A}) = 2$.

15/07 *Remark.* In each case $r(\mathcal{A}) + n(\mathcal{A}) = 3$.

4.5.2 Linear mapping view of $\mathsf{A}\mathbf{x} = \mathbf{0}$

Consider the linear map $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$, such that $\mathbf{x} \mapsto \mathbf{x}' = \mathsf{A}\mathbf{x}$, where A is the matrix of $\mathcal{A}$ with respect to a basis. From our earlier definition (3.12), the kernel of $\mathcal{A}$ is given by

$$K(\mathcal{A}) = \{\mathbf{x} \in \mathbb{R}^n : \mathsf{A}\mathbf{x} = \mathbf{0}\} . \tag{4.23}$$

15/09 The subspace $K(\mathcal{A})$ is the solution space of $\mathsf{A}\mathbf{x} = \mathbf{0}$, with a dimension denoted by $n(\mathcal{A})$.

(i) If $n(\mathcal{A}) = 0$ then $\{\mathcal{A}(\mathbf{e}_j)\}$ $(j = 1, \dots, n)$ is a linearly independent set since

$$\begin{aligned} \left\{ \sum_{j=1}^{n} \lambda_j \mathcal{A}(\mathbf{e}_j) = \mathbf{0} \right\} \quad &\Leftrightarrow \quad \left\{ \mathcal{A}\left(\sum_{j=1}^{n} \lambda_j \mathbf{e}_j \right) = \mathbf{0} \right\} \\ &\Leftrightarrow \quad \left\{ \sum_{j=1}^{n} \lambda_j \mathbf{e}_j = \mathbf{0} \right\} , \end{aligned}$$

and so $\lambda_j = 0$ $(j = 1, \dots, n)$ since $\{\mathbf{e}_j\}$ $(j = 1, \dots, n)$ is a basis. It follows that the [column] rank
15/10 of $\mathcal{A}$ is n, i.e. $r(\mathcal{A}) = n$.

(ii) Suppose that $n_{\mathcal{A}} \equiv n(\mathcal{A}) > 0$. Let $\{\mathbf{u}_i\}$ $(i = 1, \dots, n_{\mathcal{A}})$ be a basis of the subspace $K(\mathcal{A})$. Next choose $\{\mathbf{v}_j \notin K(\mathcal{A})\}$ $(j = 1, \dots, n - n_{\mathcal{A}})$ to extend this basis to form a basis of $\mathbb{R}^n$ (although not proved, this is always possible). We claim that the set $\{\mathcal{A}(\mathbf{v}_j)\}$ $(j = 1, \dots, n - n_{\mathcal{A}})$ is linearly independent. To see this note that

$$\begin{aligned} \left\{ \sum_{j=1}^{n-n_{\mathcal{A}}} \lambda_j \mathcal{A}(\mathbf{v}_j) = \mathbf{0} \right\} \quad &\Leftrightarrow \quad \left\{ \mathcal{A}\left(\sum_{j=1}^{n-n_{\mathcal{A}}} \lambda_j \mathbf{v}_j \right) = \mathbf{0} \right\} \\ &\Leftrightarrow \quad \left\{ \sum_{j=1}^{n-n_{\mathcal{A}}} \lambda_j \mathbf{v}_j = \sum_{i=1}^{n_{\mathcal{A}}} \mu_i \mathbf{u}_i \right\} , \end{aligned}$$

for some μ_i $(i = 1, \dots, n_{\mathcal{A}})$. Hence

$$- \sum_{i=1}^{n_{\mathcal{A}}} \mu_i \mathbf{u}_i + \sum_{j=1}^{n-n_{\mathcal{A}}} \lambda_j \mathbf{v}_j = 0 ,$$

and so $\mu_1 = \ldots = \mu_{n_{\mathcal{A}}} = \lambda_1 = \ldots = \lambda_{n-n_{\mathcal{A}}} = 0$ since $\{\mathbf{u}_1, \dots, \mathbf{u}_{n_{\mathcal{A}}}, \mathbf{v}_1, \dots, \mathbf{v}_{n-n_{\mathcal{A}}}\}$ is a basis for $\mathbb{R}^n$. We conclude that the set $\{\mathcal{A}(\mathbf{v}_j)\}$ $(j = 1, \dots, n - n_{\mathcal{A}})$ is linearly independent, that

$$\dim \operatorname{span}\{\mathcal{A}(\mathbf{u}_1), \dots, \mathcal{A}(\mathbf{u}_{n_{\mathcal{A}}}), \mathcal{A}(\mathbf{v}_1), \dots, \mathcal{A}(\mathbf{v}_{n-n_{\mathcal{A}}})\} = n - n_{\mathcal{A}} ,$$

and thence that

$$r(\mathcal{A}) = n - n_{\mathcal{A}} .$$

Remark. The above proves in outline the Rank-Nullity Theorem (see Theorem 3.2 on page 44), i.e. that

$$r(\mathcal{A}) + n(\mathcal{A}) = \dim \mathbb{R}^n = \text{dimension of domain} . \tag{4.24}$$

4.6 The General Solution of the Inhomogeneous Equation $\mathsf{A}\mathbf{x} = \mathbf{d}$

Again we will restrict attention to the case $m = n$.

- If $\det \mathsf{A} \neq 0$ then $n(\mathcal{A}) = 0$, $r(\mathcal{A}) = n$ and $I(\mathcal{A}) = \mathbb{R}^n$ (where $I(\mathcal{A})$ is notation for the image of $\mathcal{A}$). Since $\mathbf{d} \in \mathbb{R}^n$, there must exist $\mathbf{x} \in \mathbb{R}^n$ for which $\mathbf{d}$ is the image under $\mathcal{A}$, i.e. $\mathbf{x} = \mathsf{A}^{-1}\mathbf{d}$ exists and is unique.

- If $\det \mathsf{A} = 0$ then $n(\mathcal{A}) > 0$, $r(\mathcal{A}) < n$ and $I(\mathcal{A})$ is a subspace of $\mathbb{R}^n$. Then

  either $\mathbf{d} \notin I(\mathcal{A})$, in which there are no solutions and the equations are *inconsistent*;

  or $\mathbf{d} \in I(\mathcal{A})$, in which case there is at least one solution and the equations are *consistent*.

The latter case is described by Theorem 4.3 below.

Theorem 4.3. *If $\mathbf{d} \in I(\mathcal{A})$ then the general solution to $\mathsf{A}\mathbf{x} = \mathbf{d}$ can be written as $\mathbf{x} = \mathbf{x}_0 + \mathbf{y}$ where $\mathbf{x}_0$ is a particular fixed solution of $\mathsf{A}\mathbf{x} = \mathbf{d}$ and $\mathbf{y}$ is the general solution of $\mathsf{A}\mathbf{x} = \mathbf{0}$.*

Proof. First we note that $\mathbf{x} = \mathbf{x}_0 + \mathbf{y}$ is a solution since $\mathsf{A}\mathbf{x}_0 = \mathbf{d}$ and $\mathsf{A}\mathbf{y} = \mathbf{0}$, and thus

$$\mathsf{A}(\mathbf{x}_0 + \mathbf{y}) = \mathbf{d} + \mathbf{0} = \mathbf{d}\,.$$

Further, if

(i) $n(\mathcal{A}) = 0$, then $\mathbf{y} = \mathbf{0}$ and the solution is unique.

(ii) $n(\mathcal{A}) > 0$ then in terms of the notation of §4.5.2

$$\mathbf{y} = \sum_{j=1}^{n(\mathcal{A})} \mu_j \mathbf{u}_j\,, \tag{4.25a}$$

and

$$\mathbf{x} = \mathbf{x}_0 + \sum_{j=1}^{n(\mathcal{A})} \mu_j \mathbf{u}_j\,. \tag{4.25b}$$

□

Remark. Let us return to the three cases studied in our geometric view of §4.5.1. Suppose that the equation is inhomogeneous, i.e. (4.22a), with a particular solution $\mathbf{x} = \mathbf{x}_0$, then

(i) $n(\mathcal{A}) = 0$, $r(\mathcal{A}) = 3$, $\mathbf{y} = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_0$ is the unique solution.

(ii) $n(\mathcal{A}) = 1$, $r(\mathcal{A}) = 2$, $\mathbf{y} = \lambda\mathbf{t}$ and $\mathbf{x} = \mathbf{x}_0 + \lambda\mathbf{t}$ (a line).

(iii) $n(\mathcal{A}) = 2$, $r(\mathcal{A}) = 1$, $\mathbf{y} = \lambda\mathbf{a} + \mu\mathbf{b}$ and $\mathbf{x} = \mathbf{x}_0 + \lambda\mathbf{a} + \mu\mathbf{b}$ (a plane).

Example. Consider the (2×2) inhomogeneous case of $\mathsf{A}\mathbf{x} = \mathbf{d}$ where

$$\begin{pmatrix} 1 & 1 \\ a & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ b \end{pmatrix}\,. \tag{4.26}$$

Since $\det \mathsf{A} = (1-a)$, if $a \neq 1$ then $\det \mathsf{A} \neq 0$ and A^{-1} exists and is unique. Specifically

$$\mathsf{A}^{-1} = \frac{1}{1-a}\begin{pmatrix} 1 & -1 \\ -a & 1 \end{pmatrix}, \text{ and the unique solution is } \mathbf{x} = \mathsf{A}^{-1}\begin{pmatrix} 1 \\ b \end{pmatrix}\,. \tag{4.27}$$

If $a = 1$, then $\det \mathsf{A} = 0$, and

$$\mathsf{A}\mathbf{x} = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} = (x_1 + x_2)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\,.$$

Hence

$$I(\mathcal{A}) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \quad \text{and} \quad K(\mathcal{A}) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\},$$

and so $r(\mathcal{A}) = 1$ and $n(\mathcal{A}) = 1$. Whether there is a solution now depends on the value of b.

- If $b \neq 1$ then $\begin{pmatrix} 1 \\ b \end{pmatrix} \notin I(\mathcal{A})$, and there are no solutions because the equations are inconsistent.
- If $b = 1$ then $\begin{pmatrix} 1 \\ b \end{pmatrix} \in I(\mathcal{A})$ and solutions exist (the equations are consistent). A particular solution is

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

  The general solution is then $\mathbf{x} = \mathbf{x}_0 + \mathbf{y}$, where $\mathbf{y}$ is any vector in $K(\mathcal{A})$, i.e.

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \end{pmatrix} ,$$

  where $\mu \in \mathbb{R}$.

22/06 *Definition.* A $n \times n$ square matrix A is said to be *singular* if $\det \mathsf{A} = 0$ and *non-singular* if $\det \mathsf{A} \neq 0$.

5 Eigenvalues and Eigenvectors

5.0 Why Study This?

The energy levels in quantum mechanics are eigenvalues, ionization potentials in Hartree-Fock theory are eigenvalues, the principal moments of inertia of an inertia tensor are eigenvalues, the resonance frequencies in mechanical systems (like a violin string, or a strand of DNA held fixed by optical tweezers, or the Tacoma Narrows bridge: see http://www.youtube.com/watch?v=3mclp9QmCGs or http://www.youtube.com/watch?v=j-zczJXSxnw or ...) are eigenvalues, etc.

5.1 Definitions and Basic Results

5.1.1 The Fundamental Theorem of Algebra

Let $p(z)$ be the polynomial of degree (or order) $m \geqslant 1$,

$$p(z) = \sum_{j=0}^{m} c_j z^j \,,$$

where $c_j \in \mathbb{C}$ $(j = 0, 1, \ldots, m)$ and $c_m \neq 0$. Then the *Fundamental Theorem of Algebra* states that the equation

$$p(z) \equiv \sum_{j=0}^{m} c_j z^j = 0 \,,$$

has a solution in $\mathbb{C}$.

Remark. This will be proved when you come to study complex variable theory.

Corollary. An important corollary of this result is that if $m \geqslant 1$, $c_j \in \mathbb{C}$ $(j = 0, 1, \ldots, m)$ and $c_m \neq 0$, then we can find $\omega_1, \omega_2, \ldots, \omega_m \in \mathbb{C}$ such that

$$p(z) \equiv \sum_{j=0}^{m} c_j z^j = c_m \prod_{j=1}^{m} (z - \omega_j) \,.$$

Remarks.

(a) If $(z-\omega)^k$ is a factor of $p(z)$, but $(z-\omega)^{k+1}$ is not, we say that ω is a k times repeated root of $p(z)$, or that the root ω has a *multiplicity* of k.

(b) A complex polynomial of degree m has precisely m roots (each counted with its multiplicity).

Example. The polynomial

$$z^3 - z^2 - z + 1 = (z-1)^2(z+1) \tag{5.1}$$

16/08 has the three roots -1, $+1$ and $+1$; the root $+1$ has a multiplicity of 2.

5.1.2 Eigenvalues and eigenvectors of maps

Let $\mathcal{A} : \mathbb{F}^n \to \mathbb{F}^n$ be a linear map, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Definition. If

$$\mathcal{A}(\mathbf{x}) = \lambda \mathbf{x} \tag{5.2}$$

for some *non-zero* vector $\mathbf{x} \in \mathbb{F}^n$ and $\lambda \in \mathbb{F}$, we say that $\mathbf{x}$ is an *eigenvector* of $\mathcal{A}$ with *eigenvalue* λ.

Remarks.

(a) Let ℓ be the subspace (or line in $\mathbb{R}^n$) defined by span$\{\mathbf{x}\}$. Then $\mathcal{A}(\ell) \subseteq \ell$, i.e. ℓ is an *invariant subspace* (or *invariant line*) under $\mathcal{A}$.

(b) If $\mathcal{A}(\mathbf{x}) = \lambda \mathbf{x}$ then

$$\mathcal{A}^m(\mathbf{x}) = \lambda^m \mathbf{x}.$$

and

$$(c_0 + c_1\mathcal{A} + \ldots + c_m\mathcal{A}^m)\,\mathbf{x} = (c_0 + c_1\lambda + \ldots + c_m\lambda^m)\,\mathbf{x},$$

where $c_j \in \mathbb{F}$ $(j = 0, 1, \ldots, m)$. Let $p(z)$ be the polynomial

$$p(z) = \sum_{j=0}^{m} c_j z^j.$$

Then for any polynomial p of $\mathcal{A}$, and an eigenvector $\mathbf{x}$ of $\mathcal{A}$,

16/07 $$p(\mathcal{A})\mathbf{x} = p(\lambda)\mathbf{x}.$$

5.1.3 Eigenvalues and eigenvectors of matrices

Suppose that A is the square $n \times n$ matrix associated with the map $\mathcal{A}$ for a given basis. Then consistent with the definition for maps, if

$$\mathsf{A}\mathbf{x} = \lambda \mathbf{x} \tag{5.3a}$$

for some *non-zero* vector $\mathbf{x} \in \mathbb{F}^n$ and $\lambda \in \mathbb{F}$, we say that $\mathbf{x}$ is an *eigenvector* of A with *eigenvalue* λ. This equation can be rewritten as

$$(\mathsf{A} - \lambda \mathsf{I})\mathbf{x} = \mathbf{0}. \tag{5.3b}$$

In §4.5 we concluded that an equation of the form (5.3b) has the unique solution $\mathbf{x} = \mathbf{0}$ if $\det(\mathsf{A} - \lambda\mathsf{I}) \neq 0$. It follows that if $\mathbf{x}$ is non-zero then

$$\det(\mathsf{A} - \lambda \mathsf{I}) = 0. \tag{5.4}$$

Further, we have seen that if $\det(\mathsf{A} - \lambda\mathsf{I}) = 0$, then there is at least one solution to (5.3b). We conclude that λ is an eigenvalue of A if and only if (5.4) is satisfied.

Definition. Equation (5.4) is called the *characteristic equation* of the matrix A.

Definition. The *characteristic polynomial* of the matrix A is the polynomial

$$p_{\mathsf{A}}(\lambda) = \det(\mathsf{A} - \lambda \mathsf{I}). \tag{5.5}$$

From the definition of the determinant, e.g. (3.71a) or (3.74b), p_{A} is an nth order polynomial in λ. The roots of the characteristic polynomial are the eigenvalues of A.

Example. Find the eigenvalues of

$$\mathsf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.6a}$$

Answer. From (5.4)

$$0 = \det(\mathsf{A} - \lambda \mathsf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = (\lambda - \imath)(\lambda + \imath)\,. \tag{5.6b}$$

Whether or not A has eigenvalues now depends on the space that A maps from and to. If A (or $\mathcal{A}$) maps $\mathbb{R}^2$ to $\mathbb{R}^2$ then A has no eigenvalues, since the roots of the characteristic polynomial are complex and from our definition only real eigenvalues are allowed. However, if A maps $\mathbb{C}^2$ to $\mathbb{C}^2$ the eigenvalues of A are $\pm\imath$. 16/09

Remark. The eigenvalues of a real matrix [that maps $\mathbb{C}^n$ to $\mathbb{C}^n$] need not be real.

Property. For maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ it follows from the Fundamental Theorem of Algebra and the stated corollary that a $n \times n$ matrix has n eigenvalues (each counted with its multiplicity).

Assumption. Henceforth we will assume, unless stated otherwise, that our maps are from $\mathbb{C}^n$ to $\mathbb{C}^n$.

Property. Suppose

$$p_{\mathsf{A}}(\lambda) = c_0 + c_1\lambda + \ldots + c_n\lambda^n\,, \tag{5.7a}$$

and that the n eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

(i) $$c_0 = \det(\mathsf{A}) = \lambda_1\lambda_2\ldots\lambda_n\,, \tag{5.7b}$$

(ii) $$c_{n-1} = (-)^{n-1}Tr(\mathsf{A}) = (-)^{n-1}(\lambda_1 + \lambda_2 + \ldots + \lambda_n)\,, \tag{5.7c}$$

(iii) $$c_n = (-)^n\,. \tag{5.7d}$$

Proof. From (3.71a)

$$\det(\mathsf{A} - \lambda\mathsf{I}) = \sum_{i_1 i_2 \ldots i_n} \varepsilon_{i_1 i_2 \ldots i_n}(A_{i_1 1} - \lambda\delta_{i_1 1})\ldots(A_{i_n n} - \lambda\delta_{i_n n})\,. \tag{5.7e}$$

The coefficient of λ^n comes from the single term $(A_{11} - \lambda)\ldots(A_{nn} - \lambda)$, hence $c_n = (-)^n$. It then follows that

$$p_{\mathsf{A}}(\lambda) = (\lambda_1 - \lambda)\ldots(\lambda_n - \lambda)\,. \tag{5.7f}$$

Next, the coefficient of λ^{n-1} also comes from the single term $(A_{11} - \lambda)\ldots(A_{nn} - \lambda)$. Hence

$$\begin{aligned} c_{n-1} &= (-)^{n-1}(A_{11} + A_{22} + \ldots + A_{nn}) \\ &= (-)^{n-1}Tr(\mathsf{A}) \\ &= (-)^{n-1}(\lambda_1 + \lambda_2 + \ldots + \lambda_n)\,, \end{aligned}$$

where the final line comes by identifying the coefficient of λ^{n-1} in (5.7f). Finally from (5.5), (5.7a) and (5.7f)

$$c_0 = p_{\mathsf{A}}(0) = \det\mathsf{A} = \lambda_1\lambda_2\ldots\lambda_n\,. \qquad \square$$

5.2 Eigenspaces, Eigenvectors, Bases and Diagonal Matrices

5.2.1 Eigenspaces and multiplicity

The set of all eigenvectors corresponding to an eigenvalue λ, together with $\mathbf{0}$, is the kernel of the linear map $(\mathcal{A} - \lambda\mathcal{I})$; hence, from Theorem 3.1, it is a subspace. This subspace is called the *eigenspace* of λ, and we will denote it by E_λ.

Check that E_λ is a subspace (Unlectured). Suppose that $\mathbf{w}_i \in E_\lambda$ for $i = 1, \ldots, r$, then

$$\mathsf{A}\mathbf{w}_i = \lambda\mathbf{w}_i \quad \text{for} \quad i = 1, \ldots, r.$$

Let

$$\mathbf{v} = \sum_{i=1}^{r} c_i \mathbf{w}_i$$

where $c_i \in \mathbb{C}$, then

$$\begin{aligned}\mathsf{A}\mathbf{v} &= \sum_{i=1}^{r} c_i \mathsf{A}\mathbf{w}_i \\ &= \lambda \sum_{i=1}^{r} c_i \mathbf{w}_i \\ &= \lambda \mathbf{v}\,,\end{aligned}$$

i.e. $\mathbf{v}$ is also an eigenvector with the same eigenvalue λ, and hence $\mathbf{v} \in E_\lambda$. Thence from Theorem 2.2 E_λ is a subspace.

Definition. The multiplicity of an eigenvalue λ as a root of the characteristic polynomial is called the *algebraic multiplicity* of λ, which we will denote by M_λ. If the characteristic polynomial has degree n, then

$$\sum_\lambda M_\lambda = n\,. \tag{5.8a}$$

An eigenvalue with an algebraic multiplicity greater then one is said to be *degenerate*.

Definition. The maximum number, m_λ, of linearly independent eigenvectors corresponding to λ is called the *geometric multiplicity* of λ. From the definition of the eigenspace of λ it follows that

$$m_\lambda = \dim E_\lambda. \tag{5.8b}$$

Unproved Statement. It is possible to prove that $m_\lambda \leqslant M_\lambda$.

Definition. The difference $\Delta_\lambda = M_\lambda - m_\lambda$ is called the *defect* of λ.

Remark. We shall see below that *if* the eigenvectors of a map form a basis of $\mathbb{F}^n$ (i.e. if there is no eigenvalue with strictly positive defect), then it is possible to analyse the behaviour of that map (and associated matrices) in terms of these eigenvectors. When the eigenvectors do not form a basis then we need the concept of generalised eigenvectors (see *Linear Algebra* for details).

Definition. A vector $\mathbf{x}$ is a *generalised eigenvector* of a map $\mathcal{A} : \mathbb{F}^n \mapsto \mathbb{F}^n$ if there is some eigenvalue $\lambda \in \mathbb{F}^n$ and some $k \in \mathbb{N}$ such that

$$(\mathcal{A} - \lambda\mathcal{I})^k(\mathbf{x}) = \mathbf{0}\,. \tag{5.9}$$

Unproved Statement. The generalised eigenvectors of a map $\mathcal{A} : \mathbb{F}^n \mapsto \mathbb{F}^n$ span $\mathbb{F}^n$.

5.2.2 Linearly independent eigenvectors

Theorem 5.1. *Suppose that the linear map $\mathcal{A}$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ (i.e. $\lambda_i \neq \lambda_j$ if $i \neq j$), and corresponding non-zero eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_r$, then $\mathbf{x}_1, \ldots, \mathbf{x}_r$ are linearly independent.*

Proof. Argue by contradiction by supposing that the proposition is false, i.e. suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_r$ are linearly dependent. In particular suppose that there exist c_i $(i = 1, \ldots, r)$, not all of which are zero, such that

$$\sum_{i=1}^{r} c_i \mathbf{x}_i = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_r\mathbf{x}_r = \mathbf{0}\,. \tag{5.10a}$$

Apply the operator

$$(\mathcal{A}-\lambda_1\mathcal{I})\dots(\mathcal{A}-\lambda_{K-1}\mathcal{I})(\mathcal{A}-\lambda_{K+1}\mathcal{I})\dots(\mathcal{A}-\lambda_r\mathcal{I}) = \prod_{k=1,\dots,\overline{K},\dots,r}(\mathcal{A}-\lambda_k\mathcal{I}) \tag{5.10b}$$

to (5.10a) to conclude that

$$\begin{aligned} 0 &= \prod_{k=1,\dots,\overline{K},\dots,r}(\mathcal{A}-\lambda_k\mathcal{I})\sum_{i=1}^{r} c_i\mathbf{x}_i \\ &= \sum_{i=1}^{r}\prod_{k=1,\dots,\overline{K},\dots,r} c_i\,(\lambda_i-\lambda_k)\,\mathbf{x}_i \\ &= \left(\prod_{k=1,\dots,\overline{K},\dots,r}(\lambda_K-\lambda_k)\right) c_K\mathbf{x}_K\,. \end{aligned}$$

However, the eigenvalues are distinct and the eigenvectors are non-zero. It follows that

$$c_K = 0 \quad (K=1,\dots,r), \tag{5.10c}$$

which is a contradiction. □

Equivalent Proof (without use of the product sign: unlectured). As above suppose that the proposition is false, i.e. suppose that $\mathbf{x}_1,\dots,\mathbf{x}_r$ are linearly dependent. Let

$$d_1\mathbf{x}_1 + d_2\mathbf{x}_2 + \dots + d_m\mathbf{x}_m = \mathbf{0}\,, \tag{5.10d}$$

where $d_i \neq 0$ $(i=1,\dots,m)$, be the shortest non-trivial combination of linearly dependent eigenvectors (if necessary relabel the eigenvalues and eigenvectors so it is the first m that constitute the shortest non-trivial combination of linearly dependent eigenvectors). Apply $(\mathcal{A}-\lambda_1\mathcal{I})$ to (5.10d) to conclude that

$$d_2(\lambda_2-\lambda_1)\mathbf{x}_2 + \dots + d_m(\lambda_m-\lambda_1)\mathbf{x}_m = \mathbf{0}\,.$$

Since all the eigenvalues are assumed to be distinct we now have a contradiction since we have a shorter non-trivial combination of linearly dependent eigenvectors. □

Remark. Since a set of linearly independent vectors in $\mathbb{F}^n$ can be no larger than n, a map $\mathcal{A}:\mathbb{F}^n \mapsto \mathbb{F}^n$ can have at most n distinct eigenvalues.[27]

Property. If a map $\mathcal{A}:\mathbb{F}^n \mapsto \mathbb{F}^n$ has n distinct eigenvalues then it must have n linearly independent eigenvectors, and these eigenvectors must be a basis for $\mathbb{F}^n$.

Remark. If all eigenvalues of a map $\mathcal{A}:\mathbb{F}^n \mapsto \mathbb{F}^n$ are distinct then the eigenvectors span $\mathbb{F}^n$. If all the eigenvalues are not distinct, then sometimes it is possible to find n eigenvectors that span $\mathbb{F}^n$, and sometimes it is not (see examples (ii) and (iii) below respectively).

5.2.3 Examples

(i) From (5.6b) the eigenvalues of

$$\mathsf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are distinct and equal to $\pm\imath$. In agreement with Theorem 5.1 the corresponding eigenvectors

$$\lambda_1 = +\imath: \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ +\imath \end{pmatrix},$$

$$\lambda_2 = -\imath: \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -\imath \end{pmatrix},$$

are linearly independent, and are a basis for $\mathbb{C}^2$ (but not $\mathbb{R}^2$). We note that

$$M_\imath = m_\imath = 1,\ \Delta_\imath = 0\,, \quad M_{-\imath} = m_{-\imath} = 1,\ \Delta_{-\imath} = 0\,.$$

[27] But we already knew that.

(ii) Let A be the 3×3 matrix

$$\mathsf{A} = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}.$$

This has the characteristic equation

$$p_{\mathsf{A}} = -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0\,,$$

with eigenvalues $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$. The eigenvector corresponding to $\lambda_1 = 5$ is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

For $\lambda_2 = \lambda_3 = -3$ the equation for the eigenvectors becomes

$$(\mathsf{A} + 3\mathsf{I})\mathbf{x} = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \mathbf{x} = \mathbf{0}\,.$$

This can be reduced by elementary row operations to

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}\,,$$

with general solution

$$\mathbf{x} = \begin{pmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{pmatrix}.$$

In this case, although two of the eigenvalues are equal, a basis of linearly independent eigenvectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ can be obtained by choosing, say,

$$\mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \qquad (x_2 = 1,\ x_3 = 0)\,,$$

$$\mathbf{x}_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \qquad (x_2 = 0,\ x_3 = 1)\,.$$

We conclude that

$$M_5 = m_5 = 1,\ \Delta_5 = 0\,, \quad M_{-3} = m_{-3} = 2,\ \Delta_{-3} = 0\,.$$

(iii) Let A be the 3×3 matrix

$$\mathsf{A} = \begin{pmatrix} -3 & -1 & 1 \\ -1 & -3 & 1 \\ -2 & -2 & 0 \end{pmatrix}. \tag{5.11a}$$

This has the characteristic equation

$$p_{\mathsf{A}} = -(\lambda + 2)^3 = 0\,, \tag{5.11b}$$

with eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = -2$. Therefore there is a single equation for the eigenvectors, viz

$$(\mathsf{A} + 2\mathsf{I})\mathbf{x} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix} \mathbf{x} = \mathbf{0}\,.$$

This can be reduced by elementary row operations to

$$\begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}\,,$$

with general solution

$$\mathbf{x} = \begin{pmatrix} -x_2 + x_3 \\ x_2 \\ x_3 \end{pmatrix}.$$

In this case only two linearly independent eigenvectors can be constructed, say,

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \tag{5.11c}$$

and hence the eigenvectors do not span $\mathbb{C}^3$ (or $\mathbb{R}^3$) and cannot form a basis. We conclude that

$$M_{-2} = 3,\ m_{-2} = 2,\ \Delta_{-2} = 1\,.$$

Remarks.

- There are many alternative sets of linearly independent eigenvectors to (5.11c); for instance the orthonormal eigenvectors

$$\mathbf{x}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \tag{5.11d}$$

- Once we know that there is an eigenvalue of algebraic multiplicity 3, then there is a better way to deduce that the eigenvectors do not span $\mathbb{C}^3$ ... *if* we can change basis (see below and
17/07 *Example Sheet 3*).

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5.2.4 Diagonal matrices

Definition. A $n \times n$ matrix $\mathsf{D} = \{D_{ij}\}$ is a diagonal matrix if $D_{ij} = 0$ whenever $i \neq j$, i.e. if

$$\mathsf{D} = \begin{pmatrix} D_{11} & 0 & 0 & \dots & 0 & 0 \\ 0 & D_{22} & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & D_{n-1\,n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & D_{nn} \end{pmatrix}. \tag{5.12a}$$

Remark. $D_{ij} = D_{ii}\delta_{ij} = D_{jj}\delta_{ij}$ (no summation convention), hence

$$(D^2)_{ij} = \sum_k D_{ik}D_{kj} = \sum_k D_{ii}\delta_{ik}D_{kk}\delta_{kj} = D_{ii}^2\delta_{ij}\,, \tag{5.12b}$$

and so

$$\mathsf{D}^m = \begin{pmatrix} D_{11}^m & 0 & 0 & \dots & 0 & 0 \\ 0 & D_{22}^m & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & D_{n-1\,n-1}^m & 0 \\ 0 & 0 & 0 & \dots & 0 & D_{nn}^m \end{pmatrix}. \tag{5.12c}$$

5.2.5 Eigenvectors as a basis lead to a diagonal matrix

Suppose that the linear map $\mathcal{A} : \mathbb{F}^n \mapsto \mathbb{F}^n$ has n linearly independent eigenvectors, $(\mathbf{v}_1, \dots, \mathbf{v}_n)$, e.g. because $\mathcal{A}$ has n distinct eigenvalues. Choose the eigenvectors as a basis, and note that

$$\mathcal{A}(\mathbf{v}_i) = 0\mathbf{v}_1 + \dots + \lambda_i \mathbf{v}_i + \dots + 0\mathbf{v}_n \,.$$

It follows from (3.22) that the matrix representing the map $\mathcal{A}$ with respect to the basis of eigenvectors is diagonal, with the diagonal elements being the eigenvalues, viz.

$$\mathsf{A} = \begin{pmatrix} \mathcal{A}(\mathbf{v}_1) & \mathcal{A}(\mathbf{v}_2) & \dots & \mathcal{A}(\mathbf{v}_n) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} . \tag{5.13}$$

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Converse. If the matrix A representing the map $\mathcal{A}$ with respect to a given basis is diagonal, then the basis elements are eigenvectors and the diagonal elements of A are eigenvalues (since $\mathcal{A}(\mathbf{e}_i) = A_{ii}\mathbf{e}_i$).

Remark. If we wished to calculate powers of A it follows from (5.12c) that there would be an advantage in choosing the eigenvectors as a basis. As an example of where such a need might arise let n towns be called (rather uninterestingly) 1, 2, …, n. Write $A_{ij} = 1$ if there is a road leading directly from town i to town j and $A_{ij} = 0$ otherwise (we take $A_{ii} = 0$). If we write $\mathsf{A}^m = \{A_{ij}^{(m)}\}$ then $A_{ij}^{(m)}$ is the number of routes from i to j of length m. (A route of length m passes through $m+1$ towns including the starting and finishing towns. If you pass through the same town more than once each visit is counted separately.)

5.3 Change of Basis

We have now identified as least two types of 'nice' bases, i.e. orthonormal bases and bases of eigenvectors. A linear map $\mathcal{A}$ does not change if we change basis, but the matrix representing it does. The aim of this section, which is somewhat of a diversion from our study of eigenvalues and eigenvectors, is to work out how the elements of a matrix transform under a change of basis.

To fix ideas it may help to think of a change of basis from the standard orthonormal basis in $\mathbb{R}^3$ to a new basis which is not necessarily orthonormal. However, we will work in $\mathbb{F}^n$ and will not assume that either basis is orthonormal.

5.3.1 Transformation matrices

Let $\{\mathbf{e}_i : i = 1, \dots, n\}$ and $\{\widetilde{\mathbf{e}}_i : i = 1, \dots, n\}$ be two sets of basis vectors for $\mathbb{F}^n$. Since the $\{\mathbf{e}_i\}$ is a basis, the individual basis vectors of the basis $\{\widetilde{\mathbf{e}}_i\}$ can be written as

$$\widetilde{\mathbf{e}}_j = \sum_{i=1}^{n} \mathbf{e}_i \, P_{ij} \quad (j = 1, \dots, n)\,, \tag{5.14a}$$

for some numbers P_{ij}, where P_{ij} is the ith component of the vector $\widetilde{\mathbf{e}}_j$ in the basis $\{\mathbf{e}_i\}$. The numbers P_{ij} can be represented by a square $n \times n$ *transformation matrix* P

$$\mathsf{P} = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{e}}_1 & \widetilde{\mathbf{e}}_2 & \dots & \widetilde{\mathbf{e}}_n \end{pmatrix} , \tag{5.14b}$$

where, as in (3.22), the $\widetilde{\mathbf{e}}_j$ on the right-hand side are to be interpreted as column vectors of the components of the $\widetilde{\mathbf{e}}_j$ in the $\{\mathbf{e}_i\}$ basis. P is the matrix with respect to the basis $\{\mathbf{e}_i\}$ of the linear map $\mathcal{P}$ that transforms the $\{\mathbf{e}_i\}$ to the $\{\widetilde{\mathbf{e}}_i\}$

Similarly, since the $\{\widetilde{\mathbf{e}}_i\}$ is a basis, the individual basis vectors of the basis $\{\mathbf{e}_i\}$ can be written as

$$\mathbf{e}_i = \sum_{k=1}^{n} \widetilde{\mathbf{e}}_k \, Q_{ki} \quad (i = 1, 2, \dots, n)\,, \tag{5.15a}$$

for some numbers Q_{ki}, where Q_{ki} is the kth component of the vector $\mathbf{e}_i$ in the basis $\{\widetilde{\mathbf{e}}_k\}$. Again the Q_{ki} can be viewed as the entries of a matrix Q

$$\mathsf{Q} = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \cdots & Q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & \cdots & Q_{nn} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \end{pmatrix}\,, \tag{5.15b}$$

where in the final matrix the $\mathbf{e}_j$ are to be interpreted as column vectors of the components of the $\mathbf{e}_j$ in the $\{\widetilde{\mathbf{e}}_i\}$ basis.

5.3.2 Properties of transformation matrices

From substituting (5.15a) into (5.14a) we have that

$$\widetilde{\mathbf{e}}_j = \sum_{i=1}^{n} \left[\sum_{k=1}^{n} \widetilde{\mathbf{e}}_k Q_{ki} \right] P_{ij} = \sum_{k=1}^{n} \widetilde{\mathbf{e}}_k \left[\sum_{i=1}^{n} Q_{ki} P_{ij} \right]. \tag{5.16}$$

However, the set $\{\widetilde{\mathbf{e}}_j\}$ is a basis and so linearly independent. Thus, from noting that

$$\widetilde{\mathbf{e}}_j = \sum_{k=1}^{n} \widetilde{\mathbf{e}}_k \, \delta_{kj}\,,$$

or otherwise, it follows that

$$\sum_{i=1}^{n} Q_{ki} P_{ij} = \delta_{kj}\,. \tag{5.17}$$

Hence in matrix notation, $\mathsf{QP} = \mathsf{I}$, where I is the identity matrix. Conversely, substituting (5.14a) into (5.15a) leads to the conclusion that $\mathsf{PQ} = \mathsf{I}$ (alternatively argue by a relabeling symmetry). Thus

$$\mathsf{Q} = \mathsf{P}^{-1}\,. \tag{5.18}$$

5.3.3 Transformation law for vector components

Consider a vector $\mathbf{u}$, and suppose that in terms of the $\{\mathbf{e}_i\}$ basis it can be written in component form as

$$\mathbf{u} = \sum_{i=1}^{n} u_i \mathbf{e}_i\,. \tag{5.19a}$$

Similarly, in the $\{\widetilde{\mathbf{e}}_i\}$ basis suppose that $\mathbf{u}$ can be written in component form as

$$\begin{aligned} \mathbf{u} &= \sum_{j=1}^{n} \widetilde{u}_j \widetilde{\mathbf{e}}_j && \qquad (5.19\text{b}) \\ &= \sum_{j=1}^{n} \widetilde{u}_j \sum_{i=1}^{n} \mathbf{e}_i P_{ij} && \text{from (5.14a)} \\ &= \sum_{i=1}^{n} \mathbf{e}_i \sum_{j=1}^{n} P_{ij} \widetilde{u}_j && \text{swap summation order.} \end{aligned}$$

Since a basis representation is unique it follows from (5.19a) that

$$u_i = \sum_{j=1}^{n} P_{ij}\widetilde{u}_j \quad (i = 1, \ldots, n)\,, \tag{5.20a}$$

i.e. that

$$\mathsf{u} = \mathsf{P}\widetilde{\mathsf{u}}\,, \tag{5.20b}$$

where we have deliberately used a $\mathsf{sans\ serif}$ font to indicate the column matrices of components (since otherwise there is serious ambiguity). By applying P^{-1} to both sides of (5.20b) it follows that

$$\widetilde{\mathsf{u}} = \mathsf{P}^{-1}\mathsf{u}\,. \tag{5.20c}$$

Equations (5.20b) and (5.20c) relate the components of $\mathbf{u}$ with respect to the $\{\mathbf{e}_i\}$ basis to the components of $\mathbf{u}$ with respect to the $\{\widetilde{\mathbf{e}}_i\}$ basis.

Remark. Note from (5.14a) and (5.20a), i.e.

$$\widetilde{\mathbf{e}}_j = \sum_{i=1}^{n} \mathbf{e}_i\, P_{ij} \quad (j = 1, \ldots, n)\,,$$
$$u_i = \sum_{j=1}^{n} P_{ij}\widetilde{u}_j \quad (i = 1, \ldots, n)\,,$$

that the basis vectors and coordinates in some sense 'go opposite ways'. Compare also the definition of A_{ij} from (3.18a) with the relationship (3.19d) between the components of $\mathbf{x}' = \mathcal{A}(\mathbf{x})$ and $\mathbf{x}$

$$\mathcal{A}(\mathbf{e}_j) = \sum_{i=1}^{m} \mathbf{f}_i\, A_{ij} \quad (j = 1, \ldots, n)\,,$$
$$x'_i = (\mathcal{A}(\mathbf{x}))_i = \sum_{j=1}^{n} A_{ij}x_j \quad (i = 1, \ldots, n)\,.$$

These relations also in some sense 'go opposite ways'.

Worked Example. Let $\{\mathbf{e}_1 = (1, 0),\ \mathbf{e}_2 = (0, 1)\}$ and $\{\widetilde{\mathbf{e}}_1 = (1, 1),\ \widetilde{\mathbf{e}}_2 = (-1, 1)\}$ be two sets of basis vectors in $\mathbb{R}^2$. Find the transformation matrix P that connects them. Verify the transformation law for the components of an arbitrary vector $\mathbf{u}$ in the two coordinate systems.

Answer. We have that

$$\widetilde{\mathbf{e}}_1 = (\ \ 1, 1) = \quad (1, 0) + (0, 1) = \ \ \mathbf{e}_1 + \mathbf{e}_2\,,$$
$$\widetilde{\mathbf{e}}_2 = (-1, 1) = -1\,(1, 0) + (0, 1) = -\mathbf{e}_1 + \mathbf{e}_2\,.$$

Hence from comparison with (5.14a)

$$P_{11} = 1\,, \quad P_{21} = 1\,, \quad P_{12} = -1 \quad \text{and} \quad P_{22} = 1\,,$$

i.e.

$$\mathsf{P} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Similarly, since

$$\mathbf{e}_1 = (1, 0) = \tfrac{1}{2}(\ (1, 1) - (-1, 1)\ ) = \tfrac{1}{2}(\widetilde{\mathbf{e}}_1 - \widetilde{\mathbf{e}}_2)\,,$$
$$\mathbf{e}_2 = (0, 1) = \tfrac{1}{2}(\ (1, 1) + (-1, 1)\ ) = \tfrac{1}{2}(\widetilde{\mathbf{e}}_1 + \widetilde{\mathbf{e}}_2)\,,$$

it follows from (5.15a) that

$$\mathsf{Q} = \mathsf{P}^{-1} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Exercise. Check that $\mathsf{P}^{-1}\mathsf{P} = \mathsf{P}\mathsf{P}^{-1} = \mathsf{I}$.

Now consider an arbitrary vector $\mathbf{u}$. Then

$$\begin{aligned}\mathbf{u} &= u_1\mathbf{e}_1 + u_2\mathbf{e}_2\\ &= \tfrac{1}{2}u_1(\widetilde{\mathbf{e}}_1 - \widetilde{\mathbf{e}}_2) + \tfrac{1}{2}u_2(\widetilde{\mathbf{e}}_1 + \widetilde{\mathbf{e}}_2)\\ &= \tfrac{1}{2}(u_1 + u_2)\,\widetilde{\mathbf{e}}_1 - \tfrac{1}{2}(u_1 - u_2)\,\widetilde{\mathbf{e}}_2\,.\end{aligned}$$

Thus

$$\widetilde{u}_1 = \tfrac{1}{2}(u_1 + u_2) \quad \text{and} \quad \widetilde{u}_2 = -\tfrac{1}{2}(u_1 - u_2)\,,$$

and thus from (5.20c), i.e. $\widetilde{\mathsf{u}} = \mathsf{P}^{-1}\mathsf{u}$, we deduce that (as above)

$$\mathsf{P}^{-1} = \tfrac{1}{2}\begin{pmatrix} 1 & 1\\ -1 & 1\end{pmatrix}.$$

5.3.4 Transformation law for matrices representing linear maps from $\mathbb{F}^n$ to $\mathbb{F}^n$

Now consider a linear map $\mathcal{A} : \mathbb{F}^n \to \mathbb{F}^n$ under which $\mathbf{u} \mapsto \mathbf{u}' = \mathcal{A}(\mathbf{u})$ and (in terms of column vectors)

$$\mathsf{u}' = \mathsf{A}\mathsf{u}\,,$$

where u' and u are the component column matrices of $\mathbf{u}'$ and $\mathbf{u}$, respectively, with respect to the basis $\{\mathbf{e}_i\}$, and A is the matrix of $\mathcal{A}$ with respect to this basis.

Let $\widetilde{\mathsf{u}}'$ and $\widetilde{\mathsf{u}}$ be the component column matrices of $\mathbf{u}'$ and $\mathbf{u}$ with respect to an alternative basis $\{\widetilde{\mathbf{e}}_i\}$. Then it follows from (5.20b) that

$$\mathsf{P}\widetilde{\mathsf{u}}' = \mathsf{A}\mathsf{P}\widetilde{\mathsf{u}}\,, \quad \text{i.e.} \quad \widetilde{\mathsf{u}}' = (\mathsf{P}^{-1}\mathsf{A}\mathsf{P})\,\widetilde{\mathsf{u}}\,.$$

We deduce that the matrix of $\mathcal{A}$ with respect to the alternative basis $\{\widetilde{\mathbf{e}}_i\}$ is given by

$$\widetilde{\mathsf{A}} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}\,. \tag{5.21}$$

Example. Consider a simple shear with magnitude γ in the x_1 direction within the (x_1, x_2) plane. Then from (3.30) the matrix of this map with respect to the standard basis $\{\mathbf{e}_i\}$ is

$$\mathsf{S}_\gamma = \begin{pmatrix} 1 & \gamma & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Let $\{\widetilde{\mathbf{e}}\}$ be the basis obtained by rotating the standard basis by an angle θ about the x_3 axis. Then

$$\begin{aligned}\widetilde{\mathbf{e}}_1 &= \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2\,,\\ \widetilde{\mathbf{e}}_2 &= -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2\,,\\ \widetilde{\mathbf{e}}_3 &= \mathbf{e}_3\,,\end{aligned}$$

and thus

$$\mathsf{P}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.$$

We have already deduced that rotation matrices are orthogonal, see (3.58a) and (3.58b), and hence

$$\mathsf{P}_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} = \mathsf{P}_{-\theta}\,.$$

Alternatively, we could have deduced that $\mathsf{P}_\theta^{-1} = \mathsf{P}_{-\theta}$ by noting that the inverse of a rotation by θ is a rotation by $-\theta$. The matrix of the shear map with respect to new basis is thus given by

$$\begin{aligned}\widetilde{\mathsf{S}}_\gamma &= \mathsf{P}^{-1}\mathsf{S}_\gamma\mathsf{P}\\ &= \begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta+\gamma\sin\theta & -\sin\theta+\gamma\cos\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1+\gamma\sin\theta\cos\theta & \gamma\cos^2\theta & 0\\ -\gamma\sin^2\theta & 1-\gamma\sin\theta\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.\end{aligned}$$

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5.3.5 Transformation law for matrices representing linear maps from $\mathbb{F}^n$ to $\mathbb{F}^m$

A similar approach may be used to deduce the matrix of the map $\mathcal{A} : \mathbb{F}^n \to \mathbb{F}^m$ (where $m \neq n$) with respect to new bases of both $\mathbb{F}^n$ and $\mathbb{F}^m$.

Suppose that, as in §3.5, $\{\mathbf{e}_i\}$ $(i = 1, \dots, n)$ is a basis of $\mathbb{F}^n$, $\{\mathbf{f}_j\}$ $(j = 1, \dots, m)$ is a basis of $\mathbb{F}^m$, and A is the matrix of $\mathcal{A}$ with respect to these two bases. As before

$$\mathsf{u} \mapsto \mathsf{u}' = \mathsf{A}\,\mathsf{u} \tag{5.22a}$$

where u and u' are component column matrices of $\mathbf{u}$ and $\mathbf{u}'$ with respect to bases $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_j\}$ respectively. Now consider new bases $\{\widetilde{\mathbf{e}}_i\}$ of $\mathbb{F}^n$ and $\{\widetilde{\mathbf{f}}_j\}$ of $\mathbb{F}^m$, and let P and S be the transformation matrices for components between the $\{\mathbf{e}_i\}$ and $\{\widetilde{\mathbf{e}}_i\}$ bases, and between the $\{\mathbf{f}_j\}$ and $\{\widetilde{\mathbf{f}}_j\}$ bases, respectively. Then

$$\mathsf{P} = \begin{pmatrix}\widetilde{\mathbf{e}}_1 & \dots & \widetilde{\mathbf{e}}_n\end{pmatrix} \quad\text{and}\quad \mathsf{S} = \begin{pmatrix}\widetilde{\mathbf{f}}_1 & \dots & \widetilde{\mathbf{f}}_m\end{pmatrix},$$

where P is a $n\times n$ matrix of components (see (5.14b)) and S is a $m \times m$ matrix of components.

From (5.20a)

$$\mathsf{u} = \mathsf{P}\widetilde{\mathsf{u}}\,, \quad\text{and}\quad \mathsf{u}' = \mathsf{S}\widetilde{\mathsf{u}}'\,,$$

where $\widetilde{\mathsf{u}}$ and $\widetilde{\mathsf{u}}'$ are component column matrices of $\mathbf{u}$ and $\mathbf{u}'$ with respect to bases $\{\widetilde{\mathbf{e}}_i\}$ and $\{\widetilde{\mathbf{f}}_j\}$ respectively. Hence from (5.22a)

$$\mathsf{S}\widetilde{\mathsf{u}}' = \mathsf{A}\mathsf{P}\widetilde{\mathsf{u}}\,, \quad\text{and so}\quad \widetilde{\mathsf{u}}' = \mathsf{S}^{-1}\mathsf{A}\mathsf{P}\widetilde{\mathsf{u}}\,.$$

It follows that $\mathsf{S}^{-1}\mathsf{A}\mathsf{P}$ is the matrix of the map $\mathcal{A}$ with respect to the new bases $\{\widetilde{\mathbf{e}}_i\}$ and $\{\widetilde{\mathbf{f}}_j\}$, i.e.

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$$\widetilde{\mathsf{A}} = \mathsf{S}^{-1}\mathsf{A}\mathsf{P}\,. \tag{5.22b}$$

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5.4 Similar Matrices

Definition. The $n \times n$ matrices A and B are *similar*, or *conjugate*[28], if for some invertible matrix P

$$\mathsf{B} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}\,. \tag{5.23}$$

Remarks.

(a) Similarity is an *equivalence relation*.

(b) A map from A to $\mathsf{P}^{-1}\mathsf{A}\mathsf{P}$ is sometimes known as a *similarity transformation*.

[28] As in the *Groups* course.

(c) From (5.21) the matrices representing a map $\mathcal{A}$ with respect to different bases are similar.

(d) The identity matrix (or a multiple of it) is similar only with itself (or a multiple of it) since

$$\mathsf{P}^{-1}\mathsf{I}\mathsf{P} = \mathsf{I}\,. \tag{5.24}$$

Exercise (see Example Sheet 4). Show that a $n \times n$ matrix with a unique eigenvalue (i.e. an eigenvalue with an algebraic multiplicity of n), and with n linearly independent eigenvectors, has to be a
18/10 multiple of the identity matrix.

Properties. Similar matrices have the same determinant and trace, since

(i)
$$\begin{aligned}\det(\mathsf{P}^{-1}\mathsf{A}\mathsf{P}) &= \det \mathsf{P}^{-1}\, \det \mathsf{A}\, \det \mathsf{P}\\ &= \det \mathsf{A}\, \det(\mathsf{P}^{-1}\mathsf{P})\\ &= \det \mathsf{A}\,,\end{aligned} \tag{5.25a}$$

(ii)
$$\begin{aligned}Tr(\mathsf{P}^{-1}\mathsf{A}\mathsf{P}) &= P^{-1}_{ij} A_{jk} P_{ki}\\ &= A_{jk} P_{ki} P^{-1}_{ij}\\ &= A_{jk}\delta_{kj}\\ &= Tr(\mathsf{A})\,.\end{aligned} \tag{5.25b}$$

Remark. The determinants and traces of matrices representing a map $\mathcal{A}$ with respect to different bases are the same. We can therefore talk of the determinant and trace of a map.

Property. If A and B are similar (e.g. if they represent the same map $\mathcal{A}$ with respect to different bases), then they have the same characteristic polynomial, and hence the same eigenvalues.

Proof. Suppose that $\mathsf{B} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}$ then from (3.79) and (5.5)

$$\begin{aligned}p_{\mathsf{B}}(\lambda) &= \det(\mathsf{B} - \lambda\mathsf{I})\\ &= \det(\mathsf{P}^{-1}\mathsf{A}\mathsf{P} - \lambda\mathsf{P}^{-1}\mathsf{I}\mathsf{P})\\ &= \det(\mathsf{P}^{-1}(\mathsf{A} - \lambda\mathsf{I})\mathsf{P})\\ &= \det(\mathsf{P}^{-1})\det(\mathsf{A} - \lambda\mathsf{I})\det\mathsf{P}\\ &= \det(\mathsf{A} - \lambda\mathsf{I})\det(\mathsf{P}^{-1}\mathsf{P})\\ &= p_{\mathsf{A}}(\lambda)\,.\end{aligned} \qquad \square$$

Definition. The *characteristic polynomial* of the map $\mathcal{A}$ is the polynomial

$$p_{\mathcal{A}}(\lambda) = \det(\mathsf{A} - \lambda\mathsf{I})\,, \tag{5.25c}$$

where A is any matrix representation of $\mathcal{A}$.

Property. Suppose that A and B are similar, and that λ and u are an eigenvalue and eigenvector of A. Then $\mathsf{v} = \mathsf{P}^{-1}\mathsf{u}$ is the corresponding eigenvector of B since

$$\begin{aligned}\{\mathsf{A}\mathsf{u} = \lambda\mathsf{u}\} \quad &\Leftrightarrow \quad \left\{\mathsf{P}^{-1}\mathsf{A}\,\mathsf{I}\,\mathsf{u} = \mathsf{P}^{-1}(\lambda\mathsf{u})\right\}\\ &\Leftrightarrow \quad \left\{\mathsf{P}^{-1}\mathsf{A}\mathsf{P}\mathsf{P}^{-1}\mathsf{u} = \lambda\mathsf{P}^{-1}\mathsf{u}\right\}\\ &\Leftrightarrow \quad \{\mathsf{B}\mathsf{v} = \lambda\mathsf{v}\}\,.\end{aligned}$$

Remark. This also follows from (5.20c) and (5.21), since u and v are components of the same eigenvector, and A and B are components of the same map, in different bases.

5.5 Diagonalizable Maps and Matrices

Recall from §5.2.5 that a matrix A representing a map $\mathcal{A}$ is diagonal with respect to a basis if and only if the basis consists of eigenvectors.

Definition. A linear map $\mathcal{A} : \mathbb{F}^n \mapsto \mathbb{F}^n$ is said to be *diagonalizable* if $\mathbb{F}^n$ has a basis consisting of eigenvectors of $\mathcal{A}$.

Further, we have shown that a map $\mathcal{A}$ with n distinct eigenvalues has a basis of eigenvectors. It follows that if a matrix A has n distinct eigenvalues, then it is diagonalizable by means of a similarity transformation using the transformation matrix that changes to a basis of eigenvectors.

Definition. More generally we say that a $n \times n$ matrix A [over $\mathbb{F}$] is *diagonalizable* if A is similar with a diagonal matrix, i.e. if there exists an invertible transformation matrix P [with entries in $\mathbb{F}$] such that $\mathsf{P}^{-1}\mathsf{A}\mathsf{P}$ is diagonal.

5.5.1 When is a matrix diagonalizable?

While the requirement that a matrix A has n distinct eigenvalues is a *sufficient* condition for the matrix to be diagonalizable, it is not a *necessary* condition. The requirement is that $\mathbb{F}^n$ has a basis consisting of eigenvectors; this is possible even if the eigenvalues are not distinct as the following example shows.

Example. In example (ii) on page 84 we saw that if a map $\mathcal{A}$ is represented with respect to a given basis by a matrix

$$\mathsf{A} = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} ,$$

then the map has an eigenvalue $\lambda_1 = 5$ and a repeated eigenvalue $\lambda_2 = \lambda_3 = -3$. However, we can still construct linearly independent eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} , \quad \mathbf{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} , \quad \mathbf{x}_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} .$$

Choose $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ as new basis vectors. Then from (5.14b) with $\widetilde{\mathbf{e}}_i = \mathbf{x}_i$, it follows that the matrix for transforming components from the original basis to the eigenvector basis is given by

$$\mathsf{P} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} .$$

From the expression for an inverse (4.9)

$$\mathsf{P}^{-1} = \frac{1}{8} \begin{pmatrix} 1 & 2 & -3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{pmatrix} .$$

Hence from (5.21) the map $\mathcal{A}$ is represented with respect to the eigenvector basis by

$$\begin{aligned} \mathsf{P}^{-1}\mathsf{A}\mathsf{P} &= \frac{1}{8} \begin{pmatrix} 1 & 2 & -3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} , \end{aligned} \tag{5.26}$$

as expected from (5.13).

We need to be able to count the number of linearly independent eigenvectors.

Proposition. Suppose $\lambda_1, \ldots, \lambda_r$ are distinct eigenvalues of a linear map $\mathcal{A} : \mathbb{F}^n \mapsto \mathbb{F}^n$, and let $\mathfrak{B}_i$ denote a basis of the eigenspace E_{λ_i}. Then

$$\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \ldots \cup \mathfrak{B}_r$$

is a linearly independent set.

Proof. Argue by contradiction. Suppose that the set $\mathfrak{B}$ is linearly dependent, i.e. suppose that there exist $c_{ij} \in \mathbb{F}$, not all of which are zero, such that

$$\sum_{i=1}^{r} \sum_{j=1}^{m_{\lambda_i}} c_{ij} \mathbf{v}_{ij} = 0\,, \tag{5.27}$$

where $\mathbf{v}_{ij}$ is the j^{th} basis vector of $\mathfrak{B}_i$, and m_{λ_i} is the geometric multiplicity of $\mathfrak{B}_i$. Apply the operator (see (5.10b))

$$(\mathcal{A} - \lambda_1 \mathcal{I}) \ldots (\mathcal{A} - \lambda_{K-1} \mathcal{I})(\mathcal{A} - \lambda_{K+1} \mathcal{I}) \ldots (\mathcal{A} - \lambda_r \mathcal{I}) = \prod_{k=1,\ldots,\overline{K},\ldots,r} (\mathcal{A} - \lambda_k \mathcal{I})$$

to (5.27) to conclude that

$$\begin{aligned} 0 &= \prod_{k=1,\ldots,\overline{K},\ldots,r} (\mathcal{A} - \lambda_k \mathcal{I}) \sum_{i=1}^{r} \sum_{j=1}^{m_{\lambda_i}} c_{ij} \mathbf{v}_{ij} \\ &= \sum_{i=1}^{r} \sum_{j=1}^{m_{\lambda_i}} \prod_{k=1,\ldots,\overline{K},\ldots,r} c_{ij} \left(\lambda_i - \lambda_k\right) \mathbf{v}_{ij} \\ &= \sum_{j=1}^{m_{\lambda_K}} \left(\prod_{k=1,\ldots,\overline{K},\ldots,r} c_{Kj} \left(\lambda_K - \lambda_k\right) \right) \mathbf{v}_{Kj}\,. \end{aligned}$$

However, by definition of $\mathfrak{B}_K$ $(K = 1, \ldots, r)$, the $\{\mathbf{v}_{Kj}\}$, $(j = 1, \ldots, m_{\lambda_K})$, are linearly independent, and so we conclude (since $\lambda_K \neq \lambda_k$) that

$$c_{Kj} = 0 \quad (K = 1, \ldots, r, \quad j = 1, \ldots, m_{\lambda_K}),$$

which is a contradiction. □

Remark. If

$$\sum_i \dim E_{\lambda_i} \equiv \sum_i m_{\lambda_i} = n\,, \tag{5.28a}$$

i.e. if no eigenvalue has a non-zero defect, then $\mathfrak{B}$ is a basis [of eigenvectors], and a matrix A representing $\mathcal{A}$ is diagonalizable. However, a matrix with an eigenvalue that has a non-zero defect does not have sufficient linearly independent eigenvectors to be diagonalizable, cf. example (iii) on page 84.

Procedure (or 'recipe'). We now can construct a procedure to find if a matrix A is diagonalizable, and to diagonalize it where possible:

(i) Calculate the characteristic polynomial $p_{\mathsf{A}} = \det(\mathsf{A} - \lambda \mathsf{I})$.

(ii) Find all distinct roots, $\lambda_1, \ldots, \lambda_r$ of p_{A}.

(iii) For each λ_i find a basis $\mathfrak{B}_i$ of the eigenspace E_{λ_i}.

(iv) If no eigenvalue has a non-zero defect, i.e. if (5.28a) is true, then A is diagonalizable by a transformation matrix with columns that are the eigenvectors.

(v) However, if there is an eigenvalue with a non-zero defect, i.e. if

$$\sum_i \dim E_{\lambda_i} \equiv \sum_i m_{\lambda_i} < n\,, \tag{5.28b}$$

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then A is not diagonalizable.

5.5.2 Canonical form of 2×2 complex matrices

We claim that any 2×2 complex matrix A is similar to one of

$$\text{(i)}: \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \text{(ii)}: \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \qquad \text{(iii)}: \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \tag{5.29}$$

where $\lambda_1 \neq \lambda_2$, and no two of these are similar.

Proof. If A has distinct eigenvalues, $\lambda_1 \neq \lambda_2$ then, as shown above, A is diagonalizable by a similarity transformation to form (i). Otherwise, $\lambda_1 = \lambda_2 = \lambda$, and two cases arise according as $\dim E_\lambda = 2$ and $\dim E_\lambda = 1$. If $\dim E_\lambda = 2$ then $E_\lambda = \mathbb{C}^2$. Let $\mathfrak{B} = \{\mathbf{u}, \mathbf{v}\}$ be a basis of two linearly independent vectors. Since $\mathsf{A}\mathbf{u} = \lambda\mathbf{u}$ and $\mathsf{A}\mathbf{v} = \lambda\mathbf{v}$, A transforms to a diagonal matrix in this basis, i.e. there exists a transformation matrix P such that

$$\mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda\mathsf{I}, \tag{5.30}$$

i.e. form (ii). However, we can say more than this since it follows from (5.30) that $\mathsf{A} = \lambda\mathsf{I}$ (see also
19/09 the exercise on page 91 following (5.24)).

If $\dim E_\lambda = 1$, take non-zero $\mathbf{v} \in E_\lambda$, then $\{\mathbf{v}\}$ is a basis for E_λ. Extend this basis for E_λ to a basis $\mathfrak{B} = \{\mathbf{v}, \mathbf{w}\}$ for $\mathbb{C}^2$ by choosing $\mathbf{w} \in \mathbb{C}^2 \backslash E_\lambda$. If $\mathsf{A}\mathbf{w} = \alpha\mathbf{v} + \beta\mathbf{w}$ it follows that there exists a transformation matrix P, that transforms the original basis to $\mathfrak{B}$, such that

$$\widetilde{\mathsf{A}} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}.$$

However, if $\widetilde{\mathsf{A}}$, and thence A, is to have an eigenvalue λ of algebraic multiplicity of 2, then

$$\widetilde{\mathsf{A}} = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}.$$

Further, we note that

$$\begin{aligned} (\widetilde{\mathsf{A}} - \lambda\mathsf{I})^2 &= \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Let $\mathbf{u} = (\widetilde{\mathsf{A}} - \lambda\mathsf{I})\mathbf{w}$ ($\neq \mathbf{0}$ since $\mathbf{w}$ is neither an eigenvector of A nor $\widetilde{\mathsf{A}}$). Then, because

$$(\widetilde{\mathsf{A}} - \lambda\mathsf{I})\mathbf{u} = (\widetilde{\mathsf{A}} - \lambda\mathsf{I})^2\mathbf{w} = \mathbf{0},$$

we conclude that $\mathbf{u}$ is an eigenvector of $\widetilde{\mathsf{A}}$ (and $\mathbf{w}$ is a generalised eigenvector of $\widetilde{\mathsf{A}}$). Moreover, since

$$\widetilde{\mathsf{A}}\mathbf{w} = \mathbf{u} + \lambda\mathbf{w},$$

if we consider the alternative basis $\widetilde{\mathfrak{B}} = \{\mathbf{u}, \mathbf{w}\}$ it follows that there is a transformation matrix, say Q, such that

$$(\mathsf{P}\mathsf{Q})^{-1}\mathsf{A}(\mathsf{P}\mathsf{Q}) = \mathsf{Q}^{-1}\widetilde{\mathsf{A}}\mathsf{Q} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

We conclude that A is similar to a matrix of form (iii). □

Remarks.

(a) We observe that

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{is similar to} \quad \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix},$$

since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b) It is possible to show that *canonical forms*, or *Jordan normal forms*, exist for any dimension n. Specifically, for any complex $n \times n$ matrix A there exists a similar matrix $\widetilde{\mathsf{A}} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}$ such that

$$\widetilde{A}_{ii} = \lambda_i, \quad \widetilde{A}_{i\,i+1} = \{0, 1\}, \quad \widetilde{A}_{ij} = 0 \quad \text{otherwise,} \tag{5.31}$$

i.e. the eigenvalues are on the diagonal of $\widetilde{\mathsf{A}}$, the *super-diagonal* consists of zeros or ones, and all
19/10 other elements of $\widetilde{\mathsf{A}}$ are zero.[29]

Unlectured example. For instance, suppose that as in example (iii) on page 84,

$$\mathsf{A} = \begin{pmatrix} -3 & -1 & 1 \\ -1 & -3 & 1 \\ -2 & -2 & 0 \end{pmatrix}.$$

Recall that A has a single eigenvalue $\lambda = -2$ with an algebraic multiplicity of 3 with a defect of 1. Consider a vector $\mathbf{w}$ that is linearly independent of the eigenvectors (5.11c) (or equivalently (5.11d)), e.g.

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let

$$\mathbf{u} = (\mathsf{A} - \lambda\mathsf{I})\mathbf{w} = (\mathsf{A} + 2\mathsf{I})\mathbf{w} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix},$$

where it is straightforward to check that $\mathbf{u}$ is an eigenvector. Now form

$$\mathsf{P} = (\mathbf{u}\ \mathbf{w}\ \mathbf{v}) = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix},$$

where $\mathbf{v}$ is an eigenvector that is linearly independent of $\mathbf{u}$. We can then show that

$$\mathsf{P}^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & -2 & 1 \end{pmatrix},$$

and thence that A has a Jordan normal form

$$\mathsf{P}^{-1}\mathsf{A}\mathsf{P} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

5.5.3 Solution of second-order, constant coefficient, linear ordinary differential equations

Consider the solution of

$$\ddot{x} + b\dot{x} + cx = 0, \tag{5.32a}$$

where b and c are constants. If we let $\dot{x} = y$ then this second-order equation can be expressed as two first-order equations, namely

$$\dot{x} = y, \tag{5.32b}$$
$$\dot{y} = -cx - by, \tag{5.32c}$$

or in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5.32d}$$

[29] This is not the whole story, since $\widetilde{\mathsf{A}}$ is in fact *block diagonal*, but you will have to wait for *Linear Algebra* to find out about that.

This is a special case of the general second-order system

$$\dot{\mathbf{x}} = \mathsf{A}\mathbf{x}\,, \tag{5.33a}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \mathsf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{5.33b}$$

Let P be a transformation matrix, and let $\mathbf{z} = \mathsf{P}^{-1}\mathbf{x}$, then

$$\dot{\mathbf{z}} = \mathsf{B}\mathbf{z}\,, \quad \text{where} \quad \mathsf{B} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}\,. \tag{5.34}$$

By appropriate choice of P, it is possible to transform A to one of three canonical forms in (5.29). We consider each of the three possibilities in turn.

(i) In this case

$$\dot{\mathbf{z}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{z}\,, \tag{5.35a}$$

i.e. if

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \text{then} \quad \dot{z}_1 = \lambda_1 z_1 \quad \text{and} \quad \dot{z}_2 = \lambda_2 z_2\,. \tag{5.35b}$$

This has solution

$$\mathbf{z} = \begin{pmatrix} \alpha_1 \mathrm{e}^{\lambda_1 t} \\ \alpha_2 \mathrm{e}^{\lambda_2 t} \end{pmatrix}, \tag{5.35c}$$

where α_1 and α_2 are constants.

(ii) This is case (i) with $\lambda_1 = \lambda_2 = \lambda$, and with solution

$$\mathbf{z} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \mathrm{e}^{\lambda t}\,. \tag{5.36}$$

(iii) In this case

$$\dot{\mathbf{z}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \mathbf{z}\,, \tag{5.37a}$$

i.e.

$$\dot{z}_1 = \lambda z_1 + z_2 \quad \text{and} \quad \dot{z}_2 = \lambda z_2\,, \tag{5.37b}$$

which has solution

$$\mathbf{z} = \begin{pmatrix} \alpha_1 + \alpha_2 t \\ \alpha_2 \end{pmatrix} \mathrm{e}^{\lambda t}\,. \tag{5.37c}$$

We conclude that the general solution of (5.33a) is given by $\mathbf{x} = \mathsf{P}\mathbf{z}$ where $\mathbf{z}$ is the appropriate one of (5.35c), (5.36) or (5.37c).

Remark. If λ in case (iii) is pure imaginary, then this is a case of *resonance* in which the amplitude of an oscillation grows algebraically in time.

5.6 Cayley-Hamilton Theorem

Theorem 5.2 (Cayley-Hamilton Theorem)**.** *Every complex square matrix satisfies its own characteristic equation.*

We will not provide a general proof (see *Linear Algebra* again); instead we restrict ourselves to certain types of matrices.

5.6.1 Proof for diagonal matrices

First we need two preliminary results.

(i) The eigenvalues of a diagonal matrix D are the diagonal entries, D_{ii}. Hence if p_{D} is the characteristic polynomial of D, then $p_{\mathsf{D}}(D_{ii}) = \det(\mathsf{D} - D_{ii}\mathsf{I}) = 0$ for each i.

(ii) Further, because D^m is diagonal for $m \in \mathbb{N}$ (see (5.12c)), it follows that for *any* polynomial p,

$$\text{if} \quad \mathsf{D} = \begin{pmatrix} D_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & D_{nn} \end{pmatrix} \quad \text{then} \quad p(\mathsf{D}) = \begin{pmatrix} p(D_{11}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & p(D_{nn}) \end{pmatrix}. \tag{5.38}$$

We conclude that

$$p_{\mathsf{D}}(\mathsf{D}) = 0\,, \tag{5.39}$$

i.e. that a diagonal matrix satisfies its own characteristic equation. $\square$

5.6.2 Proof for diagonalizable matrices

First we need two preliminary results.

(i) If A and B are similar matrices with $\mathsf{A} = \mathsf{P}\mathsf{B}\mathsf{P}^{-1}$, then

$$\begin{aligned} \mathsf{A}^i &= \left(\mathsf{P}\mathsf{B}\mathsf{P}^{-1}\right)^i \\ &= \mathsf{P}\mathsf{B}\mathsf{P}^{-1}\,\mathsf{P}\mathsf{B}\mathsf{P}^{-1}\dots\mathsf{P}\mathsf{B}\mathsf{P}^{-1} \\ &= \mathsf{P}\mathsf{B}^i\mathsf{P}^{-1}\,. \end{aligned} \tag{5.40}$$

(ii) Further, on page 91 we deduced that similar matrices have the same characteristic polynomial, so suppose that for similar matrices A and B

$$p_{\mathsf{A}}(z) = p_{\mathsf{B}}(z) = \sum_{i=0}^{n} c_i z^i\,. \tag{5.41}$$

Then from using (5.40)

$$\begin{aligned} p_{\mathsf{A}}(\mathsf{A}) &= \sum_{i=0}^{n} c_i \mathsf{A}^i \\ &= \sum_{i=0}^{n} c_i \mathsf{P}\mathsf{B}^i\mathsf{P}^{-1} \\ &= \mathsf{P}\left(\sum_{i=0}^{n} c_i \mathsf{B}^i\right)\mathsf{P}^{-1} \\ &= \mathsf{P}\,p_{\mathsf{B}}(\mathsf{B})\,\mathsf{P}^{-1} \end{aligned} \tag{5.42}$$

Suppose that A is a diagonalizable matrix, i.e. suppose that there exists a transformation matrix P and a diagonal matrix D such that $\mathsf{A} = \mathsf{P}\mathsf{D}\mathsf{P}^{-1}$. Then from (5.39) and (5.42)

$$\begin{aligned} p_{\mathsf{A}}(\mathsf{A}) &= \mathsf{P}\,p_{\mathsf{D}}(\mathsf{D})\,\mathsf{P}^{-1} \\ &= \mathsf{P}0\mathsf{P}^{-1} \\ &= 0\,. \qquad \square \end{aligned} \tag{5.43}$$

Remark. Suppose that A is diagonalizable, invertible and that $c_0 \neq 0$ in (5.41). Then from (5.41) and (5.43)

$$\mathsf{A}^{-1} p_{\mathsf{A}}(\mathsf{A}) = c_0 \mathsf{A}^{-1} + c_1 \mathsf{I} \ldots + c_n \mathsf{A}^{n-1} = 0\,,$$

i.e.

$$\mathsf{A}^{-1} = -\frac{1}{c_0}\left(c_1 \mathsf{I} + \ldots + c_n \mathsf{A}^{n-1}\right)\,. \tag{5.44}$$

We conclude that it is possible to calculate A^{-1} from the positive powers of A.

5.6.3 Proof for 2×2 matrices (Unlectured)

Let A be a 2×2 complex matrix. We have already proved A satisfies its own characteristic equation if A is similar to case (i) or case (ii) of (5.29). For case (iii) A is similar to

$$\mathsf{B} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Further, $p_{\mathsf{B}}(z) = \det(\mathsf{B} - z\mathsf{I}) = (\lambda - z)^2$, and

$$p_{\mathsf{B}}(\mathsf{B}) = (\lambda \mathsf{I} - \mathsf{B})^2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 0\,.$$

The result then follows since, from (5.42), $p_{\mathsf{A}}(\mathsf{A}) = \mathsf{P}\, p_{\mathsf{B}}(\mathsf{B})\, \mathsf{P}^{-1} = 0$. □

5.7 Eigenvalues and Eigenvectors of Hermitian Matrices

5.7.1 Revision

- From (2.62a) the scalar product on $\mathbb{R}^n$ for n-tuples $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^{n} u_k v_k = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \mathsf{u}^{\mathrm{T}} \mathsf{v}\,. \tag{5.45a}$$

  Vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are said to be orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$, or if $\mathsf{u}^{\mathrm{T}}\mathsf{v} = 0$ (which is equivalent for the time being, assuming that we have an orthonormal basis).

- From (2.65) the scalar product on $\mathbb{C}^n$ for n-tuples $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^{n} u_k^* v_k = u_1^* v_1 + u_2^* v_2 + \ldots + u_n^* v_n = \mathsf{u}^{\dagger} \mathsf{v}\,, \tag{5.45b}$$

  where * denotes a complex conjugate. Vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ are said to be orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$, or if $\mathsf{u}^{\dagger}\mathsf{v} = 0$ (which is equivalent for the time being, assuming that we have an orthonormal basis).

- An alternative notation for the scalar product and associated norm is

$$\langle\, \mathbf{u} \mid \mathbf{v}\, \rangle \equiv \mathbf{u} \cdot \mathbf{v}\,, \tag{5.46a}$$

$$\|\mathbf{v}\| \equiv |\mathbf{v}| = (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}}\,. \tag{5.46b}$$

- From (3.42a) a square matrix $\mathsf{A} = \{A_{ij}\}$ is symmetric if

$$\mathsf{A}^{\mathrm{T}} = \mathsf{A}\,, \quad \text{i.e. if } \forall\, i,j \quad A_{ji} = A_{ij}\,. \tag{5.47a}$$

- From (3.42b) a square matrix $\mathsf{A} = \{A_{ij}\}$ is Hermitian if

$$\mathsf{A}^{\dagger} = \mathsf{A}\,, \quad \text{i.e. if } \forall\, i,j \quad A_{ji}^* = A_{ij}\,. \tag{5.47b}$$

  Real symmetric matrices are Hermitian.

- From (3.57a) a real square matrix Q is orthogonal if

$$\mathsf{Q}\mathsf{Q}^{\mathrm{T}} = \mathsf{I} = \mathsf{Q}^{\mathrm{T}}\mathsf{Q}\,. \tag{5.48a}$$

  The rows and columns of an orthogonal matrix each form an orthonormal set (using (5.45a) to define orthogonality).

- From (3.61) a square matrix U is said to be unitary if its Hermitian conjugate is equal to its inverse, i.e. if

$$\mathsf{U}\mathsf{U}^{\dagger} = \mathsf{I} = \mathsf{U}^{\dagger}\mathsf{U}\,. \tag{5.48b}$$

  The rows and columns of an unitary matrix each form an orthonormal set (using (5.45b) to define orthogonality). Orthogonal matrices are unitary.

5.7.2 The eigenvalues of an Hermitian matrix are real

Let H be an Hermitian matrix, and suppose that v is a non-zero eigenvector with eigenvalue λ. Then

$$\mathsf{H}\,\mathsf{v} = \lambda\,\mathsf{v}\,, \tag{5.49a}$$

and hence

$$\mathsf{v}^{\dagger}\mathsf{H}\,\mathsf{v} = \lambda\,\mathsf{v}^{\dagger}\mathsf{v}\,. \tag{5.49b}$$

Take the Hermitian conjugate of both sides; first the left hand side

$$\begin{aligned} \left(\mathsf{v}^{\dagger}\mathsf{H}\mathsf{v}\right)^{\dagger} &= \mathsf{v}^{\dagger}\mathsf{H}^{\dagger}\mathsf{v} && \text{since } (\mathsf{A}\mathsf{B})^{\dagger} = \mathsf{B}^{\dagger}\mathsf{A}^{\dagger} \text{ and } (\mathsf{A}^{\dagger})^{\dagger} = \mathsf{A} \\ &= \mathsf{v}^{\dagger}\mathsf{H}\mathsf{v} && \text{since } \mathsf{H} \text{ is Hermitian,} \end{aligned} \tag{5.50a}$$

and then the right

$$(\lambda\,\mathsf{v}^{\dagger}\mathsf{v})^{\dagger} = \lambda^{*}\mathsf{v}^{\dagger}\mathsf{v}\,. \tag{5.50b}$$

On equating the above two results we have that

$$\mathsf{v}^{\dagger}\mathsf{H}\mathsf{v} = \lambda^{*}\mathsf{v}^{\dagger}\mathsf{v}\,. \tag{5.51}$$

It then follows from (5.49b) and (5.51) that

$$(\lambda - \lambda^{*})\,\mathsf{v}^{\dagger}\mathsf{v} = 0\,. \tag{5.52}$$

However we have assumed that v is a *non-zero* eigenvector, so

$$\mathsf{v}^{\dagger}\mathsf{v} = \sum_{i=1}^{n} v_i^{*}v_i = \sum_{i=1}^{n} |v_i|^2 > 0\,, \tag{5.53}$$

and hence it follows from (5.52) that $\lambda = \lambda^{*}$, i.e. that λ is real.

Remark. Since a real symmetric matrix is Hermitian, we conclude that the eigenvalues of a real symmetric matrix are real.

5.7.3 An $n \times n$ Hermitian matrix has n orthogonal eigenvectors: Part I

$\lambda_i \neq \lambda_j$. Let v_i and v_j be two eigenvectors of an Hermitian matrix H. First of all suppose that their respective eigenvalues λ_i and λ_j are different, i.e. $\lambda_i \neq \lambda_j$. From pre-multiplying (5.49a) by $(\mathsf{v}_j)^{\dagger}$, and identifying v in (5.49a) with v_i, we have that

$$\mathsf{v}_j{}^{\dagger}\,\mathsf{H}\,\mathsf{v}_i = \lambda_i\,\mathsf{v}_j{}^{\dagger}\,\mathsf{v}_i\,. \tag{5.54a}$$

Similarly, by relabelling,

$$\mathsf{v}_i{}^\dagger\, \mathsf{H}\, \mathsf{v}_j = \lambda_j\, \mathsf{v}_i{}^\dagger\, \mathsf{v}_j\,. \tag{5.54b}$$

On taking the Hermitian conjugate of (5.54b) it follows that

$$\mathsf{v}_j{}^\dagger\, \mathsf{H}^\dagger\, \mathsf{v}_i = \lambda_j^*\, \mathsf{v}_j{}^\dagger\, \mathsf{v}_i\,.$$

However, H is Hermitian, i.e. $\mathsf{H}^\dagger = \mathsf{H}$, and we have seen above that the eigenvalue λ_j is real, hence

$$\mathsf{v}_j{}^\dagger \mathsf{H}\, \mathsf{v}_i = \lambda_j\, \mathsf{v}_j{}^\dagger\, \mathsf{v}_i\,. \tag{5.55}$$

On subtracting (5.55) from (5.54a) we obtain

$$0 = (\lambda_i - \lambda_j)\, \mathsf{v}_j{}^\dagger\, \mathsf{v}_i\,. \tag{5.56}$$

Hence if $\lambda_i \neq \lambda_j$ it follows that

$$\mathsf{v}_j{}^\dagger\, \mathsf{v}_i = \sum_{k=1}^{n} (\mathsf{v}_j)_k^* (\mathsf{v}_i)_k = 0\,. \tag{5.57}$$

20/07 Hence in terms of the complex scalar product (5.45b) the column vectors v_i and v_j are orthogonal.

20/08

$\lambda_i = \lambda_j$. The case when there is a repeated eigenvalue is more difficult. However with sufficient mathematical effort it can still be proved that orthogonal eigenvectors exist for the repeated eigenvalue (see below). Initially we appeal to *arm-waving arguments*.

An 'experimental' approach. First adopt an 'experimental' approach. In real life it is highly unlikely that two eigenvalues will be exactly equal (because of experimental error, etc.). Hence this case never arises and we can assume that we have n orthogonal eigenvectors. In fact this is a lousy argument since often it is precisely the cases where two eigenvalues are the same that results in interesting physical phenomenon, e.g. the resonant solution (5.37c). 20/09

A perturbation approach. Alternatively suppose that in the real problem two eigenvalues are exactly equal. Introduce a specific, but small, perturbation of size ε such that the perturbed problem has unequal eigenvalues (this is highly likely to be possible because the problem with equal eigenvalues is likely to be 'structurally unstable'). Now let $\varepsilon \to 0$. For all non-zero values of ε (both positive and negative) there will be n orthogonal eigenvectors, i.e. for $-1 \ll \varepsilon < 0$ there will be n orthogonal eigenvectors, and for $0 < \varepsilon \ll 1$ there will also be n orthogonal eigenvectors. We now appeal to a continuity argument claiming that there will be n orthogonal eigenvectors when $\varepsilon = 0$ because, crudely, there is nowhere for an eigenvector to 'disappear' to because of the orthogonality. We note that this is different to the case when the eigenvectors do not have to be orthogonal since then one of the eigenvectors could 'disappear' by becoming parallel to one of the existing eigenvectors.

Example. For instance consider

$$\mathsf{H} = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \quad \text{and} \quad \mathsf{J} = \begin{pmatrix} 1 & 1 \\ \varepsilon^2 & 1 \end{pmatrix}.$$

These matrices have eigenvalues and eigenvectors (see also (5.78), (5.80) and (5.82b))

$$\mathsf{H}: \quad \lambda_1 = 1+\varepsilon\,,\ \lambda_2 = 1-\varepsilon\,,\ \mathsf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \mathsf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\mathsf{J}: \quad \lambda_1 = 1+\varepsilon\,,\ \lambda_2 = 1-\varepsilon\,,\ \mathsf{v}_1 = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix},\ \mathsf{v}_2 = \begin{pmatrix} 1 \\ -\varepsilon \end{pmatrix}.$$

H is Hermitian for all ε, and there are two orthogonal eigenvectors for all ε. J has the same eigenvalues as H, and it has two linearly independent eigenvectors if $\varepsilon \neq 0$. However, the eigenvectors are not required to be orthogonal for all values of ε, which allows for the possibility that in the limit $\varepsilon \to 0$ that they can become parallel; this possibility is realised for J with the result that there is only one eigenvector when $\varepsilon = 0$.

A proof. We need some machinery first.

5.7.4 The Gram-Schmidt process

If there are repeated eigenvalues, how do we ensure that the eigenvectors of the eigenspace are orthogonal. More generally, given a finite, linearly independent set of vectors $\mathfrak{B} = \{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ how do you generate an orthogonal set $\widetilde{\mathfrak{B}} = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$?

We define the operator that projects the vector $\mathbf{w}$ orthogonally onto the vector $\mathbf{v}$ by (cf. (3.10b))

$$\mathcal{P}_{\mathbf{v}}(\mathbf{w}) = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\,. \tag{5.58}$$

The Gram-Schmidt process then works as follows. Let

$$\mathbf{v}_1 = \mathbf{w}_1\,, \tag{5.59a}$$

$$\mathbf{v}_2 = \mathbf{w}_2 - \mathcal{P}_{\mathbf{v}_1}(\mathbf{w}_2)\,, \tag{5.59b}$$

$$\mathbf{v}_3 = \mathbf{w}_3 - \mathcal{P}_{\mathbf{v}_1}(\mathbf{w}_3) - \mathcal{P}_{\mathbf{v}_2}(\mathbf{w}_3)\,, \tag{5.59c}$$

$$\vdots = \quad \vdots$$

$$\mathbf{v}_r = \mathbf{w}_r - \sum_{j=1}^{r-1}\mathcal{P}_{\mathbf{v}_j}(\mathbf{w}_r)\,. \tag{5.59d}$$

The set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ forms a system of orthogonal vectors. 20/10

Remark. We can interpret the process geometrically as follows: at stage k, $\mathbf{w}_k$ is first projected orthogonally onto the subspace generated by $\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}\}$, and then the vector $\mathbf{v}_k$ is defined to be the difference between $\mathbf{w}_k$ and this projection.

Proof. To show that these formulae yield an orthogonal sequence, we proceed by induction. First we observe that

$$\begin{aligned}\mathbf{v}_1\cdot\mathbf{v}_2 &= \mathbf{v}_1\cdot\left(\mathbf{w}_2 - \frac{\mathbf{v}_1\cdot\mathbf{w}_2}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1\right)\\ &= \mathbf{v}_1\cdot\mathbf{w}_2 - \frac{\mathbf{v}_1\cdot\mathbf{w}_2}{\mathbf{v}_1\cdot\mathbf{v}_1}\,\mathbf{v}_1\cdot\mathbf{v}_1\\ &= 0\,.\end{aligned}$$

Next we assume that $\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}\}$ are mutually orthogonal, i.e. for $1 \leqslant i, j \leqslant k-1$ we assume that $\mathbf{v}_i\cdot\mathbf{v}_j = 0$ if $i \neq j$. Then for $1 \leqslant i \leqslant k-1$,

$$\begin{aligned}\mathbf{v}_i\cdot\mathbf{v}_k &= \mathbf{v}_i\cdot\left(\mathbf{w}_k - \sum_{j=1}^{k-1}\mathcal{P}_{\mathbf{v}_j}(\mathbf{w}_k)\right)\\ &= \mathbf{v}_i\cdot\mathbf{w}_k - \sum_{j=1}^{k-1}\frac{\mathbf{v}_j\cdot\mathbf{w}_k}{\mathbf{v}_j\cdot\mathbf{v}_j}\,\mathbf{v}_i\cdot\mathbf{v}_j\\ &= \mathbf{v}_i\cdot\mathbf{w}_k - \frac{\mathbf{v}_i\cdot\mathbf{w}_k}{\mathbf{v}_i\cdot\mathbf{v}_i}\,\mathbf{v}_i\cdot\mathbf{v}_i\\ &= 0\,.\end{aligned}$$

We conclude that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ are mutually orthogonal. □

Linear independence. For what follows we need to know if the set $\widetilde{\mathfrak{B}} = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Lemma. Any set of mutually orthogonal non-zero vectors $\mathbf{v}_i$, $(i = 1, \ldots, r)$, is linearly independent.

Proof. If the $\mathbf{v}_i$, $(i = 1, \ldots, r)$, are mutually orthogonal then from (5.45b)

$$\sum_{k=1}^{n} (\mathbf{v}_i)_k^* (\mathbf{v}_j)_k = \mathbf{v}_i \cdot \mathbf{v}_j = 0 \quad \text{if} \quad i \neq j\,. \tag{5.60}$$

Suppose there exist a_i, $(i = 1, \ldots, r)$, such that

$$\sum_{j=1}^{r} a_j \mathbf{v}_j = \mathbf{0}\,. \tag{5.61}$$

Then from forming the scalar product of (5.61) with $\mathbf{v}_i$ and using (5.60) it follows that (no s.c.)

$$0 = \sum_{j=1}^{r} a_j \mathbf{v}_i \cdot \mathbf{v}_j = a_i \mathbf{v}_i \cdot \mathbf{v}_i = a_i \sum_{k=1}^{n} |(\mathbf{v}_i)_k|^2\,. \tag{5.62}$$

Since $\mathbf{v}_i$ is non-zero it follows that $a_i = 0$ $(i = 1, \ldots, r)$, and thence that the vectors are linearly independent. $\square$

Remarks.

(a) Any set of n orthogonal vectors in $\mathbb{F}^n$ is a basis of $\mathbb{F}^n$.

(b) Suppose that the Gram-Schmidt process is applied to a linearly dependent set of vectors. If $\mathbf{w}_k$ is the first vector that is a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{k-1}$, then $\mathbf{v}_k = 0$.

(c) If an orthogonal set of vectors $\widetilde{\mathfrak{B}} = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is obtained by a Gram-Schmidt process from a linearly independent set of eigenvectors $\mathfrak{B} = \{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$ with eigenvalue λ then the $\mathbf{v}_i$, $(i = 1, \ldots, r)$ are also eigenvectors because an eigenspace is a subspace and each of the $\mathbf{v}_i$ is a linear combination of the $\mathbf{w}_j$, $(j = 1, \ldots, r)$.

Orthonormal sets. Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthogonal set. For non-zero $\mu_i \in \mathbb{C}$ $(i = 1, \ldots, r)$, another orthogonal set is $\{\mu_1 \mathbf{v}_1, \ldots, \mu_r \mathbf{v}_r\}$, since

$$(\mu_i \mathbf{v}_i) \cdot (\mu_j \mathbf{v}_j) = \mu_i^* \mu_j \, \mathbf{v}_i \cdot \mathbf{v}_j = 0 \quad \text{if} \quad i \neq j\,. \tag{5.63a}$$

Next, suppose we choose

$$\mu_i = \frac{1}{|\mathbf{v}_i|}\,, \quad \text{and let} \quad \mathbf{u}_i = \frac{\mathbf{v}_i}{|\mathbf{v}_i|}\,, \tag{5.63b}$$

then

$$|\mathbf{u}_i|^2 = \mathbf{u}_i \cdot \mathbf{u}_i = \frac{\mathbf{v}_i \cdot \mathbf{v}_i}{|\mathbf{v}_i|^2} = 1\,. \tag{5.63c}$$

Hence $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is an *orthonormal* set, i.e. $\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$.

Remark. If $\mathbf{v}_i$ is an eigenvector then $\mathbf{u}_i$ is also an eigenvector (since an eigenspace is a subspace).

Unitary transformation matrices. Suppose that U is the transformation matrix between an orthonormal basis and a new orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, e.g. as defined by (5.63b). From (5.14b)

$$\mathsf{U} = \begin{pmatrix} \mathsf{u}_1 & \mathsf{u}_2 & \ldots & \mathsf{u}_n \end{pmatrix} = \begin{pmatrix} (\mathsf{u}_1)_1 & (\mathsf{u}_2)_1 & \cdots & (\mathsf{u}_n)_1 \\ (\mathsf{u}_1)_2 & (\mathsf{u}_2)_2 & \cdots & (\mathsf{u}_n)_2 \\ \vdots & \vdots & \ddots & \vdots \\ (\mathsf{u}_1)_n & (\mathsf{u}_2)_n & \cdots & (\mathsf{u}_n)_n \end{pmatrix}. \tag{5.64}$$

Then, by orthonormality,

$$(\mathsf{U}^\dagger \, \mathsf{U})_{ij} = \sum_{k=1}^{n} (\mathsf{U}^\dagger)_{ik} (\mathsf{U})_{kj} = \sum_{k=1}^{n} (\mathsf{u}_i)_k^* (\mathsf{u}_j)_k = \mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}\,. \tag{5.65a}$$

Equivalently, in matrix notation,

$$\mathsf{U}^\dagger\,\mathsf{U} = \mathsf{I}. \tag{5.65b}$$

Transformation matrices are invertible, and hence we conclude that $\mathsf{U}^\dagger = \mathsf{U}^{-1}$, i.e. that the transformation matrix, U, between an orthonormal basis and a new orthonormal basis is a *unitary* matrix.

5.7.5 An $n \times n$ Hermitian matrix has n orthogonal eigenvectors: Part II

$\lambda_i = \lambda_j$: *Proof (unlectured).*[30] Armed with the Gram-Schmidt process we return to this case. Suppose that $\lambda_1, \dots, \lambda_r$ are distinct eigenvalues of the Hermitian matrix H (reorder the eigenvalues if necessary). Let a corresponding set of *orthonormal* eigenvectors be $\mathfrak{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$. Extend $\mathfrak{B}$ to a basis $\mathfrak{B}' = \{\mathbf{v}_1, \dots, \mathbf{v}_r, \mathbf{w}_1, \dots, \mathbf{w}_{n-r}\}$ of $\mathbb{F}^n$ (where $\mathbf{w}_1, \dots, \mathbf{w}_{n-r}$ are not necessarily eigenvectors and/or orthogonal and/or normal). Next use the Gram-Schmidt process to obtain an *orthonormal* basis $\widetilde{\mathfrak{B}} = \{\mathbf{v}_1, \dots, \mathbf{v}_r, \mathbf{u}_1, \dots, \mathbf{u}_{n-r}\}$. Let P be the *unitary* $n \times n$ matrix

$$\mathsf{P} = (\mathbf{v}_1 \;\dots\; \mathbf{v}_r \;\mathbf{u}_1 \;\dots\; \mathbf{u}_{n-r})\,. \tag{5.66}$$

Then

$$\mathsf{P}^\dagger\mathsf{H}\mathsf{P} = \mathsf{P}^{-1}\mathsf{H}\mathsf{P} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_r & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & c_{11} & \dots & c_{1\;n-r} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & c_{n-r\;1} & \dots & c_{n-r\;n-r} \end{pmatrix}, \tag{5.67}$$

where C is a real symmetric $(n-r) \times (n-r)$ matrix. The eigenvalues of C are also eigenvalues of H, since

$$\begin{aligned} \det(\mathsf{H} - \lambda\mathsf{I}) &= \det(\mathsf{P}^\dagger\mathsf{H}\mathsf{P} - \lambda\mathsf{I}) \\ &= (\lambda_1 - \lambda)\dots(\lambda_r - \lambda)\det(\mathsf{C} - \lambda\mathsf{I})\,, \end{aligned} \tag{5.68}$$

and hence they are the repeated eigenvalues of H.

We now seek linearly independent eigenvectors of C; this is the problem we had before but in a $(n-r)$ dimensional space. For simplicity assume that the eigenvalues of C, λ_j $(j = r+1, \dots, n)$, are distinct (if not, recurse); denote corresponding *orthonormal* eigenvectors by $\mathbf{V}_j$ $(j = r+1, \dots, n)$. Let Q be the *unitary* $n \times n$ matrix

$$\mathsf{Q} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & V_{1\;r+1} & \dots & V_{1\;n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & V_{n-r\;r+1} & \dots & V_{n-r\;n} \end{pmatrix}, \tag{5.69a}$$

then

$$\mathsf{Q}^\dagger(\mathsf{P}^\dagger\mathsf{H}\mathsf{P})\mathsf{Q} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_r & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \lambda_{r+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \lambda_n \end{pmatrix}. \tag{5.69b}$$

[30] E.g. see *Mathematical Methods in Physics and Engineering* by *John W. Dettman* (Dover, 1988).

The required n linearly independent eigenvectors are the columns of T, where

$$\mathsf{T} = \mathsf{PQ}\,. \tag{5.70}$$

$\lambda_i = \lambda_j$: *Example.* Let A be the real symmetric matrix

$$\mathsf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{5.71a}$$

The matrix $(\mathsf{A} + \mathsf{I})$ has rank 1, so -1 is an eigenvalue with an algebraic multiplicity of at least 2. Further, from (5.7c)

$$\lambda_1 + \lambda_2 + \lambda_3 = Tr(\mathsf{A}) = 0\,,$$

and hence the other eigenvalue is 2. The eigenvectors are given by

$$\mathsf{v}_1 = \begin{pmatrix} -s_1 - t_1 \\ s_1 \\ t_1 \end{pmatrix}, \quad \mathsf{v}_2 = \begin{pmatrix} -s_2 - t_2 \\ s_2 \\ t_2 \end{pmatrix}, \quad \mathsf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{5.71b}$$

where s_i and t_i, $(i = 1, 2)$, are real numbers. We note that v_3 is orthogonal to v_1 and v_2 whatever the choice of s_i and t_i, $(i = 1, 2)$. We now wish to choose s_i and t_i, $(i = 1, 2)$ so that all three eigenvectors are mutually orthogonal.

Consider the vectors defined by the choice $s_1 = t_2 = -1$ and $s_2 = t_1 = 0$:

$$\mathsf{w}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathsf{w}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

w_1 and w_2 are not mutually orthogonal, so apply the Gram-Schmidt process to obtain

$$\mathsf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \tag{5.71c}$$

$$\mathsf{v}_2 = \mathsf{w}_2 - \frac{\mathsf{v}_1^\dagger \mathsf{w}_2}{\mathsf{v}_1^\dagger \mathsf{v}_1}\,\mathsf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}. \tag{5.71d}$$

It is straightforward to confirm that v_1, v_2 and v_3 are mutually orthogonal eigenvectors.

$\lambda_i = \lambda_j$: *Extended example (unlectured).* Suppose H is the real symmetric matrix

$$\mathsf{H} = \begin{pmatrix} 2 & 0 \\ 0 & \mathsf{A} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{5.72}$$

Then H has an eigenvalue $\lambda = -1$ with an algebraic multiplicity of 2 (like A), while the eigenvalue $\lambda = 2$ now has an algebraic multiplicity of 2 (unlike A). Consider

$$\mathsf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathsf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathsf{w}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathsf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{5.73a}$$

where v_1 and v_2 are eigenvectors of $\lambda = 2$ and $\lambda = -1$ respectively, and w_1 and w_2 have been chosen to be linearly independent of v_1 and v_2 (but no more). Applying the Gram-Schmidt process

$$\mathsf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathsf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathsf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{5.73b}$$

$$\mathsf{v}_4 = \mathsf{w}_2 - \frac{\mathsf{v}_1^\dagger \mathsf{w}_2}{\mathsf{v}_1^\dagger \mathsf{v}_1}\,\mathsf{v}_1 - \frac{\mathsf{v}_2^\dagger \mathsf{w}_2}{\mathsf{v}_2^\dagger \mathsf{v}_2}\,\mathsf{v}_2 - \frac{\mathsf{v}_3^\dagger \mathsf{w}_2}{\mathsf{v}_3^\dagger \mathsf{v}_3}\,\mathsf{v}_3 = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}. \tag{5.73c}$$

It then follows, in the notation of (5.66), (5.69a) and (5.70), that

$$\mathsf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{P}^\dagger \mathsf{H} \mathsf{P} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 1 \end{pmatrix}, \tag{5.74a}$$

$$\mathsf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \\ 0 & 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}, \quad \mathsf{T} = \mathsf{P}\mathsf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} \\ 0 & -1/\sqrt{2} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} \\ 0 & 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}. \tag{5.74b}$$

The orthonormal eigenvectors can be read off from the columns of T.

5.7.6 Diagonalization of Hermitian matrices

The eigenvectors of Hermitian matrices form a basis. Combining the results from the previous subsections we have that, whether or not two or more eigenvalues are equal, an n-dimensional Hermitian matrix has n orthogonal, and thence linearly independent, eigenvectors that form a basis for $\mathbb{C}^n$.

Orthonormal eigenvector bases. Further, from (5.63a)-(5.63c) we conclude that for Hermitian matrices it is always possible to find n eigenvectors that form an *orthonormal* basis for $\mathbb{C}^n$.

Hermitian matrices are diagonalizable. Let H be an Hermitian matrix. Since its eigenvectors, $\{\mathsf{v}_i\}$, form a basis of $\mathbb{C}^n$ it follows that H is similar to a diagonal matrix $\mathsf{\Lambda}$ of eigenvalues, and that

$$\mathsf{\Lambda} = \mathsf{P}^{-1} \mathsf{H}\, \mathsf{P}, \tag{5.75a}$$

where P is the transformation matrix between the original basis and the new basis consisting of eigenvectors, i.e. from (5.14b)

$$\mathsf{P} = \begin{pmatrix} \mathsf{v}_1 & \mathsf{v}_2 & \dots & \mathsf{v}_n \end{pmatrix} = \begin{pmatrix} (\mathsf{v}_1)_1 & (\mathsf{v}_2)_1 & \cdots & (\mathsf{v}_n)_1 \\ (\mathsf{v}_1)_2 & (\mathsf{v}_2)_2 & \cdots & (\mathsf{v}_n)_2 \\ \vdots & \vdots & \ddots & \vdots \\ (\mathsf{v}_1)_n & (\mathsf{v}_2)_n & \cdots & (\mathsf{v}_n)_n \end{pmatrix}. \tag{5.75b}$$

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Hermitian matrices are diagonalizable by unitary matrices. Suppose now that we choose to transform from an orthonormal basis to an orthonormal basis of eigenvectors $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$. Then, from (5.64) and (5.65b), we conclude that every Hermitian matrix, H, is diagonalizable by a transformation $\mathsf{U}^\dagger \mathsf{H}\, \mathsf{U}$, where U is a unitary matrix. In other words an Hermitian matrix, H, can be written in the form

$$\mathsf{H} = \mathsf{U}\mathsf{\Lambda}\mathsf{U}^\dagger, \tag{5.76}$$

where U is unitary and $\mathsf{\Lambda}$ is a diagonal matrix of eigenvalues.

Real symmetric matrices. Real symmetric matrices are a special case of Hermitian matrices. In addition to the eigenvalues being real, the eigenvectors are now real (or at least can be chosen to be so). Further the unitary matrix (5.64) specialises to an orthogonal matrix. We conclude that a real symmetric matrix, S, can be written in the form

$$\mathsf{S} = \mathsf{Q}\mathsf{\Lambda}\mathsf{Q}^{\mathrm{T}}, \tag{5.77}$$

21/08 where Q is an orthogonal matrix and $\mathsf{\Lambda}$ is a diagonal matrix of eigenvalues.

5.7.7 Examples of diagonalization

Unlectured example. Find the orthogonal matrix that diagonalizes the real symmetric matrix

$$\mathsf{S} = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \quad \text{where } \beta \text{ is real.} \tag{5.78}$$

Answer. The characteristic equation is

$$0 = \left\| \begin{matrix} 1-\lambda & \beta \\ \beta & 1-\lambda \end{matrix} \right\| = (1-\lambda)^2 - \beta^2 \,. \tag{5.79}$$

The solutions to (5.79) are

$$\lambda = \begin{cases} \lambda_+ = 1+\beta \\ \lambda_- = 1-\beta \end{cases} . \tag{5.80}$$

The corresponding eigenvectors $\mathsf{v}_\pm$ are found from

$$\begin{pmatrix} 1-\lambda_\pm & \beta \\ \beta & 1-\lambda_\pm \end{pmatrix} \begin{pmatrix} v_{\pm\,1} \\ v_{\pm\,2} \end{pmatrix} = 0 \,, \tag{5.81a}$$

or

$$\beta \begin{pmatrix} \mp 1 & 1 \\ 1 & \mp 1 \end{pmatrix} \begin{pmatrix} v_{\pm\,1} \\ v_{\pm\,2} \end{pmatrix} = 0 \,. \tag{5.81b}$$

$\beta \neq 0$**.** In this case $\lambda_+ \neq \lambda_-$, and we have that

$$v_{\pm\,2} = \pm\, v_{\pm\,1} \,. \tag{5.82a}$$

On normalising $\mathsf{v}_\pm$ so that $\mathsf{v}_\pm^\dagger \mathsf{v}_\pm = 1$, it follows that

$$\mathsf{v}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \qquad \mathsf{v}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \tag{5.82b}$$

Note that $\mathsf{v}_+^\dagger \mathsf{v}_- = 0$, as shown earlier.

$\beta = 0$**.** In this case $\mathsf{S} = \mathsf{I}$, and so any non-zero vector is an eigenvector with eigenvalue 1. In agreement with the result stated earlier, two linearly-independent eigenvectors can still be found, and we can choose them to be orthonormal, e.g. v_+ and v_- as above (if fact there is an uncountable choice of orthonormal eigenvectors in this special case).

To diagonalize S when $\beta \neq 0$ (it already is diagonal if $\beta = 0$) we construct an orthogonal matrix Q using (5.64):

$$\mathsf{Q} = \begin{pmatrix} v_{+\,1} & v_{-\,1} \\ v_{+\,2} & v_{-\,2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{5.83}$$

As a check we note that

$$\mathsf{Q}^{\mathrm{T}}\mathsf{Q} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{5.84}$$

and

$$\begin{aligned} \mathsf{Q}^{\mathrm{T}}\mathsf{S}\,\mathsf{Q} &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1+\beta & 1-\beta \\ 1+\beta & -1+\beta \end{pmatrix} \\ &= \begin{pmatrix} 1+\beta & 0 \\ 0 & 1-\beta \end{pmatrix} \\ &= \mathsf{\Lambda} \,. \end{aligned} \tag{5.85}$$

Degenerate example. Let A be the real symmetric matrix (5.71a), i.e.

$$\mathsf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

Then from (5.71b), (5.71c) and (5.71d) three orthogonal eigenvectors are

$$\mathsf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} , \quad \mathsf{v}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} , \quad \mathsf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ,$$

where v_1 and v_2 are eigenvectors of the degenerate eigenvalue $\lambda = -1$, and v_3 is the eigenvector of the eigenvalue $\lambda = 2$. By renormalising, three orthonormal eigenvectors are

$$\mathsf{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} , \quad \mathsf{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} , \quad \mathsf{u}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ,$$

and hence A can be diagonalized using the orthogonal transformation matrix

$$\mathsf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} .$$

Exercise. Check that

$$\mathsf{Q}^{\mathrm{T}}\mathsf{Q} = \mathsf{I} \quad \text{and that} \quad \mathsf{Q}^{\mathrm{T}}\mathsf{A}\mathsf{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

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5.7.8 Diagonalization of normal matrices

A square matrix A is said to be *normal* if

$$\mathsf{A}^\dagger\mathsf{A} = \mathsf{A}\,\mathsf{A}^\dagger . \tag{5.86}$$

It is possible to show that normal matrices have n linearly independent eigenvectors and can always be diagonalized. Hence, as well as Hermitian matrices, skew-Hermitian matrices (i.e. matrices such that $\mathsf{H}^\dagger = -\mathsf{H}$) and unitary matrices can always be diagonalized.

5.8 Forms

Definition: form. A map $\mathcal{F}(\mathbf{x})$

$$\mathcal{F}(\mathbf{x}) = \mathsf{x}^\dagger\mathsf{A}\mathsf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^* A_{ij} x_j \,, \tag{5.87a}$$

is called a *[sesquilinear] form*; A is called its *coefficient matrix*.

Definition: Hermitian form If $\mathsf{A} = \mathsf{H}$ is an Hermitian matrix, the map $\mathcal{F}(\mathbf{x}) : \mathbb{C}^n \mapsto \mathbb{C}$, where

$$\mathcal{F}(\mathbf{x}) = \mathsf{x}^\dagger\mathsf{H}\mathsf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^* H_{ij} x_j \,, \tag{5.87b}$$

is referred to as an *Hermitian form* on $\mathbb{C}^n$.

Hermitian forms are real. An Hermitian form is real since

$$\begin{aligned}
(\mathsf{x}^\dagger \mathsf{H} \mathsf{x})^* &= (\mathsf{x}^\dagger \mathsf{H} \mathsf{x})^\dagger && \text{since a scalar is its own transpose}\\
&= \mathsf{x}^\dagger \mathsf{H}^\dagger \mathsf{x} && \text{since } (\mathsf{AB})^\dagger = \mathsf{B}^\dagger \mathsf{A}^\dagger\\
&= \mathsf{x}^\dagger \mathsf{H} \mathsf{x}\,. && \text{since } \mathsf{H} \text{ is Hermitian}
\end{aligned}$$

Definition: quadratic form. If $\mathsf{A} = \mathsf{S}$ is a real symmetric matrix, the map $\mathcal{F}(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$, where

$$\mathcal{F}(\mathbf{x}) = \mathsf{x}^{\mathrm{T}} \mathsf{S} \mathsf{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i S_{ij} x_j\,, \tag{5.87c}$$

21/10 is referred to as a *quadratic form* on $\mathbb{R}^n$.

5.8.1 Eigenvectors and principal axes

From (5.76) the coefficient matrix, H, of a Hermitian form can be written as

$$\mathsf{H} = \mathsf{U} \Lambda \mathsf{U}^\dagger\,, \tag{5.88a}$$

where U is unitary and Λ is a diagonal matrix of eigenvalues. Let

$$\mathsf{x}' = \mathsf{U}^\dagger \mathsf{x}\,, \tag{5.88b}$$

then (5.87b) can be written as

$$\begin{aligned}
\mathcal{F}(\mathbf{x}) &= \mathsf{x}^\dagger \mathsf{U} \Lambda \mathsf{U}^\dagger \mathsf{x}\\
&= \mathsf{x}'^\dagger \Lambda \mathsf{x}' \qquad (5.88\text{c})\\
&= \sum_{i=1}^{n} \lambda_i |x'_i|^2\,. \qquad (5.88\text{d})
\end{aligned}$$

Transforming to a basis of orthonormal eigenvectors transforms the Hermitian form to a standard form with no 'off-diagonal' terms. The orthonormal basis vectors that coincide with the eigenvectors of the coefficient matrix, and which lead to the simplified version of the form, are known as *principal axes*.

Example. Let $\mathcal{F}(\mathbf{x})$ be the quadratic form

$$\mathcal{F}(\mathbf{x}) = 2x^2 - 4xy + 5y^2 = \mathsf{x}^{\mathrm{T}} \mathsf{S} \mathsf{x}\,, \tag{5.89a}$$

where

$$\mathsf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad \mathsf{S} = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}. \tag{5.89b}$$

What surface is described by $\mathcal{F}(\mathbf{x}) = \text{constant}$?

Solution. The eigenvalues of the real symmetric matrix S are $\lambda_1 = 1$ and $\lambda_2 = 6$, with corresponding unit eigenvectors

$$\mathsf{u}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathsf{u}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{5.89c}$$

The orthogonal matrix

$$\mathsf{Q} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \tag{5.89d}$$

transforms the original orthonormal basis to a basis of principal axes. Hence $\mathsf{S} = \mathsf{Q} \Lambda \mathsf{Q}^{\mathrm{T}}$, where Λ is a diagonal matrix of eigenvalues. It follows that $\mathcal{F}$ can be rewritten in the normalised form

$$\mathcal{F} = \mathsf{x}^{\mathrm{T}} \mathsf{Q} \Lambda \mathsf{Q}^{\mathrm{T}} \mathsf{x} = \mathsf{x}'^{\mathrm{T}} \Lambda \mathsf{x}' = x'^2 + 6y'^2\,, \tag{5.89e}$$

where

$$\mathsf{x}' = \mathsf{Q}^{\mathrm{T}} \mathsf{x}\,, \quad \text{i.e.} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5.89f}$$

The surface $\mathcal{F}(\mathbf{x}) = \text{constant}$ is thus an ellipse.

Hessian matrix. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that all the second-order partial derivatives exist. The *Hessian matrix* $\mathsf{H}(\mathbf{a})$ of f at $\mathbf{a} \in \mathbb{R}^n$ is defined to have components

$$H_{ij}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) \,. \tag{5.90a}$$

If f has continuous second-order partial derivatives, the mixed second-order partial derivatives commute, in which case the Hessian matrix H is symmetric. Suppose now that the function f has a critical point at $\mathbf{a}$, i.e.

$$\frac{\partial f}{\partial x_j}(\mathbf{a}) = 0 \quad \text{for} \quad j = 1, \dots, n. \tag{5.90b}$$

We examine the variation in f near $\mathbf{a}$. Then from Taylor's Theorem, (5.90a) and (5.90b) we have that near $\mathbf{a}$

$$\begin{aligned} f(\mathbf{a}+\mathbf{x}) &= f(\mathbf{a}) + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(\mathbf{a}) + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=i}^{n} x_i \, x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) + \mathrm{o}(|\mathbf{x}|^2) \\ &= f(\mathbf{a}) + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=i}^{n} x_i \, H_{ij}(\mathbf{a}) \, x_j + \mathrm{o}(|\mathbf{x}|^2) \,. \end{aligned} \tag{5.90c}$$

Hence, if $|\mathbf{x}| \ll 1$,

$$f(\mathbf{a}+\mathbf{x}) - f(\mathbf{a}) \approx \tfrac{1}{2} \sum_{i,j=1}^{n} x_i \, H_{ij}(\mathbf{a}) \, x_j = \tfrac{1}{2} \mathbf{x}^{\mathrm{T}} \mathsf{H} \mathbf{x} \,, \tag{5.90d}$$

i.e. the variation in f near $\mathbf{a}$ is given by a quadratic form. Since H is a real symmetric matrix it follows from (5.88c) that on transforming to the orthonormal basis of eigenvectors,

$$\mathbf{x}^{\mathrm{T}} \mathsf{H} \mathbf{x} = \mathbf{x}'^{\mathrm{T}} \Lambda \mathbf{x}' \tag{5.90e}$$

where Λ is the diagonal matrix of eigenvalues of H. It follows that close to a critical point

$$f(\mathbf{a}+\mathbf{x}) - f(\mathbf{a}) \approx \tfrac{1}{2} \mathbf{x}'^{\mathrm{T}} \Lambda \mathbf{x}' = \tfrac{1}{2} \left(\lambda_1 {x'_1}^2 + \ldots + \lambda_n {x'_n}^2 \right) \,. \tag{5.90f}$$

As explained in *Differential Equations*:

(i) if all the eigenvalues of $\mathsf{H}(\mathbf{a})$ are strictly positive, f has a local minimum at $\mathbf{a}$;

(ii) if all the eigenvalues of $\mathsf{H}(\mathbf{a})$ are strictly negative, f has a local maximum at $\mathbf{a}$;

(iii) if $\mathsf{H}(\mathbf{a})$ has at least one strictly positive eigenvalue and at least one strictly negative eigenvalue, then f has a saddle point at $\mathbf{a}$;

(iv) otherwise, it is not possible to determine the nature of the critical point from the eigenvalues of $\mathsf{H}(\mathbf{a})$.

5.8.2 Quadrics and conics

A *quadric*, or *quadric surface*, is the n-dimensional hypersurface defined by the zeros of a real quadratic polynomial. For co-ordinates $(x_1, \dots, x_n)$ the general quadric is defined by

$$\sum_{i,j=1}^{n} x_i A_{ij} x_j + \sum_{i=1}^{n} b_i x_i + c \equiv \mathsf{x}^{\mathrm{T}} \mathsf{A} \mathsf{x} + \mathsf{b}^{\mathrm{T}} \mathsf{x} + c = 0 \,, \tag{5.91a}$$

or equivalently

$$\sum_{i,j=1}^{n} x_j A_{ij} x_i + \sum_{i=1}^{n} b_i x_i + c \equiv \mathsf{x}^{\mathrm{T}} \mathsf{A}^{\mathrm{T}} \mathsf{x} + \mathsf{b}^{\mathrm{T}} \mathsf{x} + c = 0 \,, \tag{5.91b}$$

where A is a $n \times n$ matrix, b is a $n \times 1$ column vector and c is a constant. Let

$$\mathsf{S} = \tfrac{1}{2} \left(\mathsf{A} + \mathsf{A}^{\mathrm{T}} \right) \,, \tag{5.91c}$$

then from (5.91a) and (5.91b)

$$\mathsf{x}^{\mathrm{T}}\mathsf{S}\mathsf{x}+\mathsf{b}^{\mathrm{T}}\mathsf{x}+c=0\,. \tag{5.91d}$$

By taking the principal axes as basis vectors it follows that

$$\mathsf{x}'^{\mathrm{T}}\Lambda\mathsf{x}'+\mathsf{b}'^{\mathrm{T}}\mathsf{x}'+c=0\,. \tag{5.91e}$$

where $\Lambda=\mathsf{Q}^{\mathrm{T}}\mathsf{S}\mathsf{Q}$, $\mathsf{b}'=\mathsf{Q}^{\mathrm{T}}\mathsf{b}$ and $\mathsf{x}'=\mathsf{Q}^{\mathrm{T}}\mathsf{x}$. If Λ does not have a zero eigenvalue, then it is invertible and (5.91e) can be simplified further by a translation of the origin

$$\mathsf{x}'\to\mathsf{x}'-\tfrac{1}{2}\Lambda^{-1}\mathsf{b}'\,, \tag{5.91f}$$

to obtain

$$\mathsf{x}'^{\mathrm{T}}\Lambda\mathsf{x}'=k\,. \tag{5.91g}$$

where k is a constant. 22/07

Conic Sections. First suppose that $n=2$ and that Λ (or equivalently S) does not have a zero eigenvalue, then with

$$\mathsf{x}'=\begin{pmatrix}x'\\y'\end{pmatrix}\quad\text{and}\quad\Lambda=\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}, \tag{5.92a}$$

(5.91g) becomes

$$\lambda_1 x'^2+\lambda_2 y'^2=k\,, \tag{5.92b}$$

which is the normalised equation of a *conic section*.

$\lambda_1\lambda_2>0$. If $\lambda_1\lambda_2>0$, then k must have the same sign as the λ_j $(j=1,2)$, and (5.92b) is the equation of an *ellipse* with principal axes coinciding with the x' and y' axes.

Scale. The *scale* of the ellipse is determined by k.

Shape. The *shape* of the ellipse is determined by the ratio of eigenvalues λ_1 and λ_2.

Orientation. The *orientation* of the ellipse in the original basis is determined by the eigenvectors of S.

In the *degenerate case*, $\lambda_1=\lambda_2$, the ellipse becomes a circle with no preferred principal axes. Any two orthogonal (and hence linearly independent) vectors may be chosen as the principal axes.

$\lambda_1\lambda_2<0$. If $\lambda_1\lambda_2<0$ then (5.92b) is the equation for a *hyperbola* with principal axes coinciding with the x' and y' axes. Similar results to above hold for the scale, shape and orientation.

$\lambda_1\lambda_2=0$. If $\lambda_1=\lambda_2=0$, then there is no quadratic term, so assume that only one eigenvalue is zero; wlog $\lambda_2=0$. Then instead of (5.91f), translate the origin according to

$$x'\to x'-\frac{b'_1}{2\lambda_1}\,,\quad y'\to y'-\frac{c}{b'_2}+\frac{b'^2_1}{4\lambda_1 b'_2}\,, \tag{5.93}$$

assuming $b'_2\neq 0$, to obtain instead of (5.92b)

$$\lambda_1 x'^2+b'_2 y'=0\,. \tag{5.94}$$

This is the equation of a *parabola* with principal axes coinciding with the x' and y' axes. Similar results to above hold for the scale, shape and orientation.

Remark. If $b'_2=0$, the equation for the conic section can be reduced (after a translation) to $\lambda_1 x'^2=k$ (cf. (5.92b)), with possible solutions of zero $(\lambda_1 k<0)$, one $(k=0)$ or two $(\lambda_1 k>0)$ lines.

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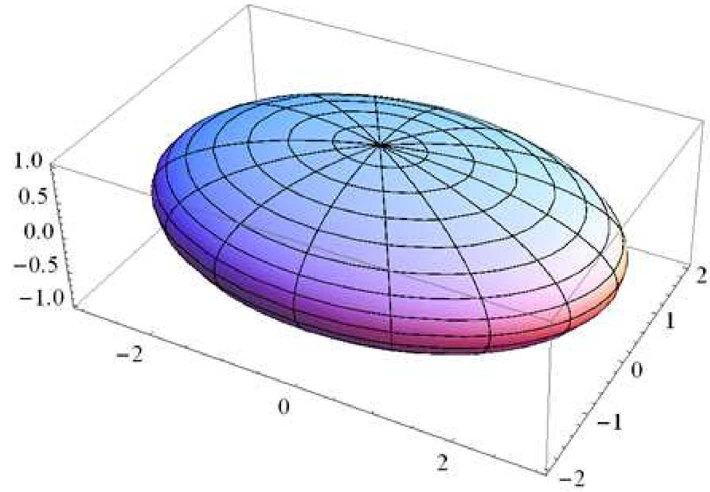


Figure 1: Ellipsoid ($\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 > 0,\ k > 0$); *Wikipedia*.

Three Dimensions. If $n = 3$ and $\mathsf{\Lambda}$ does not have a zero eigenvalue, then with

$$\mathsf{x}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \text{and} \quad \mathsf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{5.95a}$$

(5.91g) becomes

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 = k\,. \tag{5.95b}$$

Analogously to the case of two dimensions, this equation describes a number of characteristic surfaces.

Coefficients	**Quadric Surface**
$\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 > 0,\ k > 0$.	*Ellipsoid*.
$\lambda_1 = \lambda_2 > 0,\ \lambda_3 > 0,\ k > 0$.	*Spheroid*: an example of a *surface of revolution* (in this case about the z' axis). The surface is a *prolate spheroid* if $\lambda_1 = \lambda_2 > \lambda_3$ and an *oblate spheroid* if $\lambda_1 = \lambda_2 < \lambda_3$.
$\lambda_1 = \lambda_2 = \lambda_3 > 0,\ k > 0$.	*Sphere*.
$\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 = 0,\ k > 0$.	*Elliptic cylinder*.
$\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 < 0,\ k > 0$.	*Hyperboloid of one sheet*.
$\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 < 0,\ k = 0$.	*Elliptical conical surface*.
$\lambda_1 > 0,\ \lambda_2 < 0,\ \lambda_3 < 0,\ k > 0$.	*Hyperboloid of two sheets*.
$\lambda_1 > 0,\ \lambda_2 = \lambda_3 = 0,\ \lambda_1 k \geqslant 0$.	Plane.

5.9 More on Conic Sections

Conic sections arise in a number of applications, e.g. next term you will encounter them in the study of orbits. The aim of this sub-section is to list a number of their properties.

We first note that there are a number of equivalent definitions of conic sections. We have already encountered the algebraic definitions in our classification of quadrics, but conic sections can also be characterised, as the name suggests, by the shapes of the different types of intersection that a plane can make with a right circular conical surface (or double cone); namely an ellipse, a parabola or a hyperbola. However, there are other definitions.

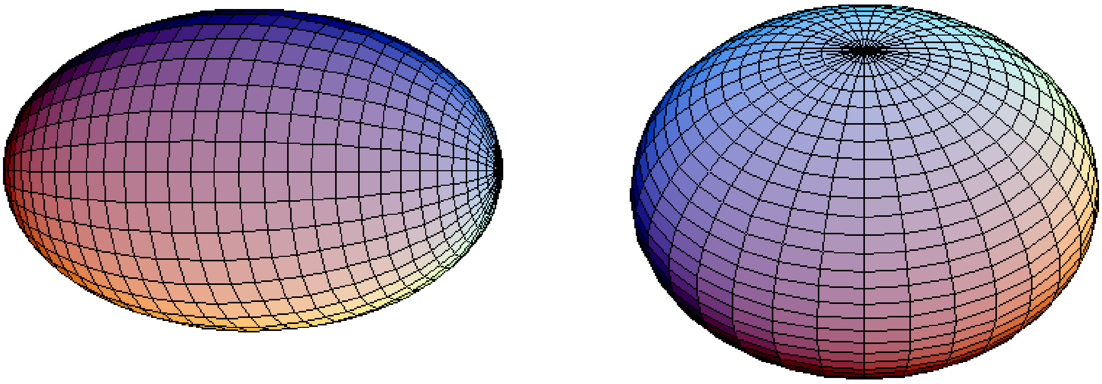


Figure 2: Prolate spheroid ($\lambda_1 = \lambda_2 > \lambda_3 > 0$, $k > 0$) and oblate spheroid ($0 < \lambda_1 = \lambda_2 < \lambda_3$, $k > 0$); *Wikipedia*.

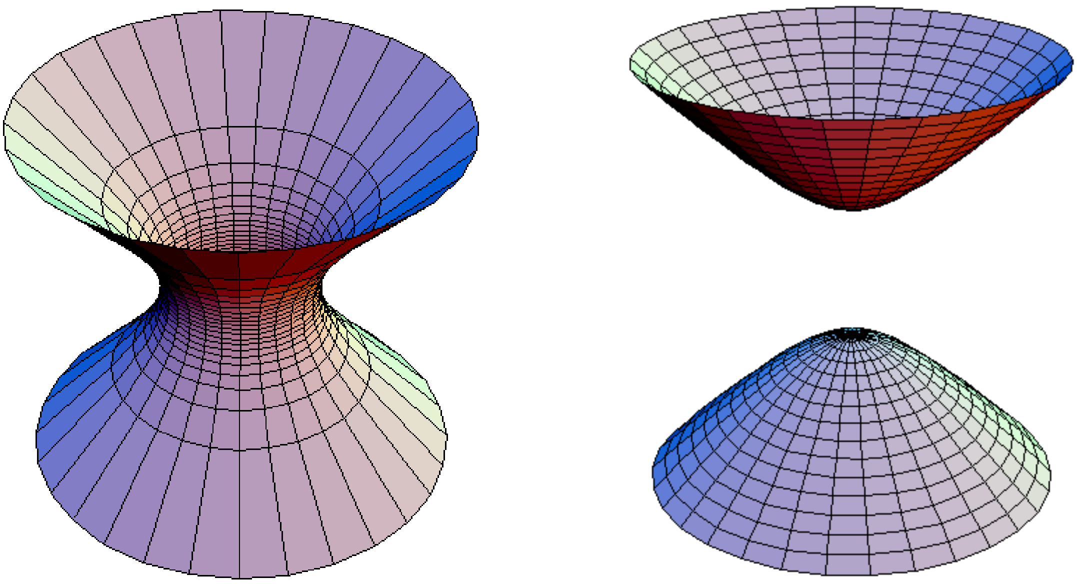


Figure 3: Hyperboloid of one sheet ($\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, $k > 0$) and hyperboloid of two sheets ($\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$, $k > 0$); *Wikipedia*.

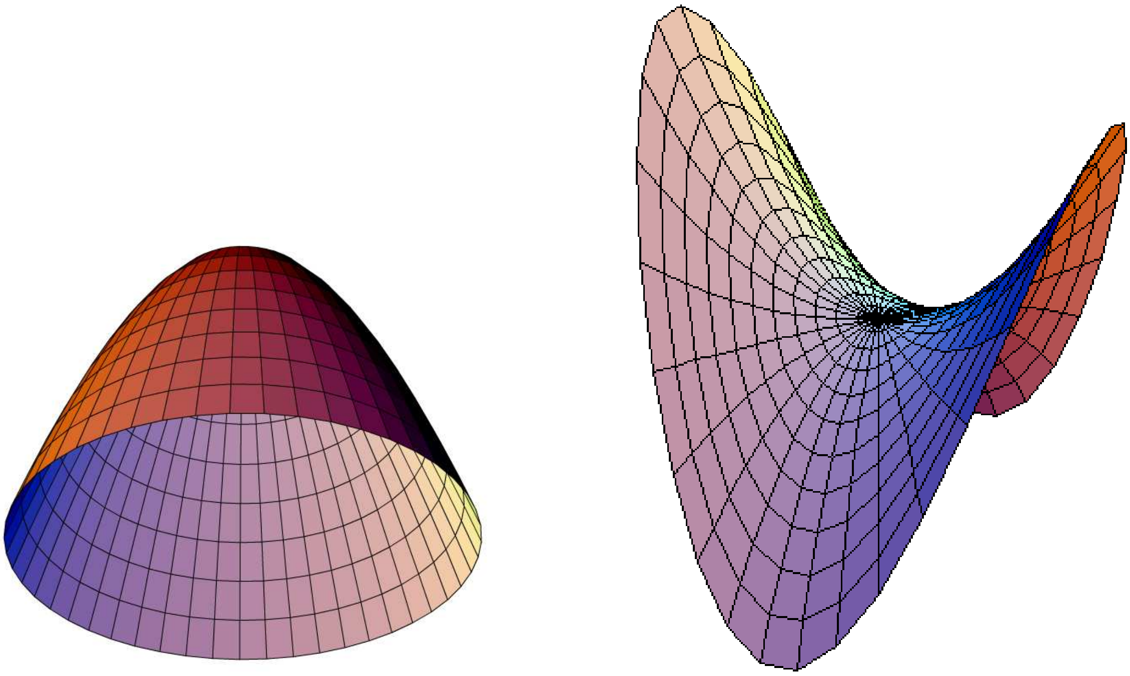


Figure 4: Paraboloid of revolution ($\lambda_1 x'^2 + \lambda_2 y'^2 + z' = 0$, $\lambda_1 > 0$, $\lambda_2 > 0$) and hyperbolic paraboloid ($\lambda_1 x'^2 + \lambda_2 y'^2 + z' = 0$, $\lambda_1 < 0$, $\lambda_2 > 0$); *Wikipedia*.

5.9.1 The focus-directrix property

Another way to define conics is by two positive real parameters: a (which determines the size) and e (the *eccentricity*, which determines the shape). The points $(\pm ae, 0)$ are defined to be the *foci* of the conic section, and the lines $x = \pm a/e$ are defined to be *directrices*. A conic section can then be defined to be the set of points which obey the *focus-directrix* property: namely that the distance from a focus is e times the distance from the directrix closest to that focus (but not passing through that focus in the case $e = 1$). 22/10

$0 < e < 1$. From the focus-directrix property,

$$\sqrt{(x-ae)^2 + y^2} = e\left(\frac{a}{e} - x\right), \tag{5.96a}$$

i.e.

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1. \tag{5.96b}$$

This is an ellipse with *semi-major axis* a and *semi-minor axis* $a\sqrt{1-e^2}$. An ellipse is the intersection between a conical surface and a plane which cuts through only one half of that surface. The special case of a circle of radius a is recovered in the limit $e \to 0$.

$e > 1$. From the focus-directrix property,

$$\sqrt{(x-ae)^2 + y^2} = e\left(x - \frac{a}{e}\right), \tag{5.97a}$$

i.e.

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1. \tag{5.97b}$$

This is a hyperbola with *semi-major axis* a. A hyperbola is the intersection between a conical surface and a plane which cuts through both halves of that surface.

$e = 1$. In this case, the relevant directrix that corresponding to the focus at, say, $x = a$ is the one at $x = -a$. Then from the focus-directrix property,

$$\sqrt{(x-a)^2 + y^2} = x + a\,, \tag{5.98a}$$

i.e.

$$y^2 = 4ax\,. \tag{5.98b}$$

This is a parabola. A parabola is the intersection of a right circular conical surface and a plane parallel to a generating straight line of that surface.

5.9.2 Ellipses and hyperbolae: another definition

Yet another definition of an ellipse is that it is the locus of points such that the *sum* of the distances from any point on the curve to the two foci is constant. Analogously, a hyperbola is the locus of points such that the *difference* in distance from any point on the curve to the two foci is constant.

Ellipse. Hence for an ellipse we require that

$$\sqrt{(x+ae)^2 + y^2} + \sqrt{(x-ae)^2 + y^2} = 2a\,,$$

where the right-hand side comes from evaluating the left-hand side at $x = a$ and $y = 0$. This expression can be rearranged, and then squared, to obtain

$$(x+ae)^2 + y^2 = 4a^2 - 4a\sqrt{(x-ae)^2 + y^2} + (x-ae)^2 + y^2\,,$$

or equivalently

$$a - ex = \sqrt{(x-ae)^2 + y^2}\,.$$

Squaring again we obtain, as in (5.96b),

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1\,. \tag{5.99}$$

Hyperbola. For a hyperbola we require that

$$\sqrt{(x+ae)^2 + y^2} - \sqrt{(x-ae)^2 + y^2} = 2a\,,$$

where the right-hand side comes from evaluating the left-hand side at $x = a$ and $y = 0$. This expression can be rearranged, and then squared, to obtain

$$(x+ae)^2 + y^2 = 4a^2 + 4a\sqrt{(x-ae)^2 + y^2} + (x-ae)^2 + y^2\,,$$

or equivalently

$$ex - a = \sqrt{(x-ae)^2 + y^2}\,.$$

Squaring again we obtain, as in (5.97b),

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1\,. \tag{5.100}$$

5.9.3 Polar co-ordinate representation

It is also possible to obtain a description of conic sections using polar co-ordinates. Place the origin of polar coordinates at the focus which has the directrix to the right. Denote the distance from the focus to the appropriate directrix by ℓ/e for some ℓ; then

$$e \neq 1: \qquad \ell/e = |a/e - ae|\,, \quad \text{i.e.} \quad \ell = a|1-e^2|\,, \tag{5.101a}$$

$$e = 1: \qquad \ell = 2a\,. \tag{5.101b}$$

From the focus-directrix property

$$r = e\left(\frac{\ell}{e} - r\cos\theta\right), \tag{5.102a}$$

and so

$$r = \frac{\ell}{1+e\cos\theta}\,. \tag{5.102b}$$

Since $r = \ell$ when $\theta = \frac{1}{2}\pi$, ℓ is this the value of y immediately above the focus. ℓ is known as the *semi-latus rectum*.

Asymptotes. Ellipses are bounded, but hyperbolae and parabolae are unbounded. Further, it follows from (5.102b) that as $r \to \infty$

$$\theta \to \pm\theta_a \quad \text{where} \quad \theta_a = \cos^{-1}\left(-\frac{1}{e}\right),$$

where, as expected, θ_a only exists if $e \geqslant 1$. In the case of hyperbolae it follows from (5.97b) or (5.100), that

$$y = \mp x\left(e^2-1\right)^{\frac{1}{2}}\left(1-\frac{a^2}{x^2}\right)^{\frac{1}{2}} \approx \pm x\tan\theta_a \mp \frac{a^2\tan\theta_a}{2x} \pm \ldots \quad \text{as} \quad |x| \to \infty\,. \tag{5.103}$$

Hence as $|x| \to \infty$ the hyperbola comes arbitrarily close to *asymptotes* at $y = \pm x\tan\theta_a$.

5.9.4 The intersection of a plane with a conical surface (Unlectured)

A right circular cone is a surface on which every point P is such that OP makes a fixed angle, say α ($0 < \alpha < \frac{1}{2}\pi$), with a given axis that passes through O. The point O is referred to as the vertex of the cone.

Let $\mathbf{n}$ be a unit vector parallel to the axis. Then from the above description

$$\mathbf{x}\cdot\mathbf{n} = |\mathbf{x}|\cos\alpha\,. \tag{5.104a}$$

The vector equation for a cone with its vertex at the origin is thus

$$(\mathbf{x}\cdot\mathbf{n})^2 = \mathbf{x}^2\cos^2\alpha\,, \tag{5.104b}$$

where by squaring the equation we have included the 'reverse' cone to obtain the equation for a right circular conical surface.

By means of a translation we can now generalise (5.104b) to the equation for a conical surface with a vertex at a general point $\mathbf{a}$:

$$\big[(\mathbf{x}-\mathbf{a})\cdot\mathbf{n}\big]^2 = (\mathbf{x}-\mathbf{a})^2\cos^2\alpha\,. \tag{5.104c}$$

Component form. Suppose that in terms of a standard Cartesian basis

$$\mathbf{x} = (x, y, z)\,, \quad \mathbf{a} = (a, b, c)\,, \quad \mathbf{n} = (l, m, n)\,,$$

then (5.104c) becomes

$$\big[(x-a)l + (y-b)m + (z-c)n\big]^2 = \big[(x-a)^2 + (y-b)^2 + (z-c)^2\big]\cos^2\alpha\,. \tag{5.105}$$

Intersection of a plane with a conical surface. Let us consider the intersection of this conical surface with the plane $z = 0$. In that plane

$$\big[(x-a)l + (y-b)m - cn\big]^2 = \big[(x-a)^2 + (y-b)^2 + c^2\big]\cos^2\alpha\,. \tag{5.106}$$

This is a curve defined by a quadratic polynomial in x and y. In order to simplify the algebra suppose that, wlog, we choose Cartesian axes so that the axis of the conical surface is in the yz plane. In that case $l = 0$ and we can express $\mathbf{n}$ in component form as

$$\mathbf{n} = (l, m, n) = (0, \sin\beta, \cos\beta)\,.$$

Further, translate the axes by the transformation

$$X = x - a\,, \qquad Y = y - b + \frac{c\sin\beta\cos\beta}{\cos^2\alpha - \sin^2\beta}\,,$$

so that (5.106) becomes

$$\left[\left(Y - \frac{c\sin\beta\cos\beta}{\cos^2\alpha - \sin^2\beta}\right)\sin\beta - c\cos\beta\right]^2 = \left[X^2 + \left(Y - \frac{c\sin\beta\cos\beta}{\cos^2\alpha - \sin^2\beta}\right)^2 + c^2\right]\cos^2\alpha\,,$$

which can be simplified to

$$X^2\cos^2\alpha + Y^2\left(\cos^2\alpha - \sin^2\beta\right) = \frac{c^2\sin^2\alpha\cos^2\alpha}{\cos^2\alpha - \sin^2\beta}\,. \tag{5.107}$$

There are now three cases that need to be considered: $\sin^2\beta < \cos^2\alpha$, $\sin^2\beta > \cos^2\alpha$ and $\sin^2\beta = \cos^2\alpha$. To see why this is, consider graphs of the intersection of the conical surface with the $X = 0$ plane, and for definiteness suppose that $0 \leqslant \beta \leqslant \frac{\pi}{2}$.

First suppose that

$$\beta + \alpha \;<\; \frac{\pi}{2}$$

i.e.

$$\beta \;<\; \tfrac{1}{2}\pi - \alpha$$

i.e.

$$\sin\beta \;<\; \sin\left(\tfrac{1}{2}\pi - \alpha\right) = \cos\alpha\,.$$

In this case the intersection of the conical surface with the $z = 0$ plane will yield a closed curve.

Next suppose that

$$\beta + \alpha \;>\; \frac{\pi}{2}$$

i.e.

$$\sin\beta \;>\; \cos\alpha\,.$$

In this case the intersection of the conical surface with the $z = 0$ plane will yield two open curves, while if $\sin\beta = \cos\alpha$ there will be one open curve.

Define

$$\frac{c^2 \sin^2 \alpha}{|\cos^2 \alpha - \sin^2 \beta|} = A^2 \quad \text{and} \quad \frac{c^2 \sin^2 \alpha \, \cos^2 \alpha}{\left(\cos^2 \alpha - \sin^2 \beta\right)^2} = B^2 \,. \tag{5.108}$$

$\sin\beta < \cos\alpha$. In this case (5.107) becomes

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1\,, \tag{5.109a}$$

which we recognise as the equation of an *ellipse* with *semi-minor* and *semi-major* axes of lengths A and B respectively (from (5.108) it follows that $A < B$).

$\sin\beta > \cos\alpha$. In this case

$$-\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1\,, \tag{5.109b}$$

which we recognise as the equation of a *hyperbola*, where B is one half of the distance between the two *vertices*.

$\sin\beta = \cos\alpha$. In this case (5.106) becomes

$$X^2 = -2c \, \cot\beta \, Y\,, \tag{5.109c}$$

where

$$X = x - a \quad \text{and} \quad Y = y - b - c\cot 2\beta\,, \tag{5.109d}$$

which is the equation of a *parabola*.

5.10 Singular Value Decomposition

We have seen that if no eigenvalue of a square matrix A has a non-zero defect, then the matrix has an *eigenvalue decomposition*

$$\mathsf{A} = \mathsf{P}\Lambda\mathsf{P}^{-1}\,, \tag{5.110a}$$

where P is a transformation matrix and Λ is the diagonal matrix of eigenvalues. Further, we have also seen that, although not all matrices have eigenvalue decompositions, a Hermitian matrix H always has an eigenvalue decomposition

$$\mathsf{H} = \mathsf{U}\Lambda\mathsf{U}^{\dagger}\,, \tag{5.110b}$$

where U is a unitary matrix.

In this section we will show that *any* real or complex $m \times n$ matrix A has a factorization, called a *Singular Value Decomposition* (SVD), of the form

$$\mathsf{A} = \mathsf{U}\Sigma\mathsf{V}^{\dagger}\,, \tag{5.111a}$$

where U is a $m \times m$ unitary matrix, Σ is an $m \times n$ rectangular diagonal matrix with non-negative real numbers on the diagonal, and V is an $n \times n$ unitary matrix. The diagonal entries σ_i of Σ are known as the singular values of A, and it is conventional to order them so that

$$\sigma_1 \geqslant \sigma_2 \geqslant \ldots\,. \tag{5.111b}$$

There are many applications of SVD, e.g. in signal processing and pattern recognition, and more specifically in measuring the growth rate of crystals in igneous rock, in understanding the reliability of seismic data, in examining entanglement in quantum computation, and in characterising the policy positions of politicians.

5.10.1 Construction

We generate a proof by constructing 'the', or more correctly 'an', SVD.

Let H be the $n \times n$ matrix

$$\mathsf{H} = \mathsf{A}^{\dagger}\mathsf{A}\,. \tag{5.112}$$

Since $\mathsf{H}^{\dagger} = (\mathsf{A}^{\dagger}\mathsf{A})^{\dagger} = \mathsf{A}^{\dagger}\mathsf{A}$, it follows that H is Hermitian. Hence there exists a $n \times n$ unitary matrix V such that

$$\mathsf{V}^{\dagger}\mathsf{A}^{\dagger}\mathsf{A}\mathsf{V} = \mathsf{V}^{\dagger}\mathsf{H}\mathsf{V} = \Lambda\,, \tag{5.113}$$

where Λ is a $n \times n$ diagonal matrix of *real* eigenvalues. Arrange the columns of V so that

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \tag{5.114}$$

Next consider the Hermitian form

$$\begin{aligned} \mathsf{x}^{\dagger}\mathsf{H}\mathsf{x} &= \mathsf{x}^{\dagger}\mathsf{A}^{\dagger}\mathsf{A}\mathsf{x} \\ &= (\mathsf{A}\mathsf{x})^{\dagger}(\mathsf{A}\mathsf{x}) \\ &\geqslant 0\,. \end{aligned} \tag{5.115}$$

By transforming to principal axes (e.g. see (5.88d)) it follows that

$$\mathsf{x}^{\dagger}\mathsf{H}\mathsf{x} = \sum_{i=1}^{n} \lambda_i |x_i'|^2 \geqslant 0\,. \tag{5.116}$$

Since the choice of x' is arbitrary, it follows that (e.g. choose $x_i' = \delta_{ij}$ for some j for $i = 1, n$)

$$\lambda_j \geqslant 0 \text{ for all } j = 1, n. \tag{5.117}$$

Definition. A matrix H with eigenvalues that are non-negative is said to be *positive semi-definite*. A matrix D with eigenvalues that are strictly positive is said to be *positive definite*.

Partition Λ so that

$$\Lambda = \begin{bmatrix} \mathsf{D} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.118}$$

where D is a $r \times r$ positive definite diagonal matrix with entries

$$D_{ii} = \sigma_i^2 = \lambda_i > 0\,. \tag{5.119}$$

Partition V into $n \times r$ and $n \times (r-n)$ matrices V_1 and V_2:

$$\mathsf{V} = \begin{bmatrix} \mathsf{V}_1 & \mathsf{V}_2 \end{bmatrix}, \tag{5.120a}$$

where, since V is unitary,

$$\mathsf{V}_1^\dagger \mathsf{V}_1 = \mathsf{I}\,, \quad \mathsf{V}_2^\dagger \mathsf{V}_2 = \mathsf{I}\,, \quad \mathsf{V}_1^\dagger \mathsf{V}_2 = 0\,, \quad \mathsf{V}_2^\dagger \mathsf{V}_1 = 0\,. \tag{5.120b}$$

With this definition it follows that

$$\begin{bmatrix} \mathsf{V}_1^\dagger \\ \mathsf{V}_2^\dagger \end{bmatrix} \mathsf{A}^\dagger \mathsf{A} \begin{bmatrix} \mathsf{V}_1 & \mathsf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathsf{V}_1^\dagger \mathsf{A}^\dagger \mathsf{A} \mathsf{V}_1 & \mathsf{V}_1^\dagger \mathsf{A}^\dagger \mathsf{A} \mathsf{V}_2 \\ \mathsf{V}_2^\dagger \mathsf{A}^\dagger \mathsf{A} \mathsf{V}_1 & \mathsf{V}_2^\dagger \mathsf{A}^\dagger \mathsf{A} \mathsf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathsf{D} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.121a}$$

and hence that

$$\mathsf{V}_1^\dagger \mathsf{A}^\dagger \mathsf{A} \mathsf{V}_1 = \mathsf{D}\,, \tag{5.121b}$$

$$\mathsf{V}_2^\dagger \mathsf{A}^\dagger \mathsf{A} \mathsf{V}_2 = (\mathsf{A}\mathsf{V}_2)^\dagger (\mathsf{A}\mathsf{V}_2) = 0 \quad \Rightarrow \quad \mathsf{A}\mathsf{V}_2 = 0\,. \tag{5.121c}$$

Define U_1 as the $m \times r$ matrix

$$\mathsf{U}_1 = \mathsf{A}\mathsf{V}_1 \mathsf{D}^{-\frac{1}{2}}\,, \tag{5.122a}$$

and note that

$$\mathsf{U}_1^\dagger \mathsf{U}_1 = (\mathsf{D}^{-\frac{1}{2}\dagger} \mathsf{V}_1^\dagger \mathsf{A}^\dagger) \mathsf{A} \mathsf{V}_1 \mathsf{D}^{-\frac{1}{2}} = \mathsf{I}\,. \tag{5.122b}$$

We deduce

- that the column vectors of U_1 are orthonormal,
- that the column vectors of U_1 are linearly independent,
- and hence that $r \leqslant m$.

Further observe that

$$\mathsf{U}_1 \mathsf{D}^{\frac{1}{2}} \mathsf{V}_1^\dagger = \mathsf{A}\mathsf{V}_1 \mathsf{D}^{-\frac{1}{2}} \mathsf{D}^{\frac{1}{2}} \mathsf{V}_1^\dagger = \mathsf{A}\,. \tag{5.122c}$$

This is almost the required result.

U_1 comprises of $r \leqslant m$ orthonormal column vectors in $\mathbb{C}^m$. If necessary extend this set of vectors to an orthonormal basis of $\mathbb{C}^m$ to form the $m \times m$ matrix U:

$$\mathsf{U} = \begin{bmatrix} \mathsf{U}_1 & \mathsf{U}_2 \end{bmatrix}, \tag{5.123a}$$

where, by construction,

$$\mathsf{U}_1^\dagger \mathsf{U}_1 = \mathsf{I}\,, \quad \mathsf{U}_2^\dagger \mathsf{U}_2 = \mathsf{I}\,, \quad \mathsf{U}_1^\dagger \mathsf{U}_2 = 0\,, \quad \mathsf{U}_2^\dagger \mathsf{U}_1 = 0\,. \tag{5.123b}$$

Hence $\mathsf{U}^\dagger \mathsf{U} = \mathsf{I}$, i.e. U is unitary. Next, recalling that $r \leqslant m$ and $r \leqslant n$, define the $m \times n$ diagonal matrix Σ by

$$\Sigma = \begin{bmatrix} \mathsf{D}^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.124}$$

where $\Sigma_{ii} = D_{ii}^{\frac{1}{2}} = \sigma_i > 0$ $(i \leqslant r)$. Then, as required,

$$\begin{aligned} \mathsf{U}\Sigma\mathsf{V}^\dagger &= \begin{bmatrix} \mathsf{U}_1 & \mathsf{U}_2 \end{bmatrix} \begin{bmatrix} \mathsf{D}^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathsf{V}_1^\dagger \\ \mathsf{V}_2^\dagger \end{bmatrix} \\ &= \mathsf{U}_1 \mathsf{D}^{\frac{1}{2}} \mathsf{V}_1^\dagger = \mathsf{A}\,. \qquad \square \end{aligned} \tag{5.125}$$

5.10.2 Remarks

Definition. The m columns of U and the n columns of V are called the *left-singular vectors* and the *right-singular vectors* of A respectively.

- The right-singular vectors are eigenvectors of the matrix $\mathsf{A}^\dagger\mathsf{A}$, and the non-zero-singular values of A are the square roots of the non-zero eigenvalues of $\mathsf{A}^\dagger\mathsf{A}$.

- For $1 \leqslant j \leqslant r$ let u_j and v_j be the j^{th} columns of U and V. Then since $\mathsf{U}\Sigma\mathsf{V}^\dagger = \mathsf{A}$

$$\mathsf{AV} = \mathsf{U}\Sigma \quad \Rightarrow \quad \mathsf{Av}_j = \sigma_j \mathsf{u}_j\,, \tag{5.126a}$$

$$\mathsf{A}^\dagger\mathsf{U} = \mathsf{V}\Sigma^\dagger \quad \Rightarrow \quad \mathsf{A}^\dagger\mathsf{u}_j = \sigma_j \mathsf{v}_j\,. \tag{5.126b}$$

It follows that

$$\mathsf{AA}^\dagger\mathsf{u}_j = \sigma_j \mathsf{Av}_j = \sigma_j^2 \mathsf{u}_j\,, \tag{5.126c}$$

and hence that the left-singular vectors are eigenvectors of the matrix $\mathsf{AA}^\dagger$, and the non-zero-singular values of A are also the square roots of the non-zero eigenvalues of $\mathsf{AA}^\dagger$.

Geometric Interpretation Consider the linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$ that takes a vector x to Ax. Then the SVD says that one can find orthonormal bases of $\mathbb{C}^n$ and $\mathbb{C}^m$ such that the linear transformation maps the j^{th} ($j = 1, \dots, r$) basis vector of $\mathbb{C}^n$ to a non-negative multiple of the j^{th} basis vector of $\mathbb{C}^m$, and sends the left-over basis vectors to zero.
If we restrict to a real vector space, and consider the sphere of radius one in $\mathbb{R}^n$, then this sphere is mapped to an ellipsoid in $\mathbb{R}^m$, with the non-zero singular values being the lengths of the semi-axes of the ellipsoid.

5.10.3 Linear Least Squares

Suppose that an $m \times n$ matrix A and a vector $\mathsf{b} \in \mathbb{R}^m$ are given. If $m < n$ the equation $\mathsf{Ax} = \mathsf{b}$ usually has an infinity of solutions. On the other hand, if there are more equations than unknowns, i.e. if $m > n$, the system $\mathsf{Ax} = \mathsf{b}$ is called *overdetermined*. In general, an overdetermined system has no solution, but we would like to have Ax and b close in a sense. Choosing the Euclidean distance $\|\mathsf{z}\| = (\sum_{i=1}^m z_i^2)^{1/2}$ as a measure of closeness, we obtain the following problem.

Problem 5.3 (Least squares in $\mathbb{R}^m$). Given $\mathsf{A} \in \mathbb{R}^{m\times n}$ and $\mathsf{b} \in \mathbb{R}^m$, find

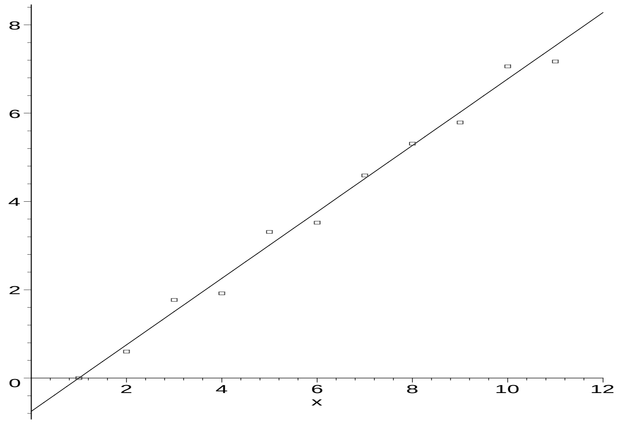
$$\mathsf{x}^* = \min_{\mathsf{x}\in\mathbb{R}^n} \|\mathsf{Ax} - \mathsf{b}\|^2\,, \tag{5.127}$$

i.e., find $\mathsf{x}^* \in \mathbb{R}^n$ which minimizes $\|\mathsf{Ax} - \mathsf{b}\|^2$. This is called the *least-squares problem.*

Example Problem. Problems of this form occur frequently when we collect m observations (x_i, y_i), which are typically prone to measurement error, and wish to exploit them to form an n-variable linear model, typically with $n \ll m$. In statistics, this is called *linear regression.*

For instance, suppose that we have m measurements of $F(x)$, and that we wish to model F with a linear combination of n functions $\phi_j(x)$, i.e.

$$F(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)\,, \quad \text{and} \quad F(x_i) \approx y_i, \quad i = 1...m.$$



x_i	y_i
1	0.00
2	0.60
3	1.77
4	1.92
5	3.31
6	3.52
7	4.59
8	5.31
9	5.79
10	7.06
11	7.17

Figure 5: Least squares straight line data fitting.

Such a problem might occur if we were trying to match some planet observations to an ellipse.[31] Hence we want to determine c such that the $F(x_i)$ 'best' fit the y_i, i.e.

$$\mathsf{Ac} = \begin{bmatrix} \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \vdots & & \vdots \\ \phi_1(x_n) & \cdots & \phi_n(x_n) \\ \vdots & & \vdots \\ \phi_1(x_m) & \cdots & \phi_n(x_m) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} F(x_1) \\ \vdots \\ F(x_n) \\ \vdots \\ F(x_m) \end{bmatrix} \approx \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ \vdots \\ y_m \end{bmatrix} = \mathsf{y}.$$

There are many ways of doing this; we will determine the c that minimizes the sum of squares of the deviation, i.e. we minimize

$$\sum_{i=1}^{m} (F(x_i) - y_i)^2 = \|\mathsf{Ac} - \mathsf{y}\|^2.$$

This leads to a *linear* system of equations for the determination of the unknown c.

Solution. We can solve the least squares problem using SVD. First use the SVD to write

$$\mathsf{Ax} - \mathsf{b} = \mathsf{U\Sigma V}^\dagger\mathsf{x} - \mathsf{b} = \mathsf{U}(\mathsf{\Sigma z} - \mathsf{d}) \quad \text{where} \quad \mathsf{z} = \mathsf{V}^\dagger\mathsf{x} \quad \text{and} \quad \mathsf{d} = \mathsf{U}^\dagger\mathsf{b}\,. \tag{5.128}$$

Then

$$\begin{aligned} \|\mathsf{Ax} - \mathsf{b}\|^2 &= (\mathsf{z}^\dagger\mathsf{\Sigma}^\dagger - \mathsf{d}^\dagger)\mathsf{U}^\dagger\mathsf{U}(\mathsf{\Sigma z} - \mathsf{d}) \\ &= \sum_1^r (\sigma_i z_i - d_i)^2 + \sum_{r+1}^m d_i^2 \\ &\geqslant \sum_{r+1}^m d_i^2\,. \end{aligned} \tag{5.129}$$

Hence the least squares solution is given by

$$z_i = \frac{d_i}{\sigma_i} \quad \text{for} \quad i = 1, \dots, r \quad \text{with} \quad z_i \text{ arbitrary for } i = r+1, \dots, m\,, \tag{5.130a}$$

or in terms of $\mathsf{x} = \mathsf{Vz}$

$$x_i = \sum_{j=1}^{r} \frac{V_{ij} d_j}{\sigma_j} + \sum_{j=r+1}^{m} V_{ij} z_j \quad \text{for} \quad i = 1, \dots, m\,. \tag{5.130b}$$

[31] Or Part III results to Part II results, or Part IA results to STEP results.

5.10.4 Film

In 1976, Cleve Moler (the inventor of MATLAB) made a 6-minute film about the singular value decomposition at what was then known as the Los Alamos Scientific Laboratory. Today the SVD is widely used in scientific and engineering computation, but in 1976 the SVD was relatively unknown. A practical algorithm for its computation had been developed only a few years earlier and the LINPACK project was in the early stages of its implementation. The 3D computer graphics involved hidden line computations (then relatively new). The computer output was 16mm celluloid film. The film has been digitized and uploaded it to YouTube, and is available at

http://youtu.be/R9UoFyqJca8.

A description of the making of the film, and its use as background graphics in the first Star Trek movie, is at:

http://blogs.mathworks.com/cleve/2012/12/10/1976-matrix-singular-value-decomposition-film/.

6 Transformation Groups

6.1 Definition[32]

A group is a set G with a binary operation $*$ that satisfies the following four axioms.

Closure: for all $a, b \in G$,

$$a * b \in G\,. \tag{6.1a}$$

Associativity: for all $a, b, c \in G$

$$(a * b) * c = a * (b * c)\,. \tag{6.1b}$$

Identity element: there exists an element $e \in G$ such that for all $a \in G$,

$$e * a = a * e = a\,. \tag{6.1c}$$

Inverse element: for each $a \in G$, there exists an element $b \in G$ such that

$$a * b = b * a = e\,, \tag{6.1d}$$

where e is an identity element.

6.2 The Set of Orthogonal Matrices is a Group

Let G be the set of all orthogonal matrices, i.e. the set of all matrices Q such that $\mathsf{Q}^{\mathrm{T}}\mathsf{Q} = \mathsf{I}$. We can check that G is a group under matrix multiplication.

Closure. If P and Q are orthogonal, then $\mathsf{R} = \mathsf{PQ}$ is orthogonal since

$$\mathsf{R}^{\mathrm{T}}\mathsf{R} = (\mathsf{PQ})^{\mathrm{T}}\mathsf{PQ} = \mathsf{Q}^{\mathrm{T}}\mathsf{P}^{\mathrm{T}}\mathsf{PQ} = \mathsf{Q}^{\mathrm{T}}\mathsf{IQ} = \mathsf{I}\,. \tag{6.2a}$$

Associativity. If P, Q and R are orthogonal (or indeed any matrices), then multiplication is associative from (3.37), i.e. because

$$\begin{aligned}((\mathsf{PQ})\mathsf{R})_{ij} &= (\mathsf{PQ})_{ik}(\mathsf{R})_{kj}\\ &= (\mathsf{P})_{i\ell}(\mathsf{Q})_{\ell k}(\mathsf{R})_{kj}\\ &= (\mathsf{P})_{i\ell}(\mathsf{QR})_{\ell j}\\ &= (\mathsf{P}(\mathsf{QR}))_{ij}\,.\end{aligned} \tag{6.2b}$$

Identity element. I is orthogonal, and thus in G, since

$$\mathsf{I}^{\mathrm{T}}\mathsf{I} = \mathsf{II} = \mathsf{I}\,. \tag{6.2c}$$

I is also the identity element since if Q is orthogonal (or indeed any matrix) then from (3.49)

$$\mathsf{IQ} = \mathsf{QI} = \mathsf{Q}\,. \tag{6.2d}$$

Inverse element. If Q is orthogonal, then Q^{T} is the required inverse element because Q^{T} is itself orthogonal (since $\mathsf{Q}^{\mathrm{TT}}\mathsf{Q}^{\mathrm{T}} = \mathsf{QQ}^{\mathrm{T}} = \mathsf{I}$) and

$$\mathsf{QQ}^{\mathrm{T}} = \mathsf{Q}^{\mathrm{T}}\mathsf{Q} = \mathsf{I}\,. \tag{6.2e}$$

Definition. The group of $n \times n$ orthogonal matrices is known as $O(n)$.

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[32] Revision for many of you, but not for those not taking *Groups*.

6.2.1 The set of orthogonal matrices with determinant +1 is a group

From (3.80) we have that if Q is orthogonal, then $\det \mathsf{Q} = \pm 1$. The set of $n \times n$ orthogonal matrices with determinant $+1$ is known as $SO(n)$, and is a group. In order to show this we need to supplement (6.2a), (6.2b), (6.2c), (6.2d) and (6.2e) with the observations that

(i) if P and Q are orthogonal with determinant $+1$ then $\mathsf{R} = \mathsf{PQ}$ is orthogonal with determinant $+1$ (and multiplication is thus closed), since

$$\det \mathsf{R} = \det(\mathsf{PQ}) = \det \mathsf{P} \, \det \mathsf{Q} = +1\,; \tag{6.3a}$$

(ii) $\det \mathsf{I} = +1$;

(iii) if Q is orthogonal with determinant $+1$ then, since $\det \mathsf{Q}^{\mathrm{T}} = \det \mathsf{Q}$ from (3.72), its inverse Q^{T} has
23/09 determinant $+1$, and is thus in $SO(n)$.

6.3 The Set of Length Preserving Transformation Matrices is a Group

The set of length preserving transformation matrices is a group because the set of length preserving transformation matrices is the set of orthogonal matrices, which is a group. This follows from the following theorem.

Theorem 6.1. *Let P be a real $n \times n$ matrix. The following are equivalent:*

(i) P *is an orthogonal matrix;*

(ii) $|\mathsf{Px}| = |\mathsf{x}|$ *for all column vectors* x *(i.e. lengths are preserved:* P *is said to be a linear isometry);*

(iii) $(\mathsf{Px})^{\mathrm{T}}(\mathsf{Py}) = \mathsf{x}^{\mathrm{T}}\mathsf{y}$ *for all column vectors* x *and* y *(i.e. scalar products are preserved);*

(iv) *if* $\{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$ *are orthonormal, so are* $\{\mathsf{Pv}_1, \ldots, \mathsf{Pv}_n\}$*;*

(v) *the columns of* P *are orthonormal.*

Proof: (i) $\Rightarrow$ (ii). Assume that $\mathsf{P}^{\mathrm{T}}\mathsf{P} = \mathsf{PP}^{\mathrm{T}} = \mathsf{I}$ then

$$|\mathsf{Px}|^2 = (\mathsf{Px})^{\mathrm{T}}(\mathsf{Px}) = \mathsf{x}^{\mathrm{T}}\mathsf{P}^{\mathrm{T}}\mathsf{Px} = \mathsf{x}^{\mathrm{T}}\mathsf{x} = |\mathsf{x}|^2\,.$$

Proof: (ii) $\Rightarrow$ (iii). Assume that $|\mathsf{Px}| = |\mathsf{x}|$ for all x, then

$$\begin{aligned} |\mathsf{P}(\mathsf{x}+\mathsf{y})|^2 &= |\mathsf{x}+\mathsf{y}|^2 \\ &= (\mathsf{x}+\mathsf{y})^{\mathrm{T}}(\mathsf{x}+\mathsf{y}) \\ &= \mathsf{x}^{\mathrm{T}}\mathsf{x} + \mathsf{y}^{\mathrm{T}}\mathsf{x} + \mathsf{x}^{\mathrm{T}}\mathsf{y} + \mathsf{y}^{\mathrm{T}}\mathsf{y} \\ &= |\mathsf{x}|^2 + 2\mathsf{x}^{\mathrm{T}}\mathsf{y} + |\mathsf{y}|^2\,. \end{aligned} \tag{6.4a}$$

But we also have that

$$\begin{aligned} |\mathsf{P}(\mathsf{x}+\mathsf{y})|^2 &= |\mathsf{Px}+\mathsf{Py}|^2 \\ &= |\mathsf{Px}|^2 + 2(\mathsf{Px})^{\mathrm{T}}\mathsf{Py} + |\mathsf{Py}|^2 \\ &= |\mathsf{x}|^2 + 2(\mathsf{Px})^{\mathrm{T}}\mathsf{Py} + |\mathsf{y}|^2\,. \end{aligned} \tag{6.4b}$$

Comparing (6.4a) and (6.4b) it follows that

$$(\mathsf{Px})^{\mathrm{T}}\mathsf{Py} = \mathsf{x}^{\mathrm{T}}\mathsf{y}\,. \tag{6.4c}$$

Proof: (iii) $\Rightarrow$ (iv). Assume $(\mathsf{Px})^{\mathrm{T}}(\mathsf{Py}) = \mathsf{x}^{\mathrm{T}}\mathsf{y}$, then

$$(\mathsf{Pv}_i)^{\mathrm{T}}(\mathsf{Pv}_j) = \mathsf{v}_i^{\mathrm{T}}\mathsf{v}_j = \delta_{ij}\,.$$ 23/10

Proof: (iv) ⇒ (v). Assume that if $\{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$ are orthonormal, so are $\{\mathsf{P}\mathsf{v}_1, \ldots, \mathsf{P}\mathsf{v}_n\}$. Take $\{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$ to be the standard orthonormal basis $\{\mathsf{e}_1, \ldots, \mathsf{e}_n\}$, then $\{\mathsf{P}\mathsf{e}_1, \ldots, \mathsf{P}\mathsf{e}_n\}$ are orthonormal. But $\{\mathsf{P}\mathsf{e}_1, \ldots, \mathsf{P}\mathsf{e}_n\}$ are the columns of P, so the result follows.

Proof: (v) ⇒ (i). If the columns of P are orthonormal then $\mathsf{P}^{\mathrm{T}}\mathsf{P} = \mathsf{I}$, and thence $(\det \mathsf{P})^2 = 1$. Thus P has a non-zero determinant and is invertible, with $\mathsf{P}^{-1} = \mathsf{P}^{\mathrm{T}}$. It follows that $\mathsf{P}\mathsf{P}^{\mathrm{T}} = \mathsf{I}$, and so P is orthogonal. □

6.3.1 $O(2)$ and $SO(2)$

All length preserving transformation matrices are thus orthogonal matrices. However, we can say more.

Proposition. Any matrix $\mathsf{Q} \in O(2)$ is one of the matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{6.5a}$$

for some $\theta \in [0, 2\pi)$.

Proof. Let

$$\mathsf{Q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.5b}$$

If $\mathsf{Q} \in O(2)$ then we require that $\mathsf{Q}^{\mathrm{T}}\mathsf{Q} = \mathsf{I}$, i.e.

$$\begin{aligned} \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{6.5c}$$

Hence

$$a^2 + c^2 = b^2 + d^2 = 1 \quad \text{and} \quad ab = -cd\,. \tag{6.5d}$$

The first of these relations allows us to choose θ so that $a = \cos\theta$ and $c = \sin\theta$ where $0 \leqslant \theta < 2\pi$. Further, $\mathsf{Q}\mathsf{Q}^{\mathrm{T}} = \mathsf{I}$, and hence

$$a^2 + b^2 = c^2 + d^2 = 1 \quad \text{and} \quad ac = -bd\,. \tag{6.5e}$$

It follows that

$$d = \pm\cos\theta \quad \text{and} \quad b = \mp\sin\theta\,, \tag{6.5f}$$

and so Q is one of the matrices

$$\mathsf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \mathsf{H}(\tfrac{1}{2}\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$ □

Remarks.

(a) These matrices represent, respectively, a rotation of angle θ, and a reflection in the line

$$y\cos\left(\tfrac{1}{2}\theta\right) = x\sin\left(\tfrac{1}{2}\theta\right). \tag{6.6a}$$

(b) Since $\det\mathsf{R} = +1$ and $\det\mathsf{H} = -1$, any matrix $\mathsf{A} \in SO(2)$ is a rotation, i.e. if $\mathsf{A} \in SO(2)$ then

$$\mathsf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{6.6b}$$

for some $\theta \in [0, 2\pi)$.

(c) Since $\det(\mathsf{H}(\frac{1}{2}\theta)\mathsf{H}(\frac{1}{2}\phi)) = \det(\mathsf{H}(\frac{1}{2}\theta))\det(\mathsf{H}(\frac{1}{2}\phi)) = +1$, we conclude (from closure of $O(2)$) that two reflections are equivalent to a rotation.

Proposition. Any matrix $\mathsf{Q} \in O(2)\backslash SO(2)$ is similar to the reflection

$$\mathsf{H}(\tfrac{1}{2}\pi) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{6.7}$$

Proof. If $\mathsf{Q} \in O(2)\backslash SO(2)$ then $\mathsf{Q} = \mathsf{H}(\frac{1}{2}\theta)$ for some θ. Q is thus symmetric, and so has real eigenvalues. Moreover, orthogonal matrices have eigenvalues of unit modulus (see *Example Sheet 4*), so since $\det \mathsf{Q} = -1$ we deduce that Q has distinct eigenvalues $+1$ and -1. Thus Q is diagonalizable from §5.5, and similar to

$$\mathsf{H}(\tfrac{1}{2}\pi) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \qquad \square$$

6.4 Metrics (or how to confuse you completely about scalar products)

Suppose that $\{\mathbf{u}_i\}$, $i = 1, \ldots, n$, is a (not necessarily orthogonal or orthonormal) basis of $\mathbb{R}^n$. Let $\mathbf{x}$ and $\mathbf{y}$ be any two vectors in $\mathbb{R}^n$, and suppose that in terms of components

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{u}_i \quad \text{and} \quad \mathbf{y} = \sum_{j=1}^{n} y_j \mathbf{u}_j \,, \tag{6.8}$$

Consider the scalar product $\mathbf{x} \cdot \mathbf{y}$. As remarked in §2.10.4, for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$,

$$\mathbf{x} \cdot (\lambda \mathbf{y} + \mu \mathbf{z}) = \lambda\, \mathbf{x} \cdot \mathbf{y} + \mu\, \mathbf{x} \cdot \mathbf{z} \,, \tag{6.9a}$$

and since $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$,

$$(\lambda \mathbf{y} + \mu \mathbf{z}) \cdot \mathbf{x} = \lambda\, \mathbf{y} \cdot \mathbf{x} + \mu\, \mathbf{z} \cdot \mathbf{x} \,. \tag{6.9b}$$

Hence from (6.8)

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= \left(\sum_{i=1}^{n} x_i \mathbf{u}_i \right) \cdot \left(\sum_{j=1}^{n} y_j \mathbf{u}_j \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \, \mathbf{u}_i \cdot \mathbf{u}_j && \text{from (6.9a) and (6.9b)} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i G_{ij} y_j \,, \end{aligned} \tag{6.10a}$$

where

$$G_{ij} = \mathbf{u}_i \cdot \mathbf{u}_j \quad (i, j = 1, \ldots, n) \,, \tag{6.10b}$$

are the scalar products of all pairs of basis vectors. In terms of the column vectors of components the scalar product can be written as

$$\mathbf{x} \cdot \mathbf{y} = \mathsf{x}^{\mathrm{T}} \mathsf{G}\, \mathsf{y} \,, \tag{6.10c}$$

where G is the symmetric matrix, or *metric*, with entries G_{ij}.

Remark. If the $\{\mathbf{u}_i\}$ form an orthonormal basis, i.e. are such that

$$G_{ij} = \mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij} \,, \tag{6.11a}$$

then (6.10a), or equivalently (6.10c), reduces to (cf. (2.62a) or (5.45a))

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i \,. \tag{6.11b}$$

Restatement. We can now restate our result from §6.3: the set of transformation matrices that preserve the scalar product with respect to the metric $\mathsf{G} = \mathsf{I}$ form a group.

6.5 Lorentz Transformations

Define the *Minkowski metric* in (1+1)-dimensional spacetime to be

$$\mathsf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.12a}$$

and a *Minkowski inner product* of two vectors (or events) to be

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathsf{x}^{\mathrm{T}} \mathsf{J} \mathsf{y}. \tag{6.12b}$$

If

$$\mathsf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \mathsf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{6.12c}$$

then

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2. \tag{6.12d}$$

6.5.1 The set of transformation matrices that preserve the Minkowski inner product

Suppose that M is a transformation matrix that preserves the Minkowski inner product. Then we require that for all vectors $\mathbf{x}$ and $\mathbf{y}$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathsf{M}\mathbf{x}, \mathsf{M}\mathbf{y} \rangle \tag{6.13a}$$

i.e.

$$\mathsf{x}^{\mathrm{T}} \mathsf{J} \mathsf{y} = \mathsf{x}^{\mathrm{T}} \mathsf{M}^{\mathrm{T}} \mathsf{J} \mathsf{M} \mathsf{y}, \tag{6.13b}$$

or in terms of the summation convention

$$x_k J_{k\ell} y_\ell = x_k M_{pk} J_{pq} M_{q\ell} y_\ell. \tag{6.13c}$$

Now choose $x_k = \delta_{ik}$ and $y_\ell = \delta_{j\ell}$ to deduce that

$$J_{ij} = M_{pi} J_{pq} M_{qj} \quad \text{i.e.} \quad \mathsf{J} = \mathsf{M}^{\mathrm{T}} \mathsf{J} \mathsf{M}. \tag{6.14}$$

Remark. Compare this with the condition, $\mathsf{I} = \mathsf{Q}^{\mathrm{T}} \mathsf{I} \mathsf{Q}$, satisfied by transformation matrices Q that preserve the scalar product with respect to the metric $\mathsf{G} = \mathsf{I}$.

Proposition. Any transformation matrix M that preserves the Minkowski inner product in (1+1)-dimensional spacetime (and keeps the past in the past and the future in the future) is one of the matrices

$$\begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}, \quad \begin{pmatrix} \cosh u & -\sinh u \\ \sinh u & -\cosh u \end{pmatrix}. \tag{6.15a}$$

for some $u \in \mathbb{R}$.

Proof. Let

$$\mathsf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.15b}$$

Then from (6.14)

$$\begin{aligned} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{6.15c}$$

Hence

$$a^2 - c^2 = 1, \quad b^2 - d^2 = -1 \quad \text{and} \quad ab = cd. \tag{6.15d}$$

If the past is to remain the past and the future to remain the future, then it turns out that we require $a > 0$. So from the first two equations of (6.15d)

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \cosh u \\ \sinh u \end{pmatrix}, \quad \begin{pmatrix} b \\ d \end{pmatrix} = \pm \begin{pmatrix} \sinh w \\ \cosh w \end{pmatrix}, \tag{6.15e}$$

where $u, w \in \mathbb{R}$. The last equation of (6.15d) then gives that $u = w$. Hence M is one of the matrices

$$\mathsf{H}_u = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}, \quad \mathsf{K}_{u/2} = \begin{pmatrix} \cosh u & -\sinh u \\ \sinh u & -\cosh u \end{pmatrix}. \qquad \square$$

Remarks.

(a) H_u and $\mathsf{K}_{u/2}$ are known as *hyperbolic rotations* and *hyperbolic reflections* respectively.

(b) Define the *Lorentz matrix*, B_v, by

$$\mathsf{B}_v = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix}, \tag{6.16a}$$

where $v \in \mathbb{R}$. A transformation matrix of this form is known as a *boost* (or a boost by a velocity v). The set of transformation boosts and hyperbolic rotations are the same since

$$\mathsf{H}_u = \mathsf{B}_{\tanh u} \quad \text{or equivalently} \quad \mathsf{H}_{\tanh^{-1} v} = \mathsf{B}_v. \tag{6.16b}$$

6.5.2 The set of Lorentz boosts forms a group

Closure. The set is closed since

$$\begin{aligned} \mathsf{H}_u \mathsf{H}_w &= \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} \cosh w & \sinh w \\ \sinh w & \cosh w \end{pmatrix} \\ &= \begin{pmatrix} \cosh(u+w) & \sinh(u+w) \\ \sinh(u+w) & \cosh(u+w) \end{pmatrix} \\ &= \mathsf{H}_{u+w}. \end{aligned} \tag{6.17a}$$

Associativity. Matrix multiplication is associative.

Identity element. The identity element is a member of the set since

$$\mathsf{H}_0 = \mathsf{I}. \tag{6.17b}$$

Inverse element. H_u has an inverse element in the set, namely H_{-u}, since from (6.17a) and (6.17b)

$$\mathsf{H}_u \mathsf{H}_{-u} = \mathsf{H}_0 = \mathsf{I}. \tag{6.17c}$$

Remark. To be continued in *Special Relativity.*

A Möbius Transformations

Consider a *map* of $\mathbb{C} \to \mathbb{C}$ ('$\mathbb{C}$ into $\mathbb{C}$') defined by

$$z \mapsto z' = f(z) = \frac{az+b}{cz+d}, \tag{A.1}$$

where $a, b, c, d \in \mathbb{C}$ are constants. We require that

(a) c and d are not both zero, so that the map is finite (except at $z = -d/c$);

(b) different points map to different points, i.e. if $z_1 \neq z_2$ then $z_1' \neq z_2'$, i.e. we require that

$$\frac{az_1+b}{cz_1+d} \neq \frac{az_2+b}{cz_2+d}, \quad \text{or equivalently} \quad (ad-bc)(z_1-z_2) \neq 0, \quad \text{i.e.} \quad (ad-bc) \neq 0.$$

Remarks.

(i) Condition (a) is a subset of condition (b), hence we need only require that $(ad - bc) \neq 0$.

(ii) $f(z)$ maps every point of the complex plane, except $z = -d/c$, into another ($z = -d/c$ is mapped to infinity).

(iii) Adding the 'point at infinity' makes f complete.

A.1 Composition

Consider a second Möbius transformation

$$z' \mapsto z'' = g(z') = \frac{\alpha z' + \beta}{\gamma z' + \delta} \quad \text{where} \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \quad \text{and} \quad \alpha\delta - \beta\gamma \neq 0.$$

Then the combined map $z \mapsto z''$ is also a Möbius transformation:

$$\begin{aligned} z'' = g(z') &= g(f(z)) \\ &= \frac{\alpha\left(\frac{az+b}{cz+d}\right) + \beta}{\gamma\left(\frac{az+b}{cz+d}\right) + \delta} \\ &= \frac{\alpha(az+b) + \beta(cz+d)}{\gamma(az+b) + \delta(cz+d)} \\ &= \frac{(\alpha a + \beta c)z + (\alpha b + \beta d)}{(\gamma a + \delta c)z + (\gamma b + \delta d)}, \end{aligned} \tag{A.2}$$

where we note that $(\alpha a + \beta c)(\gamma b + \delta d) - (\alpha b + \beta d)(\gamma a + \delta c) = (ad - bc)(\alpha\delta - \beta\gamma) \neq 0$. Therefore the set of all Möbius maps is *closed* under composition.

A.2 Inverse

For the $a, b, c, d \in \mathbb{R}$ as in (A.1), consider the Möbius map

$$z'' = \frac{-dz' + b}{cz' - a}, \tag{A.3}$$

i.e. $\alpha = -d$, $\beta = b$, $\gamma = c$ and $\delta = -a$. Then from (A.2), $z'' = z$. We conclude that (A.3) is the *inverse* to (A.1), and *vice versa*.

Remarks.

(i) (A.1) maps $\mathbb{C}\setminus\{-d/c\}$ to $\mathbb{C}\setminus\{a/c\}$, while (A.3) maps $\mathbb{C}\setminus\{a/c\}$ to $\mathbb{C}\setminus\{-d/c\}$.

(ii) The inverse (A.3) can be deduced from (A.1) by formal manipulation.

Exercise. For Möbius maps f, g and h demonstrate that the maps are *associative*, i.e. $(fg)h = f(gh)$.

A.3 Basic Maps

Translation. Put $a = 1$, $c = 0$ and $d = 1$ to obtain

$$z' = z + b\,. \tag{A.4a}$$

This map represents a translation; e.g lines map to parallel lines, while circles map to circles of the same radius but with a centre offset by b.

Dilatation and Rotation. Next put $b = 0$, $c = 0$ and $d = 1$ so that

$$z' = az = |a||z|\,e^{i(\arg a+\arg z)}\,. \tag{A.4b}$$

This map scales z by $|a|$ and rotates z by $\arg a$ about the origin O.

The line $z = z_0 + \lambda w$, where $\lambda \in \mathbb{R}$ and $w \in \mathbb{C}$, becomes

$$z' = az_0 + \lambda aw = z_0' + \lambda w'\,,$$

where $z_0' = az_0$ and $w' = aw$, which is just another line.

The circle $|z - v| = r$ becomes

$$\left|\frac{z'}{a} - v\right| = r \quad \text{or equivalently} \quad |z' - v'| = r'\,,$$

where $v' = av$ and $r' = |a|r$, which is just another circle.

Inversion and Reflection. Now put $a = 0$, $b = 1$, $c = 1$ and $d = 0$, so that $z' = \frac{1}{z}$. Thus if $z = re^{i\theta}$ then

$$z' = \frac{1}{r}e^{-i\theta}\,, \quad \text{i.e. } |z'| = |z|^{-1} \text{ and } \arg z' = -\arg z.$$

Hence this map represents *inversion in the unit circle* centred on the origin O, and *reflection in the real axis*.

The line $z = z_0 + \lambda w$, or equivalently (see (1.28b))

$$z\bar{w} - \bar{z}w = z_0\bar{w} - \bar{z}_0 w\,,$$

becomes

$$\frac{\bar{w}}{z'} - \frac{w}{\bar{z}'} = z_0\bar{w} - \bar{z}_0 w\,.$$

By multiplying by $|z'|^2$, etc., this equation can be rewritten successively as

$$\begin{aligned}
\bar{z}'\bar{w} - z'w &= (z_0\bar{w} - \bar{z}_0 w)\,z'\bar{z}' \\
z'\bar{z}' - \frac{\bar{z}'\bar{w}}{z_0\bar{w} - \bar{z}_0 w} - \frac{z'w}{\bar{z}_0 w - z_0\bar{w}} &= 0 \\
\left|z' - \frac{\bar{w}}{z_0\bar{w} - \bar{z}_0 w}\right|^2 &= \left|\frac{\bar{w}}{z_0\bar{w} - \bar{z}_0 w}\right|^2.
\end{aligned}$$

From (1.29a) this is a circle (which passes through the origin) with centre $\frac{\bar{w}}{z_0\bar{w}-\bar{z}_0 w}$ and radius $\left|\frac{\bar{w}}{z_0\bar{w}-\bar{z}_0 w}\right|$. The exception is when $z_0\bar{w}-\bar{z}_0 w = 0$, in which case the original line passes through the origin, and the mapped curve, $\bar{z}'\bar{w} - z'w = 0$, is also a straight line through the origin.

Further, under the map $z' = \frac{1}{z}$ the circle $\mid z - v \mid= r$ becomes

$$\left|\frac{1}{z'} - v\right| = r\,, \quad \text{i.e.} \quad |1 - vz'| = r|z'|\,.$$

Hence

$$(1 - vz')\left(1 - \bar{v}\bar{z}'\right) \quad = \quad r^2\bar{z}'z'\,,$$

or equivalently

$$z'\bar{z}'\left(|v|^2 - r^2\right) - vz' - \bar{v}\bar{z}' + 1 \quad = \quad 0\,,$$

or equivalently

$$z'\bar{z}' - \frac{v}{(|v|^2 - r^2)}z' - \frac{\bar{v}}{(|v|^2 - r^2)}\bar{z}' + \frac{|v|^2}{(|v|^2 - r^2)^2} \quad = \quad \frac{r^2}{(|v|^2 - r^2)^2}\,.$$

From (1.29b) this is the equation for a circle with centre $\bar{v}/(|v|^2 - r^2)$ and radius $r/(|v|^2 - r^2)$. The exception is if $|v|^2 = r^2$, in which case the original circle passed through the origin, and the map reduces to

$$vz' + \bar{v}\bar{z}' = 1\,,$$

which is the equation of a straight line.

Summary. Under inversion and reflection, circles and straight lines which do not pass through the origin map to circles, while circles and straight lines that do pass through origin map to straight lines.

A.4 The General Möbius Map

The reason for introducing the basic maps above is that the general Möbius map can be generated by composition of translation, dilatation and rotation, and inversion and reflection. To see this consider the sequence:

dilatation and rotation	$z \mapsto z_1 = cz$	$(c \neq 0)$
translation	$z_1 \mapsto z_2 = z_1 + d$	
inversion and reflection	$z_2 \mapsto z_3 = 1/z_2$	
dilatation and rotation	$z_3 \mapsto z_4 = \left(\dfrac{bc - ad}{c}\right) z_3$	$(bc \neq ad)$
translation	$z_4 \mapsto z_5 = z_4 + a/c$	$(c \neq 0)$

Exercises.

(i) Show that

$$z_5 = \frac{az + b}{cz + d}\,.$$

(ii) Construct a similar sequence if $c = 0$ and $d \neq 0$.

The above implies that a general Möbius map transforms circles and straight lines to circles and straight lines (since each constituent transformation does so).

B The Operation Count for the Laplace Expansion Formulae

On page 69 there was an exercise to find the number of operations, f_n, needed to calculate a $n \times n$ determinant using the Laplace expansion formulae. Since each $n \times n$ determinant requires us to calculate n smaller $(n-1) \times (n-1)$ determinants, plus perform n multiplications and $(n-1)$ additions,

$$f_n = n f_{n-1} + 2n - 1\,.$$

Hence

$$\frac{f_n + 2}{n!} = \frac{(f_{n-1} + 2)}{(n-1)!} + \frac{1}{n!}\,,$$

and so by recursion

$$\frac{f_n + 2}{n!} = \frac{(f_1 + 2)}{1!} + \sum_{r=2}^{n} \frac{1}{r!}$$

But $f_1 = 0$, and so

$$\begin{aligned} f_n &= n! \sum_{r=0}^{n} \frac{1}{r!} - 2 \\ &= n!e - 2 - n! \sum_{r=n+1}^{\infty} \frac{1}{r!} \\ &\to n!e - 2 \qquad\qquad \text{as } n \to \infty. \end{aligned}$$