The modified Helmholtz equation in a semi-strip

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Abstract

We study the modified Helmholtz equation in a semi-strip with Poincaré type boundary conditions. On each side of the semi-strip the boundary conditions involve two parameters and one real-valued function. Using a new transform method recently introduced in the literature we show that the above boundary-value problem is equivalent to a 2×2-matrix Riemann-Hilbert (RH) problem. If the six parameters specified by the boundary conditions satisfy certain algebraic relations this RH problem can be solved in closed form. For certain values of the parameters the solution is not unique, furthermore in some cases the solution exists only under certain restrictions on the functions specifying the boundary conditions. The asymptotics of the solution at the corners of the semi-strip is investigated. In the case that the 2×2 RH problem cannot be solved in closed form, the Carleman-Vekua method for regularising it is illustrated by analysing in details a particular case.

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1 Introduction

A new method for studying boundary value problems for *integrable* PDE's in two dimensional domains (x, y) has been introduced recently and reviewed in [1]. Examples of integrable equations are linear PDE's with constant coefficients and the usual integrable nonlinear PDE's such as the Korteweg-de Vries equation.

Let $q(x, y)$ satisfy a second order linear elliptic PDE with constant coefficients in a convex polygon in the complex z-plane, $z = x + iy$. This polygon can be either bounded with corners $z_1, \ldots, z_m,\ \ z_{m+1} = z_1$, or unbounded with corners $z_1 = \infty, z_2, \ldots, z_{m-1}, z_m = \infty$. On each side of the polygon, namely on the side (z_{j+1}, z_j) referred to as the side (j), let $q(x, y)$ satisfy Poincaré type boundary conditions

$$\left.\frac{\partial q}{\partial \nu}\right|_{\mathbf{e}_j} + \gamma_j q = g_j, \tag{1.1}$$

where $\left.\frac{\partial q}{\partial \nu}\right|_{\mathbf{e}_j} = \nabla q \cdot \mathbf{e}_j$ is the outward directional derivative in the direction $\mathbf{e}_j$ specified by the constant β_j (see Fig.1), γ_j is a real non-negative constant, and g_j is a real-valued function with appropriate smoothness and decay. The main steps of the method consists of:

(1) Construct an integral representation in the complex k-plane for $q(x, y)$ in terms of a certain function $\rho(k) = \{\rho_j(k)\}_1^m$ called the *spectral function*. The function $\rho_j(k)$ is expressed as an integral over the side (j) involving q, q_s, q_n, where q_s, q_n are the tangential and the normal derivatives of the function q. Thus from (1.1) and integration by parts it follows that each ρ_j involves one unknown boundary value.

(2) Use the fact that $\rho(k)$ satisfies a certain *global relation* to characterise the part of $\rho(k)$ involving the unknown boundary values in terms of $\{\beta_j, \gamma_j, g_j\}_1^m$. For a general m-gon with the boundary conditions (1.1), this involves the formulation of a matrix Riemann-Hilbert (RH) problem.

Regarding this method we note that the formulae for $\rho(k)$ and for $q(x, y)$ are generalised direct and inverse Fourier transforms respectively, 'custom made' for the given PDE and the given polygon. We also emphasise that for simple polygons and for a large class of boundary conditions the above RH problem can be reduced to either a triangular RH problem (which can be solved in closed form), or to two separate scalar problems; we will refer to such cases as triangular and scalar cases, respectively. In some particular cases, the scalar RH problems can be bypassed all together, and $\rho(k)$ can be obtained using only algebraic manipulations; we will refer to such cases as algebraic cases. It turns out that both the Dirichlet and the Neumann problems belong to these algebraic cases.

In this paper, we apply the above method to the Poincaré boundary value problem for the modified Helmholtz equation in the semi-strip $D = \{(x, y) \in \mathbb{R}^2 : 0 < x < \infty, 0 < y < l\}$ (Fig.1)

$$\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} - 4\alpha^2 q = 0, \quad 0 < x < \infty, \quad 0 < y < l, \tag{1.2}$$

$$\text{side 1}: \quad \cos\beta_1 q_x - \sin\beta_1 q_y + \gamma_1 q = g_1(x), \quad 0 < x < \infty, \ \ y = 0$$

$$\text{side 2}: \quad \cos\beta_2 q_y - \sin\beta_2 q_x + \gamma_2 q = g_2(y), \quad x = 0, \ \ 0 < y < l,$$

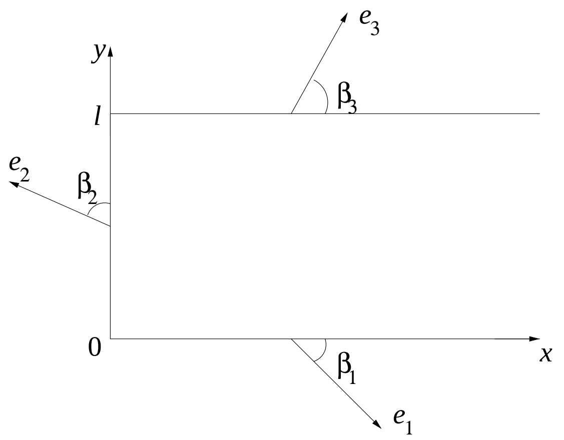

Figure 1: Geometry of the problem.

$$\text{side 3}: \quad \cos\beta_3 q_x + \sin\beta_3 q_y + \gamma_3 q = g_3(x), \quad 0 < x < \infty, \;\; y = l, \tag{1.3}$$

where α is a real constant, γ_j are real non-negative constants, and $0 < \beta_j < \pi$ $(j = 1, 2, 3)$. The functions $g_1(x)$, $g_3(x)$ vanish at the points $x = 0$ and $x = \infty$.

Let $z_1 = \infty + i0$, $z_2 = 0, z_3 = il$, $z_4 = \infty + il$. It is shown in [2] that the generalised direct and inverse Fourier transform pair associated with the modified Helmholtz equation

$$\frac{\partial^2}{\partial z \partial \bar{z}} - \alpha^2 q = 0, \tag{1.4}$$

in the semi-strip with the end-points $\{z_1, z_2, z_3, z_4\}$, are given by:

$$\rho_j(k) = \int_{z_{j+1}}^{z_j} e^{-(ikz + \frac{\alpha^2}{ik}\bar{z})} \left(q_z dz + \frac{i\alpha^2}{k} q d\bar{z} \right), \quad \mathrm{Im}(k) \leq 0 \text{ for } j = 1, 3 \text{ and } k \in \mathbb{C} \text{ for } j = 2, \tag{1.5}$$

and

$$q = \frac{1}{2\pi i}\sum_{j=1}^{3}\int_{l_j} e^{ikz+\frac{\alpha^2}{ik}\bar{z}}\rho_j(k)\frac{dk}{k}, \quad 0<x<\infty, \quad 0<y<l, \tag{1.6}$$

where the contours l_j are the rays

$$l_1 = \{k\in\mathbb{C}: \arg k = 0\}, \quad l_2 = \{k\in\mathbb{C}: \arg k = \tfrac{1}{2}\pi\}, \quad l_3 = \{k\in\mathbb{C}: \arg k = \pi\}, \tag{1.7}$$

directed away from the origin.

Furthermore, the spectral function $\rho(k) = \{\rho_j(k)\}_1^3$ satisfies the global relation

$$\sum_{j=1}^{3}\rho_j(k) = 0, \quad \mathrm{Im}(k)\leq 0. \tag{1.8}$$

For the semi-strip D and the boundary conditions (1.3), the analysis of the global relations gives rise to a matrix RH problem on the real axis.

The main aims of this paper are:

(1) To derive the associated matrix RH problem.

(2) To solve this RH problem in closed form in the triangular and scalar cases.

(3) To analyse and solve the matrix RH problem corresponding to the Laplace equation, as a particular case of the Helmholtz equation.

The paper is organised as follows. In Section 2 we review the method of [1] and use the simple case of the Dirichlet boundary conditions to illustrate the main ideas. The Poincaré boundary-value problem for the modified Helmholtz equation in the semi-strip D is reduced to a 2×2 matrix RH problem on the real axis in Section 3. In Section 4, the scalar cases are analysed, namely it is shown that if

$$\beta_1+\beta_2 = \frac{\pi}{2}m, \quad (2\alpha^2-\gamma_2^2) = (2\alpha^2-\gamma_1^2)(-1)^{m-1}, \quad m = 1,2,3, \tag{1.9}$$

and

$$\beta_2-\beta_3 = \frac{\pi}{2}n, \quad (2\alpha^2-\gamma_2^2) = (2\alpha^2-\gamma_3^2)(-1)^{n-1}, \quad n = -1,0,1, \tag{1.10}$$

then the coupled 2×2 RH problem can be reduced to two separate scalar RH problems which can be solved in closed form. In Section 5 the triangular cases are analysed, namely it is shown that if either of the conditions (1.9) or (1.10) is valid, then the matrix RH problem can be mapped to a triangular RH problem which can be solved in closed form. In Section 6.1 the matrix RH problem associated with the Laplace equation in the case $\beta_2 = \frac{1}{2}(\beta_3-\beta_1)$ is solved in closed form when all the parameters γ_j $(j=1,2,3)$ are equal to zero; in Section 6.2 it is assumed that $\gamma_1^2+\gamma_2^2+\gamma_3^2\neq 0$ and the relevant matrix RH problem is regularised by the Carleman-Vekua method [5].

2 Basic notations and the Dirichlet problem

For elliptic equations it is convenient to replace the usual Cartesian coordinates (x,y) with the complex coordinates $(z,\bar{z}) = (x+iy, x-iy)$. For example, the modified

Helmholtz equation (1.2) can be written as

$$\frac{\partial^2 q}{\partial z \partial \bar{z}} - \alpha^2 q = 0. \tag{2.1}$$

An equation in two dimensions is called integrable if and only if it can be expressed as the condition that a certain associated 1-form $W(x, y, k)$, $k \in \mathbb{C}$, is closed, i.e. $dW = 0$. A closed 1-form associated with an arbitrary linear PDE with constant coefficients is given in [4]. For example, a closed 1-form for the modified Helmholtz equation (1.2) is

$$W(z, \bar{z}, k) = e^{-ikz + \frac{i\alpha^2}{k}\bar{z}} \left(q_z dz + \frac{i\alpha^2}{k} q d\bar{z} \right), \quad k \in \mathbb{C}. \tag{2.2}$$

Indeed,

$$\begin{aligned} dW &= \left(e^{-ikz + \frac{i\alpha^2}{k}\bar{z}} q_z \right)_{\bar{z}} d\bar{z} \wedge dz + \left(\frac{i\alpha^2}{k} e^{-ikz + \frac{i\alpha^2}{k}\bar{z}} q \right)_z dz \wedge d\bar{z} \\ &= e^{-ikz + \frac{i\alpha^2}{k}\bar{z}} \left[\left(q_{z\bar{z}} + \frac{i\alpha^2}{k} q_z \right) d\bar{z} \wedge dz + \left(\frac{i\alpha^2}{k} q_z + \alpha^2 q \right) dz \wedge d\bar{z} \right]. \end{aligned} \tag{2.3}$$

Thus, using $dz \wedge d\bar{z} = -d\bar{z} \wedge dz$, it follows that

$$dW = e^{-ikz + \frac{i\alpha^2}{k}\bar{z}} \left(q_{z\bar{z}} - \alpha^2 q \right) d\bar{z} \wedge dz. \tag{2.4}$$

Hence W is closed if and only if $q(z, \bar{z})$ satisfies the modified Helmholtz equation (2.1).

Suppose that the integrable equation satisfied by $q(z, \bar{z})$ is valid in a simply connected domain D with the boundary ∂D. The equation $dW = 0$ implies

$$\int_{\partial D} W(z, \bar{z}, k) = 0, \quad k \in \mathbb{C}. \tag{2.5}$$

Following [2] we will refer to this equation as the global relation. For example, suppose that $q(z, \bar{z})$ satisfies the modified Helmholtz equation in the semi-strip D. Then the global relation (2.5) becomes equation (1.8), where $\rho_j(k)$ are defined by (1.5).

The above discussion indicates that *both* the global relation and the definition of $\rho(k)$ are a direct consequence of the closed 1-form $W(z, \bar{z}, k)$. It was shown in [2] that the global relation can be used to characterise the unknown part of $\rho(k)$, i.e. the part of $\rho(k)$ that involves the unknown boundary values. This suggests that it is desirable to express $q(z, \bar{z})$ directly in terms of $\rho(k)$ (and not in terms of the boundary values themselves). For the modified Helmholtz equation this expression is given by equation (1.6).

Such expressions can be derived either by using the spectral analysis of the closed 1-form W [2], or by using the so-called fundamental differential form (which is a slight generalisation of W) and a reformulation of Green's formula [4].

We now use the Dirichlet problem to illustrate the method [1].

Example 2.1. Let the real valued function $q(x, y)$ satisfy the modified Helmholtz equation (1.2) in the semi-strip $\{0 < x < \infty, 0 < y < l\}$, with the Dirichlet boundary conditions,

$$q(x, 0) = g_1(x), \quad q(x, l) = g_3(x), \quad 0 < x < \infty,$$

$$q(0,y)=g_2(y), \quad 0<y<l, \tag{2.6}$$

where the real valued functions g_j have appropriate smoothness and decay and are compatible at the corners $(0,0)$ and $(0,l)$. Then

$$q=\frac{1}{2\pi}\sum_{j=1}^{3}\int_{l_j} e^{ikz+\frac{\alpha^2}{ik}\bar{z}}h_j(k)\frac{dk}{k}, \quad 0<x<\infty, \quad 0<y<l, \tag{2.7}$$

where l_j are the contours (1.7), and the functions $h_j(k)$, $j=1,2,3$, are defined in terms of the given functions g_j, $j=1,2,3$, as follows. Let

$$G_1(k)=\frac{1}{2}\int_0^\infty e^{(k+\frac{\alpha^2}{k})x}\left(\frac{dg_1(x)}{dx}+\frac{2\alpha^2}{k}g_1(x)\right)dx, \quad \mathrm{Re}(k)\le 0,$$

$$G_2(k)=-\frac{1}{2}\int_0^l e^{(k+\frac{\alpha^2}{k})y}\left(\frac{dg_2(y)}{dy}+\frac{2\alpha^2}{k}g_2(y)\right)dy, \quad k\in\mathbb{C},$$

$$G_3(k)=-\frac{1}{2}\int_0^\infty e^{(k+\frac{\alpha^2}{k})x}\left(\frac{dg_3(x)}{dx}+\frac{2\alpha^2}{k}g_3(x)\right)dx, \quad \mathrm{Re}(k)\le 0, \tag{2.8}$$

$$G(k)=i[G_1(-ik)+G_2(k)+E(k)G_3(-ik)], \quad E(k)=e^{(k+\frac{\alpha^2}{k})l}, \tag{2.9}$$

$$F_1(k)=\frac{E(-k)[G(k)-\overline{G}(k)]+E(k)[G(-k)-\overline{G}(-k)]}{E(k)-E(-k)}, \tag{2.10}$$

$$F_3(k)=\frac{\overline{G}(k)-G(k)+\overline{G}(-k)-G(-k)}{E(k)-E(-k)}. \tag{2.11}$$

Then

$$h_1(k)=-F_1(k)-iG_1(-ik), \quad h_2(k)=-iG_2(k)+\overline{G}(k),$$
$$h_3(k)=-E(k)[F_3(k)+iG_3(-ik)]. \tag{2.12}$$

In order to derive formula (2.7) we first use the boundary conditions to simplify the spectral function $\rho(k)$ and then analyse the global relation.

(a) *The spectral function*

Using $q_z=\frac{1}{2}(q_x-iq_y)$ and the fact that z is x, iy and $x+il$, on the sides 1, 2, and 3, respectively, the definitions of $\rho_j(k)$ (equations (1.5)) yield

$$\rho_1(k)=\int_0^\infty e^{-(ik+\frac{\alpha^2}{ik})x}\left(\frac{1}{2}q_x-\frac{i}{2}q_y-\frac{\alpha^2}{ik}q\right)(x,0)dx, \quad \mathrm{Im}(k)\le 0,$$

$$\rho_2(k)=-i\int_0^l e^{(k+\frac{\alpha^2}{k})y}\left(\frac{1}{2}q_x-\frac{i}{2}q_y-\frac{i\alpha^2}{k}q\right)(0,y)dy, \quad k\in\mathbb{C},$$

$$\rho_3(k) = -e^{(k+\frac{\alpha^2}{k})l}\int_0^\infty e^{-(ik+\frac{\alpha^2}{ik})x}\left(\frac{1}{2}q_x - \frac{i}{2}q_y - \frac{\alpha^2}{ik}q\right)(x,l)dx, \quad \mathrm{Im}(k) \le 0. \tag{2.13}$$

By substituting the boundary conditions (2.6) into these expressions we find

$$\rho_1(k) = i\psi_1(-ik) + G_1(-ik), \quad \mathrm{Im}(k) \le 0,$$

$$\rho_2(k) = i\psi_2(k) + G_2(k), \quad k \in \mathbb{C},$$

$$\rho_3(k) = e^{(k+\frac{\alpha^2}{k})l}[i\psi_3(-ik) + G_3(-ik)], \quad \mathrm{Im}(k) \le 0, \tag{2.14}$$

where the known functions G_j are given by formulae (2.8), and the unknown functions ψ_j are defined by

$$\psi_1(k) = -\frac{1}{2}\int_0^\infty e^{(k+\frac{\alpha^2}{k})x}q_y(x,0)dx, \quad \mathrm{Re}(k) \le 0,$$

$$\psi_2(k) = -\frac{1}{2}\int_0^l e^{(k+\frac{\alpha^2}{k})y}q_x(0,y)dy, \quad k \in \mathbb{C},$$

$$\psi_3(k) = \frac{1}{2}\int_0^\infty e^{(k+\frac{\alpha^2}{k})x}q_y(x,l)dx, \quad \mathrm{Re}(k) \le 0. \tag{2.15}$$

(b) *The global relation*

Substituting equations (2.14) into the global relation (1.8) we get

$$\psi_1(-ik) + \psi_2(k) + E(k)\psi_3(-ik) = G(k), \quad \mathrm{Im}(k) \le 0, \tag{2.16}$$

where $E(k)$ and $G(k)$ are given by formulae (2.9). The complex conjugate of this equation together with the substitution $k \to \bar{k}$ yields

$$\psi_1(ik) + \psi_2(k) + E(k)\psi_3(ik) = \overline{G}(k), \quad \mathrm{Im}(k) \ge 0. \tag{2.17}$$

Both equations (2.16) and (2.17) are valid for $k \in \mathbb{R}$. Subtracting these equations we find

$$\psi_1(ik) - \psi_1(-ik) + E(k)[\psi_3(ik) - \psi_3(-ik)] = \overline{G}(k) - G(k), \quad k \in \mathbb{R}. \tag{2.18}$$

Letting $k \to -k$ we obtain

$$\psi_1(ik) - \psi_1(-ik) + E(-k)[\psi_3(ik) - \psi_3(-ik)] = G(-k) - \overline{G}(-k), \quad k \in \mathbb{R}. \tag{2.19}$$

The functions $\psi_j(ik)$, $j = 1,3$ are holomorphic for $\mathrm{Im}(k) > 0$, while the functions $\psi_j(-ik)$, $j = 1,3$ are holomorphic for $\mathrm{Im}(k) < 0$. Furthermore, the Riemann-Lebesgue lemma implies

$$\psi_j(k) = o(1), \quad k \to \infty \quad \text{or} \quad k \to 0, \quad j = 1,3. \tag{2.20}$$

Equations (2.18) and (2.19) are the boundary conditions of the following 2×2 matrix RH problem:

Find two pairs of functions $\{\psi_1(ik), \psi_3(ik)\}$ and $\{\psi_1(-ik), \psi_3(-ik)\}$ holomorphic in the upper and lower half-planes respectively, decaying at infinity, which on the real axis satisfy the conditions (2.18) and (2.19).

The above RH problem has the distinctive feature that is "doubly triangular", namely each of the combinations $\psi_j(ik) - \psi_j(-ik)$, $j = 1, 3$, can be determined independently:

$$\psi_1(ik) - \psi_1(-ik) = F_1(k), \quad k \in \mathbb{R}, \tag{2.21}$$

$$\psi_3(ik) - \psi_3(-ik) = F_3(k), \quad k \in \mathbb{R}, \tag{2.22}$$

where F_1, F_3 are defined by equations (2.10), (2.11). Each of equations (2.21, (2.22) (together with equations (2.20)) define an elementary scalar RH problem which can be solved in closed form. However, it turns out that using the representation (1.6) it is possible to *avoid* solving these RH problem.

(c) *An algebraic case*

The integral representation (1.6) of q involves the spectral functions $\rho_j(k)$, $j = 1, 2, 3$, which are given by formulae (2.14). We now concentrate on the part of q involving the unknown functions $\psi_1(-ik)$, $\psi_2(k)$, $\psi_3(-ik)$. We will show that this part can be expressed in terms of a known part as well as the unknown functions $\psi_1(ik)$, $\psi_3(ik)$. Furthermore, using the Cauchy theorem, we will show that these unknown functions *do not contribute to* q. Indeed, using equation (2.17) (which is valid for $\mathrm{Im}(k) \geq 0$) and equations (2.21) and (2.22) (which are valid for $k \in \mathbb{R}$) we can express $\psi_2(k)$, $\psi_1(-ik)$, $\psi_3(-ik)$, respectively, in terms of $\psi_1(ik)$, $\psi_3(ik)$,

$$\begin{aligned} -\psi_2(k) &= \psi_1(ik) + E(k)\psi_3(ik) - \overline{G}(k), \quad \arg k = \frac{\pi}{2}, \\ \psi_1(-ik) &= \psi_1(ik) - F_1(k), \quad k > 0, \\ \psi_3(-ik) &= \psi_3(ik) - F_3(k), \quad k > 0. \end{aligned} \tag{2.23}$$

The unknown part of $q(z, \bar{z})$ involves

$$\frac{1}{2\pi}\left(\int_{\mathcal{L}^{++}} e^{ikz + \frac{\alpha^2}{ik}\bar{z}} \psi_1(ik) \frac{dk}{k} - \int_{\mathcal{L}^{-+}} e^{ik(z-il) + \frac{\alpha^2}{ik}(\bar{z}+il)} \psi_3(ik) \frac{dk}{k}\right),$$

where $\mathcal{L}^{++} = \{(i\infty, 0) \cup (0, \infty)\}$ and $\mathcal{L}^{-+} = \{(-\infty, 0) \cup (0, i\infty)\}$ denote the positively oriented boundaries of the first and second quadrant of the complex k-plane. The function $\frac{1}{k}\exp(ikz + \frac{\alpha^2}{ik}\bar{z})$ with $x \geq 0$, $y \geq 0$, is analytic and bounded in the first quadrant of the complex k-plane. Similarly, the function $\frac{1}{k}\exp[ik(z - il) + \frac{\alpha^2}{ik}(\bar{z} + il)]$ with $x \geq 0$, $0 \leq y \leq l$, is analytic and bounded in the second quadrant. Furthermore, the functions $\psi_1(ik)$ and $\psi_3(ik)$ are analytic and bounded for $\mathrm{Im}(k) > 0$. Thus, the application of the Cauchy theorem implies that the above integrals vanish.

Recalling that $\rho_j(k)$ involve G_j, and taking into consideration equations (2.23), relation (1.6) yields (2.7).

3 Derivation of the 2×2 RH problem

We now consider the modified Helmholtz equation (1.2) with the boundary conditions (1.3) under the assumption that $\sin\beta_j \neq 0$, $j = 1,2,3$. We note that if $\sin\beta_j = 0$, $j = 1,2,3$, then after an elementary integration the boundary conditions (1.3) reduce to those considered in Example 2.1.

Proposition 3.1. Let the real value function $q(x,y)$ satisfy the modified Helmholtz equation (1.2) in the semi-strip $0 < x < \infty$, $0 < y < l$, with the boundary conditions (1.3), where $\sin\beta_j \neq 0$, $j = 1,2,3$. Then

$$q = \frac{1}{2\pi}\sum_{j=1}^{3}\int\limits_{l_j} e^{ikz+\frac{\alpha^2}{ik}\bar{z}} h_j(k)\frac{dk}{k}, \quad 0 < x < \infty, \quad 0 < y < l, \tag{3.1}$$

where the rays l_j are given by (1.7) and the functions $h_j(k)$, $j = 1,2,3$ are defined in terms of $\{\beta_j, \gamma_j, g_j\}$, $j = 1,2,3$ as follows. Let

$$J_1(k) = \frac{\gamma_1 + \frac{\alpha^2}{k}e^{-i\beta_1} + ke^{i\beta_1}}{2\sin\beta_1}, \quad J_2(k) = \frac{\gamma_2 - \frac{\alpha^2}{k}e^{i\beta_2} - ke^{-i\beta_2}}{2\sin\beta_2},$$

$$J_3(k) = \frac{\gamma_3 + \frac{\alpha^2}{k}e^{i\beta_3} + ke^{-i\beta_3}}{2\sin\beta_3}, \tag{3.2}$$

$$G_j(k) = \frac{1}{2\sin\beta_j}\int\limits_0^\infty e^{(k+\frac{\alpha^2}{k})x} g_j(x)dx, \quad j = 1,3, \quad \mathrm{Re}(k) \leq 0;$$

$$G_2(k) = \frac{1}{2\sin\beta_2}\int\limits_0^l e^{(k+\frac{\alpha^2}{k})y} g_2(y)dy, \quad k \in \mathbb{C}, \quad E(k) = e^{(k+\frac{\alpha^2}{k})l}, \tag{3.3}$$

$$G(k) = G_1(-ik) + G_2(k) + E(k)G_3(-ik)$$
$$+\frac{d_0}{2}\left(\frac{e^{i\beta_1}}{\sin\beta_1} + \frac{e^{-i\beta_2}}{\sin\beta_2}\right) - \frac{d_1}{2}E(k)\left(\frac{e^{-i\beta_2}}{\sin\beta_2} - \frac{e^{-i\beta_3}}{\sin\beta_3}\right). \tag{3.4}$$

Then

$$h_1(k) = -J_1(ik)\psi_1(-ik) + G_1(-ik) + \frac{e^{i\beta_1}d_0}{2\sin\beta_1}, \quad \arg k = 0,$$

$$h_2(k) = \frac{J_2(k)}{\overline{J_2}(k)}\left[\overline{J_1}(-ik)\psi_1(ik) + E(k)\overline{J_3}(-ik)\psi_3(ik) - \overline{G}(k)\right] + G_2(k)$$
$$-\frac{E(k)d_1 - d_0}{2e^{i\beta_2}\sin\beta_2}, \quad \arg k = \frac{\pi}{2},$$

$$h_3(k) = E(k)\left[-J_3(ik)\psi_3(-ik) + G_3(-ik) + \frac{e^{-i\beta_3}d_1}{2\sin\beta_3}\right], \quad \arg k = \pi, \tag{3.5}$$

where $d_0 = q(0,0)$, $d_1 = q(0,l)$, and the sectionally holomorphic functions $\psi_1(\pm ik)$ and $\psi_3(\pm ik)$ solve the 2×2 matrix RH problem defined by:

- $\psi_1(ik)$, $\psi_3(ik)$ *are holomorphic for* $\mathrm{Im}(k) > 0$,
- $\psi_1(k) = o(1)$, $\psi_3(k) = o(1)$, $k \to 0$ and $k \to \infty$,
- *for* $k \in \mathbb{R}$, *the functions* $\psi_j(\pm ik)$, $j = 1, 3$, *satisfy the equation*

$$\frac{\overline{J_1}(-ik)}{\overline{J_2}(k)}\psi_1(ik) - \frac{J_1(ik)}{J_2(k)}\psi_1(-ik)$$

$$+E(k)\left[\frac{\overline{J_3}(-ik)}{\overline{J_2}(k)}\psi_3(ik) - \frac{J_3(ik)}{J_2(k)}\psi_3(-ik)\right] = \frac{\overline{G}(k)}{\overline{J_2}(k)} - \frac{G(k)}{J_2(k)}, \tag{3.6}$$

together with the equation obtained from (3.6) with k *replaced by* $-k$.

Remark 3.1. If the function $q(x,0)$ has a power singularity at $x = 0$: $q(x,0) = O(x^{\delta_0})$ and $-1 < \delta_0 < 0$, then the integrals ρ_1 and ρ_2 in (1.5) are understood in the regularised sense, and $d_0 = 0$. Correspondingly, if $q(x,l) = O(x^{\delta_1})$, $x \to 0$ and $-1 < \delta_1 < 0$, then $d_1 = 0$.

Proof

The spectral functions ρ_j are defined by equations (2.13). Solving equations (1.3) for $q_y(x,0)$, $q_x(0,y)$ and $q_y(x,l)$, respectively, substituting the resulting expressions in equations (2.13), and integrating by parts, we find

$$\rho_1 = ih_1(k), \quad \rho_3(k) = ih_3(k),$$

$$\rho_2 = i\left[-J_2(k)\psi_2(k) + G_2(k) - \frac{E(k)d_1 - d_0}{2e^{i\beta_2}\sin\beta_2}\right]. \tag{3.7}$$

The unknown functions $\psi_j(k)$, $j = 1, 2, 3$ are defined by

$$\psi_1(k) = \int_0^\infty e^{(k+\frac{\alpha^2}{k})x} q(x,0)dx, \quad \mathrm{Re}(k) < 0,$$

$$\psi_2(k) = \int_0^l e^{(k+\frac{\alpha^2}{k})y} q(0,y)dy, \quad k \in \mathbb{C},$$

$$\psi_3(k) = \int_0^\infty e^{(k+\frac{\alpha^2}{k})x} q(x,l)dx, \quad \mathrm{Re}(k) < 0. \tag{3.8}$$

The abelian theorem applied to the above integrals implies that the functions $\psi_1(k)$, $\psi_3(k)$ decay as $k \to 0$ and $k \to \infty$. Next, using equations (3.7), the global relation (1.8) becomes

$$J_1(ik)\psi_1(-ik) + J_2(k)\psi_2(k) + E(k)J_3(ik)\psi_3(-ik) = G(k), \quad \mathrm{Im}(k) \le 0. \tag{3.9}$$

The complex conjugate of this equation together with the substitution $k \to \bar{k}$ yields

$$\overline{J_1}(-ik)\psi_1(ik) + \overline{J_2}(k)\psi_2(k) + E(k)\overline{J_3}(-ik)\psi_3(ik) = \overline{G}(k), \quad \mathrm{Im}(k) \ge 0. \tag{3.10}$$

The expression for q is given by equation (1.6). Using equation (3.10) to express $\psi_2(k)$ in terms of $\psi_1(ik)$ and $\psi_3(ik)$, and then substituting the resulting expression into the expression for ρ_2 given in (3.7), it follows that $\rho_2 = ih_2$. This equation together with $\rho_1 = ih_1$, $\rho_3 = ih_3$ imply that equation (1.6) gives (3.1).

Both equations (3.9) and (3.10) are valid for $k \in \mathbb{R}$. Eliminating $\psi_2(k)$ from these equations we find equation (3.6). The holomorphicity of $\psi_j(\pm ik)$, $j = 1, 3$, follows from the definition (3.8) of these functions. The proposition is proved.

In summary, the Poincaré boundary-value problem for the modified Helmholtz equation in a semi-strip $\{0 < x < \infty, 0 < y < l\}$ is equivalent to a RH problem with the boundary condition

$$\mathbf{J}(k)\begin{pmatrix}\psi_1(ik)\\ \psi_3(ik)\end{pmatrix} = \overline{\mathbf{J}}(k)\begin{pmatrix}\psi_1(-ik)\\ \psi_3(-ik)\end{pmatrix} + \begin{pmatrix}f(k)\\ -f(-k)\end{pmatrix}, \quad k \in \mathbb{R}, \tag{3.11}$$

where

$$\mathbf{J}(k) = \begin{pmatrix}\frac{\overline{J_1}(-ik)}{\overline{J_2}(k)} & E(k)\frac{\overline{J_3}(-ik)}{\overline{J_2}(k)}\\ \frac{J_1(-ik)}{J_2(-k)} & E(-k)\frac{J_3(-ik)}{J_2(-k)}\end{pmatrix}, \tag{3.12}$$

$$f(k) = \frac{\overline{G}(k)}{\overline{J_2}(k)} - \frac{G(k)}{J_2(k)}. \tag{3.13}$$

Multiplying the left- and right-hand sides of equation (3.11) by $[\mathbf{J}(k)]^{-1}$, we find the standard form:

$$\begin{pmatrix}\psi_1(ik)\\ \psi_3(ik)\end{pmatrix} = \mathbf{H}(k)\begin{pmatrix}\psi_1(-ik)\\ \psi_3(-ik)\end{pmatrix} + \boldsymbol{\mu}(k), \quad k \in \mathbb{R}, \tag{3.14}$$

with

$$\mathbf{H}(k) = \frac{1}{\det\mathbf{J}(k)}\begin{pmatrix}H_{11}(k) & H_{12}(k)\\ H_{21}(k) & H_{22}(k)\end{pmatrix}, \quad \boldsymbol{\mu}(k) = \begin{pmatrix}\mu_1(k)\\ \mu_3(k)\end{pmatrix} = [\mathbf{J}(k)]^{-1}\begin{pmatrix}f(k)\\ -f(-k)\end{pmatrix},$$

$$\begin{aligned}
H_{11}(k) &= \frac{J_1(ik)J_3(-ik)}{J_2(k)J_2(-k)}E(-k) - \frac{\overline{J_1}(ik)\overline{J_3}(-ik)}{\overline{J_2}(k)\overline{J_2}(-k)}E(k),\\
H_{12}(k) &= \frac{J_3(ik)J_3(-ik)}{J_2(k)J_2(-k)} - \frac{\overline{J_3}(ik)\overline{J_3}(-ik)}{\overline{J_2}(k)\overline{J_2}(-k)},\\
H_{21}(k) &= -\frac{J_1(ik)J_1(-ik)}{J_2(k)J_2(-k)} + \frac{\overline{J_1}(ik)\overline{J_1}(-ik)}{\overline{J_2}(k)\overline{J_2}(-k)},\\
H_{22}(k) &= -\frac{J_1(-ik)J_3(ik)}{J_2(k)J_2(-k)}E(k) + \frac{\overline{J_1}(-ik)\overline{J_3}(ik)}{\overline{J_2}(k)\overline{J_2}(-k)}E(-k).
\end{aligned} \tag{3.15}$$

It is unlikely that the general matrix RH problem (3.14), (3.15) can be solved in closed form. In Sections 4, 5 and 6.1 we consider some particular cases which admit a closed-form solution. In the general case, the above RH problem can be reduced to a system of singular integral equations with a fixed singularity. In Section 6.2 we regularise the matrix RH problem (3.14) in the case $\alpha = 0$ and $\beta_2 = \frac{1}{2}(\beta_3 - \beta_2)$.

4 Scalar cases

We will first find the conditions for the matrix RH problem (3.14) to be reduced to two separate scalar RH problems. Let

$$J_{j2}(k) = \frac{J_j(ik)\overline{J_2}(k)}{\overline{J_j}(-ik)J_2(k)}, \quad j = 1, 3. \tag{4.1}$$

Rewrite the boundary condition (3.11) as follows,

$$\frac{\overline{J_1}(-ik)}{\overline{J_2}(k)}[\psi_1(ik) - J_{12}(k)\psi_1(-ik)]$$

$$+\frac{E(k)\overline{J_3}(-ik)}{\overline{J_2}(k)}[\psi_3(ik) - J_{32}(k)\psi_3(-ik)] = \frac{\overline{G}(k)}{\overline{J_2}(k)} - \frac{G(k)}{J_2(k)}, \quad k \in \mathbb{R},$$

$$\frac{\overline{J_1}(ik)}{\overline{J_2}(-k)}J_{12}(-k)\left[\psi_1(ik) - \frac{\psi_1(-ik)}{J_{12}(-k)}\right]$$

$$+\frac{E(-k)\overline{J_3}(ik)}{\overline{J_2}(-k)}J_{32}(-k)\left[\psi_3(ik) - \frac{\psi_3(-ik)}{J_{32}(-k)}\right] = \frac{G(-k)}{J_2(-k)} - \frac{\overline{G}(-k)}{\overline{J_2}(-k)}, \quad k \in \mathbb{R}. \tag{4.2}$$

Suppose that

$$J_{j2}(k)J_{j2}(-k) = 1, \quad j = 1, 3. \tag{4.3}$$

Then the linear combinations $\psi_1(ik) - J_{12}(k)\psi_1(-ik)$ and $\psi_3(ik) - J_{32}(k)\psi_3(-ik)$ can be found explicitly,

$$\psi_1(ik) = J_{12}(k)\psi_1(-ik) + \omega_1(k), \quad k \in \mathbb{R}, \tag{4.4}$$

$$\psi_3(ik) = J_{32}(k)\psi_3(-ik) + \omega_3(k), \quad k \in \mathbb{R}. \tag{4.5}$$

These relations are scalar RH problems which define the unknown functions ψ_1, ψ_3. Here

$$\omega_1(k) = -\frac{E(k)\overline{J_3}(-ik)}{\overline{J_1}(-ik)}\omega_3(k) + \frac{1}{\overline{J_1}(-ik)}\left[\overline{G}(k) - \frac{\overline{J_2}(k)}{J_2(k)}G(k)\right],$$

$$\omega_3(k) = \left[\frac{E(-k)J_3(-ik)}{J_2(-k)} - \frac{E(k)J_1(-ik)\overline{J_3}(-ik)}{\overline{J_1}(-ik)J_2(-k)}\right]^{-1}$$

$$\times\left\{\frac{G(-k)}{J_2(-k)} - \frac{\overline{G}(-k)}{\overline{J_2}(-k)} - \frac{J_1(-ik)}{\overline{J_1}(-ik)J_2(-k)}\left[\overline{G}(k) - \frac{\overline{J_2}(k)}{J_2(k)}G(k)\right]\right\}. \tag{4.6}$$

Clearly, $\omega_3(k) = o(1)$, $k \to \pm\infty$, and $\omega_1(k) = o(1)$, $k \to -\infty$. If $k \to +\infty$, then (4.6) implies

$$\omega_1(k) = \frac{E(k)}{\overline{J_1}(-ik)}\left[-\overline{J_3}(-ik)\omega_3(k) + \overline{G_3}(ik) - \frac{\overline{J_2}(k)}{J_2(k)}G_3(-ik)\right.$$

$$\left.+\frac{d_1}{2}\left(\frac{e^{i\beta_3}}{\sin\beta_3} - \frac{e^{i\beta_2}}{\sin\beta_2}\right) - \frac{d_1\overline{J_2}(k)}{2J_2(k)}\left(\frac{e^{-i\beta_3}}{\sin\beta_3} - \frac{e^{-i\beta_2}}{\sin\beta_2}\right)\right] + o(1), \quad k \to +\infty. \tag{4.7}$$

At first glance it appears that the function $\omega_1(k)$ grows exponentially as $k \to +\infty$. However, substituting $\omega_3(k)$ from equation (4.6) into the above relation, it follows that the expression in the square brackets is equal to zero, and therefore $\omega_1(k) = o(1)$, $k \to +\infty$. As $k \to 0$, the functions $\omega_j(k)$ vanish: $\omega_j(k) = o(1)$ $(j = 1, 3)$.

The conditions (4.3) can be simplified and written in terms of β_j and γ_j as follows

$$j = 1: \qquad e^{4i(\beta_1+\beta_2)} = 1, \quad (2\alpha^2 - \gamma_2^2)\sin 2\beta_1 - (2\alpha^2 - \gamma_1^2)\sin 2\beta_2 = 0,$$

$$j = 3: \qquad e^{4i(\beta_2-\beta_3)} = 1, \quad (2\alpha^2 - \gamma_2^2)\sin 2\beta_3 + (2\alpha^2 - \gamma_3^2)\sin 2\beta_2 = 0. \tag{4.8}$$

Since $0 < \beta_j < \pi$ and $\gamma_j > 0$ $(j = 1, 2, 3)$ the above relations yield

$$j = 1: \qquad \beta_1 + \beta_2 = \frac{\pi m}{2}, \quad 2\alpha^2 - \gamma_2^2 + (-1)^m(2\alpha^2 - \gamma_1^2) = 0, \quad m = 1, 2, 3,$$

$$j = 3: \qquad \beta_2 - \beta_3 = \frac{\pi m}{2}, \quad (-1)^m(2\alpha^2 - \gamma_2^2) + 2\alpha^2 - \gamma_3^2 = 0, \quad m = -1, 0, 1. \tag{4.9}$$

Thus, both conditions (4.3) are satisfied simultaneously for the following sets of the parameters $\beta_j \in (0, \pi)$, $\gamma_j > 0$:

$$(1) \qquad \gamma_1 = \gamma_3 = \sqrt{4\alpha^2 - \gamma_2^2},\ 0 < \gamma_j < 2|\alpha|,\ \beta_1 = \pi - \beta_2,\ \beta_3 = \beta_2;$$

$$(2) \quad \gamma_1 = \sqrt{4\alpha^2 - \gamma_2^2},\ \gamma_3 = \gamma_2,\ 0 < \gamma_j < 2|\alpha|,\ \beta_1 = \pi - \beta_2,\ \beta_3 = \beta_2 \pm \frac{\pi}{2};$$

$$(3) \quad \gamma_1 = \gamma_2,\ \gamma_3 = \sqrt{4\alpha^2 - \gamma_2^2},\ 0 < \gamma_j < 2|\alpha|,\ \beta_1 = \pi - \beta_2 \pm \frac{\pi}{2},\ \beta_3 = \beta_2;$$

$$(4) \qquad \gamma_1 = \gamma_2 = \gamma_3,\ \beta_1 = \frac{\pi}{2} - \beta_2,\ \beta_3 = \frac{\pi}{2} + \beta_2,\ 0 < \beta_2 < \frac{\pi}{2};$$

$$(5) \qquad \gamma_1 = \gamma_2 = \gamma_3,\ \beta_1 = \frac{3\pi}{2} - \beta_2,\ \beta_3 = \beta_2 - \frac{\pi}{2},\ \frac{\pi}{2} < \beta_2 < \pi. \tag{4.10}$$

We now analyse the inverse transform equation (1.6) and investigate whether the solutions of the RH problems (4.4), (4.5) can be avoided. The boundary conditions (4.4) and (4.5) imply that the functions $\psi_1(-ik)$ and $\psi_3(-ik)$ can be analytically continued into $\mathbb{C}^+$,

$$\psi_1(-ik) = \frac{\psi_1(ik) - \omega_1(k)}{J_{12}(k)}, \quad k \in \mathbb{C}^+,$$

$$\psi_3(-ik) = \frac{\psi_3(ik) - \omega_3(k)}{J_{32}(k)}, \quad k \in \mathbb{C}^+. \tag{4.11}$$

The function $\psi_2(k)$ can be expressed in terms of $\psi_1(ik)$ and $\psi_3(ik)$ using equation (3.10),

$$\psi_2(k) = -\frac{\overline{J_1}(-ik)}{\overline{J_2}(k)}\psi_1(ik) - E(k)\frac{\overline{J_3}(-ik)}{\overline{J_2}(k)}\psi_3(ik) + \frac{\overline{G}(k)}{\overline{J_2}(k)}. \tag{4.12}$$

Using (4.11) and (4.12) the inverse transformation equation (1.6) yields

$$q = I_0 + I_1 + I_2 + I_3, \tag{4.13}$$

where

$$2\pi I_0 = \int_0^{\infty} \left[\frac{\overline{J_1}(-ik) J_2(k)}{\overline{J_2}(k)} \omega_1(k) + G_1(-ik) + \frac{e^{i\beta_1} d_0}{2\sin\beta_1} \right] e^{ikz + \frac{\alpha^2}{ik}\overline{z}} \frac{dk}{k}$$

$$+ \int_0^{i\infty} \left[-\frac{J_2(k)}{\overline{J_2}(k)} \overline{G}(k) + G_2(k) - \frac{E(k)d_1 - d_0}{2e^{i\beta_2}\sin\beta_2} \right] e^{ikz + \frac{\alpha^2}{ik}\overline{z}} \frac{dk}{k}$$

$$- \int_{-\infty}^{0} \left[G_3(-ik) + \frac{e^{-i\beta_3} d_1}{2\sin\beta_3} \right] E(k) e^{ikz + \frac{\alpha^2}{ik}\overline{z}} \frac{dk}{k},$$

$$I_1 = -\frac{1}{2\pi} \int_{\mathcal{L}^{++}} \frac{\overline{J_1}(-ik) J_2(k)}{\overline{J_2}(k)} \psi_1(ik) e^{ikz + \frac{\alpha^2}{ik}\overline{z}} \frac{dk}{k},$$

$$I_2 = -\frac{1}{2\pi} \int_{-\infty}^{0} \frac{\overline{J_3}(-ik) J_2(k)}{\overline{J_2}(k)} \omega_3(k) E(k) e^{ikz + \frac{\alpha^2}{ik}\overline{z}} \frac{dk}{k},$$

$$I_3 = \frac{1}{2\pi} \int_{\mathcal{L}^{-+}} \frac{\overline{J_3}(-ik) J_2(k)}{\overline{J_2}(k)} \psi_3(ik) E(k) e^{ikz + \frac{\alpha^2}{ik}\overline{z}} \frac{dk}{k}, \tag{4.14}$$

and $\mathcal{L}^{++}$, $\mathcal{L}^{-+}$ are the same contours as in Example 2.1.

We note that the integrals I_0 and I_2 are expressed in terms of the given boundary conditions, while the integrals I_1 and I_3 involve the unknown functions $\psi_1(ik)$, $\psi_3(ik)$ which are analytic in the upper half-plane $\mathbb{C}^+$. The zeros $k_1^{(2)}$, $k_2^{(2)}$ of the function

$$k\overline{J_2}(k) = -\frac{e^{i\beta_2}}{2\sin\beta_2}(k^2 - \gamma_2 k e^{-i\beta_2} + \alpha^2 e^{-2i\beta_2}) \tag{4.15}$$

are given by

$$2k_j^{(2)} = \left(\gamma_2 \cos\beta_2 + (-1)^{j-1} \sqrt{4\alpha^2 - \gamma_2^2} \sin\beta_2 \right)$$
$$+ i \left(-\gamma_2 \sin\beta_2 + (-1)^{j-1} \sqrt{4\alpha^2 - \gamma_2^2} \cos\beta_2 \right), \tag{4.16}$$

for $0 < \gamma_2 < 2|\alpha|$ and by

$$2k_j^{(2)} = \left(\gamma_2 + (-1)^{j-1} \sqrt{\gamma_2^2 - 4\alpha^2} \right) (\cos\beta_2 - i\sin\beta_2), \tag{4.17}$$

for $\gamma_2 > 2|\alpha|$.

There exist three separate cases depending on the position of the zeros of $k\overline{J_2}(k)$:

1.

$$\gamma_2 > 2|\alpha|, \tag{4.18}$$

or

$$\gamma_2 < 2|\alpha|, \quad 0 < \beta_2 < \frac{\pi}{2}, \quad \cot\beta_2 < \lambda, \tag{4.19}$$

or

$$\gamma_2 < 2|\alpha|, \quad \frac{\pi}{2} < \beta_2 < \pi, \quad -\cot\beta_2 < \lambda, \tag{4.20}$$

where $\lambda = \gamma_2(4\alpha^2 - \gamma_2^2)^{-1/2}$. Then the zeros of $\overline{J_2}(k)$ are in $\mathbb{C}^-$, thus $I_1 = I_3 = 0$, and

$$q = I_0 + I_2. \tag{4.21}$$

2.

$$\gamma_2 < 2|\alpha|, \quad 0 < \beta_2 < \tfrac{1}{2}\pi, \quad \cot\beta_2 > \lambda. \tag{4.22}$$

Then $k_2^{(2)} \in \mathbb{C}^-$, $k_1^{(2)} \in \mathbb{C}^{++} = \{k \in \mathbb{C} : \operatorname{Re} k > 0, \operatorname{Im} k > 0\}$. Thus, $I_3 = 0$ and I_1 can be computed by the residue theorem

$$q = I_0 + I_1 + I_2, \quad I_1 = -i\left[e^{ikz+\frac{\alpha^2}{ik}\bar{z}}\frac{\overline{J_1}(-ik)J_2(k)}{k\overline{J}_2'(k)}\psi_1(ik)\right]_{k=k_1^{(2)}}. \tag{4.23}$$

3.

$$\gamma_2 < 2|\alpha|, \quad \tfrac{1}{2}\pi < \beta_2 < \pi, \quad -\cot\beta_2 > \lambda. \tag{4.24}$$

Then $k_1^{(2)} \in \mathbb{C}^-$, $k_2^{(2)} \in \mathbb{C}^{-+} = \{k \in \mathbb{C} : \operatorname{Re} k < 0, \operatorname{Im} k > 0\}$. Thus,

$$q = I_0 + I_2 + I_3, \quad I_3 = i\left[e^{ikz+\frac{\alpha^2}{ik}\bar{z}}E(k)\frac{\overline{J_3}(-ik)J_2(k)}{k\overline{J}_2'(k)}\psi_3(ik)\right]_{k=k_2^{(2)}}. \tag{4.25}$$

To evaluate the values $\psi_1(ik_1^{(2)})$ and $\psi_3(ik_2^{(2)})$ one needs to solve the scalar RH problems (4.4), (4.5). In what follows we present the solution of the RH problem (4.4) in the case 2. The coefficient $J_{12}(k)$ can be factorised explicitly:

$$J_{12}(k) = -e^{2i(\beta_1+\beta_2)}\frac{(k-k_1^{(1)})(k-k_2^{(1)})(k-k_1^{(2)})(k-k_2^{(2)})}{(k-\overline{k_1^{(1)}})(k-\overline{k_2^{(1)}})(k-\overline{k_1^{(2)}})(k-\overline{k_2^{(2)}})}, \tag{4.26}$$

where

$$2k_j^{(1)} = e^{-i\beta_1}\left(i\gamma_1 + (-1)^{j-1}\sqrt{4\alpha^2-\gamma_1^2}\right), \quad j = 1, 2. \tag{4.27}$$

Equation (4.10) implies that the parameter γ_1 depends on γ_2:

(a) $\gamma_1 = \sqrt{4\alpha^2-\gamma_2^2}$, $\beta_1 = \pi - \beta_2$. Then

$$2\operatorname{Im} k_j^{(1)} = (-1)^j\gamma_2\sin\beta_2 - \sqrt{4\alpha^2-\gamma_2^2}\cos\beta_2,$$

$$2\operatorname{Im} k_j^{(2)} = -\gamma_2\sin\beta_2 - (-1)^j\sqrt{4\alpha^2-\gamma_2^2}\cos\beta_2. \tag{4.28}$$

The function $J_{12}(k)$ has one zero and three poles in $\mathbb{C}^+$. This means that the winding number (index) of the function $J_{12}(k)$ equals -2:

$$\operatorname{ind} J_{12}(k) = \frac{1}{2\pi}[\arg J_{12}(k)]_{\mathbb{R}^1} = -2. \tag{4.29}$$

Let

$$X^+(k) = \frac{(k-k_1^{(1)})(k-k_2^{(1)})(k-k_2^{(2)})}{k-\overline{k_1^{(2)}}}, \quad X^-(k) = -\overline{X^+}(k). \tag{4.30}$$

Then the functions $X^{\pm}(k)$ are analytic in $\mathbb{C}^{\pm}$, they do not vanish in $\mathbb{C}^{\pm}$ and, in addition, $J_{12}(k) = X^+(k)[X^-(k)]^{-1}$, $k \in \mathbb{R}$. Applying the Liouville theorem gives the solution of the problem (4.4)

$$\psi_1^{\pm}(ik) = X^{\pm}(k)\chi^{\pm}(k), \quad k \in \mathbb{C}^{\pm}, \tag{4.31}$$

where

$$\chi^{\pm}(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\omega_1(x)dx}{X^+(x)(x-k)}, \quad k \in \mathbb{C}^{\pm} \setminus \mathbb{R}. \tag{4.32}$$

The functions $\psi(ik)$ and $\psi(-ik)$ decay at infinity iff

$$\int_{-\infty}^{\infty} \frac{x^j \omega_1(x)dx}{X^+(x)} = 0, \quad j = 0, 1. \tag{4.33}$$

(b) $\gamma_1 = \gamma_2$, $\beta_1 = \frac{1}{2}\pi - \beta_2$. Then $J_{12}(k) \equiv 1$, and by the Sokhotski-Plemelj formulae

$$\psi_1(\pm ik) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\omega_1(x)dx}{x-k}, \quad k \in \mathbb{C}^{\pm} \setminus \mathbb{R}. \tag{4.34}$$

Finally, we show how to fix the constants $d_0 = q(0,0)$ and $d_1 = q(0,l)$. We note that in the scalar cases the function $q(x,y)$ is bounded at the corners of the semi-strip D. From (3.1) we obtain

$$d_0 = \frac{1}{2\pi}\sum_{j=1}^{3}\int_{l_j} h_j(k)\frac{dk}{k}, \quad d_1 = \frac{1}{2\pi}\sum_{j=1}^{3}\int_{l_j} e^{-\left(k+\frac{\alpha^2}{k}\right)l} h_j(k)\frac{dk}{k}. \tag{4.35}$$

On the other hand, because $h_j(k)$ $(j = 1, 2, 3)$ are linear functions of d_0 and d_1, the above relations can be rewritten as a linear system of algebraic equations

$$\begin{aligned} d_0 &= D_{00}d_0 + D_{01}d_1 + D_0, \\ d_1 &= D_{10}d_0 + D_{11}d_1 + D_1, \end{aligned} \tag{4.36}$$

where the coefficients D_{jm}, D_j $(j = 0, 1; m = 0, 1)$ are known. The system (4.36) uniquely defines the constants d_0, d_1 provided that the relevant matrix is not singular.

5 Triangular cases

We assume that equation (4.3) is valid for $j = 1$, but is not valid for $j = 3$, i.e.

$$J_{12}(k)J_{12}(-k) = 1, \quad J_{32}(k)J_{32}(-k) \neq 1. \tag{5.1}$$

The first equation in (5.1) is satisfied if:

(a) $\gamma_1 = \sqrt{4\alpha^2 - \gamma_2^2}$, $\beta_1 = \pi - \beta_2$,
or
(b) $\gamma_1 = \gamma_2$, $\beta_1 = \pi - \beta_2 \pm \frac{1}{2}\pi$.
In the case (a), $\operatorname{ind} J_{12}(k) = -2$, while in the case (b), $J_{12} = 1$, thus $\operatorname{ind} J_{12} = 0$.
The inverse transform formula (1.6) implies

$$q = I_0 + I_1 + I^*, \tag{5.2}$$

where I_0, I_1 are given by (4.14) and

$$I^* = \frac{1}{2\pi}\int_0^{i\infty} \frac{\overline{J_3}(-ik)J_2(k)}{\overline{J_2}(k)}\psi_3(ik)E(k)e^{ikz+\frac{\alpha^2}{ik}\overline{z}}\frac{dk}{k}$$
$$+\frac{1}{2\pi}\int_{-\infty}^{0} J_3(ik)\psi_3(-ik)E(k)e^{ikz+\frac{\alpha^2}{ik}\overline{z}}\frac{dk}{k}. \tag{5.3}$$

If $\frac{1}{2}\pi < \beta_2 < \pi$, then $I_1 = 0$ (see Section 4). If however, $0 < \beta_2 < \frac{1}{2}\pi$, then $I_1 \neq 0$, and in order to compute I_1 one needs to determine $\psi_1(ik)$. This function satisfies the scalar RH problem (4.4), where the function $\omega_1(k)$ is now given by

$$\omega_1(k) = -\frac{E(k)\overline{J_3}(-ik)}{\overline{J_1}(-ik)}[\psi_3(ik) - J_{32}(k)\psi_3(-ik)] + \frac{1}{\overline{J_1}(-ik)}\left[\overline{G}(k) - \frac{\overline{J_2}(k)}{J_2(k)}G(k)\right]. \tag{5.4}$$

Thus $\psi_1(ik)$ can be computed by equations (4.31) or (4.34) provided that we first compute $\psi_3(ik)$. This function satisfies the scalar RH problem

$$\psi_3(ik) = \Omega(k)\psi_3(-ik) + \omega_3(k), \quad k \in \mathbb{R}, \tag{5.5}$$

where

$$\Omega(k) = \frac{\Omega_1(k)J_{32}(k) - \Omega_2(k)}{\Omega_1(k) - \Omega_2(k)J_{32}(-k)},$$
$$\Omega_1(k) = \frac{J_1(-ik)\overline{J_3}(-ik)E(k)}{\overline{J_1}(-ik)J_2(-k)}, \quad \Omega_2(k) = \frac{\overline{J_3}(ik)E(-k)}{\overline{J_2}(-k)}.$$
$$\omega_3(k) = -[\Omega_1(k) - \Omega_2(k)J_{32}(-k)]^{-1}$$
$$\times\left\{\frac{G(-k)}{J_2(-k)} - \frac{\overline{G}(-k)}{\overline{J_2}(-k)} - \frac{J_1(-ik)}{\overline{J_1}(-ik)J_2(-k)}\left[\overline{G}(k) - \frac{\overline{J_2}(k)}{J_2(k)}G(k)\right]\right\}. \tag{5.6}$$

The coefficient $\Omega(k)$ is discontinuous at the points $k = 0$ and $k = \infty$:

$$\Omega(-\infty) = \frac{1}{J_{32}(\infty)} = -e^{-2i(\beta_2-\beta_3)}, \quad \Omega(+\infty) = J_{32}(\infty) = -e^{2i(\beta_2-\beta_3)},$$
$$\Omega(-0) = \frac{1}{J_{32}(0)} = -e^{2i(\beta_2-\beta_3)}, \quad \Omega(+0) = J_{32}(0) = -e^{-2i(\beta_2-\beta_3)}. \tag{5.7}$$

We fix the argument of the function $\Omega(k)$ as follows

$$\arg\Omega(-\infty) = \pi + 2(\beta_3 - \beta_2), \quad \arg\Omega(+0) = \pi + 2(\beta_3 - \beta_2). \tag{5.8}$$

Then

$$\arg\Omega(-0) = \pi + 2(\beta_3 - \beta_2) + \Delta^-, \quad \arg\Omega(+\infty) = \pi + 2(\beta_3 - \beta_2) + \Delta^+, \tag{5.9}$$

where Δ^- (Δ^+) is the increment of the argument of the function $\Omega(k)$ as k passes the negative (positive) semi-axis. It can be directly verified that $\Omega(k) = \overline{\Omega}(-k)$. Therefore $\Delta^+ = \Delta^- = \Delta$. Comparing (5.7) and (5.9) we find

$$\Delta = 2\pi\varkappa + 4(\beta_2 - \beta_3), \tag{5.10}$$

where $\varkappa$ is an integer. The function $\Omega(k)$ can be factorised as

$$\Omega(k) = \frac{X^+(k)}{X^-(k)}, \quad k \in \mathbb{R}, \tag{5.11}$$

where

$$X^\pm(k) = k^p \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log\Omega(k)}{x-k}dx\right\}, \quad k \in \mathbb{C}^\pm, \tag{5.12}$$

p is an integer defined by the value of the parameter Δ. The analysis of the above Cauchy integral yields

$$\begin{aligned} X(k) &\sim A_0 k^{p+\frac{\Delta}{2\pi}}, \quad k \to 0, \\ X(k) &\sim A_1 k^{p-\frac{\Delta}{2\pi}}, \quad k \to \infty, \end{aligned} \tag{5.13}$$

where A_0, A_1 are constants.

We assume

$$-3\pi < \Delta < 3\pi. \tag{5.14}$$

This is consistent with a variety of numerical experiments. Figs 2-5 present graphs of $\Omega(k)$, $0 < k < \infty$, for $\alpha = 1$ and some values of the parameters β_j, γ_j. For all these cases (5.14) is valid. Substituting equation (5.11) into (5.5) and using Liouville's theorem we find

$$\psi_3(\pm ik) = X^\pm(k)[\chi^\pm(k) + \mathcal{P}_\varkappa(k)], \quad k \in \mathbb{C}^\pm \setminus \mathbb{R}, \tag{5.15}$$

where $X^\pm$ are defined in (5.12), $\chi^\pm(k)$ are given by

$$\chi^\pm(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\omega_3(x)}{X^+(x)}\frac{dx}{x-k}, \quad k \in \mathbb{C}^\pm \setminus \mathbb{R}, \tag{5.16}$$

and $\mathcal{P}_\varkappa(k)$ is an arbitrary polynomial of degree $\varkappa$. The required asymptotics (2.20) of the function $\psi_3(k)$ defines the class of solutions and will be used to fix the integers p and $\varkappa$.

1°. If $-3\pi < \Delta \leq -2\pi$, then $p = 1$, $\mathcal{P}_\varkappa(k) \equiv 0$. The behaviour of the solution at $k = 0$ is

$$\psi_3(\pm ik) = O(k^{1+\frac{\Delta}{2\pi}}), \quad k \to 0, \quad -\frac{1}{2} < 1 + \frac{\Delta}{2\pi} \leq 0. \tag{5.17}$$

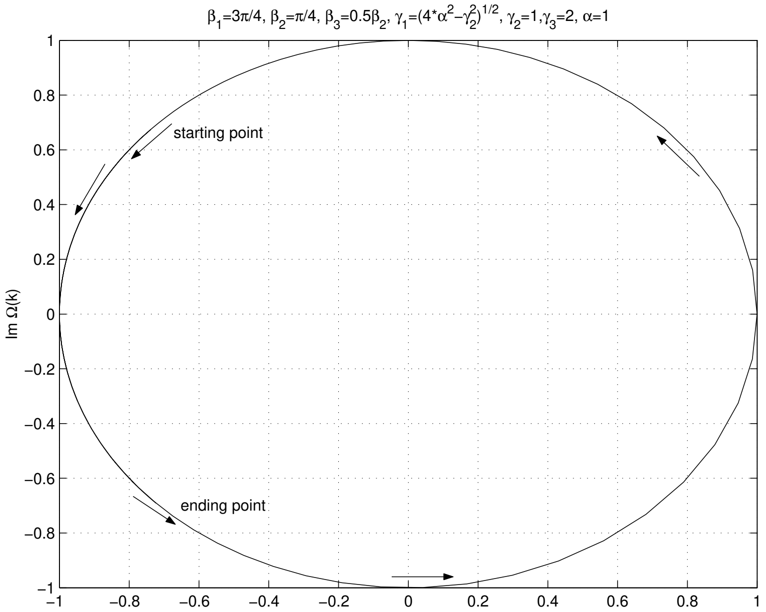

Figure 2: The set $\{\Omega(k); 0 < k < \infty\}$: $\Delta \in (2\pi, 3\pi)$.

The solution is unique, and it decays at infinity if and only if

$$\int_{-\infty}^{\infty} \frac{\omega_3(x)}{X^+(x)} x^j dx = 0, \quad j = 0, 1. \tag{5.18}$$

If these conditions are satisfied, then

$$\psi_3(\pm ik) = O(k^{-2-\frac{\Delta}{2\pi}}), \quad k \to \infty, \quad -1 \leq -2 - \frac{\Delta}{2\pi} < -\frac{1}{2}. \tag{5.19}$$

We emphasise that the integrals (5.18) are convergent at infinity since $[X^+(x)]^{-1}\omega_3(x) = O(x^{-2+\frac{\Delta}{2\pi}})$, $x \to \pm\infty$ and $-2 + \frac{\Delta}{2\pi} \in (-\frac{7}{2}, -3]$.

2°. If $-2\pi < \Delta \leq 0$, then $p = 0$, $\mathcal{P}_\varkappa(k) \equiv 0$. The solution exists and it is unique. It has the following behaviour at zero and infinity

$$\psi_3(\pm ik) = O(k^{\frac{\Delta}{2\pi}}), \quad k \to 0, \quad -1 < \frac{\Delta}{2\pi} \leq 0,$$

$$\psi_3(\pm ik) = O(k^{-1-\frac{\Delta}{2\pi}}), \quad k \to \infty, \quad -1 \leq -1 - \frac{\Delta}{2\pi} < 0. \tag{5.20}$$

3°. If $0 < \Delta \leq 2\pi$, then $p = 0$, $\mathcal{P}_\varkappa(k) = C_0$, C_0 is an arbitrary constant, and

$$\psi_3(\pm ik) = O(k^{\frac{\Delta}{2\pi}}), \quad k \to 0, \quad 0 < \frac{\Delta}{2\pi} \leq 1,$$

$$\psi_3(\pm ik) = O(k^{-\frac{\Delta}{2\pi}}), \quad k \to \infty, \quad -1 \leq -\frac{\Delta}{2\pi} < 0. \tag{5.21}$$

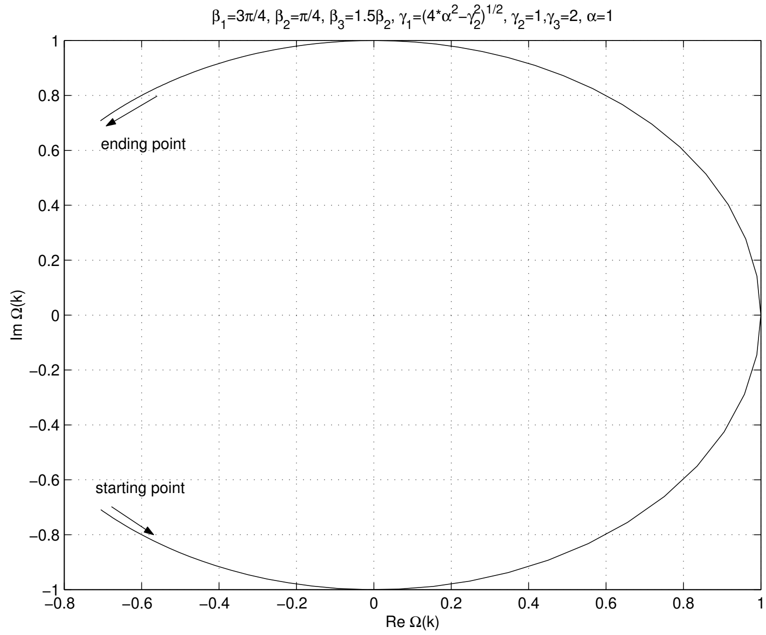

Figure 3: The set $\{\Omega(k); 0 < k < \infty\}$: $\Delta \in (3\pi/2, 2\pi)$.

4°. If $2\pi < \Delta < 3\pi$, then $p = 0$, $\mathcal{P}_\varkappa(k) = C_0 k + C_1$, C_0, C_1 are arbitrary constants, and

$$\psi_3(\pm ik) = O(k^{\frac{\Delta}{2\pi}}), \quad k \to 0, \quad 1 < \frac{\Delta}{2\pi} < \frac{3}{2},$$

$$\psi_3(\pm ik) = O(k^{1-\frac{\Delta}{2\pi}}), \quad k \to \infty, \quad -\frac{1}{2} < 1 - \frac{\Delta}{2\pi} < 0. \tag{5.22}$$

In the cases $1^\circ - 3^\circ$ the integral (5.16) is always convergent. This integral is also convergent in the case 4° provided that the function $\omega_2(x)$ satisfies the condition

$$|\omega_3(x)| \leq c|x|^{-\frac{\Delta}{2\pi}}, \quad |x| \to \infty, \quad c = \text{const}. \tag{5.23}$$

We now show how the constants d_0 and d_1 can be obtained. If $\psi_3(ik) = O(k^{-1})$, $k \to \infty$, $k \in \mathbb{C}^+$, then by the abelian theorem $q(x, l)$ is bounded at $x = 0$. Since the function $\psi_1(ik)$ solves the RH problem (4.4) the function $q(x, 0)$ is also bounded at $x = 0$. Therefore the constants d_0 and d_1 can be fixed in the same manner as in Section 4.

In the case that $\psi_3(ik) = O(k^\zeta)$, $k \to \infty$, $k \in \mathbb{C}^+$ and $-1 < \zeta < 0$, then by the abelian theorem the function $q(x, l)$ has an integrable singularity at $x = 0$: $q(x, l) = O(x^{-\zeta-1})$, $x \to 0$. This means (see Remark 3.1) that the constant d_1 vanishes. The other constant d_0 can be found from a linear algebraic equation by the method of Section 4.

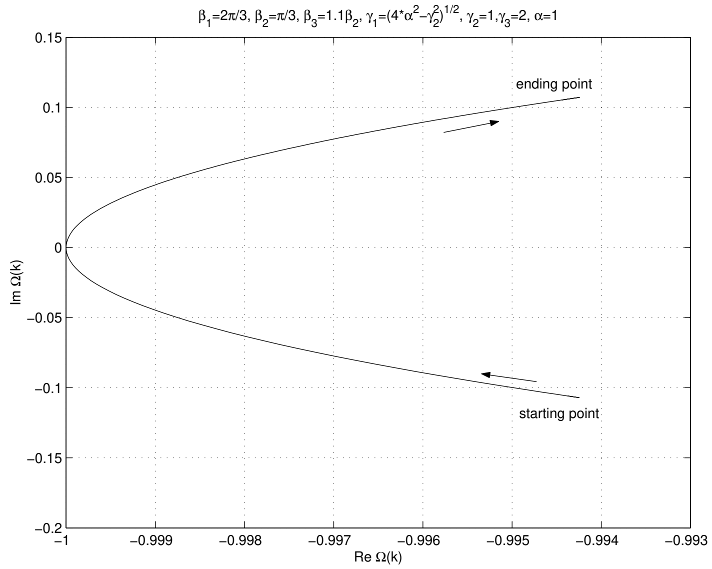

Figure 4: The set $\{\Omega(k); 0 < k < \infty\}$: $\Delta \in (-\pi, 0)$.

6 Analysis of the matrix RH problem associated with the Laplace equation

Letting $\alpha = 0$ in equation (1.6) we obtain

$$q_z = -\frac{1}{2\pi i}\sum_{j=1}^{3}\int\limits_{l_j} e^{ikz}h_j(k)dk, \quad 0 < x < \infty, \quad 0 < y < l. \tag{6.1}$$

The functions $h_j(k)$ are defined by (3.5) with $E(k) = e^{kl}$. The sectionally-holomorphic functions $\psi_1(\pm ik)$ and $\psi_3(\pm ik)$ solve the 2×2 matrix RH problem (3.14) with the functions J_j defined by

$$J_1(k) = \frac{\gamma_1 + ke^{i\beta_1}}{2\sin\beta_1}, \quad J_2(k) = \frac{\gamma_2 - ke^{-i\beta_2}}{2\sin\beta_2}, \quad J_3(k) = \frac{\gamma_3 + ke^{-i\beta_3}}{2\sin\beta_3}. \tag{6.2}$$

Note that the matrix RH problem for the Laplace equation in a semi-infinite strip was also derived in [3]. Here we analyse it and distinguish two cases : (i) $\gamma_1 = \gamma_2 = \gamma_3 = 0$ and (ii) $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 \neq 0$.

6.1 Case $\gamma_1 = \gamma_2 = \gamma_3 = 0$

In this section we analyse the matrix RH problem (3.14) in the particular case of $\alpha = 0$ and $\gamma_1 = \gamma_2 = \gamma_3 = 0$. We have

$$\boldsymbol{\psi}^+(k) = \mathbf{H}_0(k)\boldsymbol{\psi}^-(k) + \boldsymbol{\mu}(k), \quad k \in \mathbb{R}, \tag{6.3}$$

where

$$\boldsymbol{\psi}^\pm(k) = \begin{pmatrix} \psi_1(\pm ik) \\ \psi_3(\pm ik) \end{pmatrix},$$

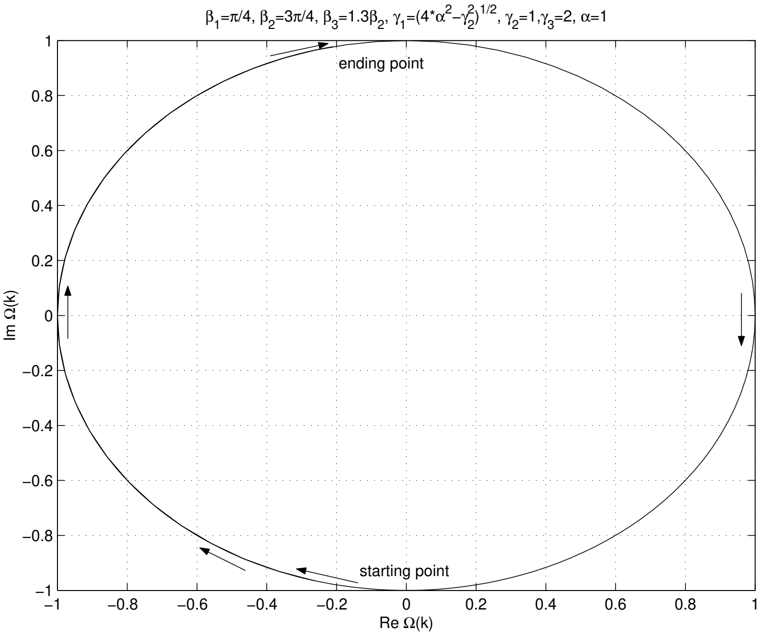

Figure 5: The set $\{\Omega(k); 0 < k < \infty\}$: $\Delta \in (-3\pi, -2\pi)$.

$$\mathbf{H}_0(k) = -\frac{1}{\sinh(kl + i\pi b)} \begin{pmatrix} \sinh(kl - i\pi a) & \frac{i \sin\beta_1}{\sin\beta_3} \sin\pi(b-a) \\ \frac{i\sin\beta_3}{\sin\beta_1}\sin\pi(b+a) & \sinh(kl + i\pi a) \end{pmatrix},$$

$$a = \frac{\beta_1 + 2\beta_2 - \beta_3}{\pi}, \quad b = \frac{\beta_1 + \beta_3}{\pi}, \tag{6.4}$$

Since $0 < \beta_j < \pi$ $(j = 1, 2, 3)$ it follows $-1 < a < 3$, $0 < b < 2$.

6.1.1 Case $\sin\pi(b \pm a) \neq 0$, $a = 0$.

We construct a closed-form solution of the matrix problem (6.3) assuming that $a = 0$ and $\sin\pi(b \pm a) \neq 0$, i.e. $\beta_2 = \frac{1}{2}(\beta_3 - \beta_1)$ and $\beta_1 \neq -\beta_2 + \frac{1}{2}\pi m$ $(m = 1, 2, 3)$, $\beta_3 \neq \beta_2 + \frac{1}{2}\pi n$ $(m = 0, \pm 1)$. In this case the system of functional equations (6.3) decouples to two separate scalar RH problems

$$\phi_j^+(k) = \lambda_j(k)\phi_j^-(k) + \eta_j(k), \quad k \in \mathbb{R}, \quad j = 1, 2. \tag{6.5}$$

Here

$$\phi_j^\pm(k) = \frac{1}{2}\left[\psi_1(\pm ik) + (-1)^{j-1}\frac{\sin\beta_1}{\sin\beta_3}\psi_3(\pm ik)\right],$$

$$\eta_j(k) = \frac{1}{2}\left[\mu_1(k) + (-1)^{j-1}\frac{\sin\beta_1}{\sin\mu_3}\mu_3(k)\right],$$

$$\lambda_j(k) = -\frac{\sinh kl + (-1)^{j-1} i \sin\pi b}{\sinh(kl + \pi i b)}, \quad j = 1, 2. \tag{6.6}$$

The functions $\lambda_j(k)$ can be represented in terms of the Γ-functions as follows

$$\lambda_1(k) = -\frac{\Gamma(1/2 + b/2 - ik')\Gamma(1/2 - b/2 + ik')}{\Gamma(1/2 + b/2 + ik')\Gamma(1/2 - b/2 - ik')},$$

$$\lambda_2(k) = \frac{\Gamma(b/2 - ik')\Gamma(1 - b/2 + ik')}{\Gamma(b/2 + ik')\Gamma(1 - b/2 - ik')}, \tag{6.7}$$

where $k' = \frac{kl}{2\pi}$, $b = \frac{2}{\pi}(\beta_1 + \beta_2)$. Next, we factorise the functions $\lambda_j(k)$

$$\lambda_j(k) = \frac{X_j^+(k)}{X_j^-(k)}, \quad k \in \mathbb{R}, \tag{6.8}$$

where

$$X_1^\pm(k) = \pm\frac{\Gamma(1/2 + b/2 \mp ik' - \delta)}{\Gamma(1/2 - b/2 \mp ik' + \delta)}, \quad X_2^\pm(k) = \frac{\Gamma(b/2 \mp ik')}{\Gamma(1 - b/2 \mp ik')},$$

$$\delta = \begin{cases} 0, & 0 < b < 1 \\ 1, & 1 < b < 2. \end{cases} \tag{6.9}$$

The functions $X_j^\pm(k)$ are analytic and do not vanish in $\mathbb{C}^\pm$. To construct the solution we need the Cauchy integral

$$\chi_j^\pm(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\eta_j(x)dx}{X_j^+(x)(x - k)}, \quad k \in \mathbb{C}^\pm \setminus \mathbb{R}. \tag{6.10}$$

Because of the asymptotics of the functions

$$X_1^\pm(k) \sim \pm(\mp ik')^{b-2\delta}, \quad X_2^\pm(k) \sim (\mp ik')^{b-1}, \quad k \to \infty, \quad k \in \mathbb{C}^\pm, \tag{6.11}$$

and $\eta_j(x) = O(x^{-1})$, $x \to \infty$, the integral (6.10) converges at infinity. By substituting formula (6.8) into (6.5) and by using (6.10) and Liouville's theorem we find:

If $0 < \beta_1 + \beta_2 < \frac{1}{2}\pi$ $(0 < b < 1)$, then

$$\phi_1^\pm(k) = X_1^\pm(k)\chi_1^\pm(k), \quad k \in \mathbb{C}^\pm,$$

$$\phi_2^\pm(k) = X_2^\pm(k)[C + \chi_2^\pm(k)], \quad k \in \mathbb{C}^\pm, \tag{6.12}$$

where C is an arbitrary constant.

If $\frac{1}{2}\pi < \beta_1 + \beta_2 < \pi$ $(1 < b < 2)$, then

$$\phi_1^\pm(k) = X_1^\pm(k)[C + \chi_1^\pm(k)], \quad k \in \mathbb{C}^\pm,$$

$$\phi_2^\pm(k) = X_2^\pm(k)\chi_2^\pm(k), \quad k \in \mathbb{C}^\pm. \tag{6.13}$$

The potentials $\psi_1(\pm ik)$ and $\psi_3(\pm ik)$ can be expressed through $\phi_1^\pm(k)$ and $\phi_2^\pm(k)$ from (6.6)

$$\psi_1(\pm ik) = \phi_1^\pm(k) + \phi_2^\pm(k),$$

$$\psi_3(\pm ik) = \frac{\sin\beta_3}{\sin\beta_1}[\phi_1^\pm(k) - \phi_2^\pm(k)]. \tag{6.14}$$

Using the definition of the functions $\psi_1(-ik)$, $\psi_3(-ik)$

$$\psi_1(-ik) = \int_0^\infty e^{-ikx} q(x,0)dx, \quad \psi_3(-ik) = \int_0^\infty e^{-ikx} q(x,l)dx, \tag{6.15}$$

the asymptotics of the above integrals

$$\psi_j(-ik) = O(k^{b-1-\delta}), \quad k \to \infty, \quad k \in \mathbb{C}^-, \tag{6.16}$$

and the abelian theorem we estimate the asymptotics of the boundary values of the function $q(x,y)$ at the corners of the semi-strip D

$$q(x,0) = O(x^{-b+\delta}), \quad q(x,l) = O(x^{-b+\delta}), \quad x \to 0, \quad -b+\delta \in (-1,0). \tag{6.17}$$

Thus, $d_0 = d_1 = 0$.

6.1.2 Scalar cases

We assume that $\sin\pi(b-a) = \sin\pi(b+a) = 0$, i.e. $\beta_1 = -\beta_2 + \frac{1}{2}\pi m$ $(m = 1,2,3)$ and $\beta_3 = \beta_2 + \frac{1}{2}\pi n$ $(n = 0, \pm 1)$. In this case the matrix $\mathbf{H}_0(k)$ becomes diagonal and constant, $\mathbf{H}_0 = \mathrm{diag}\{(-1)^{m-1}, (-1)^{n-1}\}$, and therefore the RH problem (3.14) reduces to

$$\begin{aligned} \psi_1(ik) &= (-1)^{m-1}\psi_1(-ik) + \mu_1(k), \\ \psi_3(ik) &= (-1)^{n-1}\psi_3(-ik) + \mu_3(k). \end{aligned} \tag{6.18}$$

Consequently, the solution of the above problems is easily found in terms of the Cauchy integrals. Clearly, to define the boundary values of the function $q(x,y)$ at $y=0$ and $y=l$ one can apply the Fourier transform to (6.18) using formulae (3.8) for $\alpha = 0$. Thus, for positive x,

$$q(x,0) = \hat{\mu}_1(x), \quad q(x,l) = \hat{\mu}_3(x), \tag{6.19}$$

where

$$\hat{\mu}_j(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mu_j(k) e^{-ikx} dk. \tag{6.20}$$

The functions $\hat{\mu}_j(x)$ $(j = 1,3)$ are bounded at $x = 0$ and therefore the constants d_0, d_1 can be fixed by the method of Section 4.

6.1.3 Triangular cases

We assume now that $\sin\pi(b-a) = 0$ and $\sin\pi(b+a) \neq 0$, i.e. $\beta_1 \neq -\beta_2 + \frac{1}{2}\pi m$ $(m = 1,2,3)$ and $\beta_3 = \beta_2 + \frac{1}{2}\pi n$ $(n = 0, \pm 1)$. In this case, $\mathbf{H}_0(s)$ is the following triangular matrix

$$\mathbf{H}_0(s) = (-1)^{n-1} \begin{pmatrix} \frac{\sin(ikl+\pi b)}{\sin(ikl-\pi b)} & 0 \\ -\frac{\sin\beta_3 \sin 2\pi b}{\sin\beta_1 \sin(ikl-\pi b)} & 1 \end{pmatrix}. \tag{6.21}$$

The problem of interest is the scalar RH problem

$$\psi_1(ik) = (-1)^{n-1}\frac{\sin\pi(2ik'+b)}{\sin\pi(2ik'-b)}\psi_1(-ik) + \mu_1(k), \quad k \in \mathbb{R}, \tag{6.22}$$

where as before $k' = \frac{kl}{2\pi}$. The factorization of the coefficient of the RH problem is given by

$$(-1)^{n-1}\frac{\sin\pi(2ik'+b)}{\sin\pi(2ik'-b)} = \frac{X^+(k)}{X^-(k)}, \quad k \in \mathbb{R}, \tag{6.23}$$

where

$$X^+(k) = \frac{\Gamma(b-\delta-2ik')}{\Gamma(1-b+\delta-2ik')}, \quad X^-(k) = (-1)^n\frac{\Gamma(b-\delta+2ik')}{\Gamma(1-b+\delta+2ik')}. \tag{6.24}$$

The solution to the RH problem (6.22) in the cases $0 < b < \frac{1}{2}$ and $1 < b < \frac{3}{2}$ involves an arbitrary constant:

$$\psi_1(\pm ik) = X^\pm(k)[\chi^\pm(k) + C], \quad k \in \mathbb{C}^\pm. \tag{6.25}$$

In the cases $\frac{1}{2} < b < 1$ and $\frac{3}{2} < b < 2$ it is unique:

$$\psi_1(\pm ik) = X^\pm(k)\chi^\pm(k), \quad k \in \mathbb{C}^\pm, \tag{6.26}$$

where

$$\chi^\pm(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mu_1(x)dx}{X^+(x)(x-k)}, \quad k \in \mathbb{C}^\pm \setminus \mathbb{R}. \tag{6.27}$$

We note that $b = \frac{1}{2}$ or $b = \frac{3}{2}$ imply $\sin\pi(b+a) = 0$. The asymptotics at infinity of the solution $\psi_1(\pm ik)$ follows from the analysis of formulae (6.24) to (6.27)

$$\psi_1(\pm ik) = O(k^{2b-2\delta-1-\delta'}), \quad k \to \infty, \quad k \in \mathbb{C}^\pm, \tag{6.28}$$

where

$$\delta' = \begin{cases} 0, & 0 < b < \frac{1}{2} \text{ or } 1 < b < \frac{3}{2} \\ 1, & \frac{1}{2} < b < 1 \text{ or } \frac{3}{2} < b < 2 \end{cases}. \tag{6.29}$$

The abelian theorem yields the asymptotics of the function $q(x,0)$ as $x \to 0$

$$q(x,0) = O(x^{2\delta+\delta'-2b}), \quad x \to 0. \tag{6.30}$$

Clearly, $2\delta+\delta'-2b \in (-1,0)$. The above asymptotics implies that $d_0 = 0$. The constant d_1 can be found from the corresponding linear algebraic equation following the procedure of Section 4.

The remaining possible case $\sin 2(\beta_1+\beta_2) = 0$, $\sin 2(\beta_3-\beta_2) \neq 0$, can be treated similarly.

6.2 Case $\gamma_1^2+\gamma_2^2+\gamma_3^3\neq 0$: regularisation of the RH problem

We assume that at least one of the constants γ_j $(j=1,2,3)$ is different than zero. We aim to regularise the matrix RH problem associated with the Laplace equation in the case $a=0$, $\beta_1\neq-\beta_2+\frac{1}{2}\pi m$ $(m=1,2,3)$ and $\beta_3\neq\beta_2+\frac{1}{2}\pi n$ $(n=0,\pm1)$.

According to the Carleman-Vekua method, the corresponding system of Fredholm's equations can be written explicitly since the dominant part of the matrix $\mathbf{H}(k)$, the matrix $\mathbf{H}_0(k)$, has already been factorised (Section 6.1.1). Using the asymptotics of the matrix $\mathbf{H}(k)$ as $k\to\infty$

$$\mathbf{H}(k)=-\frac{1}{\sinh(kl+i\pi b)}\begin{pmatrix}\sinh kl[1+O(k^{-1})] & \frac{i\sin\beta_1}{\sin\beta_3}\sin\pi b[1+O(k^{-1}]\\ \frac{i\sin\beta_3}{\sin\beta_1}\sin\pi b[1+O(k^{-1})] & \sinh kl[1+O(k^{-1})]\end{pmatrix}, \tag{6.31}$$

we represent the matrix $\mathbf{H}(k)$ as follows

$$\mathbf{H}(k)=\mathbf{H}_0(k)+\tilde{\mathbf{H}}(k),\quad \tilde{\mathbf{H}}(k)=\{\tilde{H}_{mj}(k)\}_{m,j=1,2}. \tag{6.32}$$

By following the procedure of Section 6.1 we obtain

$$\phi_j^+(k)=\lambda_j(k)\phi_j^-(k)+\tilde{\eta}_j(k),\quad k\in\mathbb{R},\quad j=1,2, \tag{6.33}$$

where

$$2\tilde{\eta}_j(k)=\mu_1(k)+\tilde{H}_{11}(k)\psi_1(-ik)+\tilde{H}_{12}(k)\psi_3(-ik)$$
$$-(-1)^j\frac{\sin\beta_1}{\sin\beta_3}[\mu_3(k)+\tilde{H}_{21}(k)\psi_1(-ik)+\tilde{H}_{22}(k)\psi_3(-ik)]. \tag{6.34}$$

Using the solution of Section 6.1 yields another representation of the RH problem. For example, in the case $0<b<1$ we have

$$\psi_1(-ik)=\frac{X_1^-(k)}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\eta}_1(\zeta)d\zeta}{X_1^+(\zeta)(\zeta-k)}+X_2^-(k)\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\eta}_2(\zeta)d\zeta}{X_2^+(\zeta)(\zeta-k)}+C\right].$$

$$\psi_3(-ik)=\frac{\sin\beta_3}{\sin\beta_1}\left\{\frac{X_1^-(k)}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\eta}_1(\zeta)d\zeta}{X_1^+(\zeta)(\zeta-k)}-X_2^-(k)\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\eta}_2(\zeta)d\zeta}{X_2^+(\zeta)(\zeta-k)}+C\right]\right\}. \tag{6.35}$$

Substituting (6.34) into (6.35) and then taking the limit as $k=k_R-i0$, we find a system of the Fredholm integral equations [7]. Alternatively, we can apply the inverse Fourier transform to the above equations, change the order of integration and make the substitution $\pi x=-l\log\xi$,

$$\mathbf{r}(\xi)+\int_0^1\mathbf{R}(\xi,\tau)\mathbf{r}(\tau)d\tau=\mathbf{p}(\xi),\quad 0<\xi<1, \tag{6.36}$$

where

$$r_1(\xi)=q\left(\frac{l}{\pi}\log\frac{1}{\xi},0\right),\quad r_2(\xi)=q\left(\frac{l}{\pi}\log\frac{1}{\xi},l\right), \tag{6.37}$$

$R_{mj}(\xi,\tau)$, the elements of the matrix $\mathbf{R}(\xi,\tau)$, are Fredholm's kernels. They are represented by double quadratures and can be evaluated by the residue theorem.

7 Conclusions

In this paper we have studied the modified Helmholtz equation (1.2) in a semi-strip with the Poincaré type boundary conditions (1.1). On each side of the semi-strip, the boundary conditions involve two real constants β_j, γ_j and a real-valued function g_j, $j = 1, 2, 3$.

Using the method reviewed in [1] it is straightforward to reduce the above boundary-value problem to a 2×2 matrix RH problem for the two sectionally holomorphic functions $\psi_1(\pm ik)$ and $\psi_3(\pm ik)$, $k \in \mathbb{C}^{\pm}$. The jump matrix $H(k)$ of the associated RH problem (3.14) on the real k-axis is uniquely defined in terms of the scalar functions $J_j(k)$, $j = 1, 2, 3$, which are in turn defined in terms of the constant α entering in the modified Helmholtz equation, and in terms of the constants β_j, γ_j, $j = 1, 2, 3$.

A crucial role in the investigation of the above RH problem is played by the products

$$J_{12}(k)J_{12}(-k), \quad J_{32}(k)J_{32}(-k), \tag{7.1}$$

where $J_{12}(k)$ and $J_{32}(k)$ are defined in terms of $J_j(k)$ $j = 1, 2, 3$, by equation (4.1). There exist the following three particular cases:

I. $J_{12}(k)J_{12}(-k) = 1, \quad J_{32}(k)J_{32}(-k) = 1.$

In this case the basic RH problem reduces to two separate scalar RH problems, one for the sectionally holomorphic function $\psi_1(\pm ik)$ and one for $\psi_3(\pm ik)$. Each of these RH problems can be solved in closed form; the solutions depend on the particular relations between β_j and γ_j. (a) If $\gamma_1 = \sqrt{4\alpha^2 - \gamma_2^2}$, $\beta_1 = \pi - \beta_2$, then the solution $\psi_1(\pm ik)$ exists under the conditions (4.33) and it is given by equation (4.31). (b) If $\gamma_1 = \gamma_2$, $\beta_1 = \frac{1}{2}\pi - \beta_2$, $\psi_1(\pm ik)$ is given by equation (4.34). Similar considerations are valid for $\psi_3(\pm ik)$.

The solution $q(x, y)$ of the modified Helmholtz equation also depends on the particular relations between β_j and γ_j: 1. If these parameters satisfy (4.18) or (4.19) or (4.20), then $q = I_0 + I_2$, where the integrals I_0 and I_2 depend only on the given boundary conditions, see equations (4.14) (in these cases $\psi_1(\pm ik)$ and $\psi_3(\pm ik)$ do not contribute to the solution). 2. If the parameters satisfy (4.22), then $q = I_0 + I_1 + I_2$, where I_1 depends on $\psi_1(ik_1^{(2)})$ and $k_1^{(2)}$ is known , see equation (4.16). 3. If the parameters satisfy (4.24), then $q = I_0 + I_2 + I_3$, where I_3 depends on $\psi_3(ik_2^{(2)})$, and $k_2^{(2)}$ is known, see equation (4.16).

Having constructing $\psi_1(ik)$, $\psi_3(ik)$, the values $\psi_1(ik_1^{(2)})$ and $\psi_3(ik_2^{(2)})$ follow.

II. $J_{12}(k)J_{12}(-k) = 1, \quad J_{32}(k)J_{32}(-k) \neq 1.$

In this case the basic RH problem is triangular. It can be reduced to the scalar RH problem (5.5) for the sectionally holomorphic function $\psi_3(\pm ik)$ and to a scalar RH problem for $\psi_1(\pm ik)$ whose jump depends on $\psi_3(\pm ik)$. Thus, after determining $\psi_3(\pm ik)$, $\psi_1(\pm ik)$ can be computed in closed form by solving a scalar RH problem similar to the one mentioned in I above. The scalar RH problem for $\psi_3(\pm ik)$, whose coefficient is discontinuous at $k = 0$ and $k = \infty$, is solved by equations (5.15).

The solution q is given by $q = I_0 + I_1 + I^*$, where I_1 depends of $\psi_1(\pm ik)$ (see (4.14)) and I^* depends on $\psi_3(\pm ik)$ (see (5.3).

III. $J_{12}(k)J_{12}(-k) \neq 1, \quad J_{32}(k)J_{32}(-k) = 1.$

This case is similar to II above.

We have also analysed the basic RH problem associated with the Laplace equation ($\alpha = 0$). We have shown that if $\beta_2 = \frac{1}{2}(\beta_3 - \beta_2)$ and $\gamma_1 = \gamma_2 = \gamma_3$, then the corresponding matrix RH problem can be solved in closed form. If however at least one of the parameters γ_j $(j = 1, 2, 3)$ is not equal to zero then a closed-form solution does not appear feasible. In this case the RH problem has been regularised, i.e. reduced to a system of two Fredholm integral equations.

Remark 7.1. We recall that the spectral functions (2.13) depend on the three unknown functions $\psi_1(ik)$, $\psi_2(ik)$, $\psi_3(ik)$. The basic RH problem is a consequence of eliminating $\psi_2(ik)$. It was shown in [3] that it is also possible to solve the Laplace equation in the semi-strip by eliminating either $\psi_1(ik)$ or $\psi_3(ik)$ instead of $\psi_2(k)$. This alternative approach has certain advantages since $\psi_2(k)$ is an *entire* function (see [6] for the analogous problem for the evolution equation). The implementation of this approach to the modified Helmholtz equation remains open.

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