# The Basic Elliptic Equations in an Equilateral Triangle

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May 2004

**Abstract**

In his deep and prolific investigations of heat diffusion, Lamé was led to the investigation of the eigenvalues and eigenfunctions of the Laplace operator in an equilateral triangle. In particular he derived explicit results for the Dirichlet and Neumann cases using an ingenious change of variables. The relevant eigenfunctions are complicated infinite series in terms of his variables.

Here we first show that boundary value problems with simple boundary conditions, such as the Dirichlet and the Neumann problems, can be solved in an elementary manner. In particular for these problems, the unknown Neumann and Dirichlet boundary values respectively, can be expressed in terms of a Fourier series. Our analysis is based on the so called global relation, which is an algebraic equation coupling the Dirichlet and the Neumann spectral values on the perimeter of the triangle.

As Lamé correctly pointed out, infinite series are inadequate for expressing the solution of more complicated problems, such as mixed boundary value problems. Here we show, utilizing further the global relation, that such problems can be solved in terms of *generalized Fourier integrals*.

## 1 Introduction

We first introduce some notations.

**Notations and Useful Identities**

(i) $z$ will denote the usual complex variable and $\alpha$ will denote one of the complex roots of unity,

$$z = x + iy, \quad \alpha = e^{\frac{2i\pi}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}. \tag{1.1}$$

Bar will denote complex conjugation, in particular

$$\bar{z} = x - iy, \quad \bar{\alpha} = e^{-\frac{2i\pi}{3}}.$$

$\overline{F(\bar{k})}$ will denote the *Schwarz conjugate* of the function $F(k)$.

(ii) The complex numbers

$$z_1 = \frac{l}{\sqrt{3}} e^{\frac{i\pi}{3}}, \quad z_2 = \bar{z}_1, \quad z_3 = -\frac{l}{\sqrt{3}}, \tag{1.2}$$

will denote the vertices of the equilateral triangle, and $D \subset \mathbb{C}$ will denote the interior of the triangle. The length of each side is $l$.

The sides $(z_2, z_1)$, $(z_3, z_2)$, $(z_1, z_3)$ will be referred to as sides (1), (2), (3) respectively.

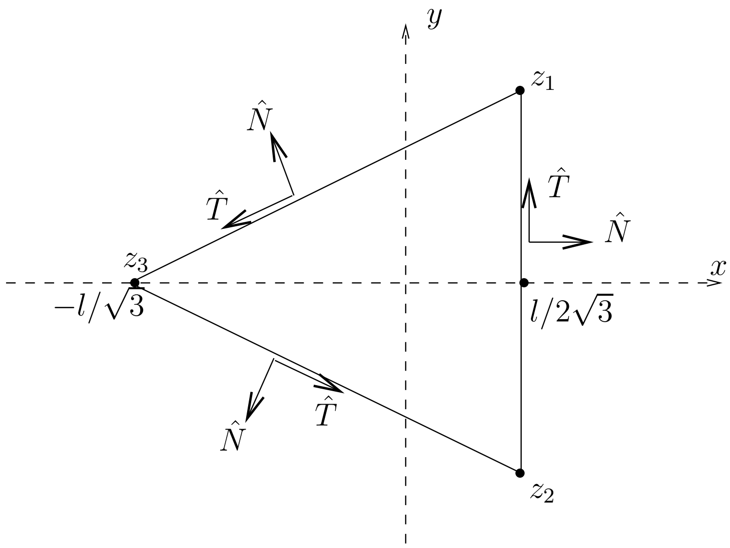


**Figure 1.1:** The fundamental domain $D$

(iii) On each side we identify the positive direction $\hat{\mathbf{T}}$ and the outward normal $\hat{\mathbf{N}}$ as in Figure 1.1. The functions

$$q^{(j)}(s), \quad q_N^{(j)}(s), \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \quad j = 1, 2, 3, \tag{1.3}$$

will denote the function $q(x, y)$, as well as the derivative of $q(x, y)$ along the outward normal $\hat{\mathbf{N}}$ respectively, for the side $(j)$.

(iv) $E(k)$ and $e(k)$ will denote the following exponential functions

$$E(k) = e^{\left(k + \frac{\lambda}{k}\right)\frac{l}{2\sqrt{3}}}, \quad e(k) = e^{\left(k + \frac{\lambda}{k}\right)\frac{l}{2}}. \tag{1.4}$$

(v) The numbers $\alpha$ and $\bar{\alpha}$ satisfy the relations

$$\alpha^2 = \bar{\alpha} = \alpha^{-1}, \quad 1 + \alpha + \bar{\alpha} = 0, \quad i\bar{\alpha} - i\alpha = \sqrt{3}, \quad i\alpha - i = \sqrt{3}\bar{\alpha}. \tag{1.5}$$

These relations imply analogous relations for $E(k)$ and $e(k)$. For example, the last three equations in (1.5) imply

$$E(k)E(\alpha k)E(\bar{\alpha} k) = 1, \quad E(i\bar{\alpha} k)E(-i\alpha k) = e(k), \quad E(i\alpha k)E(-ik) = e(\bar{\alpha} k). \tag{1.6}$$

**Formulation of the Problem**

We will investigate the basic elliptic equations in the interior of the equilateral triangle $D$, namely we will study the equation

$$q_{xx} + q_{yy} - 4\lambda q = 0, \quad (x, y) \in D, \tag{1.7}$$

where $q(x, y)$ is a real valued function and $\lambda$ is a real constant. The case of $\lambda = 0$, of $\lambda$ negative, and of $\lambda$ positive, correspond to the Laplace, the Helmholtz, and the modified Helmholtz equations, respectively. We will analyze the following problems:

(i) The Dirichlet problem

$$q^{(j)}(s) = f_j(s), \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \quad j = 1, 2, 3. \tag{1.8}$$

(ii) The oblique Robin problem,

$$\sin\beta q_N^{(j)}(s) + \cos\beta \frac{d}{ds} q^{(j)}(s) + \gamma q^{(j)}(s) = f_j(s), \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \quad j = 1, 2, 3, \tag{1.9}$$

where $\beta$ and $\gamma$ are real constants and $\sin\beta \neq 0$. The sum of the first two terms of the lhs of this equation equals the derivative of $q^{(j)}(s)$ in the direction making an angle $\beta$ with the positive direction of the side $(j)$. The Neumann and the Robin problems correspond to the following particular choices of $\beta$ and $\gamma$,

$$\text{Neumann}: \ \beta = \frac{\pi}{2}, \ \ \gamma = 0; \quad \text{Robin}: \ \beta = \frac{\pi}{2}, \ \ \gamma \neq 0. \tag{1.10}$$

(iii) The Poincaré type problem

$$\sin\beta_j q_N^{(j)}(s) + \cos\beta_j \frac{d}{ds} q_N^{(j)}(s) + \gamma_j q^{(j)}(s) = f_j(s), \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \quad j = 1, 2, 3, \tag{1.11}$$

where $\beta_1$ is a real constant such that $\sin\beta_1 \neq 0$, $\beta_2$ and $\beta_3$ satisfy $\sin\beta_2 \neq 0$, $\sin\beta_3 \neq 0$ and are given in terms of $\beta_1$ by the expressions

$$\beta_2 = \beta_1 + \frac{n\pi}{3}, \quad \beta_3 = \beta_1 + \frac{m\pi}{3}, \quad m, n \in \mathbb{Z}, \tag{1.12$a$}$$

and the real constants $\{\gamma_j\}_1^3$ satisfy the relations

$$\sin 3\beta_1 \left[\gamma_2(3\lambda - \gamma_2^2) - e^{in\pi}\gamma_1(3\lambda - \gamma_1^2)\right] = 0, \tag{1.12$b$}$$

$$\sin 3\beta_1 \left[\gamma_3(3\lambda - \gamma_3^2) - e^{im\pi}\gamma_1(3\lambda - \gamma_1^2)\right] = 0. \tag{1.12c}$$

A particular case of such a Poincaré type problem, which is solved in detail, is the modified Helmholtz equation with Neumann values on sides (2) and (3) and with Robin values on side (1), where the constant $\gamma$ is given by $\sqrt{3\lambda}$.

We *assume* that the functions $f_j$ have sufficient smoothness and that they are compatible at the corners of the triangle. The case of boundary conditions which are discontinuous at the corners will be considered elsewhere.

**The Global Relation**

The global relation, which is the fundamental algebraic relation which couples the Dirichlet and the Neumann values around the perimeter of the triangle, is [3]

$$E(-ik)\Psi_1(k) + E(-i\bar{\alpha}k)\Psi_2(\bar{\alpha}k) + E(-i\alpha k)\Psi_3(\alpha k)$$

$$= 2i\left\{E(-ik)\Phi_1(k) + E(-i\bar{\alpha}k)\Phi_2(\bar{\alpha}k) + E(-i\alpha k)\Phi_3(\alpha k)\right\}, \quad k \in \mathbb{C} - \{0\}, \tag{1.13}$$

where the exponential function $E(k)$ is defined in equation (1.4a), and $\Psi_j$ and $\Phi_j$ are the following transforms of the Neumann and Dirichlet boundary values:

$$\Psi_j(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{(k+\frac{\lambda}{k})s} q_N^{(j)}(s)ds, \quad \Phi_j(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{(k+\frac{\lambda}{k})s}\left[\frac{1}{2}\frac{d}{ds}q^{(j)}(s) + \frac{\lambda}{k}q^{(j)}(s)\right]ds, \tag{1.14}$$

for each $j = 1, 2, 3$, and every complex $k \neq 0$.

The general methodology introduced in [3] implies that the global relation must be supplemented by its Schwarz conjugate, as well as by the four equations obtained from these two equations by replacing $k$ with $\alpha k$ and with $\bar{\alpha}k$. We will refer to these six equations as the *basic algebraic relations*. In this paper we present two different techniques for solving these equations.

**Solutions via Infinite Series**

For simple problems it is possible to compute the unknown boundary values by evaluating the basic algebraic relations at particular discrete values of $k$. This yields the unknown boundary values in terms of infinite series. The Dirichlet and the Neumann problems are examples of problems which can be solved using this technique.

We use the Dirichlet problem to illustrate this approach. In this case the functions $\Phi_j$ appearing in the rhs of the global relation (1.13) can be immediately computed in terms of the given boundary conditions $f_j$, thus the global relation becomes a single equation for the three unknown functions $\{\Psi_j\}_1^3$. Multiplying this equation by $E(i\alpha k)$, and multiplying the Schwarz conjugate of the global relation by $E(-i\alpha k)$, we find the following two equations (where we have used the last two of the identities (1.6))

$$e(\bar{\alpha}k)\Psi_1(k) + e(-k)\Psi_2(\bar{\alpha}k) + \Psi_3(\alpha k) = 2iA(k), \tag{1.15}$$

$$e(-\bar{\alpha}k)\Psi_1(k) + \Psi_2(\alpha k) + e(k)\Psi_3(\bar{\alpha}k) = -2iB(k). \tag{1.16}$$

In these equations $A(k)$ and $B(k)$ are known functions and $k \in \mathbb{C} - \{0\}$.

For the general Dirichlet problem, we will supplement these two equations with the four equations obtained from these equations by replacing $k$ with $\alpha k$ and with $\bar{\alpha} k$. However, there exists a particular case for which it is sufficient to analyze only the above two equations. This is the *symmetric* Dirichlet problem, namely the problem where the functions $f_j$ are all the same, $f_j = f$, $j = 1,2,3$. Then the Neumann values $q_N^{(j)}(s)$ are also the same, $q_N^{(j)} = q_N$ and hence $\Psi_j(k) = \Psi(k)$, $j = 1,2,3$. Thus equations(1.15) and (1.16) become two equations for the three unknown functions $\Psi(k)$, $\Psi(\bar{\alpha}k)$, $\Psi(\alpha k)$. Hence, any two of them can be expressed in terms of the remaining one, for example $\Psi(\bar{\alpha}k)$ and $\Psi(\alpha k)$ can be expressed in terms of $\Psi(k)$. In particular, subtracting equations (1.15), (1.16), we find

$$(e(k) - e(-k))\,\Psi(\bar{\alpha}k) = (e(\bar{\alpha}k) - e(-\bar{\alpha}k))\,\Psi(k) - 2iG(k), \tag{1.17}$$

where $G(k) = A(k) + B(k)$ is a known function. Equation (1.17) is a single equation for the two unknown functions $\Psi(\bar{\alpha}k)$ and $\Psi(k)$. However, by evaluating this equation at those values of $k$ for which the coefficient of $\Psi(\bar{\alpha}k)$ vanishes, i.e. at $e^2(k) = 1$, or $k = s_n$,

$$s_n + \frac{\lambda}{s_n} = \frac{2in\pi}{l}, \quad n \in \mathbb{Z}, \tag{1.18}$$

it follows that $\Psi(s_n)$ can be determined. Recalling the definition of $\Psi(k)$ and evaluating equation (1.17) at $k = s_n$, we find

$$\sinh\left[\left(\bar{\alpha}s_n + \frac{\lambda}{\bar{\alpha}s_n}\right)\frac{l}{2}\right]\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{2i\pi n\frac{s}{l}} q_N(s)ds = iG(s_n), \quad n \in \mathbb{Z}. \tag{1.19}$$

Thus $q_N(s)$ can be expressed as a Fourier series.

For the general Dirichlet problem (1.8), the six basic algebraic relations couple the nine unknown functions $\{\Psi_j(k),\ \Psi_j(\bar{\alpha}k),\ \Psi_j(\alpha k)\}_1^3$. Thus any six of them can be expressed in terms of the remaining three. In particular, it is shown in section 3 that $\Psi_2(\bar{\alpha}k)$ can be expressed in terms of $\{\Psi_j(k)\}_1^3$ by the equation

$$\begin{aligned}\left(e^3(-k) - e^3(k)\right)\Psi_2(\bar{\alpha}k) &= \left[e(-\bar{\alpha}k) - e^2(-k)e(\bar{\alpha}k)\right]\left(\Psi_1(k) + e^2(k)\Psi_3(k)\right)\\ &+ e^2(-k)\left[e(-\bar{\alpha}k) - e^4(k)e(\bar{\alpha}k)\right]\Psi_2(k) + 2iX(k),\end{aligned} \tag{1.20}$$

where $X(k)$ is known. In spite of the fact that this equation is a single equation for four unknown functions, it yields all the three Neumann values $q_N^{(j)}$, $j = 1,2,3$. Indeed, by evaluating equation (1.20) at those values of $k$ for which the coefficient of $\Psi_2(\bar{\alpha}k)$ vanishes, i.e. at $e^6(k) = 1$, or $k = k_m$, where

$$k_m + \frac{\lambda}{k_m} = \frac{2im\pi}{3l}, \quad m \in \mathbb{Z}, \tag{1.21}$$

equation (1.20) yields

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\frac{2i\pi m}{3l}s}\left[q_N^{(1)}(s) + e^{-\frac{2i\pi m}{3}} q_N^{(2)}(s) + e^{\frac{2i\pi m}{3}} q_N^{(3)}(s)\right]ds = M(k_m), \quad m \in \mathbb{Z}, \tag{1.22}$$

where $M(k_m)$ is known. This equation in contrast to equation (1.19) involves *three* unknown functions. However, equation (1.21) gives *three times* as many values for $m$ as equation (1.18). Replacing in equation (1.22) $m$ by $3n$, $3n-1$, $3n-2$, and inverting the left hand sides of the resulting equations, we find

$$q_N^{(1)}(s) + q_N^{(2)}(s) + q_N^{(3)}(s) = \frac{1}{l}\sum_{-\infty}^{\infty} M(3n)e^{-\frac{2i\pi ns}{l}},$$

$$e^{-\frac{2i\pi s}{3l}}\left[q_N^{(1)}(s) + \alpha q_N^{(2)}(s) + \bar{\alpha} q_N^{(3)}(s)\right] = \frac{1}{l}\sum_{-\infty}^{\infty} M(3n-1)e^{-\frac{2i\pi ns}{l}},$$

$$e^{-\frac{4i\pi s}{3l}}\left[q_N^{(1)}(s) + \bar{\alpha} q_N^{(2)}(s) + \alpha q_N^{(3)}(s)\right] = \frac{1}{l}\sum_{-\infty}^{\infty} M(3n-2)e^{-\frac{2i\pi ns}{l}}. \qquad (1.23)$$

Thus by solving this system of three algebraic equations it follows that each one of the Neumann boundary values can be represented in terms of a Fourier series (see Proposition 3.2).

The analysis of the oblique Robin problem (equation (1.9)) is similar. However, the values of $s_n$ and of $k_m$ in general cannot be found explicitly. The values $k_m$ satisfy the transcendental equation

$$e^{\left(k_m+\frac{\lambda}{k_m}\right)l}\frac{\left(\alpha k_m e^{i\beta} + \frac{\lambda}{\alpha k_m e^{i\beta}} - \gamma\right)\left(\bar{\alpha} k_m e^{-i\beta} + \frac{\lambda}{\bar{\alpha} k_m e^{-i\beta}} - \gamma\right)}{\left(\alpha k_m e^{-i\beta} + \frac{\lambda}{\alpha k_m e^{-i\beta}} - \gamma\right)\left(\bar{\alpha} k_m e^{i\beta} + \frac{\lambda}{\bar{\alpha} k_m e^{i\beta}} - \gamma\right)} = e^{\frac{2i\pi m}{3}}, \quad m \in \mathbb{Z}. \qquad (1.24)$$

Thus in equation (1.22) instead of $\exp\{\frac{2i\pi m}{3l}\}$ we now have $\exp\left\{k_m + \frac{\lambda}{k_m}\right\}$, where $k_m$ satisfies (1.24). In the particular case of the Neumann problem, $k_m$ satisfies equation (1.21).

**Solutions Via Generalized Fourier Integrals**

For more complicated problems, such as the problem (1.11), the basic algebraic relations can be solved in terms of a generalized Fourier integral. This technique in generic, in the sense that it can also be used for the solution of simple problems.

We use such a simple problem, namely the symmetric Dirichlet problem, to illustrate this approach: It is shown in Section 4 that the integral defining $\Psi(\bar{\alpha}k)$ can be solved for $q_N(s)$. For $\lambda \geq 0$, $q_N(s)$ is given by

$$q_N(s) = \frac{i\bar{\alpha}}{4\pi}\int_{\infty e^{\frac{7i\pi}{6}}}^{\infty e^{\frac{i\pi}{6}}} e^{-\left(\bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k}\right)s}\left(1 - \frac{\lambda}{(\bar{\alpha}k)^2}\right)\Psi(\bar{\alpha}k)dk, \quad \lambda \geq 0. \qquad (1.26)$$

Replacing in this equation $\Psi(\bar{\alpha}k)$ with the expression obtained by solving equation (1.17) for $\Psi(\bar{\alpha}k)$, it follows that $q_N(s)$ involves a known integral, as well as an integral containing the unknown function $\Psi(k)$. However, using the analyticity properties of the integrant of the latter integral, it can be shown that this integral can be computed in terms of residues. Furthermore these residues can be explicitly calculated in terms of the known function $G(k)$.

The situation for more complicated problems is similar: The unknown boundary values can be expressed in terms of known integrals, as well as integrals containing the three unknown functions $\{\Psi_j(k)\}_1^3$. Exploiting the analyticity properties of the integrants of the latter integrals, it can be shown that these integrals can be computed explicitly.

**Integral Representations for $q(x, y)$**

When both the Dirichlet and the Neumann boundary values are known, the solution $q(x, y)$ can be determined either using the classical integral representation in terms of Green's functions, or using the novel integral representations constructed in [3] and [6]. For completeness, both representations are presented in Section 5.

**Organization of the Paper**

In Section 2 we derive the global relation (1.13). In Section 3 we solve the symmetric Dirichlet problem (Proposition 3.1), the general Dirichlet problem (Proposition 3.2), the general Neumann problem (Proposition 3.3), and we also discuss the oblique Robin problem. In Section 4 we discuss the basic algebraic relations associated with the Poincare boundary condition (1.11) and derive the relations (1.12b) and (1.12c). In Section 5 we obtain an alternative representation for the symmetric Dirichlet problem and then analyze the problem defined by equations (1.11) and (1.12). A particular case of this problem, which is solved in detail in Proposition 5.1, is a mixed boundary value problem for the modified Helmholtz equation. In Section 6 we discuss the associated integral representations for $q(x, y)$. Further discussion of these results is presented in Section 7.

# 2 The Global Relation

Writing the basic elliptic equation (1.7) in the complex variables $(z, \bar{z})$ we find

$$q_{z\bar{z}} - \lambda q = 0. \tag{2.1}$$

It is straightforward to verify that this equation can be rewritten in the form [3]

$$\left(e^{-ikz-\frac{\lambda}{ik}\bar{z}}q_z\right)_{\bar{z}} + \frac{\lambda}{ik}\left(e^{-ikz-\frac{\lambda}{ik}\bar{z}}q\right)_z = 0, \tag{2.2}$$

where for the rest of this section $k \in \mathbb{C} - \{0\}$. Suppose that equation (2.1) is valid in a simply connected bounded domain $\Omega \subset \mathbb{C}$ with a piecewise smooth boundary $\partial\Omega$. Then equation (2.2) and the complex form of Green's theorem imply

$$\int_{\partial\Omega} e^{-ikz-\frac{\lambda}{ik}\bar{z}}\left(q_z dz - \frac{\lambda}{ik} q d\bar{z}\right) = 0. \tag{2.3}$$

In the particular case that $\Omega$ is the triangular domain $D$, equation (2.3) becomes

$$\sum_{j=1}^{3} \tilde{\rho}_j(k) = 0, \tag{2.4}$$

where the function $\tilde{\rho}_j(k)$ is given by the following line integral along the side $(j)$ of the equilateral triangle

$$\tilde{\rho}_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz-\frac{\lambda}{ik}\bar{z}}\left(q_z dz - \frac{\lambda}{ik} q d\bar{z}\right), \quad j = 1, 2, 3. \tag{2.5}$$

In what follows we will show that

$$\tilde{\rho}_1(k) = \rho_1(k), \quad \tilde{\rho}_2(k) = \rho_2(\bar{\alpha}k), \quad \tilde{\rho}_3(k) = \rho_3(\alpha k), \tag{2.6}$$

where the functions $\rho_j(k)$ are defined in terms of the functions $\Phi_j(k)$ and $\Psi_j(k)$ by the equation

$$\rho_j(k) = E(-ik)\left[\frac{i}{2}\Psi_j(k) + \Phi_j(k)\right], \quad j = 1, 2, 3. \tag{2.7}$$

For this purpose we will use the following local parameterizations:

*Side 1:* On the side (1) the variable $z$ can be parameterized as

$$z(s) = \frac{l}{2\sqrt{3}} + is, \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right]. \tag{2.8$a$}$$

Then $z(-l/2) = z_2$ and $z(l/2) = z_1$. Since the normal and the tangential derivatives are parallel to the $x$ and to $y$ axes respectively, it follows that

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right) = \frac{1}{2}\left(\partial_N - i\partial_T\right). \tag{2.8$b$}$$

*Side 2:* If $z$ varies along the side (2) and $\zeta$ varies along the side (1), then $z = \zeta \exp\left\{-i\frac{2\pi}{3}\right\}$. Thus

$$z(s) = \left(\frac{l}{2\sqrt{3}} + is\right)e^{-i\frac{2\pi}{3}}, \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right]. \tag{2.9$a$}$$

Note again that $z(-l/2) = z_3$ and $z(l/2) = z_2$. The equation $\partial_z = \exp\left\{i\frac{2\pi}{3}\right\}\partial_\zeta$ implies that

$$\partial_z = \frac{\alpha}{2}\left(\partial_N - i\partial_T\right). \tag{2.9$b$}$$

*Side 3:* In analogy with equations (2.9), if $z$ varies along side (3) we find the equations

$$z(s) = \left(\frac{l}{2\sqrt{3}} + is\right)e^{i\frac{2\pi}{3}}, \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \tag{2.10$a$}$$

and

$$\partial_z = \frac{\bar{\alpha}}{2}\left(\partial_N - i\partial_T\right). \tag{2.10$b$}$$

Finally, $z(-l/2) = z_1$ and $z(l/2) = z_3$. Using equations (2.8)-(2.10) in the expressions (2.5) we find equations (2.6), (2.7).

# 3 The Analysis of the Global Relation for Simple Boundary Value Problems

## 3.1 The Symmetric Dirichlet Problem

We first give the details for the symmetric problem. In this case

$$q_N^{(j)}(s) = q_N(s), \quad \Phi_j(k) = F(k), \quad \Psi_j(k) = \Psi(k), \quad j = 1,2,3, \tag{3.1}$$

where the function $\Psi(k)$ is defined in terms of the unknown function $q_N(s)$ by equation (1.14a) (without the superscript $(j)$), and the function $F(k)$ is defined in terms of the given boundary condition $f(s)$ by equation (1.14b), i.e., by the equation

$$F(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\left(k+\frac{\lambda}{k}\right)s} \left[\frac{1}{2}\frac{d}{ds}f(s) + \frac{\lambda}{k}f(s)\right] ds, \quad k \in \mathbb{C} - \{0\}. \tag{3.2}$$

Using equations (3.1), the global relation (1.13) and its Schwarz conjugate yield (1.15) and (1.16), with

$$A(k) = e(\bar{\alpha}k)F(k) + e(-k)F(\bar{\alpha}k) + F(\alpha k),$$
$$B(k) = e(-\bar{\alpha}k)F(k) + F(\alpha k) + e(k)F(\bar{\alpha}k).$$

Hence, since $G = A + B$,

$$G(s_n) = 2\cosh\left[\left(\bar{\alpha}s_n + \frac{\lambda}{\bar{\alpha}s_n}\right)\frac{l}{2}\right] F(s_n) + 2e^{in\pi}F(\bar{\alpha}s_n) + 2F(\alpha s_n). \tag{3.3}$$

In summary, we have derived the following result:

**Proposition 3.1** Let the real valued function $q(x,y)$ satisfy equation (1.7) in the triangular domain $D$, with the Dirichlet conditions (1.8), where

$$f_j(s) = f(s), \quad j = 1,2,3, \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \tag{3.4}$$

and the function $f(s)$ is sufficiently smooth and satisfies the continuity condition $f(-l/2) = f(l/2)$. Then the Neumann boundary values are the same, $q_N^{(j)}(s) = q_N(s)$, $j = 1,2,3$, and are given by the Fourier series

$$q_N(s) = \frac{i}{l}\sum_{-\infty}^{\infty} e^{-\frac{2in\pi s}{l}} \frac{G(s_n)}{\sinh\left[\left(\bar{\alpha}s_n + \frac{\lambda}{\bar{\alpha}s_n}\right)\frac{l}{2}\right]}, \tag{3.5}$$

where $s_n$ is defined by equation (1.18) and $G(s_n)$ is given in terms of $f(s)$ by equations (3.2) and (3.3).

## 3.2 The General Dirichlet Problem

The global relation and its Schwarz conjugate yield equations (1.15) and (1.16), where the known functions $A(k)$ and $B(k)$ are now given by the equations

$$A(k) = e(\bar{\alpha}k)F_1(k) + e(-k)F_2(\bar{\alpha}k) + F_3(\alpha k),$$

$$B(k) = e(-\bar{\alpha}k)F_1(k) + F_2(\alpha k) + e(k)F_3(\bar{\alpha}k), \tag{3.6}$$

where

$$F_j(k) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\left(k+\frac{\lambda}{k}\right)s} \left[\frac{1}{2}\frac{d}{ds}f_j(s) + \frac{\lambda}{k}f_j(s)\right] ds, \quad j = 1, 2, 3, \quad k \in \mathbb{C} - \{0\}. \tag{3.7}$$

Replacing in equations (1.15) and (1.16) $k$ by $\bar{\alpha}k$ and then eliminating $\Psi_1(\bar{\alpha}k)$ from the resulting two equations we find

$$\begin{aligned} &e(-\alpha k)\Psi_3(k) + e(k)\Psi_2(\alpha k) - 2ie(-\alpha k)A(\bar{\alpha}k) \\ &\qquad = e(\alpha k)\Psi_2(k) + e(-k)\Psi_3(\alpha k) + 2ie(\alpha k)B(\bar{\alpha}k). \end{aligned} \tag{3.8}$$

Taking the Schwarz conjugate of this equation (or equivalently eliminating $\Psi_1(\alpha k)$ from the equations obtained from equations (1.15) and (1.16) by replacing $k$ with $\alpha k$) we find

$$\begin{aligned} &e(-\bar{\alpha}k)\Psi_3(k) + e(k)\Psi_2(\bar{\alpha}k) + 2ie(-\bar{\alpha}k)\overline{A(\bar{\alpha}\bar{k})} \\ &\qquad = e(\bar{\alpha}k)\Psi_2(k) + e(-k)\Psi_3(\bar{\alpha}k) - 2ie(\bar{\alpha}k)\overline{B(\bar{\alpha}\bar{k})}. \end{aligned} \tag{3.9}$$

Substituting $\Psi_2(\alpha k)$ from equation (3.8) and $\Psi_3(\bar{\alpha}k)$ from equation (3.9) into equation (1.16), we find an equation involving $\Psi_3(\alpha k)$, $\Psi_2(\bar{\alpha}k)$, and $\{\Psi_j(k)\}_1^3$. Eliminating $\Psi_3(\alpha k)$ from this equation and from equation (1.15) we find equation (1.20) with $X(k)$ given by the following equation,

$$\begin{aligned} X(k) \; &= \left[e^2(-k)e(\bar{\alpha}k) + e(-\bar{\alpha}k)\right]\left[F_1(k) + e^2(k)F_3(k)\right] \\ &+e^2(-k)\left[e^2(-k)e(\bar{\alpha}k)e^6(k) + e(-\bar{\alpha}k)\right]F_2(k) \\ &+2e^2(k)F_1(\alpha k) + 2F_2(\alpha k) + 2e^2(-k)F_3(\alpha k) \\ &+e^3(-k)\left[2e^2(k)F_1(\bar{\alpha}k) + (e^6(k)+1)F_2(\bar{\alpha}k) + 2e^2(-k)e^6(k)F_3(\bar{\alpha}k)\right]. \end{aligned} \tag{3.10$a$}$$

Letting $k = k_n$ we find

$$\begin{aligned} X(k_n) \; &= \left[e^2(-k_n)e(\bar{\alpha}k_n) + e(-\bar{\alpha}k_n)\right] \times \\ &\qquad \left[F_1(k_n) + e^2(-k_n)F_2(k_n) + e^2(k_n)F_3(k_n)\right] \\ &+2e^2(k_n)\left[F_1(\alpha k_n) + e^2(-k_n)F_2(\alpha k_n) + e^2(k_n)F_3(\alpha k_n)\right] \\ &+2e(-k_n)\left[F_1(\bar{\alpha}k_n) + e^2(-k_n)F_2(\bar{\alpha}k_n) + e^2(k_n)F_3(\bar{\alpha}k_n)\right] \end{aligned} \tag{3.10$b$}$$

Solving the algebraic equations (1.23) we find the following result:

**Proposition 3.2** Let the real valued function $q(x,y)$ satisfy equation (1.7) in the triangular domain $D$, with the boundary conditions (1.8), where the given functions $f_j(s)$ have sufficient smoothness and are continuous at the vertices. Then the Neumann data $q_N^{(j)}(s)$, $j=1,2,3$ can be expressed in terms of the given Dirichlet data by the Fourier series

$$q_N^{(j)}(s) = \frac{1}{3l}\sum_{n=-\infty}^{\infty}\left[M(k_{3n}) + c_1^{(j)} e^{\frac{2i\pi s}{3l}} M(k_{3n-1}) + c_2^{(j)} e^{\frac{4i\pi s}{3l}} M(k_{3n-2})\right] e^{-\frac{2i\pi n s}{l}}, \tag{3.11}$$

where $k_m$ is defined by equation (1.21),

$$c_1^{(1)} = c_2^{(1)} = 1, \quad c_1^{(2)} = c_2^{(3)} = \bar{\alpha}, \quad c_1^{(3)} = c_2^{(2)} = \alpha, \tag{3.12}$$

$$M(k_m) = \frac{2iX(k_m)}{\bar{\alpha}^n e(\bar{\alpha}k_m) - e(-\bar{\alpha}k_m)} \tag{3.13}$$

and $X(k_m)$ is defined in terms of $f_j(s)$ by equations (3.7) and (3.10).

## 3.3 The Oblique Robin Problem

Suppose that $q(x,y)$ satisfies the Poincaré boundary condition (1.11), i.e.

$$q_N^{(j)}(s) = \frac{1}{\sin\beta_j}\left(f^{(j)} - \cos\beta_j \frac{dq^{(j)}}{ds} - \gamma_j q^{(j)}\right). \tag{3.14}$$

Substituting this expression in the definition of $\rho_j(k)$, i.e. in the equation (2.7), we obtain

$$\rho_j(k) = E(-ik)\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\left(k+\frac{\lambda}{k}\right)s}\left[\frac{i}{2}q_N^{(j)}(s) + \frac{1}{2}\frac{d}{ds}q^{(j)}(s) + \frac{\lambda}{k}q^{(j)}(s)\right]ds.$$

Integrating by parts we find the following expression for $\rho_j(k)$:

$$\rho_j(k) = iE(-ik)\left[H_j(k)Y_j(k) + F_j(k) + C_j(k)\right], \quad j=1,2,3, \tag{3.15}$$

where the function $H_j(k)$ is defined by

$$H_j(k) = ke^{i\beta_j} + \frac{\lambda}{ke^{i\beta_j}} - \gamma_j, \tag{3.16}$$

the function $F_j(k)$ is defined in terms of the given boundary conditions $f_j(s)$ by the equation

$$F_j(k) = \frac{1}{2\sin\beta_j}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\left(k+\frac{\lambda}{k}\right)s} f_j(s)ds, \tag{3.17}$$

the function $C_j(k)$ involves the values of $q(x,y)$ at the vertices,

$$C_j(k) = \frac{e^{i\beta_j}}{2\sin\beta_j}\left[e(-k)q^{(j)}\left(-\frac{l}{2}\right) - e(k)q^{(j)}\left(\frac{l}{2}\right)\right], \tag{3.18}$$

and the function $Y_j(k)$ involves the unknown Dirichlet boundary values,

$$Y_j(k) = \frac{1}{2\sin\beta_j}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\left(k+\frac{\lambda}{k}\right)s} q^{(j)}(s)ds. \tag{3.19}$$

In equations (3.15)-(3.19), $k$ is complex and $k \neq 0$.

In the particular case of the oblique Robin problem (1.9), $\beta_j = \beta$ and $\gamma_j = \gamma$, $j = 1, 2, 3$, thus $H_j(k) = H(k)$, where $H(k)$ is defined by equation (3.16) without the subscript $(j)$. Substituting the expression for $\rho_j(k)$ (with $H_j = H$) in the global relation (2.4) we find

$$E(-ik)\left[H(k)Y_1(k) + F_1(k) + C_1(k)\right] + E(-i\bar{\alpha}k)\left[H(\bar{\alpha}k)Y_2(\bar{\alpha}k) + F_2(\bar{\alpha}k) + C_2(\bar{\alpha}k)\right]$$

$$+E(-i\alpha k)\left[H(\alpha k)Y_3(\alpha k) + F_3(\alpha k) + C_3(\alpha k)\right] = 0. \tag{3.20}$$

The contribution from the corner terms $C_j$ cancels. Indeed, this contribution is proportional to the following expression

$$E(-ik)\left[e(-k)q^{(1)}\left(-\frac{l}{2}\right) - e(k)q^{(1)}\left(\frac{l}{2}\right)\right] + E(-i\bar{\alpha}k)\left[e(-\bar{\alpha}k)q^{(2)}\left(-\frac{l}{2}\right) - e(\bar{\alpha}k)q^{(2)}\left(\frac{l}{2}\right)\right]$$

$$+E(-i\alpha k)\left[e(-\alpha k)q^{(3)}\left(-\frac{l}{2}\right) - e(\alpha k)q^{(3)}\left(\frac{l}{2}\right)\right]. \tag{3.21}$$

But, the assumption of continuity at the vertices implies

$$q^{(1)}\left(-\frac{l}{2}\right) = q^{(2)}\left(\frac{l}{2}\right), \quad q^{(2)}\left(-\frac{l}{2}\right) = q^{(3)}\left(\frac{l}{2}\right), \quad q^{(3)}\left(-\frac{l}{2}\right) = q^{(1)}\left(\frac{l}{2}\right). \tag{3.22}$$

Hence the terms $q^{(1)}(-l/2)$ and $q^{(2)}(l/2)$ in the expression (3.21) cancel iff

$$E(-ik)e(-k) = E(-i\bar{\alpha}k)e(\bar{\alpha}k). \tag{3.23}$$

This equation is indeed valid, and it is the consequence of the identity

$$\frac{1}{\sqrt{3}}\left(-ik + \frac{\lambda}{-ik}\right) + \left(-k + \frac{\lambda}{-k}\right) = \frac{1}{\sqrt{3}}\left(-i\bar{\alpha}k + \frac{\lambda}{-i\bar{\alpha}k}\right) + \left(\bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k}\right).$$

Using the fact that the corners term cancel, the global relation and its Schwarz conjugate yield (compare with equations (1.15) and (1.16)) the following equations:

$$e(\bar{\alpha}k)H(k)Y_1(k) + e(-k)H(\bar{\alpha}k)Y_2(\bar{\alpha}k) + H(\alpha k)Y_3(\alpha k) = -A(k),$$

$$e(-\bar{\alpha}k)\overline{H(\bar{k})}Y_1(k) + \overline{H(\bar{\alpha}\bar{k})}Y_2(\alpha k) + e(k)\overline{H(\alpha\bar{k})}Y_3(\bar{\alpha}k) = -B(k), \tag{3.24}$$

where $A(k)$ and $B(k)$ are defined by equations (3.6) in terms of $F_j$.

Let

$$P(k) = \frac{H(k)}{\overline{H(\bar{k})}}. \tag{3.25}$$

Following precisely the same steps used for the general Dirichlet problem we find the following expression for $Y_2(\bar{\alpha}k)$ in terms of $\{Y_j(k)\}_1^3$:

$$\left[e^3(-k)\frac{P^2(\bar{\alpha}k)}{P^2(\alpha k)} - e^3(k)\frac{P(\alpha k)}{P(\bar{\alpha}k)}\right]\frac{\overline{H(\alpha\bar{k})}}{\overline{H(\bar{k})}}Y_2(\bar{\alpha}k) = T(k)$$

$$+\left[e(-\bar{\alpha}k) - e^2(-k)\frac{P(\bar{\alpha}k)P(k)}{P^2(\alpha k)}e(\bar{\alpha}k)\right]\left[Y_1(k) + e^2(k)\frac{P(\alpha k)}{P(\bar{\alpha}k)}Y_3(k)\right]$$

$$+e^2(-k)\frac{P(\bar{\alpha}k)}{P(\alpha k)}\left[e(-\bar{\alpha}k) - e^4(k)\frac{P^3(\alpha k)}{P^3(\bar{\alpha}k)}\frac{P(\bar{\alpha}k)P(k)}{P^2(\alpha k)}e(\bar{\alpha}k)\right]Y_2(k), \tag{3.26}$$

where $T(k)$ is given in terms of $F_j(k)$ by the following equation

$$\overline{H(\bar{k})}T(k) = e(-k)\left[E^3(-i\alpha k) - E^3(i\alpha k)\frac{P(\bar{\alpha}k)}{P^2(\alpha k)}\right]F_1(k)$$

$$+\left[E^3(-i\bar{\alpha}k)\frac{P(\bar{\alpha}k)}{P(\alpha k)} - E^3(i\bar{\alpha}k)\frac{1}{P(\bar{\alpha}k)}\right]F_2(k) + e(k)\left[E^3(-i\alpha k)\frac{P(\alpha k)}{P(\bar{\alpha}k)} - E^3(i\alpha k)\frac{1}{P(\alpha k)}\right]F_3(k)$$

$$+e^2(k)\frac{P(\alpha k)-1}{P(\bar{\alpha}k)}F_1(\alpha k) + \frac{P(\alpha k)-1}{P(\alpha k)}F_2(\alpha k) + e^2(-k)\frac{P(\bar{\alpha}k)(P(\alpha k)-1)}{P^2(\alpha k)}F_3(\alpha k)$$

$$+e(-k)\frac{P(\bar{\alpha}k)-1}{P(\alpha k)}F_1(\bar{\alpha}k) + \left(e^3(k)\frac{P(\alpha k)}{P(\bar{\alpha}k)} - e^3(-k)\frac{P(\bar{\alpha}k)}{P^2(\alpha k)}\right)F_2(\bar{\alpha}k) + e(k)\frac{P(\bar{\alpha}k)-1}{P(\bar{\alpha}k)}F_3(\bar{\alpha}k). \tag{3.27}$$

If $k = k_m$, where $k_m$ is defined by

$$e^6(k)\frac{P^3(\alpha k)}{P^3(\bar{\alpha}k)} = 1,$$

then

$$e^2(k)\frac{P(\alpha k)}{P(\bar{\alpha}k)} = e^{\frac{2i\pi m}{3}}, \quad e^2(-k)\frac{P(\bar{\alpha}k)}{P(\alpha k)} = e^{-\frac{2i\pi m}{3}}.$$

Thus evaluating equation (3.26) at $k = k_m$, we find the following expression:

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\left(k_m + \frac{\lambda}{k_m}\right)s}\left[q^{(1)}(s) + e^{-\frac{2i\pi m}{3}}q^{(2)}(s) + e^{\frac{2i\pi m}{3}}q^{(3)}(s)\right] = G(k_m), \quad m \in \mathbb{Z} \tag{3.28}$$

where

$$G(k_m) = \frac{2T(k_m)\sin\beta}{\bar{\alpha}^m\frac{P(k_m)}{P(\alpha k_m)}e(\bar{\alpha}k_m) - e(-\bar{\alpha}k_m)}. \tag{3.29}$$

Even if one is able to invert equation (3.28) for the bracket appearing in the integrant of the lhs of equation (3.28), one will find an expression which will involve an infinite series over the transcendental values of $k_m$. Thus instead of analyzing the relevant inversion we will analyze the general Robin problem using the generalized Fourier transform approach (see Section 5).

In the case of the Neumann problem, equations (3.28) and (3.29) simplify and yield the following result.

**Proposition 3.3** Let the real valued function $q(x,y)$ satisfy (1.7) in the triangular domain $D$, with the Neumann boundary conditions

$$q_N^{(j)}(s) = f_j(s), \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \quad j = 1, 2, 3 \tag{3.30}$$

where the functions $f_j(s)$ have sufficient smoothness and are continuous at the vertices of the triangle. Then the Dirichlet data $q^{(j)}(s)$, $j = 1, 2, 3$ can be expressed in terms of the given Neumann data by the Fourier series

$$q^{(j)}(s) = \frac{1}{3l} \sum_{n=-\infty}^{\infty} \left[N(k_{3n}) + c_1^{(j)} e^{\frac{2i\pi s}{3l}} N(k_{3n-1}) + c_2^{(j)} e^{\frac{4i\pi s}{3l}} N(k_{3n-2})\right] e^{-\frac{2i\pi n s}{l}} \tag{3.31}$$

where $c_i^{(j)}$ are given by (3.12) and

$$N(k_m) = \frac{2T_N(k_m)}{\bar{\alpha}^m e(\bar{\alpha}k_m) - e(-\bar{\alpha}k_m)}. \tag{3.34}$$

The known function $T_N(k_m)$ is defined by the equation

$$\begin{aligned}
-\left(ik + \frac{\lambda}{ik}\right) T_N(k) &= e(-k)\left[E^3(-i\alpha k) + E^3(i\alpha k)\right] F_1(k) + \left[E^3(-i\bar{\alpha}k) + E^3(i\bar{\alpha}k)\right] F_2(k) \\
&+ e(k)\left[E^3(-i\alpha k) + E^3(i\alpha k)\right] F_3(k) + 2e^2(k)F_1(\alpha k) + 2F_2(\alpha k) + 2e^2(-k)F_3(\alpha k) \\
&+ 2e(-k)F_1(\bar{\alpha}k) + \left(e^3(k) + e^3(-k)\right) F_2(\bar{\alpha}k) + 2e(k)F_3(\bar{\alpha}k),
\end{aligned} \tag{3.35}$$

where $F_j(k)$ is given by (3.17) with $\sin\beta_j = 1$.

# 4 Poincaré Type Boundary Value Problems

Suppose that $q(x,y)$ satisfies the Poincaré type boundary condition (1.11). Then substituting the expression $\rho_j(k)$ from equation (3.15) into the global relation (2.4), we find an equation similar with equation (3.20), where $H(k)$, $H(\bar{\alpha}k)$ and $H(\alpha k)$ are replaced by $H_1(k)$, $H_2(\bar{\alpha}k)$ and $H_3(\alpha k)$, and $F_j$, $C_j$, $Y_j$ are defined by equations (3.17)-(3.19). Proceeding as in Section 3.3, in analogy with equation (3.26), we now find

$$D(k)H_2(\bar{\alpha}k)Y_2(\bar{\alpha}k) = \sum_{j=1}^{3} \Gamma_j(k)H_j(k)Y_j(k) + T(k) + C(k), \tag{4.1}$$

where $T(k)$ is defined in terms of the known functions $f_j(s)$, $C(k)$ involves the values of $q$ at the corners, and $D(k)$, $\{\Gamma_j(k)\}_1^3$ are defined by the following equations:

$$D(k) = \frac{P_1(\bar{\alpha}k)}{P_2(\alpha k)P_3(\alpha k)} \left[e^3(-k) - e^3(k)\frac{P_1(\alpha k)P_2(\alpha k)P_3(\alpha k)}{P_1(\bar{\alpha}k)P_2(\bar{\alpha}k)P_3(\bar{\alpha}k)}\right], \tag{4.2}$$

$$\Gamma_1(k) = \frac{1}{P_1(k)} \left[e(-\bar{\alpha}k) - e^2(-k)e(\bar{\alpha}k)\frac{P_1(k)P_1(\bar{\alpha}k)}{P_2(\alpha k)P_3(\alpha k)}\right], \tag{4.3a}$$

$$\Gamma_2(k) = e^2(-k)\frac{P_1(\bar{\alpha}k)}{P_2(k)P_2(\alpha k)}\left[e(-\bar{\alpha}k) - e^4(k)e(\bar{\alpha}k)\frac{P_2(k)P_2(\alpha k)}{P_1(\bar{\alpha}k)P_3(\bar{\alpha}k)}\right], \tag{4.3b}$$

$$\Gamma_3(k) = e^2(k)\frac{P_1(\alpha k)}{P_3(k)P_3(\bar{\alpha}k)}\left[e(-\bar{\alpha}k) - e^2(-k)e(\bar{\alpha}k)\frac{P_3(k)P_3(\bar{\alpha}k)}{P_1(\alpha k)P_2(\alpha k)}\right], \tag{4.3c}$$

with

$$P_j(k) = \frac{H_j(k)}{\overline{H_j(\bar{k})}}. \tag{4.4}$$

In order to be able to solve this problem using a generalized Fourier integral we require that when $D(k)$ vanishes, then $\Gamma_2(k)$ and $\Gamma_3(k)$ are proportional to $\Gamma_1(k)$. Actually, $\Gamma_3(k)$ is proportional to $\Gamma_1(k)$ for all complex $k$ provided that

$$P_1(k)P_1(\alpha k)P_1(\bar{\alpha}k) = P_3(k)P_3(\alpha k)P_3(\bar{\alpha}k). \tag{4.5a}$$

Equating the brackets appearing in the definitions of $\Gamma_1(k)$ and $\Gamma_2(k)$, and replacing in the resulting expression $e^6(k)$ by

$$\frac{P_1(\bar{\alpha}k)P_2(\bar{\alpha}k)P_3(\bar{\alpha}k)}{P_1(\alpha k)P_2(\alpha k)P_3(\alpha k)},$$

it follows that $\Gamma_2(k)$ is proportional to $\Gamma_1(k)$ provided that

$$P_1(k)P_1(\alpha k)P_1(\bar{\alpha}k) = P_2(k)P_2(\alpha k)P_2(\bar{\alpha}k). \tag{4.5b}$$

Equation (4.5b) is valid if the following two equations are valid:

$$\sin 3(\beta_1 - \beta_2) = 0, \quad \gamma_2(3\lambda - \gamma_2^2)\sin 3\beta_1 - \gamma_1(3\lambda - \gamma_1^2)\sin 3\beta_2 = 0. \tag{4.6}$$

Indeed, in order to simplify equation (4.5b) we first compute the product $H_1(k)H_1(\alpha k)H_1(\bar{\alpha}k)$,

$$H_1(k)H_1(\alpha k)H_1(\bar{\alpha}k) = k^3e^{3i\beta_1} + \frac{\lambda^3}{k^3e^{3i\beta_1}} + 3\lambda\gamma_1 - \gamma_1^3. \tag{4.7}$$

The function $\overline{H_1(\bar{k})}$ can be obtained from $H_1(k)$ by replacing $\beta_1$ with $-\beta_1$, thus $\overline{H_1(\bar{k})}\ \overline{H_1(\bar{\alpha}\bar{k})}\ \overline{H_1(\alpha\bar{k})}$ is given by an expression similar to (4.7) with $\beta_1$ replacing by $-\beta_1$. Hence equation (4.5b) yields

$$\frac{k^3e^{3i\beta_1} + \frac{\lambda^3}{k^3e^{3i\beta_1}} + 3\lambda\gamma_1 - \gamma_1^3}{k^3e^{-3i\beta_1} + \frac{\lambda^3}{k^3e^{-3i\beta_1}} + 3\lambda\gamma_1 - \gamma_1^3} = \frac{k^3e^{3i\beta_2} + \frac{\lambda^3}{k^3e^{3i\beta_2}} + 3\lambda\gamma_2 - \gamma_2^3}{k^3e^{-3i\beta_2} + \frac{\lambda^3}{k^3e^{-3i\beta_2}} + 3\lambda\gamma_2 - \gamma_2^3}.$$

This equation simplifies to the equation

$$\left(k^6 - \frac{\lambda^6}{k^6}\right)\sin 3(\beta_1 - \beta_2) + \left(k^3 - \frac{\lambda^3}{k^3}\right)\left[(3\lambda\gamma_2 - \gamma_2^3)\sin 3\beta_1 - (3\lambda\gamma_1 - \gamma_1^3)\sin 3\beta_2\right] = 0,$$

which is valid for all $k$ iff equations (4.6) are valid.

Equation (4.6a) implies $\beta_2 = \beta_1 + n\pi/3$, then $\sin 3\beta_2 = \sin 3\beta_1 \exp\{in\pi\}$ and we find equation (1.12b). Similarly equation (4.5a) yields equation (1.12c).

**The Case that the Corner Terms Cancel**

The definition of the corner terms $C_j(k)$, i.e. equation (3.18), shows that $C_j(k)$ involves $\exp[i\beta_j]/\sin\beta_j$. Thus the contribution of the corner terms in the global relation (3.20) vanishes iff

$$e^{2i\beta_1} = e^{2i\beta_2} = e^{2i\beta_3}. \tag{4.8}$$

**Example 1.**

$$\beta_1 = \beta_2 = \beta_3 = \frac{2\pi}{3}, \quad \gamma_j \text{ arbitrary}. \tag{4.9}$$

In this case

$$H_j(k) = k\alpha + \frac{\lambda}{k\alpha} - \gamma_j, \quad P_j(k) = \frac{k\alpha + \frac{\lambda}{k\alpha} - \gamma_j}{k\bar{\alpha} + \frac{\lambda}{k\bar{\alpha}} - \gamma_j}. \tag{4.10}$$

**Example 2.**

$$\beta_1 = \beta_2 = \beta_3 = \beta, \quad \beta \text{ arbitrary}, \quad \gamma_2 = \gamma_3 = 0, \quad \gamma_1 = (3\lambda)^{\frac{1}{2}}, \quad \lambda > 0. \tag{4.11}$$

In this case

$$H_1(k) = ke^{i\beta} + \frac{\lambda}{ke^{i\beta}} - \gamma_1, \quad H_2(k) = H_3(k) = ke^{i\beta} + \frac{\lambda}{ke^{i\beta}}. \tag{4.12}$$

In particular if $\beta = \pi/2$, then

$$H_1(k) = i\left(k - \frac{\lambda}{k}\right) - \gamma_1, \quad H_2(k) = H_3(k) = i\left(k - \frac{\lambda}{k}\right). \tag{4.13}$$

Thus

$$P_1(k) = \frac{i\left(k - \frac{\lambda}{k}\right) - \gamma_1}{-i\left(k - \frac{\lambda}{k}\right) - \gamma_1}, \quad P_2(k) = P_3(k) = -1. \tag{4.14}$$

Hence

$$D(k) = P_1(\bar{\alpha}k)\left[e^3(-k) - e^3(k)\frac{P_1(\alpha k)}{P_1(\bar{\alpha}k)}\right], \tag{4.15a}$$

$$\Gamma_1(k) = \frac{1}{P_1(k)}\left[e(-\bar{\alpha}k) - e^2(-k)e(\bar{\alpha}k)P_1(k)P_1(\bar{\alpha}k)\right], \tag{4.15b}$$

$$\Gamma_2(k) = e^2(-k)P_1(\bar{\alpha}k)\left[e(-\bar{\alpha}k) + \frac{e^4(k)e(\bar{\alpha}k)}{P_1(\bar{\alpha}k)}\right] \tag{4.15c}$$

$$\Gamma_3(k) = e^2(k)P_1(\alpha k)\left[e(-\bar{\alpha}k) + \frac{e^2(-k)e(\bar{\alpha}k)}{P_1(\alpha k)}\right]. \tag{4.15d}$$

If $P_1(k)$ is defined by (4.14) with $\gamma_1^2 = 3\lambda$, it can be verified that

$$P_1(k)P_1(\bar{\alpha}k) = -\frac{1}{P_1(\alpha k)}.$$

Hence

$$\Gamma_3(k) = -\frac{e^2(k)}{P_1(\bar{\alpha}k)}\Gamma_1(k), \quad \Gamma_2(k)|_{D(k)=0} = -\frac{e^2(-k)\Gamma_1(k)}{P_1(\alpha k)}. \tag{4.16}$$

**The Laplace Equation**

In the particular case of the Laplace equation with $\gamma_j = 0$, $P_j = e^{2i\beta_j}$, i.e. $P_j$ is independent of $k$.

# 5 The Analysis of the Global Relation Via Fourier Integrals

In this section we restrict $\lambda$ to be non-negative. Slightly more complicated formulae can be derived for $\lambda < 0$.

We first derive equation (1.26). Letting $k = |k|e^{\frac{i\pi}{6}}$, the definition of $\Psi(\bar{\alpha}k)$ yields

$$\Psi(-i|k|) = \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{\left(-i|k| + \frac{\lambda}{-i|k|}\right)s} q_N(s) ds. \tag{5.1}$$

Suppose that $\lambda > 0$. Letting $t(|k|) = |k| - \lambda/|k|$, it follows that if $|k| \in (0, +\infty)$, then $t \in (-\infty, \infty)$. Thus inverting equation (5.1) we find

$$q_N(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{its} \Psi(-i|k|) dt, \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \quad q_N(s) = 0 \text{ elsewhere},$$

where $|k|$ (in the argument of $\Psi$) is a function of $t$. Rewriting $t$ in terms of $|k|$ we find

$$q_N(s) = \frac{1}{2\pi} \int_0^{\infty} e^{i\left(|k| - \frac{\lambda}{|k|}\right)s} \Psi(-i|k|) \left(1 + \frac{\lambda}{|k|^2}\right) d|k|.$$

Letting $|k| = ke^{-i\pi/6}$ we obtain

$$q_N(s) = \frac{1}{2\pi} \int_0^{\infty e^{\frac{i\pi}{6}}} e^{-\left(\bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k}\right)s} \Psi(\bar{\alpha}k) \left(ke^{-\frac{i\pi}{6}} + \frac{\lambda}{ke^{-\frac{i\pi}{6}}}\right) \frac{dk}{k}. \tag{5.2}$$

The rhs of this equation equals

$$\frac{1}{2\pi} \int_{-\infty e^{\frac{i\pi}{6}}}^{0} e^{-\left(\bar{\alpha}\xi + \frac{\lambda}{\bar{\alpha}\xi}\right)s} \Psi(\bar{\alpha}\xi) \left(\xi e^{-\frac{i\pi}{6}} + \frac{\lambda}{\xi e^{-\frac{i\pi}{6}}}\right) \frac{d\xi}{\xi}. \tag{5.3}$$

Indeed, for the derivation of (5.3) we first observe that the function $\Psi(\bar{\alpha}k)$ remains invariant under the transformation $k \to \bar{\alpha}\lambda/k$. Thus making the change of variables $k = \bar{\alpha}\lambda/\xi$ in the rhs of equation (5.2), and using

$$ke^{-\frac{i\pi}{6}} + \frac{\lambda}{ke^{-\frac{i\pi}{6}}} \to -\left(\xi e^{-\frac{i\pi}{6}} + \frac{\lambda}{\xi e^{-\frac{i\pi}{6}}}\right), \quad \frac{dk}{k} = -\frac{d\xi}{\xi},$$

we find the expression (5.3). Combining (5.2) and (5.3) we obtain

$$q_N(s) = \frac{1}{4\pi} \int_{-\infty e^{\frac{i\pi}{6}}}^{\infty e^{\frac{i\pi}{6}}} e^{-\left(\bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k}\right)s} \Psi(\bar{\alpha}k) \left(ke^{-\frac{i\pi}{6}} + \frac{\lambda}{ke^{-\frac{i\pi}{6}}}\right) \frac{dk}{k}. \tag{5.4}$$

Using $e^{-\frac{i\pi}{6}} = i\bar{\alpha}$, this equations becomes equation (1.26).
If $\lambda = 0$, we set $k = t\exp\{i\frac{7\pi}{6}\}$, $t \in \mathbb{R}$, and rewrite (1.14a) as

$$\Psi(it) = \int_{-l/2}^{l/2} e^{its} q_N(s) ds \tag{5.5}$$

which is inverted to

$$q_N(s) = \frac{l}{2\pi} \int_{-\infty}^{+\infty} e^{-its} \Psi(it) dt. \tag{5.6}$$

Replacing in (5.6) $\{it\}$ with $\{\bar{\alpha}k\}$, we arrive at

$$q_N(s) = \frac{i\bar{\alpha}}{2\pi} \int_{\infty e^{\frac{i7\pi}{6}}}^{\infty e^{\frac{i\pi}{6}}} e^{-\bar{\alpha}ks} \Psi(\bar{\alpha}k) dk. \tag{5.7}$$

Note that (5.7), in comparison with (1.26) misses a factor of 1/2; this is due to the linearity of the used transformation in this case.

## 5.1 The Symmetric Dirichlet Problem

Solving equation (1.17) for $\Psi(\bar{\alpha}k)$ and substituting the resulting expression in equation (1.26) we find

$$q_N(s) = \frac{i\bar{\alpha}}{4\pi} \int_{\infty e^{\frac{7i\pi}{6}}}^{\infty e^{\frac{i\pi}{6}}} \frac{e^{-\left(\bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k}\right)s}}{\Delta(k)} \left[(e(\bar{\alpha}k) - e(-\bar{\alpha}k))\Psi(k) - 2iG(k)\right] \left(1 - \frac{\lambda}{(\bar{\alpha}k)^2}\right) dk, \tag{5.5}$$

where $\Delta(k)$ denotes the coefficient of $\Psi(\bar{\alpha}k)$ in equation (1.17), i.e.

$$\Delta(k) = e(k) - e(-k). \tag{5.6}$$

The line $\left(\infty e^{\frac{7i\pi}{6}}, \infty e^{\frac{i\pi}{6}}\right)$ splits the complex $k$-plane into the two half planes,

$$\mathcal{D}^+ = \left\{ k \in \mathbb{C}, \quad \frac{\pi}{6} < \arg k < \frac{7\pi}{6} \right\},$$
$$\mathcal{D}^- = \left\{ k \in \mathbb{C}, \quad \frac{7\pi}{6} < \arg k < \frac{13\pi}{6} \right\}. \tag{5.7}$$

We observe that

$$e^{-\left(\bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k}\right)s} e(\bar{\alpha}k) \quad \text{is bounded for} \quad k \in \mathcal{D}^-,$$
$$e^{-\left(\bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k}\right)s} e(-\bar{\alpha}k) \quad \text{is bounded for} \quad k \in \mathcal{D}^+. \tag{5.8}$$

Indeed the exponential of (5.8a) involves $\bar{\alpha}k\left(\frac{l}{2} - s\right)$, and since $l/2 - s \geq 0$, the exponential in (5.8a) is bounded for $\mathrm{Re}(\bar{\alpha}k) \leq 0$, i.e. in $\mathcal{D}^-$. Similarly, the exponential of (5.8b) involves $-\bar{\alpha}k\left(\frac{l}{2} + s\right)$, which since $l/2 + s \geq 0$, is bounded for $\mathrm{Re}(\bar{\alpha}k) \geq 0$, i.e. in $\mathcal{D}^+$.

We also note that $\Psi(k)/\Delta(k)$ is bounded for all $k \in \mathbb{C}$, $k \neq s_n$. Indeed, for Re $k > 0$, $\Delta(k)$ is dominated by $e(k)$, while for Re $k < 0$, $\Delta(k)$ is dominated by $e(-k)$, hence

$$\frac{\Psi(k)}{\Delta(k)} \sim \begin{cases} \Psi(k)e(-k), & \mathrm{Re}k > 0 \\ \\ -\Psi(k)e(k), & \mathrm{Re}k < 0. \end{cases} \tag{5.9}$$

Furthermore $\Psi(k)e(-k)$ involves $k(s - l/2)$ which is bounded for Re $k \geq 0$, while $\Psi(k)e(k)$ involves $k(s + l/2)$ which is bounded for Re $k \leq 0$ (recall that $-\frac{l}{2} \leq s \leq \frac{l}{2}$).

The above considerations imply that the parts of the integral (5.5) containing $e(\bar{\alpha}k)\Psi(k)$ and $e(-\bar{\alpha}k)\Psi(k)$ can be computed by using Cauchy's theorem in $\mathcal{D}^-$ and $\mathcal{D}^+$ respectively. The associated residues can be computed as follows: Let $s_n^+$ and $s_n^-$ denote the subsets of $s_n$ in $\mathcal{D}^+$ and $\mathcal{D}^-$, respectively. Evaluating equation (1.17) at $k = s_n^\pm$ we find

$$e(\bar{\alpha}s_n^-)\Psi(s_n^-) = \frac{2iG(s_n^-)}{-e^2(-\bar{\alpha}s_n^-) + 1}, \quad -e(-\bar{\alpha}s_n^+)\Psi(s_n^+) = \frac{2iG(s_n^+)}{1 - e^2(\bar{\alpha}s_n^+)}. \tag{5.10}$$

Thus

$$\begin{aligned} q_N(s) = &-\frac{\bar{\alpha}}{2\pi}\int_{\infty e^{\frac{7i\pi}{6}}}^{\infty e^{\frac{i\pi}{6}}} e^{-\left(\bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k}\right)s}\frac{G(k)}{\Delta(k)}\left[1 - \frac{\lambda}{(\bar{\alpha}k)^2}\right]dk \\ &-i\bar{\alpha}\sum_{s_n^+} e^{-\left(\bar{\alpha}s_n^+ + \frac{\lambda}{\bar{\alpha}s_n^+}\right)s}\frac{G(s_n^+)}{\Delta'(s_n^+)\left[1 - e^2(\bar{\alpha}s_n^+)\right]}\left[1 - \frac{\lambda}{(\bar{\alpha}s_n^+)^2}\right] \\ &+i\bar{\alpha}\sum_{s_n^-} e^{-\left(\bar{\alpha}s_n^- + \frac{\lambda}{\bar{\alpha}s_n^-}\right)s}\frac{G(s_n^-)}{\Delta'(s_n^-)\left[1 - e^2(-\bar{\alpha}s_n^-)\right]}\left[1 - \frac{\lambda}{(\bar{\alpha}s_n^-)^2}\right]. \end{aligned} \tag{5.11}$$

## 5.2 The Poincaré Problem

Evaluating equation (4.1) at $k = k_m$, where $k_m$ is a zero of $D(k)$, it follows that the unknown terms $Y_j(k)$ appear in the form

$$\begin{aligned} \Gamma_1(k_m)H_1(k_m)\Bigg\{&Y_1(k_m) + e^2(-k_m)\frac{P_1(\bar{\alpha}k_m)P_1(k_m)}{P_2(k_m)P_2(\bar{\alpha}k_m)}\frac{H_2(k_m)}{H_1(k_m)}Y_2(k_m) \\ &+ e^2(k_m)\frac{P_1(\alpha k_m)P_1(k_m)}{P_3(k_m)P_3(\bar{\alpha}k_m)}\frac{H_3(k_m)}{H_1(k_m)}Y_3(k_m)\Bigg\}. \end{aligned}$$

The crucial difference of this general case, as compared with the oblique Robin case (1.9), is the following: Using the definition of $Y_j(k_m)$ we find that the coefficients of $q^{(2)}(s)$ and of $q^{(3)}(s)$ involve in general $k_m$-dependent expressions, thus it is not clear how the associated integral can be inverted. In contrast, equation (4.1) can be solved using the approach of section (5.1). The definition of $Y_2(\bar{\alpha}k)$, i.e. equation (3.19), and equation(1.26), imply

$$q^{(2)}(s) = \frac{i\bar{\alpha}\sin\beta_2}{2\pi}\int_{\infty e^{\frac{7i\pi}{6}}}^{\infty e^{\frac{i\pi}{6}}} e^{-\left(\bar{\alpha}k + \frac{\lambda}{\bar{\alpha}k}\right)s}\left(1 - \frac{\lambda}{(\bar{\alpha}k)^2}\right)Y_2(\bar{\alpha}k)dk. \tag{5.11}$$

Solving equation (4.1) for $Y_2(\bar{\alpha}k)$ and substituting the resulting expression in equation (5.11) we find an integral involving the three unknown functions $\{Y_j(k)\}_1^3$. The known part of this integral involves the factors (5.8) analyzed already, as well as factors of the type

$$e^2(-k)\frac{Y_j(k)}{D(k)}, \quad \frac{e^2(k)Y_j(k)}{D(k)}.$$

These terms are bounded for all $k \neq k_m$. Indeed, ignoring the terms involving $P_j(k)$ we find

$$e^2(-k)\frac{Y_j(k)}{D(k)} \sim \begin{cases} Y_j(k)e(-k) \cdot e^4(-k), & \mathrm{Re}k > 0 \\ \\ Y_j(k)e(k), & \mathrm{Re}k < 0, \end{cases}$$

$$e^2(k)\frac{Y_j(k)}{D(k)} \sim \begin{cases} Y_j(k)e(-k), & \mathrm{Re}k > 0 \\ \\ Y_j(k)e(k) \cdot e^4(k), & \mathrm{Re}k < 0, \end{cases}$$

which are identical with the expressions (5.9) except for the occurrence of the factors $e^4(-k)$ and $e^4(k)$ for Re $k > 0$ and Re $k < 0$, which are obviously bounded.

The above discussion implies that the integral involving

$$\frac{e(-\bar{\alpha}k)}{D(k)}\left[\frac{H_1(k)}{P_1(k)}Y_1(k) + e^2(-k)\frac{P_1(\bar{\alpha}k)H_2(k)}{P_2(k)P_2(\alpha k)}Y_2(k) + e^2(k)\frac{P_1(\alpha k)H_3(k)}{P_3(k)P_3(\bar{\alpha}k)}Y_3(k)\right], \tag{5.12}$$

can be computed by using Cauchy's theorem in $\mathcal{D}^+$. Evaluating equation (4.1) at $k_m^+$ it follows that the associated residue equals

$$\frac{-[T(k_m^+) + C(k_m^+)]}{1 - \frac{P_1(k_m^+)P_1(\bar{\alpha}k_m^+)}{P_2(\alpha k_m^+)P_3(\alpha k_m^+)}e^2(-k_m^+)e^2(\bar{\alpha}k_m^+)}. \tag{5.13}$$

Similarly, the integral involving

$$-\frac{P_1(k)P_1(\bar{\alpha}k)}{P_2(\alpha k)P_3(\alpha k)}\frac{e(\bar{\alpha}k)}{D(k)}\left[\frac{e^2(-k)}{P_1(k)}H_1(k)Y_1(k) + \frac{e^2(k)P_2(\alpha k)P_3(\alpha k)}{P_1(k)P_1(\bar{\alpha}k)P_3(\bar{\alpha}k)}H_2(k)Y_2(k)\right.$$
$$\left. + \frac{P_3(\alpha k)}{P_1(\bar{\alpha}k)P_1(\alpha k)}H_3(k)Y_3(k)\right] \tag{5.14}$$

can be computed by using Cauchy's theorem in $\mathcal{D}^-$. Evaluating equation (4.1) at $k_m^-$, it follows that the associated residue equals

$$\frac{[T(k_m^-) + C(k_m^-)]}{1 - \frac{P_2(\alpha k_m^-)P_3(\alpha k_m^-)}{P_1(k_m^-)P_1(\bar{\alpha}k_m^-)}e^2(k_m^-)e^2(-\bar{\alpha}k_m^-)}. \tag{5.15}$$

In what follows we give the details for a mixed Neumann-Robin problem.

We will consider Example 2, as it is described by (4.11) with $\beta = \frac{\pi}{2}$. On side (1) we assume the Robin condition

$$q_N^{(1)}(s) + \sqrt{3}\lambda q^{(1)}(s) = f_1(s), \tag{5.16}$$

and on sides (2) and (3) we assume the Neumann conditions

$$q_N^{(2)}(s) = f_2(s) \tag{5.17}$$

and

$$q_N^{(3)}(s) = f_3(s). \tag{5.18}$$

Then $H_j(k)$, $P_j(k)$, $D(k)$ and $\Gamma_j(k)$, are given by (4.13), (4.14), (4.15a) and (4.15b,c,d) respectively. Furthermore, $\Gamma_3(k)$ is proportional to $\Gamma_1(k)$ (see equation (4.16a)), while $\Gamma_2(k)$ becomes proportional to $\Gamma_1(k)$ only on those $k_m$'s for which $D(k)$ vanishes. These are roots of the transcendental equation

$$e^{3\left(k+\frac{\lambda}{k}\right)} = \frac{\left(k+\frac{\lambda}{k}-\sqrt{\lambda}\right)\left(k+\frac{\lambda}{k}-2\sqrt{\lambda}\right)}{\left(k+\frac{\lambda}{k}+\sqrt{\lambda}\right)\left(k+\frac{\lambda}{k}+2\sqrt{\lambda}\right)}. \tag{5.19}$$

For $k = k_m$, equation (4.1), in view of (4.16), implies

$$\Gamma_1(k_m)\left[H_1(k_m)Y_1(k_m) - \frac{e^2(-k_m)}{P_1(\alpha k_m)}H_2(k_m)Y_2(k_m) - \frac{e^2(k_m)}{P_1(\bar{\alpha}k_m)}H_3(k_m)Y_3(k_m)\right] = -T(k_m). \tag{5.20}$$

By virtue of (4.15b) and the identity

$$P_1(k_m)P_1(\alpha k_m)P_1(\bar{\alpha}k_m) = -1 \tag{5.21}$$

equation (5.20) is written as

$$\left[e(-\bar{\alpha}k_m) + e(\bar{\alpha}k_m)\frac{e^2(-k_m)}{P_1(\alpha k_m)}\right]\left[\frac{H_1(k_m)Y_1(k_m)}{P_1(k_m)}\right.$$

$$\left. + e^2(-k_m)P_1(\bar{\alpha}k_m)H_2(k_m)Y_2(k_m) + e^2(k_m)P_1(\alpha k_m)H_3(k_m)Y_3(k_m)\right] = -T(k_m). \tag{5.22}$$

Since the corner terms $C(k)$ vanish, the representation (5.11) and equation (4.1) provide

$$q^{(2)}(s) = \frac{i\bar{\alpha}}{2\pi}\int_{\infty e^{i\frac{7\pi}{6}}}^{\infty e^{i\frac{\pi}{6}}} e^{-\left(\bar{\alpha}k+\frac{\lambda}{\bar{\alpha}k}\right)s}\left(1-\frac{\lambda}{(\bar{\alpha}k)^2}\right)\frac{1}{H_2(\bar{\alpha}k)D(k)}\left[\sum_{j=1}^{3}\Gamma_j(k)H_j(k)Y_j(k) + T(k)\right]dk. \tag{5.23}$$

Utilizing the expression (5.22) we arrive at the following result.

**Proposition 5.1.** Let the real valued function $q(x,y)$ satisfy equation (1.7) with $\lambda > 0$ in the triangular domain $D$, with the Robin boundary condition (5.16) on side (1) and the Neumann boundary conditions (5.17) and (5.18) on sides (2) and (3), where the given functions $f_j(s)$ have sufficient smoothness and are continuous at the vertices. Then the Dirichlet value on side (2) is given by

$$q^{(2)}(s) = \frac{i\bar{\alpha}}{2\pi}\int_{\infty e^{i\frac{7\pi}{6}}}^{\infty e^{i\frac{\pi}{6}}} e^{-\left(\bar{\alpha}k+\frac{\lambda}{\bar{\alpha}k}\right)s}\left(1-\frac{\lambda}{(\bar{\alpha}k)^2}\right)\frac{T(k)}{H_2(\bar{\alpha}k)D(k)}dk$$

$$+\bar{\alpha}\sum_{k_m^+}\frac{e^{-\left(\bar{\alpha}k_m^+ + \frac{\lambda}{\bar{\alpha}k_m^+}\right)s}}{H_2(\bar{\alpha}k_m^+)D'(k_m^+)}\left(1-\frac{\lambda}{(\bar{\alpha}k_m^+)^2}\right)\frac{T(k_m^+)}{1+\frac{E^6(i\alpha k_m^+)}{P_1(\alpha k_m^+)}}$$

$$-\bar{\alpha}\sum_{k_m^-}\frac{e^{-\left(\bar{\alpha}k_m^- + \frac{\lambda}{\bar{\alpha}k_m^-}\right)s}}{H_2(\bar{\alpha}k_m^-)D'(k_n^-)}\left(1-\frac{\lambda}{(\bar{\alpha}k_m^-)^2}\right)\frac{T(k_m^-)}{1+P_1(\alpha k_m^-)E^6(-i\alpha k_m^-)}, \tag{5.24}$$

where $D'(k_n^\pm)$ denotes the derivative of $D(k)$ evaluated at $k = k_n^\pm$. The summations are taken over all $k_m^+ \in \mathcal{D}^+$ and $k_m^- \in \mathcal{D}^-$ respectively, and $T$, $H_2$, $P_1$ are defined by equations (3.27), (3.16), (4.4) respectively.

There exist similar formulas for the Dirichlet values on sides (1) and (3).

# 6 The Integral Representations

If $\lambda \geq 0$ the classical Green's representation is given by [2]

$$q(\mathbf{r}) = \frac{1}{2\pi}\int_{\partial D}\left[K(2\sqrt{\lambda}|\mathbf{r}-\mathbf{r}'|)\partial_{n'}q(\mathbf{r}') - q(\mathbf{r}')\partial_{n'}K(2\sqrt{\lambda}|\mathbf{r}-\mathbf{r}'|)\right]dl(\mathbf{r}') \tag{6.1}$$

where the integration is over the boundary $\partial D$ of the triangle in the positive direction, $\partial_{n'}$ denotes the outward normal derivative on $\partial D$, $dl(\mathbf{r}')$ is the line element along $\partial D$, and $K(x)$ is the modified Bessel function of the zeroth order and of the second kind for the modified Helmholtz equation. For the case of the Helmholtz equation $K(x)$ is proportional to the Hankel function of the zeroth order and of the first kind, while for the Laplace's equation $K(x)$ is proportional to the logarithm of $x$.

For the Laplace equation, the integral representation constructed in [3] is defined as follows:

$$\frac{\partial q}{\partial z} = \frac{1}{2\pi}\sum_{j=1}^{3}\int_{l_j} e^{ikz}\tilde{\rho}_j(k)dk, \quad z \in D, \tag{6.2}$$

where the contours $l_j$ are the rays from 0 to $\infty$ specified by the arguments $-\pi/2$, $\pi/6$, $5\pi/6$ respectively, and the functions $\tilde{\rho}_j(k)$ are defined by equations (2.6) in terms of $\rho_j(k)$, where the latter functions are defined by equations (2.7), (1.14) with $\lambda = 0$.

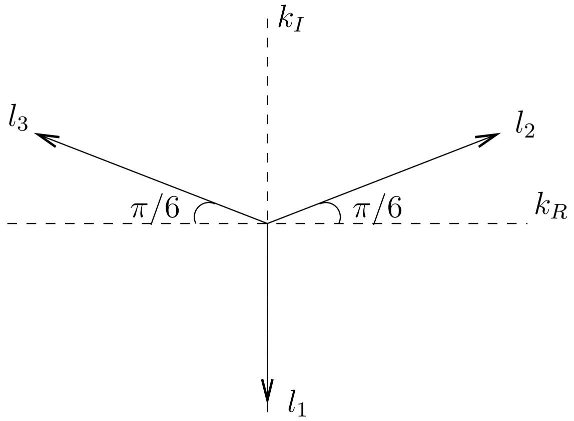


**Figure 6.1:** The rays $l_j$ in the complex $k$-plane

For the modified Helmholtz equation, the analogue of equation (6.2) is [3]

$$q(z, \bar{z}) = \frac{1}{2\pi i} \sum_{j=1}^{3} \int_{l_j} e^{ikz + \frac{\lambda}{ik}\bar{z}} \tilde{\rho}_j(k) \frac{dk}{k}, \quad z \in D, \tag{6.3}$$

where the rays $l_j$ are the same as in (6.2) and $\tilde{\rho}_j(k)$ are defined by equations (2.6) and (2.7).

There exists a similar representation for the Helmholtz equation, which however, in addition to rays, it also involves circular arcs [3].

## 6.1 The Symmetric Dirichlet Problem

Using the integral representation (6.3) it is possible to compute directly $q(z, \bar{z})$, bypassing the computation of the unknown boundary values. For brevity of presentation we will only give details for the symmetric Dirichlet problem. The analysis of the more general boundary value problems (1.8)-(1.11) is similar.

Recalling the definitions of $\tilde{\rho}_j$, i.e. equations (2.6) and (2.7), it follows that the representation $q(z, \bar{z})$ given by equation (6.3) involves the known function $F(k)$ defined by equation (3.2), as well as the unknown function $\Psi(k)$ which on the rays $l_j$ appears as:

$$l_1 : E(-ik)\Psi(k), \quad l_2 : E(-i\bar{\alpha}k)\Psi(\bar{\alpha}k), \quad l_3 : E(-i\alpha k)\Psi(\alpha k). \tag{6.4}$$

Solving equation (1.17) for $\Psi(\bar{\alpha}k)$ in terms of $\Psi(k)$, and then using the Schwarz conjugation of the resulting equations in order to express $\Psi(\alpha k)$ in terms of $\Psi(k)$, it follows that the expressions in (6.4) involve the unknown function $\Psi(k)/\Delta(k)$ (where $\Delta(k)$ is defined by equation (5.6)) times the following expressions:

$$l_1 : E(-ik)[e(k) - e(-k)], \quad l_2 : E(-i\bar{\alpha}k)[e(\bar{\alpha}k) - e(-\bar{\alpha}k)], \quad l_3 : E(-i\alpha k)[e(\alpha k) - e(-\alpha k)]. \tag{6.5}$$

The third of the relations in (1.5) implies $e(k) = E(i\bar{\alpha}k)E(-i\alpha k)$, thus

$$E(-ik)e(k) = E^2(i\bar{\alpha}k), \quad E(-ik)e(-k) = E^2(i\alpha k).$$

Replacing $k$ by $\bar{\alpha}k$ and by $\alpha k$ in these identities, we find

$$E(-i\bar{\alpha}k)e(\bar{\alpha}k) = E^2(i\alpha k), \quad E(-i\bar{\alpha}k)E(-\bar{\alpha}k) = E^2(ik),$$

$$E(-i\alpha k)e(\alpha k) = E^2(ik), \quad E(-i\alpha k)e(-\alpha k) = E^2(i\bar{\alpha}k).$$

Thus the expressions in (6.5) become

$$l_1 : E^2(i\bar{\alpha}k) - E^2(i\alpha k), \quad l_2 : E^2(i\alpha k) - E^2(ik), \quad l_3 : E^2(ik) - E^2(i\bar{\alpha}k).$$

Hence, the unknown part of $q(z, \bar{z})$ involves the following integral

$$J(z, \bar{z}) = \sum_{j=1}^{3} J_j(z, \bar{z}), \tag{6.6}$$

$$J_3(z,\bar z)=\frac{1}{4\pi}\int_{\{-l_1\}\cup\{l_2\}}e^{ikz+\frac{\lambda}{ik}\bar z}E^2(i\alpha k)\frac{\Psi(k)dk}{k\Delta(k)}, \tag{6.7a}$$

$$J_1(z,\bar z)=\frac{1}{4\pi}\int_{\{-l_2\}\cup\{l_3\}}e^{ikz+\frac{\lambda}{ik}\bar z}E^2(ik)\frac{\Psi(k)dk}{k\Delta(k)}, \tag{6.7b}$$

$$J_2(z,\bar z)=\frac{1}{4\pi}\int_{\{-l_3\}\cup\{l_1\}}e^{ikz+\frac{\lambda}{ik}\bar z}E^2(i\bar\alpha k)\frac{\Psi(k)dk}{k\Delta(k)}. \tag{6.7c}$$

Each of the above integrals can be computed in terms of residues. Indeed, it was shown in Section 5 that $\Psi(k)/k\Delta(k)$ is bounded as $k\to 0$ and as $k\to\infty$. Furthermore, it will be verified below that the exponentials,

$$e^{ikz+\frac{\lambda}{ik}\bar z}E^2(i\alpha k),\quad e^{ikz+\frac{\lambda}{ik}\bar z}E^2(ik),\quad e^{ikz+\frac{\lambda}{ik}\bar z}E^2(i\bar\alpha k), \tag{6.8}$$

are bounded as $k\to 0$ and $k\to\infty$, for $\arg k$ in

$$\left[-\frac{\pi}{2},\frac{\pi}{6}\right],\quad\left[\frac{\pi}{6},\frac{5\pi}{6}\right],\quad\left[\frac{5\pi}{6},\frac{3\pi}{2}\right],$$

respectively, provided that $(z,\bar z)\in D$. We first consider the first exponential in (6.8); since $z_2=-\bar\alpha\frac{l}{\sqrt3}$, this exponential can be written as $\exp\{ik(z-z_2)+\lambda(\bar z-\bar z_2)/ik\}$. If $z$ is in the triangular domain then

$$\frac{\pi}{2}\le\arg(z-z_2)\le\frac{5\pi}{6}.$$

Thus if $-\frac{\pi}{2}\le\arg k\le\frac{\pi}{6}$, we find

$$0\le\arg[k(z-z_2)]\le\pi.$$

Hence $\exp\{ik(z-z_2)\}$ is bounded as $|k|\to\infty$ and $\exp\{\lambda(\bar z-\bar z_2)\bar k/i|k|^2\}$ is bounded as $|k|\to 0$. Hence there is no contribution from zero and from infinity.

The results for the second and for the third integrals in (6.8) follows from the above result by using appropriate rotations. The roots of $\Delta(k)=0$ lie on the imaginary axis. Denote by $s_n^+$ those with positive imaginary part and by $s_n^-$ those with negative imaginary part. Obviously, the residue from each $s_n^+$ has a full contribution to $J_1(z,\bar z)$, while the residue contribution from each $s_n^-$ is split in two halfs, one half is contributed to $J_2(z,\bar z)$ and one half to $J_3(z,\bar z)$.

Tedious but straightforward calculations lead to the expression

$$J(z,\bar z)=\sum_{s_n^+}e^{is_n^+z+\frac{\lambda}{is_n^+}\bar z}\frac{E^2(is_n^+)G(s_n^+)}{s_n^+\Delta'(s_n^+)\Delta(\bar\alpha s_n^+)}$$

$$+\frac12\sum_{s_n^-}e^{is_n^-z+\frac{\lambda}{is_n^-}\bar z}\frac{\left(E^2(i\alpha s_n^-)+E^2(i\bar\alpha s_n^-)\right)G(s_n^-)}{s_n^-\Delta'(s_n^-)\Delta(\bar\alpha s_n^-)}. \tag{6.9}$$

In the above calculations, the value of $\Psi_1(s_n^\pm)$ is obtained from (1.18).

We observe that for each $n\in\mathbb{Z}$,

$$e^{is_nz+\frac{\lambda}{is_n}\bar z}=e^{\pm2\sqrt{(\frac{n\pi}{l})^2+\lambda}x}\cdot e^{-2i\frac{n\pi}{l}y} \tag{6.10}$$

thus this expression shows that the equilateral triangle accepts separable solutions. It is clear that each eigensolution in (6.10), solves equation (1.7).

# 7 Conclusion

Eigenvalues and eigenfunctions for equation (1.7) with homogeneous Dirichlet, Neumann, and Robin boundary conditions were constructed in the classical works of Lamé [9]-[11]. Some of these results have been rederived by several authors. In particular the Dirichlet problem is discussed in the recent review [12] and the Robin problem in [13]. It is remarkable that Lamé argued, using physical considerations, that it is impossible to solve certain problems using infinite series as opposed to integrals. Indeed Lamé writes [11 p.191]: "The series should therefore express the fact that the temperature remains zero on strips of constant width separated by other strips of double width, in which the temperature may vary. The analytic interpretation of this sort of discontinuity demands the introduction of terms where the variables appear *inside integrals*[1]. These terms, of a nature that we will not consider here, cannot disappear from the total series unless the discontinuity disappears".

In this paper, we first present a straightforward treatment of simple boundary value problems. This treatment, which is based on the evaluation of the basic algebraic relations (see the Introduction) at particular values of $k$, expresses the unknown boundary values in terms of infinite series. The Dirichlet, Neumann and Robin problems can be solved using this approach. We then show that, in agreement with the above remarks of Lamé, more complicated boundary value problems apparently require the use of generalized Fourier integrals as opposed to infinite series. Proposition 5.1 presents the solution of such a problem.

In this paper we have concentrated on the generalized Dirichlet to Neumann map, i.e. we have concentrated on determining the unknown boundary values as opposed to determining $q(x,y)$ itself. In this respect we note that: (a) In some applications one requires precisely these unknown boundary values. (b) When both the Dirichlet and the Neumann boundary values are known, it is straightforward to compute $q(x,y)$.

We emphasize however, that the approach of Section 5 can be used to construct directly $q(x,y)$. Indeed, if one uses the novel integral representations for $q(x,y)$ obtained in [2], instead for the representation (1.26) for $q_N$, and if one follows the approach of Section 5, one can again compute explicitly the contribution of the unknown functions $Y_j(k)$. This latter approach is illustrated in Section 6.1 for the symmetric Dirichlet problem. More complicated problems using this approach are solved in [4].

In order to compute $q(x,y)$ from the knowledge of both the Dirichlet and the Neumann boundary values one can use either the classical Green's formulae or the representations of [3]. Regarding the latter representations we note that they provide a tailor-made transform for the particular problem at hand. In fact the exponential $\exp\{ikz + (\lambda/ik)\bar{z}\}$ reflects the structure of the PDE, the contours $l_j$ in the complex $k$-plane reflect the geometry of the domain, and the functions $\rho_j(k)$ describe the boundary conditions.

Both the Dirichlet and the Neumann problems involve elementary trigonometric functions. It is interesting that the analysis of the global condition yields these separable solutions without the direct use of separation of variables.

For arbitrary values of the constants $\beta_j$ and $\gamma_j$, the Poincaré problem (1.11) gives rise to a matrix Riemann-Hilbert problem. For the particular case that equations (1.12) are valid, it is possible to avoid this Riemann-Hilbert problem and to solve the problem in closed form.

[1]These words were not italic in the original text

Although equations (1.12) impose severe restrictions on $\beta_j$ and $\gamma_j$, some of the resulting cases appear interesting. These cases include the following:

(1) $\beta_1 = \beta_2 = \beta_3 = \frac{2\pi}{3}$, $\gamma_j$ arbitrary.

In this case the angles are specified, but $\gamma_j$ are arbitrary.

(2) The mixed Neumann-Robin problem analyzed in Section 5.

(3) $\beta_2 = \beta + \frac{4\pi}{3}, \quad \beta_3 = \beta + \frac{2\pi}{3}, \quad \gamma_1 = \gamma_2 = \gamma_3 = \gamma.$

In this case all derivatives are computed along a direction making an angle $\beta$ with the positive vertical axis. This follows from the fact that the angles $\beta_2$ and $\beta_3$ specified in Figure 7.1 equal $\pi - \beta - \pi/3$ and $\pi/3 - \beta$ respectively.

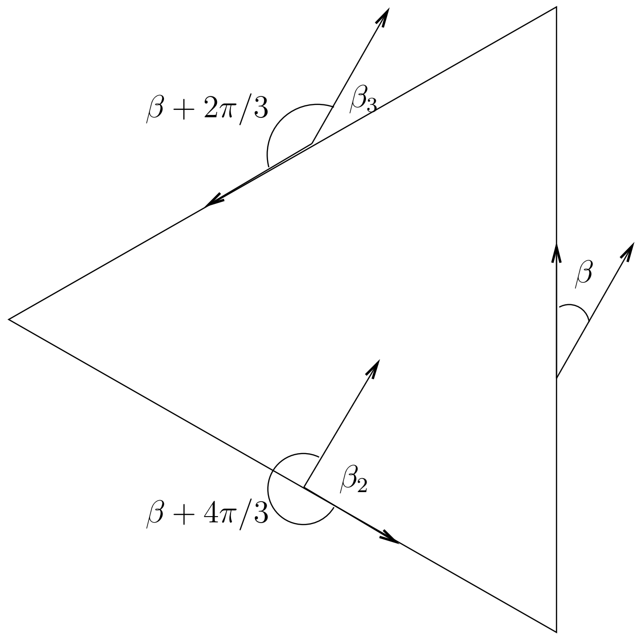


**Figure 7.1**

Equation (1.7) appears in diverse physical applications. For example the modified Helmholtz equation appears in the study of pattern formation [8].

The results presented here can be made rigorous, following a formalism similar to the one used in [5] for evolution PDEs and in [7] for the Laplace equation.

Several problems remain open which include the following:

1. The investigation of singularities associated with discontinuous boundary conditions.

2. If the $\beta_j$'s differ, then the global relation (3.20) contains a contribution from $q$ at the three corners. Several approaches for determining these terms are presented in [4] and [1]. However, the optimal treatment of these terms remains open.

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