1 Constraints

(a) Reproduce the constraint equations by varying the action

$$S = \int d^4x\sqrt{h}N\left\{\frac{M_{\rm Pl}^2}{2}\left[{}^{(3)}R + K_{ij}K^{ij} - K^2\right] + P(X,\phi)\right\}.$$

with respect to N and N^i.
(b) Solve the $\delta S/\delta N^i$ constraint to find δN, working in flat gauge to linear order.

2 Gauge Transformations

(a) Derive the linear-order gauge transformations of A, B, ψ and h_{00}, for a generic change of coordinates $x^\mu \to x^\mu + \epsilon^\mu$.
(b) Derive the gauge transformation from Newtonian gauge to flat gauge and viceversa. In particular, given some generic perturbations $\{A^N, h_{00}^N, \varphi^N\}$ in Newtonian gauge, determine teh corresponding perturbations $\{h_{00}^f, \psi^f, \varphi^f\}$ in flat gauge.

3 Conservation of $\mathcal{R}$

In the lecture, we prove the conservation of $\mathcal{R}$ and γ_{ij} on superHubble scales in the presence of a generic energy-momentum tensor $T_{\mu\nu}$ by working in comoving gauge. Prove again the conservation of $\mathcal{R}$ by working in Newtonian gauge. In particular, you might want to start with the change of coordinates

$$\epsilon^\mu = \left\{\epsilon(t), \lambda x^i\right\}.$$

and show that the gauge transformations are

$$\begin{aligned}
&\Phi = -\dot\epsilon\,, && \Psi = H\epsilon - \frac{\lambda}{3}\,. && \\
&\delta\rho = -\dot{\bar\rho}\epsilon\,, && \delta u = \epsilon\,, && \pi^S = 0\,, \\
&\delta p = -\dot{\bar p}\epsilon\,, && \varphi = -\epsilon\dot{\bar\phi}\,. &&
\end{aligned}$$

Then use the scalar part of the ij components of the Einstein's equation,

$$k_ik_j\left(\Phi - \Psi\right) = 0,$$

to impose the physicality condition on $\epsilon(t)$. Your final result should be

$$\mathcal{R} = \frac{\lambda}{3}\,, \qquad \varphi = -\dot{\bar\phi}\frac{\mathcal{R}}{a}\int_T^t a(t')dt'\,, \qquad \Phi = \Psi = \mathcal{R}\left[-1 + \frac{H}{a}\int_T^t a(t')dt'\right].$$

4 Power Spectrum and Correlation Function

(a) Writing the Fourier transform of $\delta(\boldsymbol{k})$ and $\delta(\boldsymbol{k}')$, derive the expression for $\langle\delta(\boldsymbol{k})\delta(\boldsymbol{k}')\rangle$ in terms of the correlation function $\xi(r)$.
(b) Writing the Fourier transform of $\delta(\boldsymbol{x})$ and $\delta(\boldsymbol{x}'+\boldsymbol{r})$, derive the expression for $\langle\delta(\boldsymbol{x})\delta(\boldsymbol{x}'+\boldsymbol{r})\rangle$ in terms of the power spectrum $P(k)$.
(c) The power spectrum and correlation function are related to each other through spherical Bessel transformations $P(k) = 4\pi\int \mathrm{d}r r^2\xi(r)j_0(kr)$ and $\xi(r) = 1/(2\pi^2)\int \mathrm{d}k k^2 P(k)j_0(kr)$. Show that closure of this relation, i.e., $P \to \xi \to P$ yields the closure relation for spherical Bessel functions

$$\int_0^\infty \mathrm{d}x x^2 j_\alpha(ux)j_\alpha(vx) = \frac{\pi}{2u^2}\delta^{(\mathrm{D})}(u-v)\,.$$

5 Gaussian PDF and Edgeworth expansion

Consider the d_y dimensional random vector $\boldsymbol{y}$ described by the multivariate Gaussian PDF

$$\mathbb{P}_{\mathrm{G}}(\boldsymbol{y}) = \frac{1}{\sqrt{(2\pi)^{d_y}|M|}}\exp\left[-\frac{1}{2}\boldsymbol{y}^{\mathrm{T}}M^{-1}\boldsymbol{y}\right].$$

Now split the random vector $\boldsymbol{y}$ into two components $\boldsymbol{y} = (\boldsymbol{m}, \boldsymbol{n})$ of dimensions d_m and d_n with covariance matrix

$$M = \begin{pmatrix} \langle m,m\rangle & \langle m,n\rangle \\ \langle n,m\rangle & \langle n,n\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B^{\rm T} & D \end{pmatrix} = \begin{pmatrix} S & T \\ T^{\rm T} & V \end{pmatrix}^{-1} .$$

(a) *Marginalization and conditional probability* – By explicitly integrating out the variables n, show that

$$\mathbb{P}_{\rm G}(\boldsymbol{m}) = \frac{1}{\sqrt{(2\pi)^{d_m}|A|}} \exp\left[-\frac{1}{2}\boldsymbol{m}^{\rm T}A^{-1}\boldsymbol{m}\right] .$$

It will be useful to use the formulae for block inversion of the matrix

$$M^{-1} = \begin{pmatrix} S & T \\ T^{\rm T} & V \end{pmatrix} = \begin{pmatrix} A^{-1} + A^{-1}B(D - B^{\rm T}A^{-1}B)^{-1}B^{\rm T}A^{-1} & -A^{-1}B(D - B^{\rm T}A^{-1}B)^{-1} \\ -(D - B^{\rm T}A^{-1}B)^{-1}B^{\rm T}A^{-1} & (D - B^{\rm T}A^{-1}B)^{-1} \end{pmatrix}$$

and to complete the squares. For the determinant one has $|M| = |A|/|V|$ with $V = (D - B^{\rm T}A^{-1}B)^{-1}$. Now show that the conditional PDF of variables n given m can be expressed as

$$\mathbb{P}_{\rm G}(\boldsymbol{n}|\boldsymbol{m}) = \frac{1}{\sqrt{(2\pi)^{d_n}|\Sigma|}} \exp\left[-\frac{1}{2}(\boldsymbol{n} - \boldsymbol{\mu})^{\rm T}\Sigma^{-1}(\boldsymbol{n} - \boldsymbol{\mu})\right] ,$$

where $\boldsymbol{\mu} = B^{\rm T}A^{-1}\boldsymbol{m}$ and $\Sigma = V^{-1} = D - B^{\rm T}A^{-1}B$.

(b) *Edgeworth expansion* – Similar to what is discussed in the lecture notes for the one-point PDF, the Edgeworth expansion of a Gaussian random field with three point function $\mathcal{R}_{ijl} \ll 1$ can be expressed as

$$\mathbb{P}(\boldsymbol{y}) = \left(1 - \frac{\mathcal{R}_{ijl}}{3!}\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_l}\right)\mathbb{P}_{\rm G}(\boldsymbol{y}) .$$

Evaluate the PDF explicitly and then simplify the result for the case of a diagonal covariance matrix $C_{ij} = \delta^{\rm (K)}_{ij}C_i$. Show that the characteristic function is given by

$$\mathcal{M}({\rm i}\boldsymbol{J}) = \left(1 - (-{\rm i})^3\frac{\mathcal{R}_{ijl}}{3!}J_iJ_jJ_l\right)\exp\left[-\frac{1}{2}J_mC_{mn}J_n\right] .$$

Use the characteristic function to evaluate the expectation values $\langle f_i\rangle$, $\langle f_if_j\rangle$ and $\langle f_if_jf_l\rangle$.

6 Primordial non-Gaussianity

In the local non-Gaussian model, higher order corrections to the linear solution $\mathcal{R}^{(1)}$ arise simply from powers of $\mathcal{R}^{(1)}$, that is, at second-order

$$\mathcal{R}(\boldsymbol{x}) = \mathcal{R}^{(1)}(\boldsymbol{x}) + \mathcal{R}^{(2)}(\boldsymbol{x}) + \ldots = \mathcal{R}^{(1)} + \tfrac{3}{5}f_{\rm NL}\left((\mathcal{R}^{(1)})^2 - \langle(\mathcal{R}^{(1)})^2\rangle\right) + \ldots ,$$

where we assume a Gaussian isotropic model for the linear solution $\mathcal{R}^{(1)}$, that is, its Fourier transform satisfies $\langle\mathcal{R}^{(1)}(\boldsymbol{k})\,\mathcal{R}^{(1)}(\boldsymbol{k}')\rangle = (2\pi)^3P_{\mathcal{R}}(k)\,\delta^{\rm (D)}(\boldsymbol{k}+\boldsymbol{k}')$ with power spectrum $P(k)$. Using the convolution theorem, show that the solution for the second-order solution $\mathcal{R}^{(2)}$ can be expressed in the form,

$$\mathcal{R}^{(2)}(\boldsymbol{k}) = \int\frac{{\rm d}^3\boldsymbol{k}'}{(2\pi)^3}\int\frac{{\rm d}^3\boldsymbol{k}''}{(2\pi)^3}\left[\mathcal{R}^{(1)}(\boldsymbol{k}')\,\mathcal{R}^{(1)}(\boldsymbol{k}'') - \langle\mathcal{R}^{(1)}(\boldsymbol{k}')\,\mathcal{R}^{(1)}(\boldsymbol{k}'')\rangle\right]\,\delta^{\rm (D)}(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'')$$

Hence, find the bispectrum for the local model $\langle\mathcal{R}(\boldsymbol{k}_1)\,\mathcal{R}(\boldsymbol{k}_2)\,\mathcal{R}(\boldsymbol{k}_3)\rangle = (2\pi)^3\delta^{\rm (D)}(\boldsymbol{k}_1{+}\boldsymbol{k}_2{+}\boldsymbol{k}_3)\,B(k_1,\,k_2,\,k_3)$,

$$B^{\rm loc}(k_1,k_2,k_3) = \tfrac{6}{5}f_{\rm NL}\left[P_{\mathcal{R}}(k_1)P_{\mathcal{R}}(k_2) + P_{\mathcal{R}}(k_2)P_{\mathcal{R}}(k_3) + P_{\mathcal{R}}(k_3)P_{\mathcal{R}}(k_1)\right] .$$

7 Projections of statistically-homogeneous and isotropic random fields

Consider a random field $F(\boldsymbol{x})$ that is statistically homogeneous and isotropic and has zero mean. The radial projection of the gradient of this field on the surface of the sphere $|\boldsymbol{x}| = r$ defines a field $f(\hat{\boldsymbol{n}}) \equiv \hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}F(r\hat{\boldsymbol{n}})$.

(a) Using the Rayleigh plane-wave expansion,

$$e^{{\rm i}\boldsymbol{k}\cdot\boldsymbol{x}} = 4\pi\sum_{lm}{\rm i}^l j_l(kr)Y^*_{lm}(\hat{\boldsymbol{k}})Y_{lm}(\hat{\boldsymbol{x}}) ,$$

where $j_l(kr)$ are spherical Bessel functions, show that the spherical multipoles of $f(\hat{\boldsymbol{n}})$ are related to the Fourier transform of $F(\boldsymbol{x})$ by

$$f_{lm} = 4\pi \mathrm{i}^l \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} F(\boldsymbol{k}) k j_l'(kr) Y_{lm}^*(\hat{\boldsymbol{k}}) \,,$$

where $j_l'(x) = \mathrm{d}j_l/\,\mathrm{d}x$.
(b) Hence show that $f(\hat{\boldsymbol{n}})$ is statistically isotropic with angular power spectrum

$$C_l = 4\pi \int \mathrm{d}\ln k\, \Delta_F^2(k) k^2 [j_l'(kr)]^2 \,,$$

where $\Delta_F^2(k)$ is the power spectrum of $F(\boldsymbol{x})$. Sketch $[j_l'(kr)]^2$ and so deduce that the mapping between k and l is less sharp for the projection of the radial gradient of a field compared to the projection of the field itself.