1 Boltzmann Equation for Harmonic Oscillators

(a) The full time derivative of the distribution function can be written as

$$\frac{\mathrm{d}f(t,x,p)}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial f}{\partial x} + \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\partial f}{\partial p}\,.$$

By the definition of momentum, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p}{m}\,,$$

and the equation of motion of the harmonic oscillator can be written as

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\kappa x\,.$$

The collisionless Boltzmann equation for the harmonic oscillator then is

$$\frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} - \kappa x\frac{\partial f}{\partial p} = 0 \tag{1}$$

An equilibrium distribution function satisfies $\partial f/\partial t = 0$. It is claimed that the distribution function is then only a function of the energy. To show that $f(E)$ is indeed a solution, consider

$$\begin{aligned}\frac{p}{m}\frac{\partial f(E)}{\partial x} - \kappa x\frac{\partial f(E)}{\partial p} &= \frac{\mathrm{d}f}{\mathrm{d}E}\left[\frac{p}{m}\frac{\partial E}{\partial x} - \kappa x\frac{\partial E}{\partial p}\right]\\ &= 0\,.\end{aligned}$$

(b) First, integrate Eq. (1) over all momentum. The $\partial f/\partial p$ term vanishes after integrating by parts and noticing that $f = 0$ at $p = \pm\infty$ (there are no particles with infinite momentum!). Hence, we find

$$\frac{\partial n}{\partial t} + \frac{\partial(nv)}{\partial x} = 0\,.$$

This is the continuity equation. To get the Euler equation, first multiply by p/m and then integrate over all momentum. This gives

$$\frac{\partial(nv)}{\partial t} + \frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}\frac{p^2}{m^2}f + \frac{\kappa x}{m}n = 0\,,$$

where the last term follows from integration by parts. The integral over p^2 yields two terms: *i*) a bulk velocity term, nv^2, and *ii*) a pressure term, P. Using the continuity equation, we get

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{n}\frac{\partial P}{\partial x} + \frac{\kappa x}{m} = 0\,.$$

2 Boltzmann Hierarchy

We start from the Boltzmann equation in Fourier space

$$\Theta' + \mathrm{i}k\mu\Theta = \Phi' - \mathrm{i}k\mu\Psi + \Gamma\left[\Theta_0 - \Theta + \mathrm{i}\mu v_\mathrm{e}\right]\,. \tag{2}$$

Let us remind ourselves of the dependencies of the variables:

temperature perturbation $\Theta(\eta, \boldsymbol{k}, \mu = \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}})$ The temperature perturbation depends on wavevector $\boldsymbol{k}$ and on the relative angle between photon propagation direction $\hat{\boldsymbol{p}}$ and wavevector $\hat{\boldsymbol{k}}$ quantified by $\mu = \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}$.

temperature monopole $\Theta_0(\eta, \boldsymbol{k})$ The temperature monopole is the angular average of the above temperature perturbation and thus independent of the direction cosine μ.

metric perturbations $\Psi(\eta, \boldsymbol{k})$ **and** $\Phi(\eta, \boldsymbol{k})$ The metric perturbations only depend on the wavevector of the perturbation but not on the photons direction of propagation. They are thus independent of μ.

electron velocity $v_e(\eta, \boldsymbol{k})$

Let us furthermore remind ourselves of the definition of the multipole moments

$$\Theta_l(\eta, \boldsymbol{k}) = \frac{1}{(-\mathrm{i})^l} \int \frac{\mathrm{d}\mu}{2} \mathcal{P}_l(\mu) \Theta(\eta, \boldsymbol{k}, \mu)\,.$$

Let us first consider the general case $l \geq 2$ and consider the monopole and dipole explicitly later. We first multiply the Boltzmann Equation (2) with $\mathcal{P}_l(\mu),\ l \geq 2$ and integrate over μ. The $\mu\Psi$ and μv_e terms are dipoles and don't contribute. The Φ' and $\Gamma\Theta_0$ terms are monopoles and don't contribute either. For the $\mu\mathcal{P}_(\mu)\Theta(\eta, \boldsymbol{k}, \mu)$ term we can now use the recursion relation

$$\mu P_l(\mu) = \frac{(l+1)}{(2l+1)} P_{l+1}(\mu) + \frac{l}{(2l+1)} P_{l-1}(\mu)\,.$$

$$\Theta_l' + k \left(\frac{l+1}{2l+1} \Theta_{l+1} - \frac{l}{2l+1} \Theta_{l-1} \right) = -\Gamma\Theta_l\,.$$

For the $l = 0$ case, we simply integrate the Boltzmann Equation (2) over μ. The $\mathrm{i}k\mu\Theta$ term leads to a monopole, the dipole terms $\mu\Psi$ and μv_e do not contribute and the $\Gamma[\Theta_0 - \Theta]$ terms cancel. In summary, we have

$$\Theta_0' + k\Theta_1 = +\Phi'\,.$$

Finally, for the $l = 1$ case, we multiply the Boltzmann equation with $\mathcal{P}_1(\mu) = \mu$ and integrate. Now, the dipole terms $\mu\Psi$ and μv_e do do contribute, but the monopoles vanish. The μ^2 term can be rewritten as a sum of a monopole and quadrupole

$$\mu^2 = \mu\mathcal{P}_1 = \frac{2}{3}\mathcal{P}_2 + \frac{1}{3}\mathcal{P}_0\,,$$

which follows either from the recursion relation or the explicit form of the Legendre polynomials. We thus have

$$3\Theta_1' + 2k\Theta_2 = k\Theta_0 - k\Psi - \Gamma(3\Theta_1 + v_\mathrm{b})\,.$$

Based on the tight coupling approximation, we can furthermore ignore the quadrupole Θ_2 and arrive at the desired result.

3 Anisotropies from Tensors

(a) With $P^\mu = dx^\mu/d\lambda$, it follows that

$$\frac{\mathrm{d}\eta}{\mathrm{d}\lambda} = \frac{\epsilon}{a^2}\,, \qquad \frac{\mathrm{d}x^i}{\mathrm{d}\eta} = \hat{p}^i - \frac{1}{2} h^i_j \hat{p}^j\,.$$

The geodesic equation in conformal time then is

$$\frac{\epsilon}{a^2} \frac{\mathrm{d}P^\mu}{\mathrm{d}\eta} + \Gamma^\mu_{\nu\rho} P^\nu P^\rho = 0\,.$$

The 0-component is

$$\frac{\epsilon}{a^2} \frac{\mathrm{d}}{\mathrm{d}\eta} \left(\frac{\epsilon}{a^2} \right) + \frac{\epsilon^2}{a^4} \left[\Gamma^0_{00} + \Gamma^0_{ij} \left(\hat{p}^i - \frac{1}{2} h^i_k \hat{p}^k \right) \left(\hat{p}^j - \frac{1}{2} h^j_l \hat{p}^l \right) \right] = 0\,.$$

Substituting the given expressions for Γ^0_{00} and Γ^0_{ij}, we get

$$\begin{aligned} \frac{\mathrm{d}\ln\epsilon}{\mathrm{d}\eta} &= +\mathcal{H} - \left(\mathcal{H}\delta_{ij} + \mathcal{H} h_{ij} + \frac{1}{2} h'_{ij} \right) \left(\hat{p}^i - \frac{1}{2} h^i_k \hat{p}^k \right) \left(\hat{p}^j - \frac{1}{2} h^j_l \hat{p}^l \right) \\ &= -\frac{1}{2} h'_{ij} \hat{p}^i \hat{p}^j\,, \end{aligned}$$

where in the second line we have dropped all terms beyond linear order in h_{ij}.

(b) Assuming that tight-coupling holds until recombination, we can neglect the temperature quadrupole at last scattering. The only tensor-induced contribution to the temperature anisotropies then comes from the free-streaming effect:

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\eta} = \frac{\mathrm{d}\ln\epsilon}{\mathrm{d}\eta} = -\frac{1}{2}h'_{ij}\hat{p}^i\hat{p}^j \,.$$

Integrating the Boltzmann equation along the line-of-sight, we find

$$\tilde{\Theta}(\hat{\boldsymbol{n}}) \equiv \Theta(\eta_0, \boldsymbol{x}_0 \equiv \boldsymbol{0}, \hat{\boldsymbol{n}} = -\hat{\boldsymbol{p}}) = -\frac{1}{2}\int_{\eta_*}^{\eta_0} \mathrm{d}\eta'\, h'_{ij}(\eta', \boldsymbol{x}(\eta'))\, \hat{n}^i\hat{n}^j \,,$$

where $\boldsymbol{x}(\eta') \equiv \chi(\eta')\hat{\boldsymbol{n}}$. Writing h'_{ij} in terms of its Fourier transform, we get

$$\tilde{\Theta}(\hat{\boldsymbol{n}}) = -\frac{1}{2}\int_{\eta_*}^{\eta_0} \mathrm{d}\eta' \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\, h'_{ij}(\eta', \boldsymbol{k})\, \hat{n}^i\hat{n}^j\, e^{-ik\chi(\eta')\,\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}} \qquad (\star)\,.$$

(c) It is convenient to expand the tensor mode in its two helicity components

$$h_{ij}(\eta, \boldsymbol{k}) = \sum_{\lambda=\pm} \frac{1}{\sqrt{2}} h_\lambda(\eta, \boldsymbol{k})\, \epsilon^\lambda_{ij}(\hat{\boldsymbol{k}})\,.$$

We first consider the special case $\boldsymbol{k} = k\hat{\boldsymbol{z}}$. In that case, the explicit form of the polarization tensor is

$$\boldsymbol{\epsilon}^\pm(\hat{\boldsymbol{z}}) = \frac{1}{2}\begin{pmatrix} 1 & \pm\mathrm{i} & 0 \\ \pm\mathrm{i} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Contracting this with $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we find

$$\begin{aligned}
\epsilon^\pm_{ij}(\hat{\boldsymbol{z}})\, \hat{n}^i\hat{n}^j &= \frac{1}{2}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\begin{pmatrix} 1 & \pm\mathrm{i} & 0 \\ \pm\mathrm{i} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \\
&= \frac{1}{2}\sin^2\theta\, e^{\pm 2i\phi} = \sqrt{\frac{8\pi}{15}}Y_{2\pm2}(\hat{\boldsymbol{n}})\,.
\end{aligned}$$

The contribution of the Fourier mode $\boldsymbol{k} = k\hat{\boldsymbol{z}}$ to the integral in $(\star)$ therefore is

$$\sqrt{\frac{4\pi}{15}}\, h'_\pm(\eta, k\hat{\boldsymbol{z}})\, Y_{2\pm2}(\hat{\boldsymbol{n}})\, e^{-\mathrm{i}k\chi\cos\theta}\,.$$

Using the Rayleigh plane-wave expansion for the exponential and expressing $P_l(\cos\theta)$ in terms of $Y_{L0}(\hat{\boldsymbol{n}})$, this can be written as

$$\frac{4\pi}{\sqrt{15}}\, h'_\pm(\eta, k\hat{\boldsymbol{z}}) \times \sum_{L\geq 0}(-\mathrm{i})^L\sqrt{2L+1}\, j_L(k\chi) Y_{2\pm2}(\hat{\boldsymbol{n}}) Y_{L0}(\hat{\boldsymbol{n}})\,.$$

Next, we replace the product of the two spherical harmonics by a sum over spherical harmonics weighted by Wigner $3j$ symbols

$$\sqrt{\frac{4\pi}{3}}\, h'_\pm(\eta, k\hat{\boldsymbol{z}}) \sum_{L\geq 0}\left[(-\mathrm{i})^L(2L+1) j_L(k\chi) \times \sum_{l\geq 2}\sqrt{2l+1}\begin{pmatrix} 2 & L & l \\ \mp 2 & 0 & \pm 2 \end{pmatrix}\begin{pmatrix} 2 & L & l \\ 0 & 0 & 0 \end{pmatrix} Y_{l\pm2}(\hat{\boldsymbol{n}})\right].$$

Writing out the $3j$ symbols explicitly and using a recursion relations for Bessel functions to express $j_{l\pm2}$ in terms of j_l, gives

$$\sqrt{\frac{4\pi}{15}}\, h'_\pm(\eta, k\hat{\boldsymbol{z}})\, Y_{2\pm2}(\hat{\boldsymbol{n}})\, e^{-ik\chi\cos\theta} = -\sqrt{\frac{\pi}{2}} h'_\pm(\eta, k\hat{\boldsymbol{z}}) \sum_l \alpha_l\, \frac{j_l(k\chi)}{(k\chi)^2}\, Y_{l\,\pm2}(\hat{\boldsymbol{n}})\,,$$

where $\alpha_l \equiv (-\mathrm{i})^l\sqrt{2l+1}\sqrt{(l+2)!/(l-2)!}$.

So far, we have only considered Fourier modes with $\boldsymbol{k}$ parallel to $\hat{\boldsymbol{z}}$. The contribution from a general Fourier mode is obtained by rotating the previous result. We are told that this gives

$$h^{\pm\prime}_{ij}(\eta, \boldsymbol{k})\, \hat{n}^i\hat{n}^j\, e^{-\mathrm{i}k\chi\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}} = -\sqrt{\frac{\pi}{2}} h'_\pm(\eta, \boldsymbol{k}) \sum_l \alpha_l\, \frac{j_l(k\chi)}{(k\chi)^2} \sum_m D^l_{m\pm2}(\hat{\boldsymbol{k}})\, Y_{lm}(\hat{\boldsymbol{n}}) \qquad (\star\star)\,,$$

and that an explicit form of the Wigner D matrices won't be needed.

(d) We write the temperature anisotropy in a harmonic expansion

$$\tilde{\Theta}(\hat{\boldsymbol{n}}) = a_{lm} Y_{lm}(\hat{\boldsymbol{n}}) \,.$$

The angular power spectrum is then defined as $\langle a_{lm} a^*_{l'm'} \rangle = C_l \, \delta_{ll'} \delta_{mm'}$. We find the a_{lm}'s by substituting ($\star\star$) into ($\star$):

$$a_{lm} = \frac{1}{\sqrt{2}} \sum_\lambda \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} (-\mathrm{i})^l \Theta_l^\lambda(\boldsymbol{k}) \sqrt{\frac{4\pi}{2l+1}} D^l_{m\pm 2}(\hat{\boldsymbol{k}}) \,,$$

where

$$\boxed{\Theta_l^\lambda(\boldsymbol{k}) = \frac{2l+1}{4} \sqrt{\frac{(l+2)!}{(l-2)!}} \int_{\eta_*}^{\eta_0} \mathrm{d}\eta' \, h'_\lambda(\eta', \boldsymbol{k}) \, \frac{j_l(k\chi)}{(k\chi)^2}} \quad (\dagger) \,.$$

To obtain the power spectrum, we compute

$$\langle a_{lm} a^*_{l'm'} \rangle = (-i)^{l-l'} \frac{4\pi}{\sqrt{(2l+1)(2l'+1)}} \frac{1}{2} \sum_{\lambda\lambda'} \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3} \left\{ \left[\frac{\Theta_l^\lambda(\boldsymbol{k})}{h_\lambda(\boldsymbol{k})} \right] \left[\frac{\Theta_{l'}^{\lambda'}(\boldsymbol{k}')}{h_{\lambda'}(\boldsymbol{k}')} \right] \right.$$
$$\times \langle h_\lambda(\boldsymbol{k}) h^*_{\lambda'}(\boldsymbol{k}') \rangle \, D^l_{m\lambda}(\hat{\boldsymbol{k}}) D^{l'}_{m'\lambda'}(\hat{\boldsymbol{k}}') \,.$$

Using

$$\langle h_\lambda(\boldsymbol{k}) h^*_{\lambda'}(\boldsymbol{k}') \rangle = \frac{2\pi^2}{k^3} \Delta_h^2(k) \, \delta_D(\boldsymbol{k} - \boldsymbol{k}') \, \delta_{\lambda\lambda'} \quad \text{and} \quad \Theta_l^\lambda(k) \equiv \frac{\Theta_l^\lambda(\boldsymbol{k})}{h_\lambda(\boldsymbol{k})} \,,$$

we get

$$\langle a_{lm} a^*_{l'm'} \rangle = \frac{(-i)^{l-l'}}{\sqrt{(2l+1)(2l'+1)}} \int \mathrm{d}\ln k \left| \Theta_l^{(\lambda)}(k) \right|^2 \Delta_h^2(k) \underbrace{\int \mathrm{d}\hat{\boldsymbol{k}} \, D^l_{m\lambda}(\hat{\boldsymbol{k}}) D^{l'}_{m'\lambda'}(\hat{\boldsymbol{k}})}_{\frac{4\pi}{2l+1} \delta_{ll'} \delta_{mm'}}$$

$$= \delta_{ll'} \delta_{mm'} \frac{4\pi}{(2l+1)^2} \int \mathrm{d}\ln k \left| \Theta_l^\lambda(k) \right|^2 \Delta_h^2(k) \,.$$

The final result for the angular power spectrum therefore is

$$\boxed{C_l = \frac{4\pi}{(2l+1)^2} \int \mathrm{d}\ln k \left| \Theta_l^\lambda(k) \right|^2 \Delta_h^2(k)} \quad (\dagger\dagger) \,.$$

To evaluate ($\dagger\dagger$) we need an expression for the tensor mode function in ($\dagger$). The universe is matter dominated at recombination and thereafter. The tensor mode functions then satisfy

$$h_\lambda(\eta, \boldsymbol{k}) = 3 h_\lambda(\eta_i, \boldsymbol{k}) \frac{j_1(k\eta)}{k\eta} \quad \rightarrow \quad h'_\lambda(\eta, \boldsymbol{k}) = -3k h_\lambda(\eta_i, \boldsymbol{k}) \frac{j_2(k\eta)}{k\eta} \,.$$

Substituting this into ($\dagger$), we get

$$\boxed{\Theta_l^\lambda(k) = -\frac{3}{4}(2l+1) \sqrt{\frac{(l+2)!}{(l-2)!}} \int_{\eta_*}^{\eta_0} k \, \mathrm{d}\eta' \, \frac{j_2(k\eta')}{k\eta'} \, \frac{j_l(k(\eta_0 - \eta'))}{(k(\eta_0 - \eta'))^2}} \quad (\ddagger) \,.$$

(e) Since gravitational waves decay inside horizon, we expect the tensor-induced CMB spectrum to drop off on small scales. Let's see how this is reflected in the result ($\ddagger$). This integral is non-negligible only when $k\eta' \sim 2$ and $k(\eta_0 - \eta') \sim l$ simultaneously, since otherwise at least one of the two Bessel functions is close to zero. Both conditions can be satisfied simultaneously only if

$$l \lesssim 2 \frac{\eta_0}{\eta_*} \sim 2\sqrt{z_* + 1} \sim 60 \,.$$

Finally, let us show that our result (‡) corresponds to $l(l+1)C_l = \text{const.}$ on large scales. Let us approximate $j_2(x)/x$ by $\delta_D(x-2)$. We then get

$$\Theta_l^\lambda(k) \sim -\frac{3}{4}(2l+1)\sqrt{\frac{(l+2)!}{(l-2)!}}\,\frac{j_l(k\eta_0-2)}{(k\eta_0-2)^2} \quad \text{for} \quad \eta_* < k^{-1} < \eta_0 .$$

The restriction on wavenumbers is such that only gravitational waves that enter the horizon between last-scattering and today contribute. Assuming a scale-invariant primordial tensor spectrum, $\Delta_h^2 = A_t$, the angular power spectrum becomes

$$\begin{aligned} C_l &\sim \frac{\pi}{4}\frac{(l+2)!}{(l-2)!}\,A_t \int_{1/\eta_0}^{1/\eta_*} \mathrm{d}\ln k\, \frac{j^2(k\eta_0-2)}{(k\eta_0-1)^4} \\ &= \frac{\pi}{4}\frac{(l+2)!}{(l-2)!}\,A_t \int_0^{\eta_0/\eta_*-1} \frac{\mathrm{d}x}{x+2}\,\frac{j_l^2(x)}{x^4}\,, \end{aligned}$$

where we substituted $x \equiv k\eta_0 - 2$ in the second line. For $l \gg 1$, but $l \ll 60$, the integral is dominated by $x \sim l$, and we can replace $x+2$ by x in the integrand and extend the upper limit of integration to ∞. Using

$$\int_0^\infty \mathrm{d}x\,\frac{j_l^2(x)}{x^5} = \frac{4}{15}\frac{(l-2)!}{(l+2)!}\frac{1}{(l+3)(l-2)}\,,$$

we find

$$C_l \sim \frac{\pi}{15}\frac{A_t}{(l+3)(l-2)}\,.$$

Hence, $l(l+1)C_l \approx \textit{const.}$, as expected.

4 Recombination, last scattering, and the visibility function

(a) The optical depth to Thomson scattering back to redshift z is

$$\tau(z) = \int_{t(z)}^{t_0} c\bar{n}_e\sigma_\mathrm{T}\,\mathrm{d}t' = \int_0^z \frac{c\bar{n}_e\sigma_\mathrm{T}}{(1+z')H(z')}\,\mathrm{d}z'\,, \tag{3}$$

where we have used $1+z = a_0/a \Rightarrow \mathrm{d}z/\mathrm{d}t = -(1+z)H(z)$. Since $\bar{n}_e\sigma_\mathrm{T}$ is such a rapidly-falling function of z, the optical depth is dominated by the upper end of the integration range. The universe is well approximated as being matter dominated then, so that

$$H(z) \approx H_0\sqrt{\Omega_m}(1+z)^{3/2}\,, \tag{4}$$

where Ω_m is the *current* matter density parameter. Substituting into Eq. (3), we have

$$\tau(z) \approx 9.2\times 10^4 \int_0^z \left(\frac{z'}{1000}\right)^{12.1} z'^{-5/2}\,\mathrm{d}z' \tag{5}$$

$$= 0.27(z/1000)^{10.6}\,. \tag{6}$$

Note that we have approximated $1+z$ as z when substituting for $H(z)$ in the integrand.
(b) The derivative of the optical depth with respect to z is

$$\frac{\mathrm{d}\tau}{\mathrm{d}z} = \frac{0.27\times 10.6}{1000}\left(\frac{z}{1000}\right)^{9.6}\,, \tag{7}$$

so that the redshift visibility is

$$g(z) = 2.86\times 10^{-3}\left(\frac{z}{1000}\right)^{9.6}\exp[-0.27(z/1000)^{10.6}]\,. \tag{8}$$

This is plotted in Fig. 1.
(c) Solving $\tau(z_*) = 1$ gives

$$z_* = 1000\exp\left[-\frac{\ln 0.27}{10.6}\right] = 1130\,. \tag{9}$$

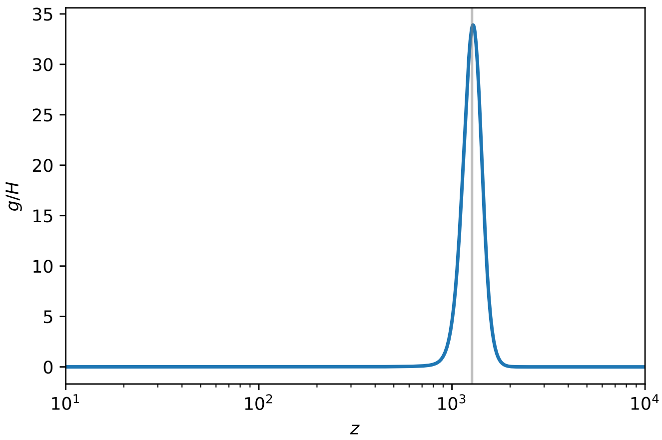

Figure 1: Redshift visibility function, $g(z) = (\mathrm{d}\tau/\mathrm{d}z)e^{-\tau}$.

To quantify the width of the visibility function, σ_z, we evaluate the second derivative of $-\ln g$:

$$-\ln g = \text{const.} - 9.6 \ln z + 0.27 \left(\frac{z}{1000}\right)^{10.6}$$
$$\Rightarrow \quad -\frac{\mathrm{d}^2 \ln g}{\mathrm{d}z^2} = \frac{9.6}{z^2} + \frac{10.6 \times 9.6 \times \tau(z)}{z^2} \,. \tag{10}$$

Evaluating this at z_*, and equating this to $1/\sigma_z^2$, we have

$$\frac{1}{\sigma_z^2} = \frac{9.6 \times 11.6}{z_*^2} \quad \Rightarrow \quad 2\sigma_z = 214 \,. \tag{11}$$

(d) Taking the duration of last scattering as the redshift interval $\Delta z = 2\sigma_z = 214$, we can convert to a magnitude of conformal time, $\Delta\eta$, using

$$a\, \mathrm{d}\eta = \mathrm{d}t = -\frac{\mathrm{d}z}{(1+z)H(z)} \quad \Rightarrow \quad \Delta\eta \approx \frac{\Delta z}{H(z_*)} \,, \tag{12}$$

where we have set $a_0 = 1$. Assuming matter domination around z_*, we use Eq. (4) to find

$$\Delta\eta = \frac{\Delta z}{H_0 \sqrt{\Omega_m}(1+z_*)^{3/2}} = 46\,\mathrm{Mpc} \,, \tag{13}$$

where we have used $H_0 = 67\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $\Omega_m = 0.3$ (and have been careful to reinstate the required factor of c).

5 Non-Gaussianity

First we need the spherical multipoles g_{lm}:

$$g_{lm} = 4\pi i^l \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} F(\boldsymbol{k}) j_l(kr) Y_{lm}^*(\hat{\boldsymbol{k}}) \,. \tag{14}$$

If we now form the three-point correlator, we find

$$\begin{aligned}\langle g_{l_1m_1} g_{l_2m_2} g_{l_3m_3}\rangle &= (4\pi)^3 i^{l_1+l_2+l_3} \int \prod_{i=1}^{3} \left[\frac{\mathrm{d}^3\boldsymbol{k}_i}{(2\pi)^3} j_{l_i}(k_i r) Y_{l_i m_i}^*(\hat{\boldsymbol{k}}_i)\right] \langle F(\boldsymbol{k}_1)F(\boldsymbol{k}_2)F(\boldsymbol{k}_3)\rangle \\ &= \frac{(4\pi)^3}{(2\pi)^3} i^{l_1+l_2+l_3} \int \prod_{i=1}^{3} \left[\frac{\mathrm{d}^3\boldsymbol{k}_i}{(2\pi)^3} j_{l_i}(k_i r) Y_{l_i m_i}^*(\hat{\boldsymbol{k}}_i)\right] \\ &\qquad \times B_F(k_1,k_2,k_3)\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) \,. \end{aligned} \tag{15}$$

Following the hint, we write the delta-function in integral form,

$$\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{x}\, e^{i(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\cdot\boldsymbol{x}}\,, \tag{16}$$

and expand the plane waves with the Rayleigh expansion to find

$$\begin{aligned}\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3) &= 2^3\int \mathrm{d}x\, x^2 \prod_{i=1}^{3}\left[\sum_{l_i' m_i'} i^{l_i'} j_{l_i'}(k_i x) Y_{l_i' m_i'}(\hat{\boldsymbol{k}}_i) Y^*_{l_i' m_i'}(\hat{\boldsymbol{x}})\right] \\ &= 2^3\int \mathrm{d}x\, x^2 \sum_{l_1' m_1'}\sum_{l_2' m_2'}\sum_{l_3' m_3'}\prod_{i=1}^{3}\left[i^{l_i'} j_{l_i'}(k_i x) Y_{l_i' m_i'}(\hat{\boldsymbol{k}}_i)\right]\mathcal{G}^{m_1' m_2' m_3'}_{l_1' l_2' l_3'}\,. \end{aligned}\tag{17}$$

In the last line, we have performed the integral over $\hat{\boldsymbol{x}}$ using the definition of the Gaunt integral given in the question. (We note that this integral is real, and vanishes unless $m_1+m_2+m_3=0$ and $l_1+l_2+l_3$ is even. The latter condition follows from parity.) Finally, we substitute back into Eq. (15). Isolating the angular integrals over $\hat{\boldsymbol{k}}_i$, and using orthonormality of the spherical harmonics,

$$\int \mathrm{d}\hat{\boldsymbol{k}}\, Y_{l'm'}(\hat{\boldsymbol{k}}) Y^*_{lm}(\hat{\boldsymbol{k}}) = \delta_{ll'}\delta_{mm'}\,, \tag{18}$$

we find (after sorting out all the factors of 2π)

$$\begin{aligned}\langle g_{l_1 m_1} g_{l_2 m_2} g_{l_3 m_3}\rangle = \left(\frac{2}{\pi}\right)^3 (-1)^{l_1+l_2+l_3}\mathcal{G}^{m_1 m_2 m_3}_{l_1 l_2 l_3} \\ \times \int \mathrm{d}x\, x^2 \int \prod_{i=1}^{3}\left[\mathrm{d}k_i\, k_i^2 j_{l_i}(k_i r) j_{l_i}(k_i r)\right] B_F(k_1,k_2,k_3)\,. \end{aligned}\tag{19}$$

The prefactor $(-1)^{l_1+l_2+l_3}=1$ due to Gaunt integral, so we can read off the reduced bispectrum

$$b_{l_1 l_2 l_3} = \left(\frac{2}{\pi}\right)^3 \int \mathrm{d}x\, x^2 \int \prod_{i=1}^{3}\left[\mathrm{d}k_i\, k_i^2 j_{l_i}(k_i r) j_{l_i}(k_i r)\right] B_F(k_1,k_2,k_3)\,. \tag{20}$$

Under rotation of $g(\hat{\boldsymbol{n}})$, we have $g_{lm}\to\sum_{m'} D^l_{mm'} g_{lm'}$. If the correlator is to be invariant under such rotations, we must have

$$\langle g_{l_1 m_1} g_{l_2 m_2} g_{l_3 m_3}\rangle = \sum_{m_1' m_2' m_3'} D^{l_1}_{m_1 m_1'} D^{l_2}_{m_2 m_2'} D^{l_3}_{m_3 m_3'}\langle g_{l_1 m_1'} g_{l_2 m_2'} g_{l_3 m_3'}\rangle \tag{21}$$

$$\Rightarrow\quad \mathcal{G}^{m_1 m_2 m_3}_{l_1 l_2 l_3} = \sum_{m_1' m_2' m_3'} D^{l_1}_{m_1 m_1'} D^{l_2}_{m_2 m_2'} D^{l_3}_{m_3 m_3'}\mathcal{G}^{m_1' m_2' m_3'}_{l_1 l_2 l_3}\,. \tag{22}$$

Taking the complex conjugate of both sides, recalling that the Gaunt integral is real, and using unitarity of the rotation matrices – $D^{l*}_{mm'}(R) = D^l_{m'm}(-R)$, where $-R$ is the inverse rotation to R – Eq. (22) is equivalent to the requirement that

$$\begin{aligned}\int \mathrm{d}\hat{\boldsymbol{n}} \prod_{i=1}^{3} Y_{l_i m_i}(\hat{\boldsymbol{n}}) &= \int \mathrm{d}\hat{\boldsymbol{n}} \prod_{i=1}^{3}\left[\sum_{m_i'} D^{l_i}_{m_i' m_i}(-R) Y_{l_i m_i'}(\hat{\boldsymbol{n}})\right] \\ &= \int d\hat{\boldsymbol{n}} \prod_{i=1}^{3} \hat{D}^{-1} Y_{l_i m_i}(\hat{\boldsymbol{n}})\,. \end{aligned}\tag{23}$$

The value of actively-rotated function at $\hat{\boldsymbol{n}}$ is the same as the value of the original function at the back-rotated position, i.e., $\hat{D}^{-1}Y_{lm}(\hat{\boldsymbol{n}}) = Y(\hat{D}\hat{\boldsymbol{n}})$. Using this in the right-hand side of Eq. (23), and noting that since rotations are orthogonal transformations $d(\hat{D}\hat{\boldsymbol{n}}) = d\hat{\boldsymbol{n}}$, establishes the truth of this requirement.

6 CMB Polarization – Correlation Functions

In Fourier space we can invert the relations for $E(\boldsymbol{l})$ and $B(\boldsymbol{l})$ to get

$$Q(\boldsymbol{l}) = E(\boldsymbol{l})\cos(2\phi_{\boldsymbol{l}}) - B(\boldsymbol{l})\sin(2\phi_{\boldsymbol{l}}) \tag{24}$$

$$U(\boldsymbol{l}) = E(\boldsymbol{l})\sin(2\phi_{\boldsymbol{l}}) + B(\boldsymbol{l})\cos(2\phi_{\boldsymbol{l}}) \tag{25}$$

Taking the Fourier transform yields

$$Q(\boldsymbol{\theta}) = \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2} \left[E(\boldsymbol{l})\cos(2\phi_{\boldsymbol{l}}) - B(\boldsymbol{l})\sin(2\phi_{\boldsymbol{l}})\right] e^{i\boldsymbol{l}\cdot\boldsymbol{\theta}} \tag{26}$$

$$U(\boldsymbol{\theta}) = \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2} \left[E(\boldsymbol{l})\sin(2\phi_{\boldsymbol{l}}) + B(\boldsymbol{l})\cos(2\phi_{\boldsymbol{l}})\right] e^{i\boldsymbol{l}\cdot\boldsymbol{\theta}} \tag{27}$$

From the definition of the correlation function we get

$$\begin{aligned}
C_Q(\boldsymbol{\theta}) &\equiv \langle Q(\boldsymbol{\theta})Q(\boldsymbol{\theta}')\rangle \\
&= \Big\langle \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{l}'}{(2\pi)^2} \left[E(\boldsymbol{l})\cos(2\phi_{\boldsymbol{l}}) - B(\boldsymbol{l})\sin(2\phi_{\boldsymbol{l}})\right] \times \\
&\quad \left[E(\boldsymbol{l})\cos(2\phi_{\boldsymbol{l}'}) - B(\boldsymbol{l}')\sin(2\phi_{\boldsymbol{l}'})\right] e^{i\boldsymbol{l}\cdot\boldsymbol{\theta}} e^{i\boldsymbol{l}'\cdot\boldsymbol{\theta}'} \Big\rangle \\
&= \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{l}'}{(2\pi)^2} \left[\langle E(\boldsymbol{l})E(\boldsymbol{l}')\rangle \cos(2\phi_{\boldsymbol{l}})\cos(2\phi_{\boldsymbol{l}'}) + \right. \\
&\quad \left. \langle B(\boldsymbol{l})B(\boldsymbol{l}')\rangle \sin(2\phi_{\boldsymbol{l}})\sin(2\phi_{\boldsymbol{l}'})\right] e^{i\boldsymbol{l}\cdot\boldsymbol{\theta}} e^{i\boldsymbol{l}'\cdot\boldsymbol{\theta}'} \\
&= \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2}\frac{\mathrm{d}^2\boldsymbol{l}'}{(2\pi)^2} \left[(2\pi)^2 C_{El}\delta(\boldsymbol{l}+\boldsymbol{l}')\cos(2\phi_{\boldsymbol{l}})\cos(2\phi_{\boldsymbol{l}'}) + \right. \\
&\quad \left. (2\pi)^2 C_{Bl}\delta(\boldsymbol{l}+\boldsymbol{l}')\sin(2\phi_{\boldsymbol{l}})\sin(2\phi_{\boldsymbol{l}'})\right] e^{i\boldsymbol{l}\cdot\boldsymbol{\theta}} e^{i\boldsymbol{l}'\cdot\boldsymbol{\theta}'} \\
&= \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2} \left[C_{El}\cos(2\phi_{\boldsymbol{l}})^2 + C_{Bl}\sin(2\phi_{\boldsymbol{l}})^2\right] e^{i\boldsymbol{l}\cdot(\boldsymbol{\theta}+\boldsymbol{\theta}')} \\
&= \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2} \left[C_{El}\cos(2\phi_{\boldsymbol{l}})^2 + C_{Bl}\sin(2\phi_{\boldsymbol{l}})^2\right] e^{il\theta\cos\phi_{\boldsymbol{l}}},
\end{aligned} \tag{28}$$

where we used the fact that E- and B-modes are not correlated. We also fixed the frame to have $\boldsymbol{\theta}+\boldsymbol{\theta}' \parallel \boldsymbol{e}_x$ yielding $\boldsymbol{l}\cdot\boldsymbol{\theta} = l\theta\cos\phi_{\boldsymbol{l}}$ and redefined $\boldsymbol{\theta}+\boldsymbol{\theta}' \to \boldsymbol{\theta}$. Similarly, for $U(\boldsymbol{\theta})$ we find

$$C_U(\boldsymbol{\theta}) = \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2} \left[C_{El}\sin(2\phi_{\boldsymbol{l}})^2 + C_{Bl}\cos(2\phi_{\boldsymbol{l}})^2\right] e^{il\theta\cos\phi_{\boldsymbol{l}}} \tag{29}$$

For the linear combinations of the two correlation functions we get

$$\begin{aligned}
C_Q(\boldsymbol{\theta}) + C_U(\boldsymbol{\theta}) &= \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2}(C_{El}+C_{Bl})e^{il\theta\cos\phi_{\boldsymbol{l}}} \\
&= \frac{1}{(2\pi)^2}\int_0^\infty l\,\mathrm{d}l \int_0^{2\pi} \mathrm{d}\phi_{\boldsymbol{l}}(C_{El}+C_{Bl})e^{il\theta\cos\phi_{\boldsymbol{l}}} \\
&= \frac{1}{2\pi}\int_0^\infty l\,\mathrm{d}l(C_{El}+C_{Bl})J_0(l\theta),
\end{aligned} \tag{30}$$

$$\begin{aligned}
C_Q(\boldsymbol{\theta}) - C_U(\boldsymbol{\theta}) &= \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2}(C_{El}-C_{Bl})\left[\cos(2\phi_{\boldsymbol{l}})^2 - \sin(2\phi_{\boldsymbol{l}})^2)\right] e^{il\theta\cos\phi_{\boldsymbol{l}}} \\
&= \frac{1}{(2\pi)^2}\int_0^\infty l\,\mathrm{d}l \int_0^{2\pi} \mathrm{d}\phi_{\boldsymbol{l}}(C_{El}-C_{Bl})e^{il\theta\cos\phi_{\boldsymbol{l}}}e^{-4i\phi_{\boldsymbol{l}}} \\
&= \frac{1}{2\pi}\int_0^\infty l\,\mathrm{d}l(C_{El}-C_{Bl})J_4(l\theta),
\end{aligned} \tag{31}$$

where we used the following formula for Bessel functions:

$$\frac{(-i)^n}{2\pi}\int_0^{2\pi} e^{iy\cos\phi}e^{-in\phi}\,\mathrm{d}\phi = J_n(y) \tag{32}$$

For the correlation between Q-modes and temperature we have

$$C_C(\boldsymbol{\theta}) \equiv \langle Q(\boldsymbol{\theta})T(\boldsymbol{\theta}')\rangle \tag{33}$$

$$= \int \frac{\mathrm{d}^2\boldsymbol{l}}{(2\pi)^2} C_{Cl}\cos(2\phi_{\boldsymbol{l}})e^{il\theta\cos\phi_{\boldsymbol{l}}} \tag{34}$$

Here we made use of Eq. (25) and used the fact that $\langle B(\boldsymbol{l})T(\boldsymbol{l}')\rangle = 0$. We can now express $\cos(2\phi_{\boldsymbol{l}}) = e^{2i\phi_{\boldsymbol{l}}}/2 + e^{-2i\phi_{\boldsymbol{l}}}/2$ and use the relation $J_n(y) = (-1)^n J_{-n}(y)$ for the Bessel functions to obtain

$$C_C(\boldsymbol{\theta}) = \frac{1}{2\pi}\int l\,\mathrm{d}l C_{Cl} J_2(l\theta). \tag{35}$$

7 Stokes parameters (quick question)

i)

$$(E_x, E_y) = \mathcal{E}(\cos\theta, \sin\theta)$$

$$\begin{aligned} Q =& E_x^2 - E_y^2 = \mathcal{E}^2(\cos^2\theta - \sin^2\theta) = \mathcal{E}^2\cos 2\theta \\ U =& 2E_xE_y = \mathcal{E}^2 \sin\theta\cos\theta = \mathcal{E}^2\sin 2\theta \end{aligned}$$

ii)

$$(E_x, E_y) = \mathcal{E}(1, \pm \mathrm{i})$$

$$\begin{aligned} Q =& |E_x|^2 - |E_y|^2 = 0 \\ U =& 2\Re(E_xE_y^*) = 0 \\ V =& 2\Im(E_xE_y^*) = \mp 1 \end{aligned}$$

8 Lensing Deflection

Let us consider a photon geodesic parametrized by the affine parameter λ. The photon momentum is then given by $P^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$. For the relation between path length and conformal time, we have $\mathrm{d}\eta = (1-2\Phi)\,\mathrm{d}l$ and thus

$$P^i = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^i}{\mathrm{d}l}\frac{\mathrm{d}l}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\lambda} = P^0(1+2\Phi)n^i. \tag{36}$$

Calculating the derivative wrt the affine parameter λ, we obtain

$$\frac{\mathrm{d}P^i}{\mathrm{d}\lambda} = \frac{\mathrm{d}n^i}{\mathrm{d}l}(1+4\Phi)(P^0)^2 + \frac{\mathrm{d}P^0}{\mathrm{d}\lambda}n^i(1+2\Phi) + 2n^i(P^0)^2\left(\frac{\partial\Phi}{\partial\eta} + n^j\frac{\partial\Phi}{\partial x^j}\right) \tag{37}$$

We can now employ the geodesic equation $\mathrm{d}P^\mu/\mathrm{d}\lambda = -\Gamma^\mu_{\alpha\beta}P^\alpha P^\beta$ and solve for the derivative of the photon direction. We have

$$\frac{\mathrm{d}n^i}{\mathrm{d}l} = -(1-4\Phi)\Gamma^i_{\mu\nu}\frac{P^\mu}{P^0}\frac{P^\nu}{P^0} + n^i(1-2\Phi)\Gamma^0_{\mu\nu}\frac{P^\mu}{P^0}\frac{P^\nu}{P^0} - 2n^i\partial_0\Phi - 2n^in^j\partial_j\Phi \tag{38}$$

From the definition of the metric

$$\mathrm{d}s^2 = a^2(\eta)\left\{-(1+2\Phi)\,\mathrm{d}\eta^2 + (1-2\Phi)\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\right\}. \tag{39}$$

we obtain for the Christoffel symbols

$$\begin{aligned} \Gamma^0_{00} =& \mathcal{H} + \partial_0\Phi \\ \Gamma^0_{j0} =& \partial_j\Phi \\ \Gamma^0_{jl} =& (1-4\Phi)\gamma_{jl}\mathcal{H} - \gamma_{jl}\partial_0\Phi \\ \Gamma^i_{00} =& \gamma^{ij}\partial_j\Phi \\ \Gamma^i_{j0} =& \delta^i_j\mathcal{H} - \partial_0\Phi\delta^i_j \\ \Gamma^i_{jl} =& \frac{1}{2}\gamma^{im}\left(\partial_j\gamma_{lm} + \partial_l\gamma_{jm} - \partial_m\gamma_{jl}\right) \\ & - \left(\delta^i_l\partial_j\Phi + \delta^i_j\partial_l\Phi - \gamma^{im}\gamma_{jl}\partial_m\Phi\right) \end{aligned}$$

Using **flat space** $\gamma_{ij} = \delta_{ij}$ and plugging into the above expression and using $P^i/P^0 = n^i(1+2\Phi)$ it follows

$$\begin{aligned} \frac{\mathrm{d}n^i}{\mathrm{d}l} =& -(1-4\Phi)\Gamma^i_{00} - (1-2\Phi)\Gamma^i_{j0}n^j - \Gamma^i_{jl}n^jn^l \\ & + (1-2\Phi)\Gamma^0_{00}n^i + \Gamma^0_{j0}n^in^j + (1+2\Phi)\Gamma^0_{jl}n^in^jn^l \\ & - 2n^i\partial_0\Phi - 2n^in^j\partial_j\Phi \end{aligned} \tag{40}$$

$$= 2n^in^j\partial_j\Phi - 2\partial_i\Phi \tag{41}$$

Writing the above result in vector form and using $\boldsymbol{n}\cdot\boldsymbol{n} = 1$, we finally obtain

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}l} = 2\boldsymbol{n}\;\boldsymbol{n}\cdot\boldsymbol{\nabla}\Phi - 2\boldsymbol{\nabla}\Phi = 2\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{\nabla}\Phi). \tag{42}$$

9 Lensing Bispectrum

Using the convolution theorem we have that in wavenumber spacce

$$\tilde{\Theta}(\boldsymbol{l}) = \Theta(\boldsymbol{l}) + \delta\Theta(\boldsymbol{l}) = \Theta(\boldsymbol{l}) + \int \frac{\mathrm{d}^2\boldsymbol{l}'}{(2\pi)^2}\boldsymbol{l}' \cdot (\boldsymbol{l} - \boldsymbol{l}')\phi(\boldsymbol{l}')\Theta(\boldsymbol{l} - \boldsymbol{l}') \tag{43}$$

$$\begin{aligned} \left\langle \tilde{\Theta}(\boldsymbol{l}_1)\tilde{\Theta}(\boldsymbol{l}_2)\tilde{\Theta}(\boldsymbol{l}_3) \right\rangle &= \langle \delta\Theta(\boldsymbol{l}_1)\Theta(\boldsymbol{l}_2)\Theta(\boldsymbol{l}_3)\rangle + 2 \text{ cyc.} \\ &= \int \frac{\mathrm{d}^2\boldsymbol{l}'}{(2\pi)^2}\boldsymbol{l}' \cdot (\boldsymbol{l}_1 - \boldsymbol{l}') \langle \phi(\boldsymbol{l}')\Theta(\boldsymbol{l}_1 - \boldsymbol{l}')\Theta(\boldsymbol{l}_2)\Theta(\boldsymbol{l}_3)\rangle + 2 \text{ cyc.} \\ &= \left[\boldsymbol{l}_2 \cdot \boldsymbol{l}_3 P^{\phi\Theta}_{l_2} P^{\Theta\Theta}_{l_3} + \boldsymbol{l}_2 \cdot \boldsymbol{l}_3 P^{\phi\Theta}_{l_3} P^{\Theta\Theta}_{l_2} + 2 \text{ cyc.}\right] (2\pi)^2 \delta^{(\mathrm{D})}(\boldsymbol{l}_1 + \boldsymbol{l}_2 + \boldsymbol{l}_3) \\ &\approx 0 \end{aligned} \tag{44}$$

$P^{\phi\Theta}_l$, vanishes for the Doppler and Sachs-Wolfe contributions which are created at recombination but has some large scale contributions from the integrated Sachs-Wolfe effect.

$$\begin{aligned} \left\langle \phi(\boldsymbol{l}_1)\tilde{\Theta}(\boldsymbol{l}_2)\tilde{\Theta}(\boldsymbol{l}_3) \right\rangle &= \langle \phi(\boldsymbol{l}_1)\delta\Theta(\boldsymbol{l}_2)\Theta(\boldsymbol{l}_3)\rangle + \langle \phi(\boldsymbol{l}_1)\Theta(\boldsymbol{l}_2)\delta\Theta(\boldsymbol{l}_3)\rangle \\ &= \int \frac{\mathrm{d}^2\boldsymbol{l}'}{(2\pi)^2}\boldsymbol{l}' \cdot (\boldsymbol{l}_2 - \boldsymbol{l}') \langle \phi(\boldsymbol{l}_1)\phi(\boldsymbol{l}')\Theta(\boldsymbol{l}_2 - \boldsymbol{l}')\Theta(\boldsymbol{l}_3)\rangle + \dots \\ &= \left[\boldsymbol{l}_1 \cdot \boldsymbol{l}_3 P^{\phi\phi}_{l_1} P^{\Theta\Theta}_{l_3} + \boldsymbol{l}_1 \cdot \boldsymbol{l}_2 P^{\phi\phi}_{l_1} P^{\Theta\Theta}_{l_2}\right] (2\pi)^2 \delta^{(\mathrm{D})}(\boldsymbol{l}_1 + \boldsymbol{l}_2 + \boldsymbol{l}_3) \end{aligned} \tag{45}$$