1 Power spectrum of randomly-placed halos

(a) The 2-point correlation function is

$$\begin{aligned}\xi(\boldsymbol{x},\boldsymbol{x}') &= \langle \delta\rho(\boldsymbol{x})\delta\rho(\boldsymbol{x}')\rangle \\ &= \langle \rho(\boldsymbol{x})\rho(\boldsymbol{x}')\rangle - \langle \rho(\boldsymbol{x})\rangle^2\,,\end{aligned} \tag{1}$$

where we have used statistical homogeneity ($\langle\rho(\boldsymbol{x})\rangle = \langle\rho(\boldsymbol{x}')\rangle$). For the mean of the density, we have

$$\begin{aligned}\langle\rho(\boldsymbol{x})\rangle &= \sum_N \mathbb{P}(N)\left(\prod_{i=1}^N \int \frac{d^3\boldsymbol{x}_i}{V}\right)\sum_{i=1}^N \kappa(\boldsymbol{x}-\boldsymbol{x}_i) \\ &= \sum_N \mathbb{P}(N) N \int \frac{d^3\boldsymbol{y}}{V}\kappa(\boldsymbol{y}) \\ &= m\frac{\langle N\rangle}{V} = mn\,,\end{aligned} \tag{2}$$

where $\mathbb{P}(N)$ is the Poisson probability of having N halos in the volume V. For $\langle\rho(\boldsymbol{x})\rho(\boldsymbol{x}')\rangle$, we have

$$\begin{aligned}\langle\rho(\boldsymbol{x})\rho(\boldsymbol{x}')\rangle &= \sum_N \mathbb{P}(N)\left(\int\prod_{i=1}^N \frac{\mathrm{d}^3\boldsymbol{x}_i}{V}\right)\sum_{i=1}^N\sum_{j=1}^N \kappa(\boldsymbol{x}-\boldsymbol{x}_i)\kappa(\boldsymbol{x}'-\boldsymbol{x}_j) \\ &= \sum_N \mathbb{P}(N)\int\left(\prod_{i=1}^N \frac{\mathrm{d}^3\boldsymbol{x}_i}{V}\right)\left[\sum_{i=1}^N \kappa(\boldsymbol{x}-\boldsymbol{x}_i)\kappa(\boldsymbol{x}'-\boldsymbol{x}_i) + \sum_{i\neq j}\kappa(\boldsymbol{x}-\boldsymbol{x}_i)\kappa(\boldsymbol{x}'-\boldsymbol{x}_j)\right] \\ &= \sum_N \mathbb{P}(N)\left[N\int\frac{d^3\boldsymbol{y}}{V}\kappa(\boldsymbol{x}-\boldsymbol{y})\kappa(\boldsymbol{x}'-\boldsymbol{y})\right. \\ &\qquad\left. + N(N-1)\left(\int\frac{d^3\boldsymbol{y}}{V}\kappa(\boldsymbol{y})\right)\left(\int\frac{d^3\boldsymbol{y}'}{V}\kappa(\boldsymbol{y}')\right)\right] \\ &= n\int d^3\boldsymbol{y}\,\kappa(\boldsymbol{x}-\boldsymbol{y})\kappa(\boldsymbol{x}'-\boldsymbol{y}) + m^2\frac{\langle N(N-1)\rangle}{V^2}\,.\end{aligned} \tag{3}$$

Using

$$\begin{aligned}\langle N(N-1)\rangle &= \mathrm{var}(N) + \langle N\rangle^2 - \langle N\rangle \\ &= \langle N\rangle^2\,,\end{aligned} \tag{4}$$

since for a Poisson-distributed random variable $\mathrm{var}(N) = \langle N\rangle$, and combining with Eq. (2), we find

$$\xi(\boldsymbol{x},\boldsymbol{x}') = n\int \mathrm{d}^3\boldsymbol{y}\kappa(\boldsymbol{x}-\boldsymbol{y})\kappa(\boldsymbol{x}'-\boldsymbol{y})\,. \tag{5}$$

(b) Statistical homogeneity would imply that $\xi(\boldsymbol{x},\boldsymbol{x}')$ depends only on the relative separation $\boldsymbol{x}-\boldsymbol{x}'$; this is clearly true since changing integration variables gives

$$\xi(\boldsymbol{x},\boldsymbol{x}') = n\int \mathrm{d}^3\boldsymbol{y}\kappa(\boldsymbol{x}-\boldsymbol{x}'+\boldsymbol{y})\kappa(\boldsymbol{y})\,. \tag{6}$$

Statistical isotropy would imply that $\xi(\boldsymbol{x},\boldsymbol{x}')$ is invariant under rotation of $\boldsymbol{x}$ and $\boldsymbol{x}'$. We have

$$\begin{aligned}\xi(\mathsf{R}\boldsymbol{x},\mathsf{R}\boldsymbol{x}') &= n\int d^3\boldsymbol{y}\kappa(\mathsf{R}\boldsymbol{x}-\boldsymbol{y})\kappa(\mathsf{R}\boldsymbol{x}'-\boldsymbol{y}) \\ &= n\int d^3\boldsymbol{y}\kappa(\mathsf{R}\boldsymbol{x}-\mathsf{R}\boldsymbol{y})\kappa(\mathsf{R}\boldsymbol{x}'-\mathsf{R}\boldsymbol{y})\,,\end{aligned} \tag{7}$$

where we have rotated the integration variable with R. Provided that the profile $\kappa(\boldsymbol{x})$ is spherically symmetric, so that $\kappa(\mathsf{R}\boldsymbol{x}) = \kappa(\boldsymbol{x})$, we see that $\xi(\mathsf{R}\boldsymbol{x},\mathsf{R}\boldsymbol{x}') = \xi(\boldsymbol{x},\boldsymbol{x}')$ and the correlation function respects statistical isotropy. (Statistical isotropy would be recovered for asymmetric halo profiles if we further assumed their orientations were random.)

(c) Taking the Fourier transform of $\delta\rho(\boldsymbol{x})$ and forming its 2-point correlator, we have

$$\begin{aligned}\langle \delta\rho(\boldsymbol{k})\delta\rho^*(\boldsymbol{k}')\rangle &= \int \frac{\mathrm{d}^3\boldsymbol{x}}{(2\pi)^3}\int \frac{\mathrm{d}^3\boldsymbol{x}'}{(2\pi)^3}\xi(\boldsymbol{x},\boldsymbol{x}')e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}e^{i\boldsymbol{k}'\cdot\boldsymbol{x}'}\\ &= n\int \frac{\mathrm{d}^3\boldsymbol{x}}{(2\pi)^3}\int \frac{\mathrm{d}^3\boldsymbol{x}'}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{y}\kappa(\boldsymbol{x}-\boldsymbol{y})\kappa(\boldsymbol{x}'-\boldsymbol{y})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}e^{i\boldsymbol{k}'\cdot\boldsymbol{x}'}\\ &= n\kappa(\boldsymbol{k})\kappa^*(\boldsymbol{k}')\int \mathrm{d}^3\boldsymbol{y}\, e^{i\boldsymbol{y}\cdot(\boldsymbol{k}'-\boldsymbol{k})}\\ &= (2\pi)^3 n|\kappa(k)|^2\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\,. \end{aligned} \tag{8}$$

Comparing with the definition of the power spectrum,

$$\langle \delta\rho(\boldsymbol{k})\delta\rho^*(\boldsymbol{k}')\rangle = P_{\delta\rho}(k)(2\pi)^3\delta^{(\mathrm{D})}(\boldsymbol{k}-\boldsymbol{k}')\,, \tag{9}$$

we find

$$P_{\delta\rho}(k) = n|\kappa(k)|^2\,. \tag{10}$$

(d) The Fourier transform of the halo profile is

$$\kappa(\boldsymbol{k}) = \int \frac{\mathrm{d}^3\boldsymbol{x}}{(2\pi)^3}\kappa(\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{11}$$

If the profile is compact with support of extent R, $\kappa(k)$ tends to a constant for $kR \ll 1$ and the power spectrum goes like k^0 on such scales.

2 Continuity and Euler Equation

(a) Let us integrate the Vlaslov equations over momentum

$$\int \mathrm{d}^3p\,\frac{\partial f}{\partial\eta} + \int \mathrm{d}^3p\,\frac{p_i}{am}\frac{\partial f}{\partial x^i} - am\frac{\partial\phi}{\partial x^i}\int \mathrm{d}^3p\,\frac{\partial f}{\partial p^i} = 0 \tag{12}$$

The last term is a total derivative integrated over the full space, yielding an integral over the function at infity. As well defined f need to vanish at infinity, this term is zero.

$$\frac{\partial}{\partial\eta}\int \mathrm{d}^3p\, f + \frac{\partial}{\partial x^i}\int \mathrm{d}^3p\,\frac{p_i}{am}f = 0 \tag{13}$$

With the definitions of the moments:

$$\rho(\boldsymbol{x},\eta) = \frac{m}{a^3}\int \mathrm{d}^3p f(\boldsymbol{x},\boldsymbol{p},\eta), \tag{14}$$

$$v_i(\boldsymbol{x},\eta) = \int \mathrm{d}^3p\frac{p_i}{am}f(\boldsymbol{x},\boldsymbol{p},\eta)\left(\int \mathrm{d}^3p f(\boldsymbol{x},\boldsymbol{p},\eta)\right)^{-1} \tag{15}$$

$$\sigma_{ij}(\boldsymbol{x},\eta) = \int \mathrm{d}^3p\frac{p_i}{am}\frac{p_j}{am}f(\boldsymbol{x},\boldsymbol{p},\eta)\left(\int \mathrm{d}^3p f(\boldsymbol{x},\boldsymbol{p},\eta)\right)^{-1} - v_i(\boldsymbol{x})v_j(\boldsymbol{x}) \tag{16}$$

we have

$$\frac{\partial}{\partial\eta}\left(\frac{a^3}{m}\rho(\boldsymbol{x},\eta)\right) + \frac{\partial}{\partial x_i}\left(\frac{a^3}{m}\rho v_i\right) \tag{17}$$

with $\rho = \bar{\rho}(1+\delta) = \bar{\rho}_0 a^{-3}(1+\delta)$ we thus have

$$\frac{a^3}{m}\bar{\rho}\frac{\partial}{\partial\eta}\delta + \frac{a^3}{m}\bar{\rho}\frac{\partial}{\partial x_i}\left[v_i(1+\delta)\right] \tag{18}$$

which is the desired continuity equation

$$\boxed{\delta' + \nabla\cdot[\boldsymbol{v}(1+\delta)] = 0} \tag{19}$$

Let us integrate the weighted Vlaslov equations over momentum with weight one:

$$\begin{aligned}\int \mathrm{d}^3p\ \frac{p_i}{ma}\frac{\partial f}{\partial \eta} + \int \mathrm{d}^3p\ \frac{p_i}{am}\frac{p_j}{am}\frac{\partial f}{\partial x^j} - \frac{\partial \phi}{\partial x^j}\int \mathrm{d}^3p\ p_i\frac{\partial f}{\partial p^j} &=0\\ \frac{\partial}{\partial \eta}\int \mathrm{d}^3p\ \frac{p_i}{ma}f + \mathcal{H}\int \mathrm{d}^3p\ \frac{p_i}{ma}f + \int \mathrm{d}^3p\ \frac{p_i}{am}\frac{p_j}{am}\frac{\partial f}{\partial x^j} + \frac{\partial \phi}{\partial x^j}\int \mathrm{d}^3p\ f\frac{\partial p_i}{\partial p^j} &=0\\ \frac{\partial}{\partial \eta}\left(\frac{a^3}{m}\rho v_i\right) + \mathcal{H}\frac{a^3}{m}\rho v_i + \frac{\partial}{\partial x^j}\left(\frac{a^3}{m}\rho\sigma_{ij} + \frac{a^3}{m}\rho v_i v_j\right) + \delta_{ij}\frac{\partial \phi}{\partial x^j}\int \mathrm{d}^3p f &=0\end{aligned} \tag{20}$$

devoloping the first term above and using the previously derived continuity equation:

$$\begin{aligned}\frac{\partial}{\partial \eta}\left(\frac{a^3}{m}\rho v_i\right) &= v_i\frac{a^3\bar{\rho}}{m}\delta' - \frac{a^3\rho}{m}v_i'\\ &= -v_i\frac{a^3}{m}\bar{\rho}\frac{\partial}{\partial x^j}\left(v_j(1+\delta)\right) + \frac{a^3}{m}\rho v_i'\\ &= -v_i v_j\frac{\partial}{\partial x^j}\left(\frac{a^3}{m}\rho\right) - \frac{a^3}{m}\rho v_i\frac{\partial}{\partial x^j}v_j + \frac{a^3}{m}\rho v_i'\end{aligned} \tag{21}$$

such that we obtain

$$\frac{a^3}{m}\rho\frac{\partial}{\partial \eta}v_i + \mathcal{H}\frac{a^3}{m}\rho v_i + \frac{a^3}{m}\frac{\partial}{\partial x^i}\left(\rho\sigma_{ij}\right) + \frac{a^3}{m}v_i\frac{\partial}{\partial x^i}v_j + \frac{a^3}{m}\rho\frac{\partial \phi}{\partial x^i} = 0 \tag{22}$$

Dividing by $a^3\rho/m$ we obtain the Euler equation:

$$\boxed{v_i' + \mathcal{H}v_i + \boldsymbol{v}\cdot\nabla v_i + \nabla_i\phi = -\frac{1}{\rho}\nabla_i(\rho\sigma_{ij})} \tag{23}$$

(b) Let us take the Fourier tranform of the continuity equation (19)

$$\int \mathrm{d}^3x\left(\delta'(\boldsymbol{x}) + \nabla\cdot\boldsymbol{v}(\boldsymbol{x})\right)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = -\int \mathrm{d}^3x\nabla\cdot(\boldsymbol{v}(\boldsymbol{x})\delta(\boldsymbol{x}))e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{24}$$

using the definition the velocity divergence θ (assuming vanishing curl):

$$v(\boldsymbol{k}) = -i\frac{\boldsymbol{k}}{k^2}\theta(\boldsymbol{k}) \tag{25}$$

the left-hand side of the Fourier transformed continuity equation is simply

$$\delta'(\boldsymbol{k}) + \theta(\boldsymbol{k}) = -\int \mathrm{d}^3x\nabla\cdot(\boldsymbol{v}(\boldsymbol{x})\delta(\boldsymbol{x}))e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{26}$$

Let's focus on the Fourier trasnform of the RHS:

$$\begin{aligned}\int \mathrm{d}^3x\nabla\cdot(\boldsymbol{v}(\boldsymbol{x})\delta(\boldsymbol{x}))e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} &= \int \mathrm{d}^3x\,\mathrm{d}^3y\left[\delta(\boldsymbol{y})\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x})\nabla_{\boldsymbol{y}}\delta(\boldsymbol{y})\right]\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\\ &= \int \mathrm{d}^3x\,\mathrm{d}^3y\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\int\frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3}e^{i\boldsymbol{q}_1\cdot\boldsymbol{x}}e^{i\boldsymbol{q}_2\cdot\boldsymbol{y}}i\left(\boldsymbol{q}_1+\boldsymbol{q}_2\right)\delta(\boldsymbol{q}_1)\boldsymbol{v}(\boldsymbol{q}_2)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\\ &= \int\frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3}\delta(\boldsymbol{q}_1)\boldsymbol{v}(\boldsymbol{q}_2)i(\boldsymbol{q}_1+\boldsymbol{q}_2)(2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)\\ &= \int\frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)\alpha(\boldsymbol{q}_2,\boldsymbol{q}_1)\delta(\boldsymbol{q}_1)\theta(\boldsymbol{q}_2)\,,\end{aligned} \tag{27}$$

where we found the coupling kernel:

$$\alpha(\boldsymbol{k},\boldsymbol{k}') = \frac{\boldsymbol{k}\cdot(\boldsymbol{k}+\boldsymbol{k}')}{k^2}\,, \tag{28}$$

as expected.
Let's now look at the Euler equation (23). After taking the divergence of the equation and Fourier transforming we easily obtain:

$$\theta'(\boldsymbol{k}) + \mathcal{H}\theta(\boldsymbol{k}) + \frac{3}{2}\Omega_m(a)\mathcal{H}^2\delta(\boldsymbol{k}) = -\int \mathrm{d}^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\nabla_j\left(v_i\cdot\nabla^i v^j\right), \tag{29}$$

where we used Poisson equation to relate the divergence of the gravitational potential to the density perturbation:

$$\nabla^2\phi = \frac{3}{2}\Omega_m(a)\delta\,. \tag{30}$$

We focus again on the RHS of the Euler equation:

$$\begin{aligned}\int \mathrm{d}^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\nabla_j\left(v_i\cdot\nabla^i v^j\right) &= \int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{y}\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\left[\nabla_j v_i(\boldsymbol{x})\nabla^i v^j(\boldsymbol{y}) + \frac{1}{2}v_i(\boldsymbol{x})\nabla^i\nabla_j v^j(\boldsymbol{y}) + \frac{1}{2}v_i(\boldsymbol{y})\nabla^i\nabla_j v^j(\boldsymbol{x})\right]\\
&= \int \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{y}\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\int\frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3}e^{i\boldsymbol{q}_1\cdot\boldsymbol{x}}e^{i\boldsymbol{q}_2\cdot\boldsymbol{y}}\Big[q_1^j v_i(\boldsymbol{q}_1) + \frac{1}{2}v_i(\boldsymbol{q}_1)q_{2,j}q_2^i v^j(\boldsymbol{q}_2)\\
&\quad + \frac{1}{2}v_i(\boldsymbol{q}_2)q_{1,j}q_1^i v^j(\boldsymbol{q}_1)\Big]\\
&= \int \frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)\theta(\boldsymbol{q}_1)\theta(\boldsymbol{q}_2)\frac{1}{2q_1^2q_2^2}\left(2(\boldsymbol{q}_1\cdot\boldsymbol{q}_2)^2 + q_2^2\boldsymbol{q}_1\cdot\boldsymbol{q}_2 + q_1^2\boldsymbol{q}_1\cdot\boldsymbol{q}_2\right)\\
&= \int \frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{q}_2)\beta(\boldsymbol{q}_1,\boldsymbol{q}_2)\theta(\boldsymbol{q}_1)\theta(\boldsymbol{q}_2)\,.\end{aligned} \tag{31}$$

Where we denoted the second coupling kernel:

$$\beta(\boldsymbol{k},\boldsymbol{k}') = \frac{(\boldsymbol{k}\cdot\boldsymbol{k}')^2}{k^2k'^2} + \frac{1}{2}\frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{kk'}\left(\frac{k}{k'}+\frac{k'}{k}\right) = \frac{1}{2}(\boldsymbol{k}+\boldsymbol{k}')^2\frac{\boldsymbol{k}\cdot\boldsymbol{k}'}{k^2k'^2}\,. \tag{32}$$

3 Loop Diagrams

(a) As we have derived in class, the continuity and Euler equation in Fourier space can be written as:

$$\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}\ln a}\delta(\boldsymbol{k}) + \theta(\boldsymbol{k}) = -\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3q'}{(2\pi)^3}\delta^{(\mathrm{D})}(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{q}')\alpha(\boldsymbol{q},\boldsymbol{q}')\theta(\boldsymbol{q})\delta(\boldsymbol{q}')\,,$$

$$\mathcal{H}\frac{\mathrm{d}}{\mathrm{d}\ln a}\theta(\boldsymbol{k}) + \mathcal{H}\theta(\boldsymbol{k}) + \frac{3}{2}\Omega_\mathrm{m}(a)\mathcal{H}^2\delta(\boldsymbol{k}) = -\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3q'}{(2\pi)^3}\delta^{(\mathrm{D})}(\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{q}')\beta(\boldsymbol{q},\boldsymbol{q}')\theta(\boldsymbol{q})\theta(\boldsymbol{q}')\,.$$

In a matter-only Einstein de Sitter Universe, we can solve the equations with the perturbative ansatz

$$\delta(\boldsymbol{k},\eta) = \sum_{i=1}^{\infty}a^i(\eta)\delta^{(i)}(\boldsymbol{k}) \qquad\qquad \theta(\boldsymbol{k},\eta) = -\mathcal{H}(\eta)\sum_{i=1}^{\infty}a^i(\eta)\,\tilde{\theta}^{(i)}(\boldsymbol{k})$$

Using the ansatz in the Fourier version of the Euler and continuity equation we obtain

$$\mathcal{H}\sum_j ja^j\delta^{(j)} - \mathcal{H}\sum_j a^j\tilde{\theta}^{(j)} = \mathcal{H}\sum_{l,m}a^la^m\alpha\tilde{\theta}^{(l)}\delta^{(m)}$$

$$-\mathcal{H}^2\sum_j ja^j\tilde{\theta}^{(j)} + \frac{1}{2}\mathcal{H}^2\sum_j a^j\tilde{\theta}^{(j)} - \mathcal{H}^2\sum_j a^j\tilde{\theta}^{(j)} + \frac{3}{2}\mathcal{H}^2\sum a^j\delta^{(j)} = -\mathcal{H}^2\sum_{l,m}a^la^m\beta\tilde{\theta}^{(l)}\tilde{\theta}^{(m)}$$

We can now sort this equation by orders, factoring out the sum over j and require that the equation is satisfied order by order

$$ja^j\delta^{(j)} - a^j\tilde{\theta}^{(j)} = \sum_{l=1}^{j-1}a^la^{j-l}\alpha\tilde{\theta}^{(l)}\delta^{(j-l)}$$

$$-ja^j\tilde{\theta}^{(j)} + \frac{1}{2}a^j\tilde{\theta}^{(j)} - a^j\tilde{\theta}^{(j)} + \frac{3}{2}\sum a^j\delta^{(j)} = -\sum_{l=1}^{j-1}a^la^{j-l}\beta\tilde{\theta}^{(l)}\tilde{\theta}^{(j-l)}$$

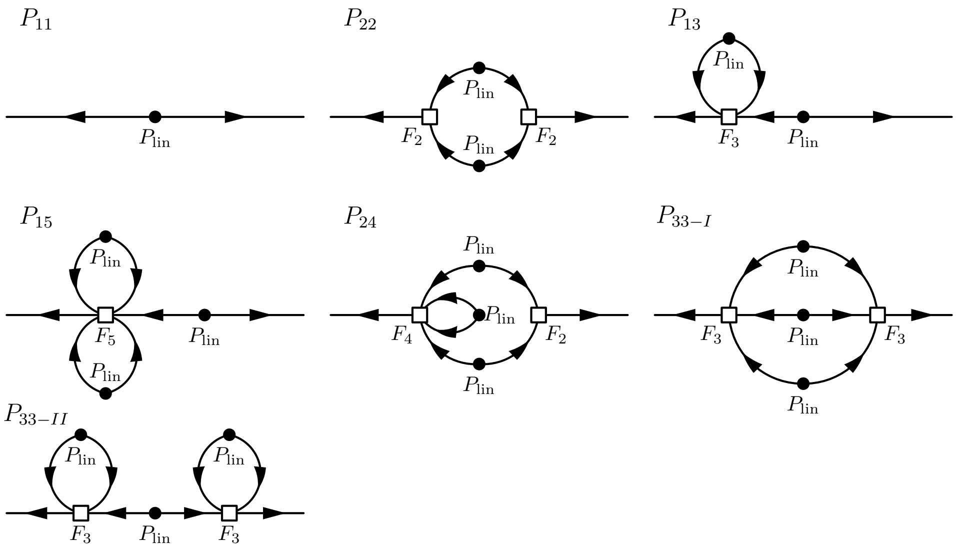

Figure 1: Diagrams for the tree-level, one-loop and two-loop power spectrum.

Let us now express the n-th order solution as a convolution over the Gaussian initial field and the coupling kernels F_n and G_n

$$\delta^{(n)}(\boldsymbol{k}) = \prod_{m=1}^{n} \left\{ \int \frac{\mathrm{d}^3 q_m}{(2\pi)^3} \delta^{(1)}(\boldsymbol{q}_m) \right\} F_n(\boldsymbol{q}_1, \dots, \boldsymbol{q}_n)(2\pi)^3 \delta^{(\mathrm{D})}(\boldsymbol{k} - \boldsymbol{q}|_1^n)$$
$$\tilde{\theta}^{(n)}(\boldsymbol{k}) = \prod_{m=1}^{n} \left\{ \int \frac{\mathrm{d}^3 q_m}{(2\pi)^3} \delta^{(1)}(\boldsymbol{q}_m) \right\} G_n(\boldsymbol{q}_1, \dots, \boldsymbol{q}_n)(2\pi)^3 \delta^{(\mathrm{D})}(\boldsymbol{k} - \boldsymbol{q}|_1^n)$$

The resulting integral equation has to be satisfied at every point in k-space such that we can write down the following system for the kernels

$$jF_j - G_j = \sum_{l=1}^{j-1} \alpha G_l F_{j-l}$$
$$\frac{3}{2} F_j - \left(\frac{1}{2} + j \right) G_j = - \sum_{l=1}^{j-1} \beta G_l G_{j-l}$$

Solving for F_n and G_n we can finally derive the recursion relations

$$F_n(\boldsymbol{q}_1, \dots, \boldsymbol{q}_n) = \sum_{m=1}^{n-1} \frac{G_m(\boldsymbol{q}_1, \dots, \boldsymbol{q}_m)}{(2n+3)(n-1)} \Big[(2n+1)\alpha \left(\boldsymbol{q}|_1^m, \boldsymbol{q}|_{m+1}^n \right) F_{n-m}\left(\boldsymbol{q}_{m+1}, \dots, \boldsymbol{q}_n \right)$$
$$+ 2\beta \left(\boldsymbol{q}|_1^m, \boldsymbol{q}|_{m+1}^n \right) G_{n-m}\left(\boldsymbol{q}_{m+1}, \dots, \boldsymbol{q}_n \right) \Big]$$
$$G_n(\boldsymbol{q}_1, \dots, \boldsymbol{q}_n) = \sum_{m=1}^{n-1} \frac{G_m(\boldsymbol{q}_1, \dots, \boldsymbol{q}_m)}{(2n+3)(n-1)} \Big[3\alpha \left(\boldsymbol{q}|_1^m, \boldsymbol{q}|_{m+1}^n \right) F_{n-m}\left(\boldsymbol{q}_{m+1}, \dots, \boldsymbol{q}_n \right)$$
$$+ 2n\beta \left(\boldsymbol{q}|_1^m, \boldsymbol{q}|_{m+1}^n \right) G_{n-m}\left(\boldsymbol{q}_{m+1}, \dots, \boldsymbol{q}_n \right) \Big]$$

where $\boldsymbol{q}|_1^m = \boldsymbol{q}_1 + \ldots + \boldsymbol{q}_m$.

(b) As we discussed in the lectures at one-loop or next-to-leading (NLO) order in the power spectrum has four density fields leading to a a 22 and a 13 contribution. The third order 12 correlator is odd and thus vanishes due to Wick theorem. At two-loop level, i.e., NNLO we have contributions from sixth order, leading to the combinations 15, 24 and 33. The diagrams for the tree-level, one-loop and two-loop

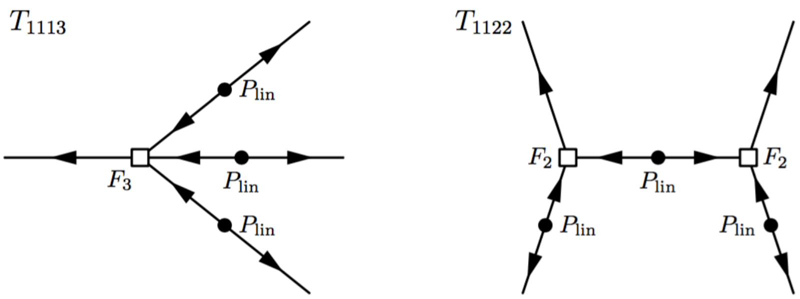

Figure 2: Diagrams for the tree-level trispectrum.

power spectrum are given in Fig. 1 and the corresponding Fourier space expressions are given by

$$P_{15}(k) = 5\times 3\times 2P(k)\int\frac{\mathrm{d}^3q}{(2\pi)^3}\int\frac{\mathrm{d}^3q'}{(2\pi)^3}F_5(\boldsymbol{q},-\boldsymbol{q},\boldsymbol{q}',-\boldsymbol{q}',-\boldsymbol{k})P(q)P(q')$$
$$P_{33-I}(k) = 6\int\frac{\mathrm{d}^3q}{(2\pi)^3}\int\frac{\mathrm{d}^3q'}{(2\pi)^3}|F_3(\boldsymbol{q},\boldsymbol{q}',\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{q}')|^2P(q)P(q')P(|\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{q}'|)$$
$$P_{33-II}(k) = P(k)\left[3\int\frac{\mathrm{d}^3q}{(2\pi)^3}P(q)F_3(\boldsymbol{k},\boldsymbol{q},-\boldsymbol{q})\right]^2$$
$$P_{24}(k) = 4\times 3\times 2\int\frac{\mathrm{d}^3q}{(2\pi)^3}\int\frac{\mathrm{d}^3q'}{(2\pi)^3}F_2(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q})F_4(\boldsymbol{q}',-\boldsymbol{q}',-\boldsymbol{q},-\boldsymbol{k}+\boldsymbol{q})P(q')P(q)P(|\boldsymbol{k}-\boldsymbol{q}|)$$

Let us briefly discuss the behaviour of these diagrams when the loop momentum becomes large. The UV-sensitivity can be troublesome and lead to UV-divergencies for particular classes of initial power spectra and thus needs to be regularized with appropriate counterterms. To quickly assess the UV-behaviour of the power spectra it is useful to consider the general behaviour of the kernels given in the lectures.

- P_{15} - single kernel scales as k^2 - speed of sound renormalization
- P_{24} - if the connecting loop is large k^4, if the daisy diagram is large k^2
- P_{33-II} - this is just P_{13}^2/P_{lin}, which means the speed of sound counterterm is known $k^4c_s^4P(k)$
- P_{33-I} - we have two kernels in the loop and it thus scales as k^4 - stochastic term

The *connected* trispectrum has no contribution just from linear fields, i.e. at order $\mathcal{O}(\delta^4)$. The leading contributions thus arise at order $\mathcal{O}(\delta^6)$, i.e., from 1113 and 1122. The diagrams for the tree-level trispectrum are given in Fig. 2.

$$T_{1113}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = 3\times 2F_{3,s}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)P(k_1)P(k_2)P(k_3)+\text{perm.}$$
$$T_{1122}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = 4F_{2,s}(-\boldsymbol{k}_3,\boldsymbol{k}_1+\boldsymbol{k}_3)F_{2,s}(-\boldsymbol{k}_4,\boldsymbol{k}_2+\boldsymbol{k}_4)P(\boldsymbol{k}_1+\boldsymbol{k}_3)P(k_3)P(k_4)+\text{perm.}$$

4 Equivalence of Lagrangian and Eulerian PT

Let us use the continuity equation between Eulerian and Lagrangian coordinates $\boldsymbol{x}=\boldsymbol{q}+\boldsymbol{\Psi}(\boldsymbol{q})$

$$[1+\delta(\boldsymbol{x})]\,\mathrm{d}^3x = \mathrm{d}^3q$$

The transform to Fourier space can be reexpressed in terms of Lagrangian quantities

$$\begin{aligned}
\delta(\boldsymbol{k}) &= \int \mathrm{d}^3x \ \exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}]\}\delta(\boldsymbol{x}) \\
&= \int \mathrm{d}^3q \exp\{[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}+\boldsymbol{\Psi}(\boldsymbol{q}))]\} - \int \mathrm{d}^3q \exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}]\} \\
&= \int \mathrm{d}^3q \exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}]\} \left[\exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Psi}(\boldsymbol{q})]\} - 1\right] \\
&\approx \int \mathrm{d}^3q \exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}]\} \left[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Psi}(\boldsymbol{q}) - \frac{1}{2}\left(\boldsymbol{k}\cdot\boldsymbol{\Psi}(\boldsymbol{q})\right)^2\right] \\
&= \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Psi}^{(1)}(\boldsymbol{k}) + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Psi}^{(2)}(\boldsymbol{k}) - \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\boldsymbol{k}\cdot\boldsymbol{\Psi}^{(1)}(\boldsymbol{p})\boldsymbol{k}\cdot\boldsymbol{\Psi}^{(1)}(\boldsymbol{k}-\boldsymbol{p}) \\
&= \delta^{(1)}(\boldsymbol{k}) + \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3} l_2(\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p})\delta^{(1)}(\boldsymbol{q})\delta^{(1)}(\boldsymbol{k}-\boldsymbol{p}) + \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{p^2}\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{p})}{(\boldsymbol{k}-\boldsymbol{p})^2}\delta^{(1)}(\boldsymbol{q})\delta^{(1)}(\boldsymbol{k}-\boldsymbol{p}) \\
&= \delta^{(1)}(\boldsymbol{k}) + \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3} K_2(\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p})\delta^{(1)}(\boldsymbol{q})\delta^{(1)}(\boldsymbol{k}-\boldsymbol{p})
\end{aligned}$$

In the third to last line we have used that the displacement field can be expanded as $\boldsymbol{\Psi} = \boldsymbol{\Psi}^{(1)} + \boldsymbol{\Psi}^{(2)}$ and that at first order $\boldsymbol{\Psi} = -\mathrm{i}\boldsymbol{k}/k^2\delta(\boldsymbol{k})$ and $\boldsymbol{L}_2 = \boldsymbol{k}/k^2 l_2$. Adding some zeros, i.e., adding and subtracting $\boldsymbol{p}$'s and $\boldsymbol{k}$'s, the second order kernel can be rewritten as

$$\frac{\boldsymbol{k}\cdot\boldsymbol{p}}{p^2}\frac{\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{p})}{(\boldsymbol{k}-\boldsymbol{p})^2} = 1 + (\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{p}\left[\frac{1}{p^2} + \frac{1}{(\boldsymbol{k}-\boldsymbol{p})^2}\right] + \frac{[(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{p}]^2}{p^2(\boldsymbol{k}-\boldsymbol{p})^2}$$

The total quadratic kernel is thus given by

$$\begin{aligned}
K_2(\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p}) &= \frac{3}{7} - \frac{3}{7}\frac{[(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{p}]^2}{p^2(\boldsymbol{k}-\boldsymbol{p})^2} + \frac{7}{7} + (\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{p}\left[\frac{1}{p^2} + \frac{1}{(\boldsymbol{k}-\boldsymbol{p})^2}\right] + \frac{7}{7}\frac{[(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{p}]^2}{p^2(\boldsymbol{k}-\boldsymbol{p})^2} \\
&= \frac{10}{7} + (\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{p}\left[\frac{1}{p^2} + \frac{1}{(\boldsymbol{k}-\boldsymbol{p})^2}\right] + \frac{4}{7}\frac{[(\boldsymbol{k}-\boldsymbol{p})\cdot\boldsymbol{p}]^2}{p^2(\boldsymbol{k}-\boldsymbol{p})^2} = 2F_{2,s}(\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p})
\end{aligned}$$

Thus we have

$$\delta(\boldsymbol{k}) = \delta^{(1)}(\boldsymbol{k}) + \int \frac{\mathrm{d}^3p}{(2\pi)^3} F_{2,s}(\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p})\delta^{(1)}(\boldsymbol{q})\delta^{(1)}(\boldsymbol{k}-\boldsymbol{p}) = \delta^{(1)}(\boldsymbol{k}) + \delta^{(2)}(\boldsymbol{k})$$

Which proves that perturbatively to second order the Lagrangian and Eulerian treatments agree. There is however a slight difference if Lagrangian calculations are left unexpanded, since the advedction effect of long wavelength perturbative motions on small scale fluctuations is resummed and thus provides a better description of the evolved BAO feature.

5 Redshift Space Distortions

$$\boldsymbol{s} = \boldsymbol{x} + \frac{\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\ \hat{\boldsymbol{n}}}{\mathcal{H}}$$

Obviously the displacement along the line of sight does not change the total number of objects between the physical configuration space and the observed redshift space. The true overdensity $\delta(\boldsymbol{x})$ and the inferred overdensity $\delta_\mathrm{s}(\boldsymbol{s})$ are thus related by a continuity equation

$$[1+\delta(\boldsymbol{x})]\ \mathrm{d}^3x = [1+\delta_\mathrm{s}(\boldsymbol{s})]\ \mathrm{d}^3s$$

In Fourier space we have

$$
\begin{aligned}
\delta_{\rm s}(\boldsymbol{k}) =& \int \mathrm{d}^3 s \;\exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{s}]\}\delta_{\rm s}(\boldsymbol{s}) \\
=& \int \mathrm{d}^3 s \;\exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{s}]\}\left[1+\delta_{\rm s}(\boldsymbol{s})\right] - \int \mathrm{d}^3 x \exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}]\} \\
=& \int \mathrm{d}^3 x \;\exp\big\{\big[\mathrm{i}\boldsymbol{k}\cdot\big(\boldsymbol{x}+\boldsymbol{v}_\parallel/\mathcal{H}\big)\big]\big\}\left[1+\delta(\boldsymbol{x})\right] - \int \mathrm{d}^3 x \exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}]\}\left[1+\delta(\boldsymbol{x})-\delta(\boldsymbol{x})\right] \\
=& \delta(\boldsymbol{k}) + \int \mathrm{d}^3 x \;\exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}]\}\left[\exp\big\{\big[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}_\parallel/\mathcal{H}\big]\big\}-1\right]\left[1+\delta(\boldsymbol{x})\right] \\
\approx& \delta(\boldsymbol{k}) + \int \mathrm{d}^3 x \;\exp\{[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}]\}\left\{\mathrm{i}\left(\frac{k_\parallel}{\mathcal{H}}\right)v_\parallel(\boldsymbol{x}) + \frac{\mathrm{i}^2}{2}\left(\frac{k_\parallel}{\mathcal{H}}\right)^2 v_\parallel^2(\boldsymbol{x}) + \frac{\mathrm{i}^3}{3!}\left(\frac{k_\parallel}{\mathcal{H}}\right)^3 v_\parallel^3(\boldsymbol{x})\right\}\left[1+\delta(\boldsymbol{x})\right] \\
=& \delta(\boldsymbol{k}) + \mathrm{i}\frac{k_\parallel}{\mathcal{H}}v_\parallel(\boldsymbol{k}) + \frac{\mathrm{i}^2}{2}\left(\frac{k_\parallel}{\mathcal{H}}\right)^2\left[v_\parallel\star v_\parallel\right](\boldsymbol{k}) + \frac{\mathrm{i}^3}{3!}\left(\frac{k_\parallel}{\mathcal{H}}\right)^3\left[v_\parallel\star v_\parallel\star v_\parallel\right](\boldsymbol{k}) \\
&+\mathrm{i}\frac{k_\parallel}{\mathcal{H}}\left[\delta\star v_\parallel\right](\boldsymbol{k}) + \frac{\mathrm{i}^2}{2!}\left(\frac{k_\parallel}{\mathcal{H}}\right)^2\left[\delta\star v_\parallel\star v_\parallel\right](\boldsymbol{k})
\end{aligned}
$$

In the last line we have expanded to third order (where velocities are counted as first order quantities) and used that the powers in configuration space translate to convolutions in Fourier space. This expansion up to third order in conjunction with perturbation theory for the velocity field $v^{(n)} \propto \theta^{(n)}$ and density field can be used to calculate the one-loop power spectrum in redshift space or if restricted to second order the tree-level bispectrum.

At linear order we have with $\boldsymbol{v} = -\mathrm{i}\mathcal{H}f\boldsymbol{k}\delta(\boldsymbol{k})/k^2$ and $k_\parallel = \mu k$ where $\mu = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{z}}$

$$\delta_{\rm s}(\boldsymbol{k}) = \delta(\boldsymbol{k}) + \mathrm{i}\frac{k_\parallel}{\mathcal{H}}v_\parallel(\boldsymbol{k}) = \delta(\boldsymbol{k})\left[1+f\mu^2\right]$$

In the tree-level power spectrum this leads to

$$P_s(k) = P(k)\left[1+f\mu^2\right]^2$$

Transverse to the line of sight $\mu = 0$ and the redshift space power spectrum agrees with the configuration space result. Note that velocities are unbiased on large scales such that we have for galaxies with linear bias b_1 in $\delta_{\rm g} = b_1\delta$

$$P_{g,s}(k) = b_1^2 P(k)\left[1+\frac{f}{b_1}\mu^2\right]^2 = P_{\rm g}(k)\left[1+\beta\mu^2\right]^2$$

with $\beta = f/b_1$.

6 Non-local Bias

Let us consider the local Lagrangian bias model up to second order

$$\delta_{\rm g}^{(\rm L)}(\boldsymbol{q}) = b_1^{(\rm L)}\delta(\boldsymbol{q}) + \frac{b_2^{(\rm L)}}{2}\left(\delta^2(\boldsymbol{q}) - \left\langle\delta^2\right\rangle\right)\,,$$

where $\delta(\boldsymbol{q})$ is the Gaussian density field in Lagrangian space and the Lagrangian and Eulerian coordinates are related by the displacement field $\boldsymbol{\Psi}(\boldsymbol{q})$ as $\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{\Psi}(\boldsymbol{q})$. Note that the displacement field can be expanded perturbatively in Largangian perturbation theory and that at first order $\boldsymbol{\Psi}^{(1)}(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}/k^2\delta^{(1)}(\boldsymbol{k})$ The continuity equations for galaxies and dark matter are given by

$$\left[1+\delta(\boldsymbol{x})\right]\mathrm{d}^3x = \mathrm{d}^3q\,, \qquad \left[1+\delta_{\rm g}^{(\rm E)}(\boldsymbol{x})\right]\mathrm{d}^3x = \left[1+\delta_{\rm h}^{(\rm L)}(\boldsymbol{q})\right]\mathrm{d}^3q\,.$$

which can be combined to yield

$$\delta_{\rm h}^{(\rm E)}(\boldsymbol{x}) = \delta_{\rm h}^{(\rm L)}(\boldsymbol{q}) + \delta_{\rm h}^{(\rm L)}(\boldsymbol{q})\delta(\boldsymbol{x}) + \delta(\boldsymbol{x})$$

At first order we obtain the well known relation between Lagrangian and Eulerian linear bias

$$\delta_{\mathrm{h}}^{(\mathrm{E},1)}(\boldsymbol{x}) = \delta_{\mathrm{h}}^{(\mathrm{L},1)}(\boldsymbol{x}) + \delta^{(1)}(\boldsymbol{x}) = \left[b_1^{(\mathrm{L})} + 1\right]\delta^{(1)}(\boldsymbol{x}) = b_1^{(\mathrm{E})}\delta^{(1)}(\boldsymbol{x})$$

At second order the continuity equation yields

$$\delta_{\mathrm{h}}^{(\mathrm{E},2)}(\boldsymbol{x}) = \delta_{\mathrm{h}}^{(\mathrm{L},2)}(\boldsymbol{x}) - \boldsymbol{\Psi}^{(1)}\cdot\boldsymbol{\nabla}\delta_{\mathrm{h}}^{(\mathrm{L},1)}(\boldsymbol{x}) + \delta_{\mathrm{h}}^{(\mathrm{L},1)}(\boldsymbol{x})\delta^{(1)}(\boldsymbol{x}) + \delta^{(2)}(\boldsymbol{x}) \tag{33}$$

The second order density field can be written in terms of the squared field, the shift term and the tidal term (omitting the position argument for the sake of readability)

$$\delta^{(2)}(\boldsymbol{x}) = \frac{17}{21}\delta^2 - \boldsymbol{\Psi}\boldsymbol{\nabla}\delta + \frac{2}{7}s^2$$

We thus have

$$\begin{aligned}
\delta_{\mathrm{h}}^{(\mathrm{E},2)}(\boldsymbol{x}) =& \frac{b_2^{(\mathrm{L})}}{2}\delta^2 - b_1^{(\mathrm{L})}\boldsymbol{\Psi}\cdot\boldsymbol{\nabla}\delta + b_1^{(\mathrm{L})}\delta^2 + \left[\frac{17}{21}\delta^2 - \boldsymbol{\Psi}\boldsymbol{\nabla}\delta + \frac{2}{7}s^2\right] \\
=& \frac{b_2^{(\mathrm{L})}}{2}\delta^2 - b_1^{(\mathrm{L})}\boldsymbol{\Psi}\cdot\boldsymbol{\nabla}\delta + b_1^{(\mathrm{L})}\delta^2 + (b_1^{(\mathrm{L})}+1)\left[\frac{17}{21}\delta^2 - \boldsymbol{\Psi}\boldsymbol{\nabla}\delta + \frac{2}{7}s^2\right] - b_1^{(\mathrm{L})}\left[\frac{17}{21}\delta^2 - \boldsymbol{\Psi}\boldsymbol{\nabla}\delta + \frac{2}{7}s^2\right] \\
=& \left(\frac{b_2^{(\mathrm{L})}}{2} + \frac{4}{21}b_1^{(\mathrm{L})}\right)\delta^2 + (b_1^{(\mathrm{L})}+1)\delta^{(2)} - \frac{2}{7}b_1^{(\mathrm{L})}s^2 \\
=& \frac{b_2^{(\mathrm{E})}}{2}\delta^2(\boldsymbol{x}) + b_1^{(\mathrm{E})}\delta^{(2)}(\boldsymbol{x}) - \frac{2}{7}b_1^{(\mathrm{L})}s^2(\boldsymbol{x})
\end{aligned} \tag{34}$$

Here we have defined the Eulerian bias parameters

$$b_1^{(\mathrm{E})} = b_1^{(\mathrm{L})} + 1 \qquad\qquad b_2^{(\mathrm{E})} = b_2^{(\mathrm{L})} + \frac{8}{21}b_1^{(\mathrm{L})}$$

Starting from (33) we can derive the same result in Fourier space where the products become convolutions $\boldsymbol{\Psi}^{(1)}(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}/k^2\delta^{(1)}(\boldsymbol{k})$ and $\boldsymbol{\nabla} \to -\mathrm{i}\boldsymbol{k}$:

$$\begin{aligned}
\delta_{\mathrm{h}}^{(\mathrm{E},2)}(\boldsymbol{k}) =& \int\frac{\mathrm{d}^3q}{(2\pi)^3}\left\{\frac{b_2^{(\mathrm{L})}}{2} + b_1^{(\mathrm{L})}\frac{\boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})}{q^2} + b_1^{(\mathrm{L})} + F_{2,s}(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q})\right\}\delta(\boldsymbol{q})\delta(\boldsymbol{k}-\boldsymbol{q}) \\
=& \int\frac{\mathrm{d}^3q}{(2\pi)^3}\left\{\frac{b_2^{(\mathrm{L})}}{2} + b_1^{(\mathrm{L})}\frac{\boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})}{2}\left(\frac{1}{q^2}+\frac{1}{|\boldsymbol{k}-\boldsymbol{q}|^2}\right) + b_1^{(\mathrm{L})}\right. \\
&\left. + \frac{5}{7} + \frac{\boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})}{2}\left(\frac{1}{q^2}+\frac{1}{|\boldsymbol{k}-\boldsymbol{q}|^2}\right) + \frac{2}{7}\frac{[\boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})]^2}{q^2|\boldsymbol{k}-\boldsymbol{q}|^2}\right\}\delta(\boldsymbol{q})\delta(\boldsymbol{k}-\boldsymbol{q}) \\
=& \int\frac{\mathrm{d}^3q}{(2\pi)^3}\left\{\frac{b_2^{(\mathrm{L})}}{2} + b_1^{(\mathrm{L})}\frac{\boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})}{2}\left(\frac{1}{q^2}+\frac{1}{|\boldsymbol{k}-\boldsymbol{q}|^2}\right) + b_1^{(\mathrm{L})}\right. \\
&\left. + \frac{17}{21} + \frac{\boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})}{2}\left(\frac{1}{q^2}+\frac{1}{|\boldsymbol{k}-\boldsymbol{q}|^2}\right) + \frac{2}{7}\left(\frac{[\boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})]^2}{q^2|\boldsymbol{k}-\boldsymbol{q}|^2} - \frac{1}{3}\right)\right\}\delta(\boldsymbol{q})\delta(\boldsymbol{k}-\boldsymbol{q}) \\
=& \int\frac{\mathrm{d}^3q}{(2\pi)^3}\left\{\frac{b_2^{(\mathrm{L})}}{2} + b_1^{(\mathrm{L})} + b_1^{(\mathrm{E})}F_{2,s}(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}) - \frac{17}{21}b_1^{(\mathrm{L})} - \frac{2}{7}b_1^{(\mathrm{L})}S_2(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q})\right\}\delta(\boldsymbol{q})\delta(\boldsymbol{k}-\boldsymbol{q}) \\
=& \int\frac{\mathrm{d}^3q}{(2\pi)^3}\left\{\frac{1}{2}b_2^{(\mathrm{L})} + \frac{4}{21}b_1^{(\mathrm{L})} + b_1^{(\mathrm{E})}F_{2,s}(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}) - \frac{2}{7}b_1^{(\mathrm{L})}S_2(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q})\right\}\delta(\boldsymbol{q})\delta(\boldsymbol{k}-\boldsymbol{q})\,.
\end{aligned}$$

Which agrees with (34) upon identifying:

$$s^2(\boldsymbol{k}) = \int\frac{\mathrm{d}^3q}{(2\pi)^3}S_2(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q})\delta^{(1)}(\boldsymbol{q})\delta^{(1)}(\boldsymbol{k}-\boldsymbol{q})\,, \tag{35}$$

as the Fourier transform of the square of the tidal tensor.