1 Power spectrum of randomly-placed halos*

(a) A crude model for the non-Gaussian distribution of dark matter at late times considers all dark matter to be located in halos with identical density profiles $\kappa(\boldsymbol{x})$ and mass $m = \int d^3\boldsymbol{x}\,\kappa(\boldsymbol{x})$. Imagine laying these down at random with an average number density $\bar{n}$. If, in some realisation of this random process, there are a total of N such haloes in some large volume V, the mass density within the volume is

$$\rho(\boldsymbol{x}) = \sum_{i=1}^{N} \kappa(\boldsymbol{x} - \boldsymbol{x}_i)\,,$$

where $\boldsymbol{x}_i$ is the position of the centre of the ith halo. Taking the $\{\boldsymbol{x}_i\}$ to be uniformly distributed in V, and the total number N to be a Poisson process with mean nV, show that the two-point correlation function of $\delta\rho(\boldsymbol{x}) \equiv \rho(\boldsymbol{x}) - \langle \rho(\boldsymbol{x}) \rangle$ is

$$\xi(\boldsymbol{x}, \boldsymbol{x}') = \bar{n} \int d^3\boldsymbol{y}\kappa(\boldsymbol{x} - \boldsymbol{y})\kappa(\boldsymbol{x}' - \boldsymbol{y})\,.$$

(b) Show that the two-point correlation function is consistent with statistical homogeneity and, for spherically-symmetric $\kappa(\boldsymbol{x})$, statistical isotropy.
(c) By expressing the two-point correlator in Fourier space in terms of $\xi(\boldsymbol{x}, \boldsymbol{x}')$, show that the power spectrum of $\delta\rho(\boldsymbol{x})$ is

$$P_{\delta\rho}(k) = n|\kappa(k)|^2\,,$$

for spherical profiles with Fourier transform $\kappa(k)$.
(d) Argue that the power spectrum behaves as k^0 on scales large compared to the characteristic size of the haloes. (Such a power spectrum describes white noise.)

2 Continuity and Euler Equation

(a) Derive the continuity and Euler equations from the Vlaslov equation using the definitions of density, velocity and velocity dispersion given in the lecture notes.
(b) Using the real space formulation of continuity and Euler equation in the absence of velocity dispersion $\sigma_{ij} = 0$, derive the coupling kernels α and β given in the Lecture notes.

3 Perturbation Theory Loop Diagrams*

(a) From the Fourier space versions of the continuity and Euler equations in a matter-only, Einstein-de-Sitter universe and the power series ansatz for the perturbative solution, derive the recursion relations for the kernels F_n and G_n given in the lecture notes. For this purpose, first rewrite the time derivatives as derivative with respect to the scale factor.
(b) Draw the diagrams for the tree-level trispectrum, one-loop bispectrum and two-loop power spectrum. Write down the expressions in terms of the gravitational coupling kernels $F_{n,\mathrm{s}}$.

4 Equivalence of Lagrangian and Eulerian PT

Consider the expression for the density field $\delta(\boldsymbol{k})$ in terms of the displacement field $\boldsymbol{\Psi} = \boldsymbol{\Psi}^{(1)} + \boldsymbol{\Psi}^{(2)}$:

$$\delta(\boldsymbol{k}) = \int \mathrm{d}^3q \exp\left[-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q} + \boldsymbol{\Psi}(\boldsymbol{q}))\right] - \int \mathrm{d}^3q \exp\left[-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}\right],$$

to show the perturbative equivalence of Lagrangian and Eulerian perturbation theory up to second order. Expand the exponential up to this order and use that the displacement field in Fourier-space can be expressed as

$$\boldsymbol{\Psi}^{(n)}(\boldsymbol{k}) = \frac{\mathrm{i}}{n!}\prod_{i=1}^{n}\left\{\int \frac{\mathrm{d}^3q_i}{(2\pi)^3}\delta^{(1)}(\boldsymbol{q}_i)\right\}\boldsymbol{L}_n(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n)(2\pi)^3\delta^{(\mathrm{D})}(\boldsymbol{k} - \boldsymbol{q}_1 \ldots - \boldsymbol{q}_n)$$

with $\boldsymbol{L}_1 = \boldsymbol{k}/k^2$ and

$$\boldsymbol{L}_2(\boldsymbol{q}_1, \boldsymbol{q}_2) = \frac{3}{7}\frac{\boldsymbol{q}_1 + \boldsymbol{q}_2}{(\boldsymbol{q}_1 + \boldsymbol{q}_2)^2}\left(1 - \frac{(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)^2}{q_1^2 q_2^2}\right).$$

Finally reorder the terms to show that the result agrees with the expression of the first and second order density fields in terms of F_1 and F_2.

5 Redshift Space Distortions

When observing galaxies, we infer their radial distance using the redshift assuming they are comoving with the homogeneous Hubble flow. The inhomogeneities arising from density perturbations are causing a peculiar motion around the Hubble flow. Thus, the inferred distance is distorted $\boldsymbol{s} = \boldsymbol{x} + \boldsymbol{v}\cdot\hat{\boldsymbol{n}}\ \hat{\boldsymbol{n}}/\mathcal{H}$. Using continuity between real and redshift space

$$[1+\delta_{\mathrm{s}}(\boldsymbol{s})]\ \mathrm{d}^3s = [1+\delta(\boldsymbol{x})]\ \mathrm{d}^3x\,,$$

show that up to third order in the density and velocity fields one has for the density field in redshift space

$$\begin{aligned}\delta_s(\boldsymbol{k}) = \delta(\boldsymbol{k}) &- \mathrm{i}\frac{k_\parallel}{\mathcal{H}}v_\parallel(\boldsymbol{k}) + \frac{\mathrm{i}^2}{2}\left(\frac{k_\parallel}{\mathcal{H}}\right)^2[v_\parallel\star v_\parallel](\boldsymbol{k}) - \frac{\mathrm{i}^3}{3!}\left(\frac{k_\parallel}{\mathcal{H}}\right)^3[v_\parallel\star v_\parallel\star v_\parallel](\boldsymbol{k})\\ &- \mathrm{i}\frac{k_\parallel}{\mathcal{H}}[\delta\star v_\parallel](\boldsymbol{k}) + \frac{\mathrm{i}^2}{2!}\left(\frac{k_\parallel}{\mathcal{H}}\right)^2[\delta\star v_\parallel\star v_\parallel](\boldsymbol{k})\,.\end{aligned}$$

Here $k_\parallel = \boldsymbol{k}\cdot\hat{\boldsymbol{n}}$, $v_\parallel = \boldsymbol{v}\cdot\hat{\boldsymbol{n}}$ and $[a\star b](\boldsymbol{k})$ stands for a convolution in Fourier-space. *Hint:* Write $\delta_s(\boldsymbol{k}) = \int \mathrm{d}^3s e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{s}}\delta_{\mathrm{s}}(\boldsymbol{s})$.

6 Non-local Bias

Consider a local Lagrangian bias model up to second order

$$\delta_{\mathrm{g}}^{(\mathrm{L})}(\boldsymbol{q}) = b_1^{(\mathrm{L})}\delta(\boldsymbol{q}) + \frac{b_2^{(\mathrm{L})}}{2}\left(\delta^2(\boldsymbol{q}) - \langle\delta^2\rangle\right)\,,$$

where $\delta(\boldsymbol{q})$ is the Gaussian density field in Lagrangian space. The continuity equations for galaxies and dark matter are given by

$$[1+\delta(\boldsymbol{x})]\ \mathrm{d}^3x = \mathrm{d}^3q\,,\qquad \left[1+\delta_{\mathrm{g}}^{(\mathrm{E})}(\boldsymbol{x})\right]\ \mathrm{d}^3x = \left[1+\delta_{\mathrm{g}}^{(\mathrm{L})}(\boldsymbol{q})\right]\ \mathrm{d}^3q\,.$$

Show that the bias in Eulerian space can be written as

$$\delta_{\mathrm{g}}^{(\mathrm{E})}(\boldsymbol{x}) = \left(b_1^{(\mathrm{L})}+1\right)\left(\delta^{(1)}(\boldsymbol{x}) + \delta^{(2)}(\boldsymbol{x})\right) + \left(\frac{1}{2}b_2^{(\mathrm{L})} + \frac{4}{21}b_1^{(\mathrm{L})}\right)[\delta^{(1)}(\boldsymbol{x})]^2 - \frac{2}{7}b_1^{(\mathrm{L})}s^2(\boldsymbol{x})\,,$$

where s^2 is the square of the tidal tensor, which is the Fourier transform of

$$s^2(\boldsymbol{k}) = \int\frac{\mathrm{d}^3q}{(2\pi)}S_2(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q})\delta^{(1)}(\boldsymbol{q})\delta^{(1)}(\boldsymbol{k}-\boldsymbol{q})\,,$$

with

$$S_2(\boldsymbol{q}_1,\boldsymbol{q}_2) = \frac{(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)^2}{q_1^2q_2^2} - \frac{1}{3}\,.$$

Hint: Write down the second order matter field $\delta^{(2)}$ in terms of squared field δ^2, advection term $\boldsymbol{\Psi}\cdot\boldsymbol{\nabla}\delta$ and tidal tensor s^2.