1 Boltzmann Equation for Harmonic Oscillators

Consider a one-dimensional harmonic oscillator with energy

$$E = \frac{p^2}{2m} + \frac{1}{2}\kappa x^2 \,.$$

The distribution function of the harmonic oscillator depends on time t, position x, and momentum p. Show that the collisionless Boltzmann equation is given by

$$\frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} - \kappa x \frac{\partial f}{\partial p} \,,$$

and prove that the equilibrium distribution function is a function of the energy E only.
Using the moments of the distribution function

$$n = \int \mathrm{d}p \; f \,, \quad v = \frac{1}{n}\int \mathrm{d}p \; \frac{p}{m} f \,, \quad P = \int \mathrm{d}p \; \frac{p^2}{m^2} f - n v^2 \,,$$

derive the fluid equations from the Boltzmann equation.

2 Boltzmann Hierarchy

Using the multipole expansion of the temperature perturbations

$$\Theta(\eta, \boldsymbol{k}, \mu = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{n}}) = \sum_l (-\mathrm{i})^l (2l+1)\Theta_l(\eta, \boldsymbol{k})\mathcal{P}_l(\mu) \,,$$

derive the continuity ($l = 0$) and Euler ($l = 1$) equations from the Boltzmann equation. Derive the generic equation for higher order moments ($l > 2$)

$$\Theta_l' + k\left(\frac{l+1}{2l+1}\Theta_{l+1} - \frac{l}{2l+1}\Theta_{l-1}\right) = -\Gamma\Theta_l \,.$$

Remind yourself why this equation tells us that moments higher than $l = 2$ are suppressed.
Using $T^0_{\ 0} = \bar{\rho}(1+\delta)$ and $T^i_{\ 0} = (\bar{\rho} + \bar{P})v^i$, show that from

$$T^\mu_{\ \nu} = \int \mathrm{d}^3 p \; f P^\mu P_\nu$$

it follows that $\delta_\gamma = 4\Theta_0$, $v_\gamma = -3\Theta_1$.

3 Anisotropies from Tensors

Consider a spacetime with tensor perturbations described by the metric

$$\mathrm{d}s^2 = a^2\left[-\mathrm{d}\eta^2 + (\delta^{(\mathrm{K})}_{ij} + h_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j\right]$$

where h_{ij} is transverse and traceless.
a) Considering the geodesic equation for the above metric, show that the comoving energy ϵ of a photon evolves as

$$\frac{\mathrm{d}\ln\epsilon}{\mathrm{d}\eta} = -\frac{1}{2}h'_{ij}\hat{p}^i\hat{p}^j \,.$$

b) Assuming tight-coupling and instantaneous recombination, show that the line-of-sight solution for the photon temperature anisotropy is

$$\Theta(\hat{\boldsymbol{n}}) = -\frac{1}{2}\int \mathrm{d}\eta' \int_{\boldsymbol{k}} h'_{ij}(\eta', \boldsymbol{k})\hat{n}^i\hat{n}^j \exp\left[-\mathrm{i}k\chi'\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}\right] \qquad (\dagger)$$

where $\chi' = \eta_0 - \eta'$.
c) Consider a single gravitational wave with momentum $\boldsymbol{k}$ pointing the in $\hat{\boldsymbol{z}}$-direction and expand h_{ij} into its two polarization modes

$$h_{ij}(\eta, \boldsymbol{k}) = \sum_{\lambda=\pm 2}\frac{1}{\sqrt{2}}h_\lambda(\eta\,\boldsymbol{k})\epsilon^\lambda_{ij}(\hat{\boldsymbol{k}}), \quad \text{with} \quad \epsilon^\lambda_{ij}(\hat{\boldsymbol{z}}) = \frac{1}{2}\begin{pmatrix} 1 & \pm\mathrm{i} & 0 \\ \pm\mathrm{i} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,.$$

Show that

$$\epsilon^{\pm 2}_{ij}(\hat{\boldsymbol{z}})\hat{n}^i\hat{n}^j = \sqrt{\frac{8\pi}{15}}Y_{2\pm 2}(\hat{\boldsymbol{n}})\,.$$

With this is can be shown that the contribution of the Fourier mode $\boldsymbol{k} = k\hat{\boldsymbol{z}}$ to the integral in Eq. (†) is

$$\sqrt{\frac{4\pi}{15}}h'_{\pm 2}(\eta, k\hat{\boldsymbol{z}})Y_{2,\pm 2}(\hat{\boldsymbol{n}})\exp\left[-\mathrm{i}k\chi\cos\theta\right] = -\sqrt{\frac{\pi}{2}}h'_{\pm 2}(\eta, k\hat{\boldsymbol{z}})\sum_l \alpha_l \frac{j_l(k\chi)}{(k\chi)^2}Y_{l,\pm 2}(\hat{\boldsymbol{n}})\,,$$

where

$$\alpha_l = (-\mathrm{i})^l\sqrt{2l+1}\sqrt{\frac{(l+2)!}{(l-2)!}}\,.$$

To generalize to generic Fourier modes $\boldsymbol{k}$, we can rotate the above result, yielding

$$h^{\pm 2\prime}_{ij}(\eta, \boldsymbol{k})\hat{n}^i\hat{n}^j\exp\left[-\mathrm{i}k\chi\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}\right] = -\sqrt{\frac{\pi}{2}}h'_{\pm 2}(\eta, \boldsymbol{k})\sum_l \alpha_l\frac{j_l(k\chi)}{(k\chi)^2}\sum_m D^l_{m\pm 2}(\hat{\boldsymbol{k}})Y_{lm}(\hat{\boldsymbol{n}})\,.$$

d) Show that the angular power spectrum of the tensor-induced anisotropies

$$C_l = \frac{4\pi}{2l+1}\int \mathrm{d}\ln k\left|\Theta^\lambda_l(k)\right|^2\Delta^2_h(k)\,,$$

where with $\chi' = \eta_0 - \eta'$.

$$\Theta^\lambda_l(k) = -\frac{3}{4}(2l+1)\sqrt{\frac{(l+2)!}{(l-2)!}}\int_{\eta_*}^{\eta_0}\mathrm{d}\eta'\; k\frac{j_2(k\eta')}{k\eta'}\frac{j_l(k\chi')}{(k\chi')^2}\,.$$

e) Discuss the shape of the spectrum. Show that a scale-invariant tensor spectrum $\Delta^2_h(k) \sim$ const. leads to $l(l+1)C_l$ on large scales.
Useful results:

$$\Gamma^0_{00} = \mathcal{H} \qquad\qquad \Gamma^0_{ij} = \mathcal{H}\delta^{(\mathrm{K})}_{ij} + \mathcal{H}h_{ij} + \frac{1}{2}h'_{ij}$$

$$\int \mathrm{d}^2\hat{\boldsymbol{k}}\; D^l_{mn}(\hat{\boldsymbol{k}})D^{l'}_{mn'}(\hat{\boldsymbol{k}}) = \frac{4\pi}{2l+1}\delta^{(\mathrm{K})}_{ll'}\delta^{(\mathrm{K})}_{mm'}$$

$$\int_0^\infty \mathrm{d}x\;\frac{j_l^2(x)}{x^5} = \frac{4}{15}\frac{(l-2)!}{(l+2)!}\frac{1}{(l+3)(l-2)}$$

4 Recombination, last scattering, and the visibility function

Hydrogen recombination occurs out of equilibrium once the recombination time (which is proportional to $1/n_e$) exceeds the Hubble time. An approximate expression for the inverse of the (proper) mean-free time of Thomson scattering in the redshift range $800 < z < 1200$ is

$$cn_e\sigma_\mathrm{T} = 9.2\times 10^4\left(\Omega_m H_0^2\right)^{1/2}(z/1000)^{12.1}\,.$$

a) Calculate the optical depth $\tau(z)$ back to redshift z for z in this redshift range, assuming matter domination.
b) Calculate the redshift visibility function $g(z) \equiv (d\tau/dz)e^{-\tau}$.
c) Determine the redshift of photon decoupling z_*(where the optical depth is unity) and the (redshift) width of the visibility function. [*Hint: use the second derivative of* $-\ln g$ *as a proxy for the inverse squared width, i.e., fit a Gaussian.*]
d) Using your results in part (c), find the approximate duration of last scattering in terms of conformal time.
If (chemical) equilibrium held during hydrogen recombination, the fraction of free electrons x_e (i.e., the number of free electrons divided by the the number of protons including those in atomic hydrogen) would be given by the Saha formula (as you learnt in the *Cosmology* course):

$$\frac{1-X_e}{X_e^2} = 4\eta_p\zeta(3)\sqrt{\frac{2}{\pi}}\left(\frac{k_\mathrm{B}T}{m_ec^2}\right)^{3/2}\exp\left(\frac{B}{k_\mathrm{B}T}\right)\,,$$

where T is the temperature, $B \approx 13.6\,\mathrm{eV}$ is the binding energy of hydrogen, $\eta_p \approx 8.1 \times 10^{-10}$ is the proton-to-photon ratio, and $\zeta(3) \approx 1.202$. In practice, equilibrium cannot be maintained during recombination once the inverse recombination time $n_e\langle\sigma v\rangle$ (where $\langle\sigma v\rangle$ is the thermally-averaged cross section for recombination) falls below the expansion rate H. Given that the thermally-averaged cross section can be approximated by

$$\frac{\langle\sigma v\rangle}{c} = 3.06 \times 10^{-27}\left(\frac{B}{k_{\mathrm{B}}T}\right)^{1/2}\ln\left(\frac{B}{k_{\mathrm{B}}T}\right)\,\mathrm{m}^2\,,$$

estimate the temperature at which equilibrium fails and the ionization fraction at that time. (You should assume that $x_e \ll 1$ and note that the current number density of protons, excluding those in helium, is approximately $0.332\,\mathrm{m}^{-3}$.)

5 CMB Polarization – Correlation Functions

We have defined $E(\boldsymbol{l})$ and $B(\boldsymbol{l})$ as

$$\begin{aligned} E(\boldsymbol{l}) &= Q(\boldsymbol{l})\cos(2\phi_{\boldsymbol{l}}) + U(\boldsymbol{l})\sin(2\phi_{\boldsymbol{l}}), &(1)\\ B(\boldsymbol{l}) &= U(\boldsymbol{l})\cos(2\phi_{\boldsymbol{l}}) - Q(\boldsymbol{l})\sin(2\phi_{\boldsymbol{l}}), &(2)\end{aligned}$$

whose power spectra are

$$\begin{aligned} \langle E(\boldsymbol{l})E(\boldsymbol{l}')\rangle &= (2\pi)^2 C_{El}\delta^{(\mathrm{D})}(\boldsymbol{l}+\boldsymbol{l}'), &(3)\\ \langle B(\boldsymbol{l})B(\boldsymbol{l}')\rangle &= (2\pi)^2 C_{Bl}\delta^{(\mathrm{D})}(\boldsymbol{l}+\boldsymbol{l}'), &(4)\\ \frac{1}{2}\left(\langle E(\boldsymbol{l})T(\boldsymbol{l}')\rangle + \mathrm{c.c.}\right) &= (2\pi)^2 C_{Cl}\delta^{(\mathrm{D})}(\boldsymbol{l}+\boldsymbol{l}'). &(5)\end{aligned}$$

Show that the correlation functions of Q and U are given by

$$\begin{aligned} C_Q(\theta) + C_U(\theta) &= \int \frac{l\,\mathrm{d}l}{2\pi}(C_{El} + C_{Bl})J_0(l\theta), &(6)\\ C_Q(\theta) - C_U(\theta) &= \int \frac{l\,\mathrm{d}l}{2\pi}(C_{El} - C_{Bl})J_4(l\theta), &(7)\\ C_C(\theta) &= -\int \frac{l\,\mathrm{d}l}{2\pi}C_{Cl}J_2(l\theta), &(8)\end{aligned}$$

where J_ν are cylindrical Bessel functions.

6 Non-Gaussianity

Suppose that the field $F(\boldsymbol{x})$ is non-Gaussian, with a 3D bispectrum

$$\langle F(\boldsymbol{k}_1)F(\boldsymbol{k}_2)F(\boldsymbol{k}_3)\rangle = (2\pi)^3 B_F(k_1,k_2,k_3)\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,,$$

and consider the projection $g(\hat{\boldsymbol{n}}) = F(r\hat{\boldsymbol{n}})$, with spherical multipoles g_{lm}. Show that

$$\langle g_{l_1m_1}g_{l_2m_2}g_{l_3m_3}\rangle = b_{l_1l_2l_3}\mathcal{G}^{m_1m_2m_3}_{l_1l_2l_3}\,, \qquad (*)$$

where the *reduced bispectrum*

$$b_{l_1l_2l_3} = \left(\frac{2}{\pi}\right)^3\int \mathrm{d}x\,x^2\int\prod_{i=1}^{3}\left[\mathrm{d}k_i\,k_i^2 j_{l_i}(k_ir)j_{l_i}(k_ix)\right]B_F(k_1,k_2,k_3)\,,$$

and the *Gaunt integral*

$$\mathcal{G}^{m_1m_2m_3}_{l_1l_2l_3} \equiv \int \mathrm{d}\hat{\boldsymbol{n}}\,Y_{l_1m_1}Y_{l_2m_2}Y_{l_3m_3}\,.$$

[*Hint:* express the 3D delta function in integral form and expand the resulting product of three plane waves with the Rayleigh expansion.]
Show explicitly that the 3-point function in $(*)$ is invariant under rotation of the g_{lm}.

7 Stokes parameters (quick question)

Derive the Stokes parameters for radiation linearly-polarized at an angle θ to the x-axis, i.e., $(E_x, E_y) = \mathcal{E}(\cos\theta, \sin\theta)$. Do the same for right and left-circularly-polarized radiation with $(E_x, E_y) = \mathcal{E}(1, \mp i)$, respectively.

8 Lensing Deflection

Consider the first order perturbed metric in a Universe without anisotropic stress $\Phi = \Psi$

$$\mathrm{d}s^2 = a^2(\tau)\left\{-(1+2\Phi)\,\mathrm{d}\tau^2 + (1-2\Phi)\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\right\}.$$

While the spatial part of the above metric $\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \mathrm{d}\chi^2 + S_K^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$ can account for spatial curvature, the photon deflection can be calculated in a locally flat coordinate system. Using this simplification, derive the spatial geodesic equation for photons

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}l} = 2\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\nabla}\Phi) = -2\boldsymbol{\nabla}_\perp\Phi$$

Here $\mathrm{d}l \equiv \sqrt{\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j}$ and $n^i \equiv \mathrm{d}x^i/\,\mathrm{d}l$.

9 Lensing Bispectrum

The lensed CMB temperature perturbations are given in terms of the deflected primordial CMB perturbations as

$$\tilde{\Theta}(\hat{\boldsymbol{n}}) = \Theta(\hat{\boldsymbol{n}}) + \nabla_{\hat{\boldsymbol{n}}}\phi\cdot\nabla_{\hat{\boldsymbol{n}}}\Theta\,.$$

In the flat-sky approximation the power spectra are defined as

$$\begin{aligned}\langle\phi(\boldsymbol{l})\Theta(\boldsymbol{l}')\rangle &= 0\,,\\ \langle\phi(\boldsymbol{l})\phi(\boldsymbol{l}')\rangle &= (2\pi)^2\delta^{(\mathrm{D})}(\boldsymbol{l}+\boldsymbol{l}')C_l^{\phi\phi}\,,\\ \langle\Theta(\boldsymbol{l})\Theta(\boldsymbol{l}')\rangle &= (2\pi)^2\delta^{(\mathrm{D})}(\boldsymbol{l}+\boldsymbol{l}')C_l^{TT}\,.\end{aligned}$$

The cross-correlation between lensing potential and primordial CMB temperature fluctuations vanishes because the deflection potential is dominated by low-redshift fluctuations. Calculate the bispectrum induced by CMB lensing

$$\left\langle\tilde{\Theta}(\boldsymbol{l}_1)\tilde{\Theta}(\boldsymbol{l}_2)\tilde{\Theta}(\boldsymbol{l}_3)\right\rangle\,,$$

as well as the cross-bispectrum between the lensing potential and the lensed CMB temperature

$$\left\langle\phi(\boldsymbol{l}_1)\tilde{\Theta}(\boldsymbol{l}_2)\tilde{\Theta}(\boldsymbol{l}_3)\right\rangle\,.$$