Mathematical Tripos Part IA: Introductory Examples Sheet

The material on this sheet is not particularly related to any course. The intention is to give you something to do until the lecture courses have progressed far enough for you to tackle the first proper examples sheet. Some of the questions have hard bits, so don't expect to do everything!

1 Sketch the graph of the function $y(x)$ given by

$$y(x) = \frac{x-3}{(x+1)(x-2)},$$

indicating the positions of the turning points.

Prove that there is a range of values which y cannot take if x is real.

2 (i) Find dy/dx when $\tan y = \left(\frac{x-1}{2-x}\right)^{\frac{1}{2}}$, where $1 < x < 2$. Hence integrate

$$\int \frac{1}{\sqrt{(x-1)(2-x)}}\,dx \qquad (1 < x < 2).$$

(ii) Guess the integral

$$\int \frac{1}{\sqrt{(t-a)(b-t)}}\,dt \qquad (a < t < b)$$

and verify your guess by differentiation.

(iii) Do the integral of part (ii) by writing $(t-a)(b-t)$ in the form $A^2 - (t-B)^2$ and using the new variable θ defined by $(t-B) = A\sin\theta$. Check using trigonometry that your answers agree.

3 A curve is given parametrically by

$$x = a(\theta - \sin\theta), \quad y(\theta) = a(1+\cos\theta),$$

where a is a positive constant. Show that the gradient of the curve is $-\cot\frac{1}{2}\theta$. Sketch the curve in the x-y plane, explaining carefully how you discovered its main features.

A small smooth bead is slides on the curve. Show that its velocity $\big(\dot{x}, \dot{y}\big)$, where the dot denotes differentiation with respect to time, t, is

$$2a\dot{\theta}\sin\tfrac{1}{2}\theta\,\Big(\sin\tfrac{1}{2}\theta,\ -\cos\tfrac{1}{2}\theta\Big).$$

Show also, by conservation of energy, that if the bead started at rest with $\theta = \theta_0$, where $0 < \theta < \pi$, then

$$\dot{\theta} = \frac{\sqrt{(g/2a)(\cos\theta_0 - \cos\theta)}}{\sin\frac{1}{2}\theta}.$$

Hence find the time for the bead to reach the point with $\theta = \pi$. Comment on your result.

4 Let

$$I_n = \int_0^\infty \operatorname{sech}^n u \,\mathrm{d}u.$$

By integrating by parts, show that for $n > 0$

$$\int_0^\infty (\operatorname{sech}^{n+2} u\ \sinh u)\sinh u\,\mathrm{d}u = (n+1)^{-1} I_n$$

and deduce that $(n+1)I_{n+2} = nI_n$. Find the value of I_6.
[NB: $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$, $\operatorname{sech} x = (\cosh x)^{-1}$. You will need to use the identity $\sinh^2 u = \cosh^2 u - 1$.]

5 Show that $\int_0^{2\pi} e^{in\theta}\mathrm{d}\theta = 0$ where n is a non-zero integer. Express $\cos\theta$ in terms of $e^{i\theta}$.

Write down the coefficient of the term which is independent of x in binomial expansion of $(x+x^{-1})^{2n}$, where n is a positive integer. By taking $x = e^{i\theta}$ and using the results of the first paragraph, evaluate

$$\int_0^{2\pi} \cos^{2n}\theta \,\mathrm{d}\theta \qquad \text{and} \qquad \int_0^{2\pi} \cos^{2n} 2\theta \,\mathrm{d}\theta.$$

6 A uniform wire is bent into the shape of a circular arc of radius a which subtends an angle 2α at the centre of the circle. The distance, d, of the centre of mass of the wire from the centre of the circle may be written as $d = f(\alpha)$. By cutting the arc into two similar arcs, or otherwise, show that

$$f(\alpha) = f(\alpha/2)\cos(\alpha/2). \qquad (*)$$

Show that $f(\alpha) = (A\sin\alpha)/\alpha$ satisfies this equation. Assume that this is the correct form for $f(\alpha)$. By considering the case when α is very small, show that $d = (a\sin\alpha)/\alpha$. Show that the corresponding result for a lamina in the shape of a sector of a circle is $d = (2a\sin\alpha)/(3\alpha)$.

Try solving the functional equation $(*)$ by considering the function $g(x)$ defined by $g(x) = \dfrac{\sin x}{xf(x)}$.

7 (i) Show that $\displaystyle\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \frac{\pi^2}{8}$.

(ii) By writing $\sin^{-1} x = \displaystyle\int_0^x \frac{dy}{\sqrt{1-y^2}}$, show that $\sin^{-1} x = \displaystyle\sum_0^\infty \frac{(2n)!}{2^{2n}(n!)^2}\,\frac{x^{2n+1}}{2n+1}$.

(iii) Let $I_n = \displaystyle\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx$. Show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$ and hence that $I_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$.

Deduce that $\displaystyle\sum_0^\infty \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

Noting that $4\displaystyle\sum_1^\infty \frac{1}{(2n)^2} = \sum_1^\infty \frac{1}{n^2}$, deduce also that $\zeta(2) = \dfrac{\pi^2}{6}$, where $\zeta(k) = \displaystyle\sum_{n=1}^\infty \frac{1}{n^k}$.

8 For any given polynomial f, the factor theorem states that $f(a) = 0$ if and only if $f(x) = (x-a)g(x)$ where $g(x)$ is a polynomial. By comparison with the factor theorem, show that the (correct) result

$$\sin\pi x = \pi x\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\cdots \qquad (*)$$

is plausible. Can $(*)$ be derived from the factor theorem?

(i) Use $(*)$ to derive the Wallis's formula

$$\frac{\pi}{2} = \frac{2}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\frac{8}{7}\times\frac{8}{9}\times\cdots.$$

Use $(*)$ and the double angle formula for $\sin 2\pi x$ to obtain an infinite product for $\cos\pi x$ and hence obtain an infinite product for $\sqrt{2}$.

(ii) By considering the Taylor series for $\sin\pi x$, obtain the expression in the previous question for $\zeta(2)$. Obtain the corresponding expression for $\zeta(4)$.

9 The aim of this question is to prove that π is irrational by contradiction. Suppose $\pi = p/q$ where p and q are positive integers and let

$$I_n = \int_0^{\pi} \mathrm{f}(x) \sin x \, \mathrm{d}x \,,$$

where $\mathrm{f}(x) = \dfrac{q^n x^n (\pi - x)^n}{n!}$ and q is a positive integer.

(i) Show that

$$I_n = \sum_{j=0}^{n} (-1)^j \left(\mathrm{f}^{(2j)}(\pi) + \mathrm{f}^{(2j)}(0) \right)$$

where $\mathrm{f}^k(x)$ denotes the kth derivative of $\mathrm{f}(x)$. Deduce that I_n is an integer.

(ii) Show that

$$I_n \leq \frac{\pi}{n!} \left(\frac{q\pi^2}{4} \right)^n ,$$

and deduce that $I_n \to 0$ as $n \to \infty$.

(iii) Deduce that π is irrational.