Localised coherent solutions of the DSI and DSII equations - a numerical study

M. M^{c}Connell, A.S. Fokas* and B. Pelloni [1]

Department of Mathematics
University of Reading
Reading, RG6 6AX, UK.

* DAMTP
University of Cambridge
Cambridge, CB3 0WA, UK.

November 17, 2003

Abstract

We implement several numerical methods for computing the solution of the Cauchy problems for two well-known integrable equations in two space dimensions, known as the Davey-Stewartson I and II equations. Most of our computations confirm earlier theoretical predictions, including a recent result implying the instablity of a class of special solutions of the DSII equation. These solutions, known as lumps, had been previously considered natural candidates as coherent asymptotic structures of the evolution. In contrast to all other cases, when the initial condition has sufficiently large energy no global existence result is known for the DSII equation, in the focusing regime. Our preliminary computations indicate in this case the possibility that the solution blows up, hence that no global existence result can hold.

1 Introduction

The Davey-Stewartson I and II equations (denoted DSI and DSII) are two well-known examples of integrable equations in two space dimensions, which arise as higher dimensional generalisations of the nonlinear Schrödinger (NLS) equation, as well as from physical considerations. Indeed, they appear in many applications, for example in the description of gravity-capillarity surface wave packets in the limit of shallow water. The following results for these equations have been established by several authors:
(i) The Cauchy problem with decaying initial conditions for the DSI equation admits a global solution. This solution disperses away after a long time. The analogous problem supplemented with nontrivial boundary conditions at infinity also admits a global solution, but in this case this solution decomposes into a number of *dromions* after a long time [18].

[1] Corresponding author. **E-mail:** b.pelloni@rdg.ac.uk

(ii) The Cauchy problem with decaying initial conditions for the *defocusing* DSII equation admits a global solution, which disperses away after a long time[19].
(iii) The Cauchy problem with decaying initial conditions for the *focusing* DSII equation admits a global solution provided that a certain norm of the initial data is small [19]. At present, it is *not* known if there exists a global solution without this small norm assumption. In general, the DSII equation admits certain localised solutions, known as *lumps*, which would seem natural candidates for coeherent asymptotic structures. However, it has recently been shown by Pelinovski and Sulem [26] that these solution are spectrally unstable. Thus, contrary to initial expectations, lumps do not appear to be relevant in the long time asymptotic description of the solution.

In this paper, we use a spectral code, developed using exclusively routines within the MATLAB symbolic package, to compute numerically the solution of the Cauchy problem for both the DSI and DSII equations. The emphasis is not on obtaining a very accurate code, but rather on being able to obtain a simple code that nevertheless captures all the relevant properties of the solution. Indeed, using our code, we verify (i) and (ii) numerically. We note that these verifications have been done by a number of authors before [5, 24, 25, 29].
With respect to dromions, we also show that although localised stable coherent structures emerge from any initial disturbance for the DSI equation, these structures do not preserve their form upon interaction. This is consistent with the analytical description of the inelastic interaction of dromions.
The novel indication emerging from our numerical simulations concerns (iii). Our computations indicate that, if one takes as initial condition a 1-lump solution of the DSII equation, evaluated at $t = 0$, the corresponding solution is extremely sensitive to the relevant parameters. This observation is consistent with the analytical result concerning the instability of lumps. We show that for small initial data, the solution disperses away. However, for sufficiently large initial data, our computations indicate the possibility of blow-up. The evidence given here is not conclusive. In particular, we see the numerical scheme reaching a certain height, then stabilising before oscillating around the peak reached. This could be real PDE behaviour (see e.g. [23]). However it has been suggested [10] that this is precisely a phenomenon that accompanies discrete approximations of PDEs that exhibit blowup. To investigate this claim accurately, a more sophisticated adaptive code is probably called for, going beyond the scope of the present work.

2 The DS equations

The Davey-Stewartson (DS) equations for the function $q = q(x, y, t)$ are given by

$$\begin{aligned} &iq_t + \frac{1}{2}\sigma^2(q_{xx} + \sigma^2 q_{yy}) + \lambda|q|^2 q - \Phi_x q = 0, \\ &\Phi_{xx} - \sigma^2\Phi_{yy} - 2\lambda(|q|^2)_x = 0, \qquad \lambda = \pm 1, \;\; \sigma^2 = \pm 1. \end{aligned} \tag{2.1}$$

The case $\sigma = 1$ is called the DSI equation, while $\sigma = i$ is the DSII equation. The parameter λ characterises the focusing or defocusing case.
Davey and Stewartson first derived their model in the context of water waves, from purely physical considerations. In this context, $q(x, y, t)$ is the amplitude of a surface wave packet, while $\Phi(x, y)$ is the velocity potential of the mean flow interacting with the surface wave.
These equations can also be considered as the integrable generalisation to two space dimensions of the nonlinear Schrödinger (NLS) equation, one of the fundamental one-dimensional integrable equations. The NLS equation is

$$iq_t + q_x + 2\lambda|q|^2 q = 0, \quad \lambda = \pm 1 \tag{2.2}$$

where $\lambda = 1$ corresponds to the focusing case. In this case, there exist solutions that do not decay in time. These special solutions, which take the form of localised, stable peaks, are called *solitons* if they travel, and *breathers* if they are stationary. Their importance stems from the fact that they are the coherent asymptotic structures emerging from the evolution, under the focusing NLS, of *any* initially localised waveform.
We emphasise that equations (2.1) are not the only possible integrable generalisation of NLS to two dimensions; however, these particular equations were derived as models of physical phenomena in their own right, and hence have particular relevance in mathematical physics. In addition, for the DSI and focusing DSII equations, special localised solutions can be explicitly found. They are called *dromions* and *lumps*, respectively. Pursuing the analogy with the localised solutions of the NLS equation, it is natural to investigate whether these structure are stable, and if they characterise the asymptotic behaviour of all solutions.
In general, for such integrable equations in two dimensions, with given decaying initial data, the Cauchy problem posed on the infinite plane $(x, y) \in \mathbb{R} \times \mathbb{R}$ can be expressed through the solution of a Fredholm linear integral equation of the second kind, defined in terms of the initial data $q(x, y, 0)$.
For the case of the DSI and of the defocusing DSII equations, it has been proved by Fokas and Sung [19], Beals and Coifman [2], and Fokas and Santini [18] that the solution exists globally, and can be expressed through a Fredholm equation uniquely defined in terms of $q(x, y, 0)$, for any decaying smooth $q(x, y, 0)$. For the case of focusing DSII, this has been proven so far only if a certain norm of $q(x, y, 0)$ is small enough [19]. Hence in the latter case, it is not possible to derive a global existence result for the solution, at least by the same approach. To our knowledge, the question of the global existence of the solution for focusing DSII is still open.
The structural instability of one-lump solutions of the DSII equation, which had been observed earlier [22], has been recently proved by Pelinovski and Sulem [26]. Namely, these authors first associate the lumps with eigenvalues embedded into a two-dimensional essential spectrum for the associated differential system (the spatial part of the Lax pair, called the Dirac system). They then show that these eigenvalues are structurally unstable under local perturbations of

the initial condition. This implies that, for small data at least, localized initial conditions decay into radiation unless these data happen to be initial conditions correspoding exactly to lumps.
The instability of the lumps, which we clearly observe numerically, means that no other initial condition will eventualy resolve into lumps. Hence the lumps cannot play the role of generic, coherent asymptotic structures: they are entirely special, non-generic solutions.
If the amplitude is sufficiently small, a generic initial waveform will radiate away. However, if the small data assumption is not satisfied, we cannot predict that there will be only radiation. Even starting with an initial lump, the numerical disturbance will be sufficient to destroy the balance, and then the initial data will not be stable. What we observe is that, if the amplitude of the initial localised wave (which we take to be a two-dimensional Gaussian) is sufficiently large, then the amplitude of the solution quickly and steeply grows. Hence we conjecture that, for large data, the focusing DSII exhibits blow-up.

3 The numerical schemes

The numerical schemes used in this study are those developed by White and Weidemann [29], by Nishinari and Yajima [24], and by the authors, based on standard techniques. The method developed by the authors uses an integrating factor to remove the linear term, then a fourth-order Runge-Kutta time stepping to solve the resulting non-linear equation. White and Weideman use a split-step Fourier method, whereas Nishinari and Yajima use a pseudo-spectral method with a Runge-Kutta or Burilsh-Store time stepping. Here we only use Runge-Kutta time stepping as a matter of convenience, although it is generally accepted that Burilsh-Store stepping is more accurate.
The several numerical approaches alllow us to use two methods for solving the DSII equation, and two methods for solving the DSI equation. Obtaining similar time evolution by different methods yields greater confidence in the results.
We now describe briefly the methods used for the computations presented.

Split-step Fourier method for the DSII equation

We start by considering the splitting of the DSII equation into a linear and a nonlinear part, as described in White and Weideman [29]. This yields

$$\mathrm{L}: iq_t - \frac{1}{2}(q_{xx} - q_{yy}) \quad = \quad 0, \tag{3.3}$$

$$\begin{aligned} iq_t + \sigma|q|^2 q - q\Phi_x &= 0, && (3.4)\\ \mathrm{N}: \Phi_{xx} + \Phi_{yy} - 2\lambda(|q|^2)_x &= 0. && (3.5)\end{aligned}$$

To solve the linear part, we approximate the discrete function q by the two dimensional discrete finite Fourier series

$$q_{jk} = \sum_m \sum_n \widehat{q}_{mn} e^{i(\mu_m x_j + \nu_n y_k)}, \tag{3.6}$$

where $q_{jk} \approx q(x_j, y_k)$, x_j and y_k are a discretisation of the domain $[-P, -P] \times [P, P]$, P some large positive constant, into $j = -M/2, \ldots 0, \ldots M/2 - 1$ and $k = -N/2, \ldots 0, \ldots N$ points, μ_m, ν_n are the corresponding wavenumbers.
We can then transform equation (3.3) to Fourier space, and write it in the form

$$i \frac{d\widehat{q}_{mn}}{dt} + \frac{1}{2}(\mu_m^2 - \nu_n^2)\widehat{q}_{mn} = 0. \tag{3.7}$$

This is a separable ODE with solution

$$\widehat{q}_{mn}(t) = \widehat{q}_{mn}(0) e^{(\frac{i}{2}(\mu_m^2 - \nu_n^2)t)}. \tag{3.8}$$

Therefore we obtain the incremental solution

$$\widehat{q}_{mn}(t + \Delta t) = \widehat{q}_{mn}(t) e^{(\frac{i}{2}(\mu_m^2 - \nu_n^2)\Delta t)}, \tag{3.9}$$

and an inverse Fourier transform yields the solution $q_{jk}(t + \Delta t)$.
To solve the nonlinear part, we note that equation (3.5) is just the Laplace equation. It is well known that this equation can be efficiently solved using spectral methods. In this particular case, the Fourier representations

$$|q_{jk}|^2 = \sum_m \sum_n \widehat{|q_{mn}|^2} e^{i(\mu_m x_j + \nu_n y_k)}, \tag{3.10}$$

$$\Phi_{jk} = \sum_m \sum_n \widehat{\Phi}_{mn} e^{i(\mu_m x_j + \nu_n y_k)}, \tag{3.11}$$

are used to solve for $\Phi(x, y, t)$ to transform (3.5) to Fourier space:

$$(\mu_m^2 + \nu_n^2)\widehat{\Phi}_{mn} + 2\lambda i \mu_m \widehat{|q_{mn}|^2} = 0. \tag{3.12}$$

Equation (3.12) is rearranged for $\widehat{\Phi}_{mn}$ and substituted into the following expression for Φ_x, which is then used in (3.4):

$$(\Phi_x)_{jk} = \sum_{m \neq 0} \sum_{n \neq 0} i \mu_m \widehat{\Phi}_{mn} e^{i(\mu_m x_j + \nu_n y_k)}. \tag{3.13}$$

To solve equation (3.4) we follow the suggestion in [1] and use the strong coupling limit of the DS equations. This method yields the solution

$$q_{jk}(t + \Delta t) = q_{jk}(t) e^{(i(\lambda |q_{jk}(t)|^2 - (\Phi_x)_{jk}(t))\Delta t)}. \tag{3.14}$$

Now equations (3.14), (3.13) and (3.9) can be used to solve for $q(x, y, t)$.

Integrating factor with Runge-Kutta method for the DSII equation

In this case we solve

$$iq_t - \frac{1}{2}(q_{xx} - q_{yy}) + (\lambda|q|^2 - \Phi_x)q = 0, \tag{3.15}$$

where Φ_x is found using (3.13). The Fourier transform of equation (3.15) yields

$$\widehat{q}_t + \frac{1}{2}i(\mu_m^2 - \nu_n^2)\widehat{q} - \mathcal{F}\left[i\left(\lambda|q|^2 - \Phi_x\right)q\right] = 0. \tag{3.16}$$

Multiplying this by the integrating factor $e^{\frac{i}{2}(\mu_m^2-\nu_n^2)t}$ we obtain

$$\widehat{q}_t e^{\frac{i}{2}(\mu_m^2-\nu_n^2)t} + \frac{1}{2}i(\nu_n^2 - \mu_m^2)\widehat{q}e^{\frac{i}{2}(\mu_m^2-\nu_n^2)t} - \mathcal{F}\left[i\left(\lambda|q|^2 - \Phi_x\right)q\right]e^{\frac{i}{2}(\mu_m^2-\nu_n^2)t} = 0. \tag{3.17}$$

Defining $\widehat{Q} = e^{\frac{i}{2}(\mu_m^2-\nu_n^2)t}\widehat{q}$ and differentiating with respect to time finally leads to

$$\widehat{q}_t = \widehat{Q}_t e^{-\frac{i}{2}(\mu_m^2-\nu_n^2)} - \frac{i}{2}(\mu_m^2 - \nu_n^2)\widehat{q}.$$

Substituting in (3.17), using $u = \mathcal{F}^{-1}\left[\widehat{Q}e^{-\frac{i}{2}(\mu_m^2-\nu_n^2)}\right]$, we find after some manipulations

$$\widehat{Q}_t = \mathcal{F}\left[i\left(\lambda\left|\mathcal{F}^{-1}\left[\widehat{Q}e^{-\frac{i}{2}(\mu_m^2-\nu_n^2)}\right]\right|^2 - \Phi_x\right)\mathcal{F}^{-1}\left[\widehat{Q}e^{-\frac{i}{2}(\mu_m^2-\nu_n^2)}\right]\right]e^{\frac{i}{2}(\mu_m^2-\nu_n^2)t}, \tag{3.18}$$

which is in a suitable form for solving using the Runge-Kutta algorithm.

Split step Fourier method for the DSI equation

On account of its hyperbolicity, the DSI equation is more difficult to solve than the DSII equation. In addition, non-trivial boundary conditions on Φ need to be specified and imposed at infinity in order to generate non-decaying solutions. The change of variables $\xi = x + y$, $\eta = x - y$, puts the DSI system in the form

$$\begin{aligned} iq_t &+ (q_{\xi\xi} + q_{\eta\eta}) - uV = 0, && (3.19)\\ V &= \frac{1}{2}\lambda\left(\int_{-\infty}^{\xi}\left(|q|^2\right)_\eta d\xi + \int_{-\infty}^{\xi}\left(|q|^2\right)_\xi d\eta\right) + \\ &\quad \Phi_\xi(\xi, -\infty, t) + \Phi_\eta(-\infty, \eta, t). && (3.20)\end{aligned}$$

Equation (3.19) can now be split into its linear and nonlinear parts

$$\begin{aligned} \mathrm{L} : iq_t + (q_{\xi\xi} + q_{\eta\eta}) &= 0, && (3.21)\\ \mathrm{N} : iq_t - uV &= 0, && (3.22)\end{aligned}$$

and solved using the split-step method, advancing the linear and nonlinear parts separately in time.
The linear part is solved by transforming it to Fourier space and advancing in time. This yields the incremental solution

$$\widehat{q}_{mn}(t+\Delta t) = \widehat{q}_{mn}(t) e^{\frac{i}{2}(\mu_m^2+\nu_n^2)\Delta t}, \tag{3.23}$$

and q_{jk} is computed by an inverse transform.
The nonlinear part (3.22) is solved by using the strong coupling limit [18, 29]. Using the approximation $\int_t^{t+\Delta t} V dt = V(t)\Delta t + O(\Delta t^2)$, this yields the solution

$$q(t+\Delta t) = q(t) e^{-V(t)\Delta t}. \tag{3.24}$$

The first order split-step method is progressed by firstly using (3.24) which provides the initial data for the linear problem which is solved using (3.23). The second order method requires half a nonlinear step using (3.24) followed by a full linear step with (3.23) and finally a half nonlinear time step using (3.24) again.
The difficult part of this computation is the numerical evaluation of the function V. In practice, we consider the computational domain $[-P, P]\times[-P, P]$, for some large positive P, as an approximation to $\mathbb{R}\times\mathbb{R}$. Thus the boundary conditions are assumed to be supplied at ξ, $\eta = -P$.

Nishinari and Yajima 's method for the DSI Equation

Nishinari and Yajima [24] write the DSI equation in the form

$$iq_t + (q_{\xi\xi} + q_{\eta\eta}) + (U+V)q = 0, \tag{3.25}$$

$$U_\eta = \left(|q|^2\right)_\xi, \tag{3.26}$$

$$V_\xi = \left(|q|^2\right)_\eta, \tag{3.27}$$

where $U = \Phi_x - |q|^2$ and $V = \Phi_y - |q|^2$ (note that V here is different to the V given in the previous section, compare equations (3.19) and (3.25)).
For the case of the one-dromion solution, the cross sections of U and V at the boundary $\xi, \eta = -\infty$ are given explicitly in [24] as

$$U|_{\eta=-\infty} = \frac{8k_r^2 \exp(\zeta_1+\zeta_1^*)}{\left[1+\exp(\zeta_1+\zeta_1^*)\right]^2}, \tag{3.28}$$

$$V|_{\xi=-\infty} = \frac{8l_r^2 \exp(\zeta_2+\zeta_2^*)}{\left[1+\exp(\zeta_2+\zeta_2^*)\right]^2}. \tag{3.29}$$

where ζ_1 and ζ_2 depend on constants γ, k_r, k_i, l_r and l_i. These constants are all real, and constitute essentially free parameters in the one-dromion solution. The constant γ determines an amplitude, k_r the width of the pulse in the ξ

direction, and l_r that in the η direction. The quantities k_i and l_i are the ξ and η components of velocity, respectively.
The numerical method involves solving (3.26) and (3.27) by a fourth order Runge-Kutta method, using interpolation to calculate the solution at fractional grid points. Then equation (3.25) is solved by first computing the pseudo-spectral space derivative and then integrating in time with both Burilsh-Store and fourth order Runge-Kutta schemes.
We encountered problems in approximating the $U+V$ term numerically. Nishinari and Yajima use a Runge-Kutta method to step forward from $-P$ using interpolation to calculate at fractional grid points, so the grid resolution limits the accuracy. We note that the limitation encountered here is due to our implementation of the method and is not inherent to their scheme.

Comparison of the numerical schemes

DSII: The split-step method performs better when increasing the number of time steps. This is to be attributed to the superior accuracy of spectral methods over Runge-Kutta time stepping. Using Burilsh-Store time stepping may yield an improvement, but because of its faster running time, the majority of results in this paper use the split-step method.

DSI: The numerical evaluation of the $U+V$ term is difficult. Howeverm, grid and time step refinement reduce the global error, and we also used a cubic filter to minimise reflections and high frequencies, thus eliminating most of the numerical noise from the results. Replacing the $U+V$ calculation with the analytic expression given by Nishinari and Yajima for the rational soliton, leads to very clean results in both the White and Weideman and Nishinari and Yajima models that closely match the analytical results. Improvements in our calculation of $U+V$ are not investigated here due to our present emphasis on the DSII equation, for which the evaluation of $U+V$ is not required.

4 Numerical results for DSI

To test our methods, we reproduce the well known results for the DSI equation. In figure (1) an initial dromion waveform disperses when the boundary values are zero. In figure (2) the initial Gaussian waveform sheds then breathes, reproducing the result in [25]. Figure (3) shows the interaction of two dromions, rerpoducing the result in [24]. The influence of the boundary conditions, which come through as tracks, is to preferentially attract the field q. The initial wave can be seen to evolve along the tracks. As expected, where tracks cross there is a higher amplitude due to the superposition of the tracks.

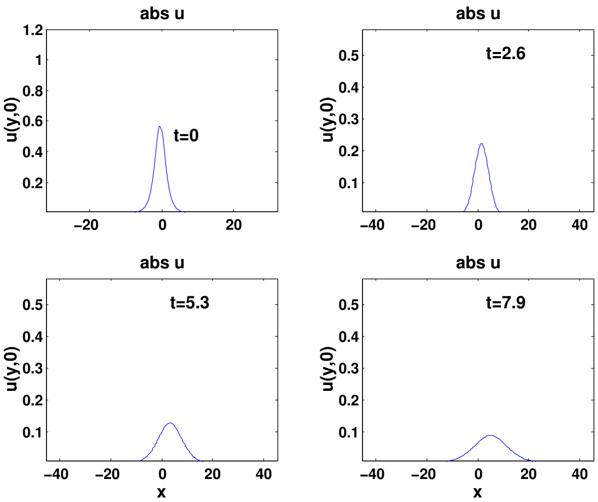

Figure 1: DSI equation. Any initial wave form disperses when the boundary conditions are zero. (Nishinari and Yajima method).

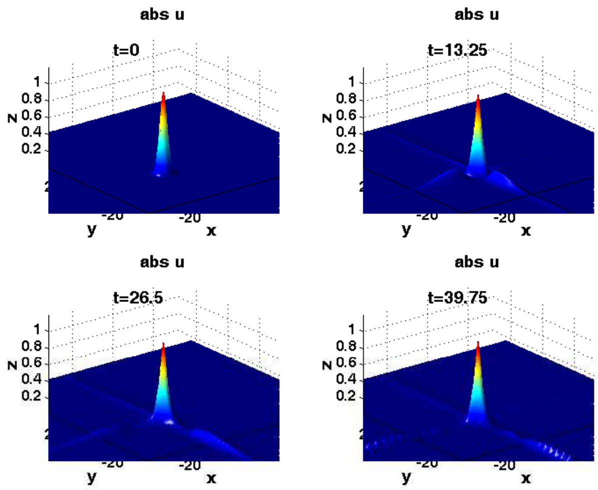

Figure 2: DSI equation. Any initial wave form sheds along the tracks to take on the form of a dromion when the boundary conditions are soliton solutions of the KdV equation. (Nishinari and Yajima method).

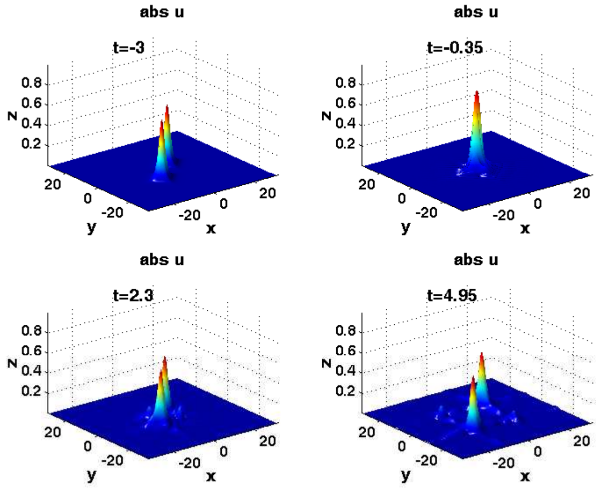

Figure 3: DSI equation.The interaction of two dromions is inelastic, and the initial forms are not preserved after the collision. (Nishinari and Yajima method).

5 Numerical results for DSII

The behaviour of the solutions of the DSII equation is well understood for all cases except, in the focusing case, ($\lambda = -1$), if the initial localised wave has sufficiently large energy. To test and validate the numerical models used, we first present numerical results for the cases that are well understood theoretically. We then turn to the results for large energy initial conditions.

Figure (4) shows an example of arbitrary initial data decaying away to zero when $\lambda = 1$ (defocusing case). In this case the initial data is the lump solution

$$q(z, \bar{z}, t) = \frac{2\beta \exp(i(p^2 + \bar{p}^2)t + pz - \bar{p}\bar{z})}{|z + \alpha + 2ipt|^2 + |\beta|^2}, \tag{5.30}$$

where α, β and p are complex parameters and $z = x + iy$. Here $\alpha = 2 + 0i$, $\beta = 2 + 0i$ and $p = 1 + 0i$.
When $\lambda = -1$ (focusing case) the same initial data, an exact lump, is preserved for all time, figure (5). In fact, any choice of parameter leads to a lump that is preserved for all time, irrespective of amplitude.
When the initial amplitude is sufficiently small, e.g. less than one, all initial data that are not exactly lumps, rational solitons, or other known asymptotic solutions, decay to zero, figure (6). These results are in agreement with the theoretical analysis, and indicate that our codes are computing the correct approximate solution.

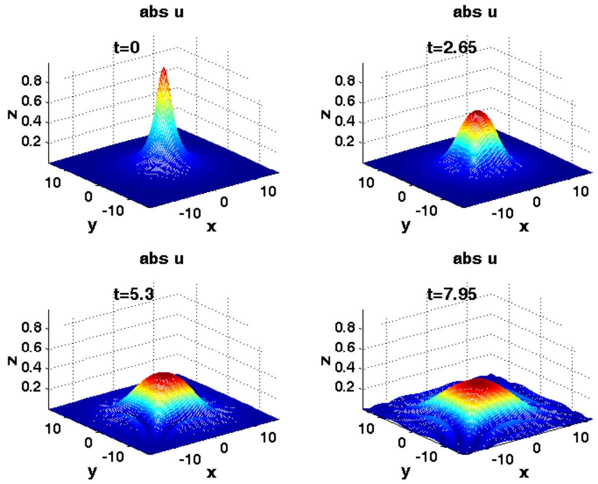

Figure 4: DSII equation. For the defocusing case $\lambda = 1$, all initial wave forms decay. Split-step method.

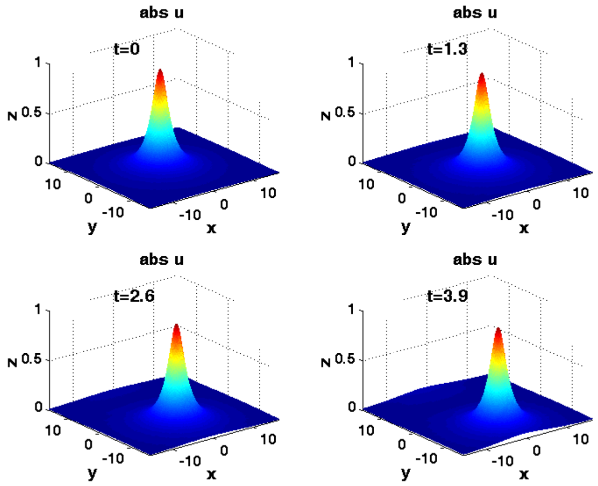

Figure 5: DSII equation. For the focusing case $\lambda = -1$, lump initial wave forms propogate without decay. Split-step method.

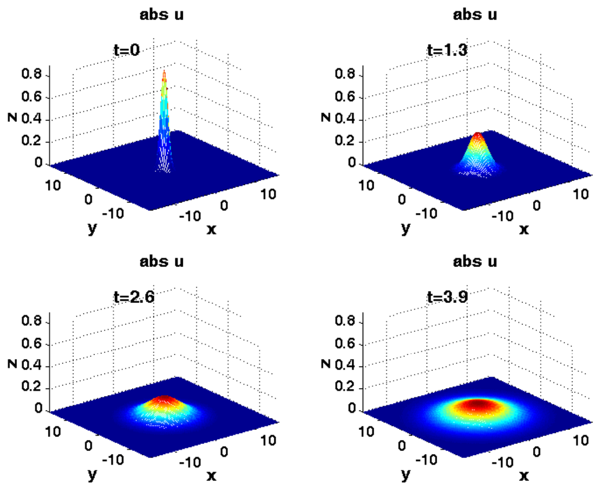

Figure 6: DSII equation. When the initial amplitude is less than one all incoherent data decay. The initial data here is a Gaussian. Split step method.

The DSII focusing evolution of large initial conditions

Coherent structures such as lumps and rational solitons,

$$q(x,y,t) = \frac{2e^{2i(y-t)}}{1+(x+1)^2+(y-2t)^2}, \tag{5.31}$$

are preserved without change, regardless of their amplitude. In addition, when lumps and rational solitons interact, the maximum amplitude, $\max(|q|^2)$, rises significantly during the collision, then reduces back to the initial amplitude as the lumps or rational solitons continue to travel unaffected by the interaction. This behaviour is shown in figures (7) and (8), where the result of the numerical collision of eight lumps is illustrated. Figures (9) and (10) show a soliton with amplitude $A = 5$ progressing without decay or growth.

In constrast to this behaviour, the modelling of large energy initial conditions such as Gaussians leads to growth in $|q|^2$ and the solution appears to blow up, figure (11). It is this behaviour that is interesting.

Fokas and Sung [19] proved that the solution exists globally if the amplitude of the initial waveform is small enough. In this case, blow-up cannot occur. However, no global existence results are available to our knowledge without this small norm assumption.

We tested these results for Gaussian initial conditions, equation (5.32), and also for initial conditions *close to* exact lumps, equation (5.33), and *close to* exact rational solitons, equation (5.34):

$$q(x,y,0) = Ae^{-\mu(x^2+y^2)}, \tag{5.32}$$

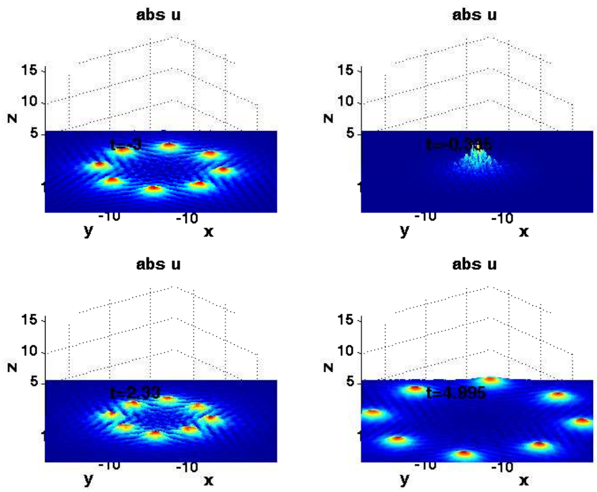

Figure 7: DSII equation. Tip height (L^∞ norm) as eight lumps move together, interact and continue as if there had been no interaction. Split-step method.

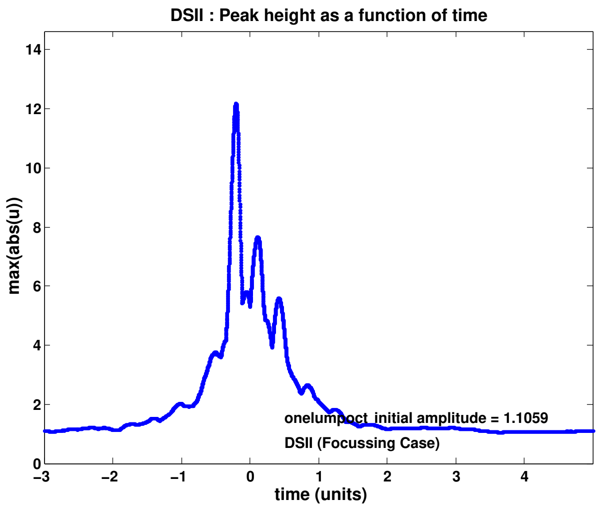

Figure 8: DSII equation. Amplitude (L^∞ norm) as eight lumps move together, interact and continue as if there had been no interaction. Split-step method.

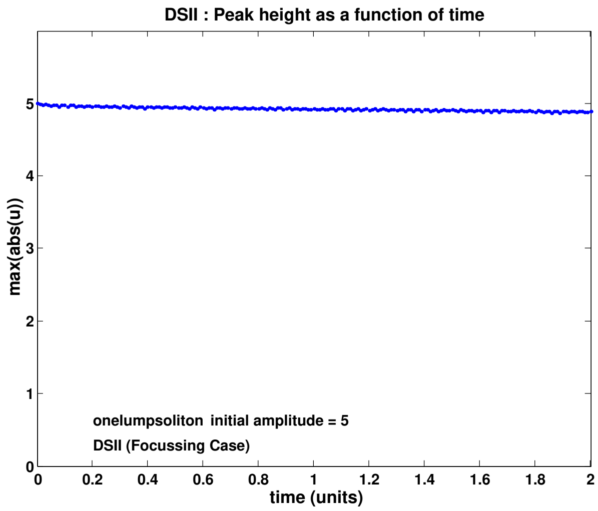

Figure 9: DSII equation. When the amplitude of a coherent structures such as lumps is large the coherent structure propogates without change. Split-step method.

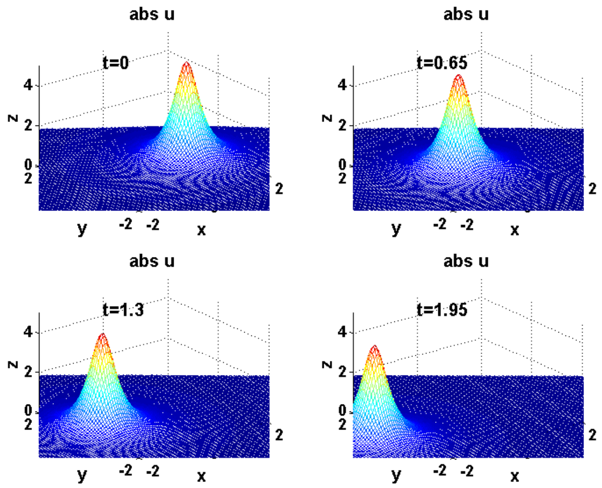

Figure 10: DSII equation. The maximum amplitude as a function of time for a lump with amplitude $A = 5$. The coherent structure propogates without change. Split-step method.

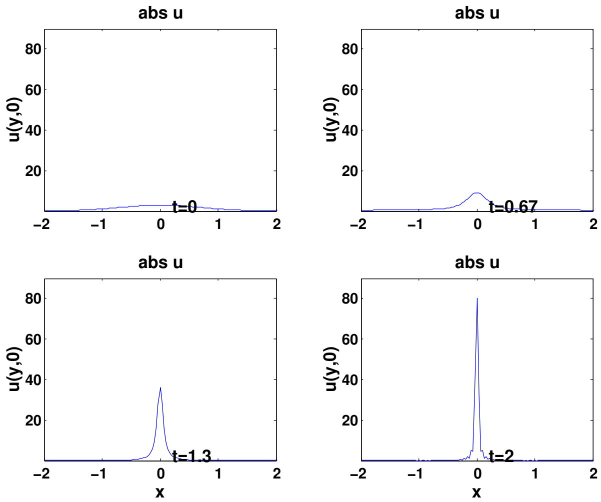

Figure 11: DSII equation. When the amplitude is sufficiently large, all initial data other than exact asymptotic solutions, appear to blow up. Here the initial condition is a Gaussian wave form with $A = 3$. Split-step method.

$$q(z, \bar{z}, t) = A \frac{2\beta \exp(i(p^2 + \bar{p}^2)t + pz - \bar{p}\bar{z})}{|z + \alpha + 2ipt|^2 + |\beta|^2}, \tag{5.33}$$

$$q(x, y, t) = A \frac{2e^{2i(y-t)}}{1 + (x+1)^2 + (y-2t)^2}. \tag{5.34}$$

In our experiments, $A = 3$, $\mu = 1$, and the other parameters are as before, Teh results are shown in figure (11).

The apparent blow-up process stops at an amplitude dependent on the value of δx and δt. Figure (12) shows the progression of the maximum amplitude for three different values of δx and δt, with $\delta x/\delta t$ fixed.

We make a few observations based on these results.

1. The height reached is a function of δx and δt.

2. The computational grid limits the possibility of capturing the growth of the peak of the wave, particularly when the peak becomes tall and narrow.

3. As the time of the simulation is increased, the tip height begins to oscillate, figure (13)

4. The blow up is not universal. There exist special initial data that lead to large amplitudes which do not blow up. Indeed, the interaction of lumps progresses as expected - which gives us confidence in the numerics to model large amplitudes, figures (8) and (9).

5. It is the authors' opinion that the blow up is real, while the plateau and oscillations are pathologies due to the numerical method stepping over the blow up,

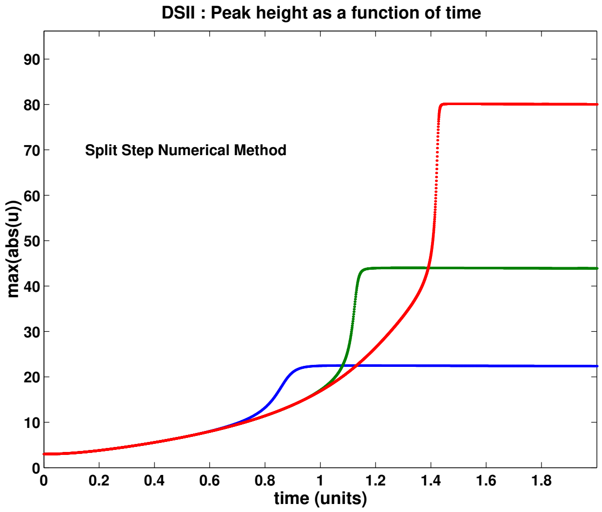

Figure 12: DSII equation. The onset of blow up with a Gaussian initial data of amplitude $A = 3$, for three different δx and δt, $\delta x/\delta t$ fixed: $\delta x = 0.125$, $\delta t = 0.002$; $\delta x = 0.0625$, $\delta t = 0.001$; $\delta x = 0.03125$, $\delta t = 0.0005$. The largest amplitude is for the $\delta x = 0.03125$, $\delta t = 0.0005$ case. Split-step method.

a known phenomenon in numerical modelling. The amplitude of the numerical solution does not always stabilise; travelling waveforms rise sharply, apperently blowing up, then shrink away rapidly suggesting that the fixed time stepping numerics have stepped over the fine structure of the blow up.

6 Conclusion

We have illustrated several numerical techniques for the DSI and DSII equations and produced the results predicted by the known theory using these models. The unexpected result of what appears to be blow-up for the DSII focusing equation has been investigated. Given that global existence for this equation has only been proven for small norm, it is not known whether this blow up is an artifact of the numerical method, or a genuine characteristic of the DSII equation. We believe it to be genuine because of the recent result by [26] which showed that lumps are not asymptotic attractors of the DSII equation, as previously believed. Our numerical models capture higher amplitudes in the blow up as the grid spacing and time stepping is refined, and the models produce the expected results, see figures (8) and (9), for certain large amplitude initial conditions corresponding to known exact solutions.

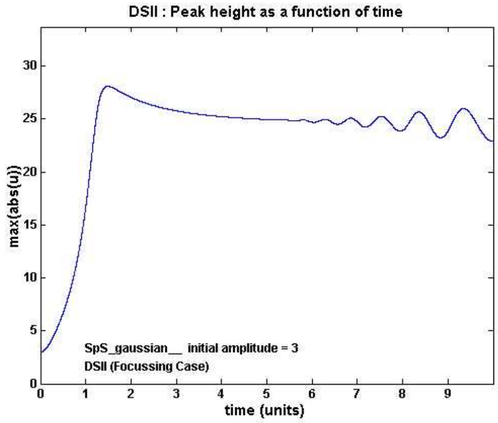

Figure 13: DSII equation. When the time of the simulation is increased the solution oscillates. Split-step method.

7 Acknowledgements

M.M^{c}C. wishes to thank the Department of Mathematics of the University of Reading for its support during the course of this research.
B.P. gratefully acknowledges the support of the Nuffield foundation grant NAL/00608/GR.

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, 1991.

[2] R. Beals and R.R. Coifman, *Inv. Prb.*, **5** (1989), 87.

[3] D. ben-Avraham and A.S. Fokas. The solution of the modified Helmholtz equation in a wedge and an application to diffusion-limited coalescence. *Phys. Lett. A*, 263, 1999.

[4] D.J. Benney and G.L. Roskes, *Wave Instabilities*, Stud. Appl. Math., **48** (1969), 377-385.

[5] C. Besse and C.H. Bruneau, Numerical study of elliptic-hyperbolic Davey-Stewartso system: dromions simulation adn blow-up, *Math. Mod. Meth. Appl. Sci.*, **8**(8) (1998), 1363-1386.

[6] M. Boiti, J.J.-P. Leon, L. Martina and F. Pempinelli, *Scattering of Localized Solitons in the Plane*, Phys. Lett. A, **132** (1988), 432-439.

[7] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, 1988

[8] A. Davey and K. Stewartson, *On Three-Dimensional Packets of Surface Waves*, Proc. Royal Soc. London Ser. A, **338** (1974), 101-110.

[9] P. Deift and X. Zhou, *A Steepest Descent Method for Oscillatory Riemann-Hilbert Problems - Asymptotics for the MKdV Equation*, Ann. of Math., **137** (1993), 295-368.

[10] G.Fibich and B.Ilan, Discretization effects in the nonlinear Schroedinger equation,*Appl. Num. Math* **44**, (2003), 63-75.

[11] A.S. Fokas, On the integrability of linear and nonlinear PDEs. *J. Math. Phys.*, 41:4188, 2000.

[12] A.S. Fokas and A.R. Its. The generalised Ehrenpreis-Palamodov representation and the long time asymptotics of evolution equations. preprint, 2000.

[13] A.S. Fokas and A.R. Its. The Nonlinear Schrödinger equation on a finite interval. preprint, 2000.

[14] A.S. Fokas, A.R. Its, and L.Y. Sung. The Nonlinear Schrödinger equation on the half line. preprint, 2000.

[15] A.S. Fokas, D.E. Pelinovsky and C. Sulem, *The Interaction of Lumps with a Line Soliton for the DSII Equation*, *Physica D*,152/153 (2001), 189–198.

[16] A.S. Fokas and B. Pelloni. A moving boundary value problems for linear evolution equations. *Phys. Rev. Lett.*, 84:4785, 2000.

[17] A.S. Fokas and B. Pelloni. Two-point boundary value problems for linear evolution equations. Math. Proc. Camb. Phil. Soc. (in press), 2000.

[18] A.S. Fokas and P.M. Santini, *Dromions and a Boundary Value Problem for the Davey-Stewartson I Equation*, Physica D, **44** (1990), 99-130.

[19] A.S. Fokas and and L.Y. Sung, *On the Solvability of the N-Wave, the Davey-Stewartson and the Kadomtsev-Petviashvili Equation*, Inverse Problems, **8** (1992), 673-708.

[20] A.S. Fokas and L.Y. Sung. Initial boundary value problems for linear dispersive evolution equations on the half-line. preprint, 1999.

[21] A.S. Fokas, D. Tsoubelis, and L.Y. Sung. The inverse spectral method for colliding gravitational waves. *Math. Phys. Anal. and Geom.*, 1:313–330, 1999.

[22] R.R. Gadylshin and O.M. Kyselev, Structural instabiulity of soliton for the DSII equation, *Teor. Mat. Fiz.*, **118** (1999) 354-361.

[23] Le Mesurier et al., *Physica D* **31**, (1988) 78-102.

[24] K. Nishinari and T. Yajima, *Numerical Analyses of the collision of localised structures in the Davey-Stewartson equations*, Phys. Rev. E **51** (1995), 4986-4993.

[25] K. Nishinari, T. Yajima and T. Nakao, *Time Evolution of Gaussian Type Initial Conditions Associated With The Davey-Stewartson Equations*, J. Phys. A: Math. Gen. **29** (1996), 4237-4245.

[26] D.E. Pelinovsky and C. Sulem, *Spectral decomposition for the Dirac system associated to the DSII Equation*, Inv. Prob. **16** (2000), 59-74.

[27] T.R. Taha and M.J. Ablowitz, *Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations II. Numerical, Nonlinear Schrödinger Equation*, J. Comp. Phys., **55** (1984), 203-230.

[28] J.A.C. Weideman and B.M. Herbst, *Split-Step Methods for the Solution of the Nonlinear Schrödinger Equation*, SIAM J. Numer. Anal., **23** (1986), 485-507.

[29] P.W. White and J.A.C. Weideman, *Numerical Simulation of Solitons and Dromions in the Davey-Stewartson System*, Math. Comp. Simul., **37** (1994), 469-479.