The Generalized Dirichlet to Neumann Map for Certain Nonlinear Evolution PDEs

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Abstract

Let $q(x,t)$ satisfy a nonlinear integrable evolution PDE whose highest spatial derivative is of order n. An initial-boundary value problem on the half-line for such a PDE is at least linearly well posed if one prescribes initial conditions, as well as N boundary conditions at $x = 0$, where for n even N equals $n/2$ and for n odd, depending on the sign of the highest derivative, N equals either $\frac{n-1}{2}$ or $\frac{n+1}{2}$. For example, for the nonlinear Schrödinger (NLS) and the sine-Gordon (sG) $N = 1$, while for the modified Korteweg-deVries (mKdV) $N = 1$ or $N = 2$ depending on the sign of the third derivative. Constructing the generalized Dirichlet to Neumann map means determining those boundary values at $x = 0$ which are not prescribed as boundary conditions, in terms of the given initial and boundary conditions. A general methodology is presented which constructs this map in terms of the solution of a system of two nonlinear ODEs. This formulation implies that for the focusing NLS, for the sG, and for the two focusing versions of the mKdV, this map is global in time. It appears that this is the first time in the literature that such a characterization for nonlinear PDEs is explicitly described. It is also shown here that for particular choices of the boundary conditions the above map can be linearized.

1 Introduction

A nonlinear evolution PDE is called integrable iff it admits a Lax pair formulation [1], i.e. iff it can be expressed as the compatibility condition of two linear eigenvalue equations. The most well known such PDEs are the Korteweg-deVries (KdV), the nonlinear Schrödinger (NLS), the sine-Gordon (sG), and the modified KdV (mKdV). The latter three equations are:

$$\frac{i\partial q}{\partial t}+\frac{\partial^2 q}{\partial x^2}-2\rho|q|^2q=0, \quad \rho=\pm 1, \tag{1.1}$$

$$\frac{\partial^2 q}{\partial t^2}-\frac{\partial^2 q}{\partial x^2}+\sin q=0, \quad q \ \text{ real}, \tag{1.2}$$

$$\frac{\partial q}{\partial t}\pm\frac{\partial^3 q}{\partial x^3}\mp 6\rho q^2\frac{\partial q}{\partial x}=0, \quad q \ \text{ real}, \quad \rho=\pm 1. \tag{1.3$^\pm$}$$

Notations and Assumptions

- Subscripts with respect to x and t denote partial derivatives, for example

$$q_t=\frac{\partial q}{\partial t}, \quad q_x=\frac{\partial q}{\partial x}.$$

- $\overline{f(k)}$ denotes the complex conjugate of $f(k)$.
- σ_3 denotes the third Pauli matrix, i.e. $\sigma_3=\mathrm{diag}(1,-1)$.
- $q_0(x)=q(x,0)$ denotes the initial condition; for the sG in addition to $q_0(x)$ one also needs the initial condition $q_1(x)=q_t(x,0)$. It is assumed that the initial conditions belong to the Schwartz space on the half-line which will be denoted by $S(\mathbb{R}^+)$.
- $g_j(t)=\partial_x^j q(0,t)$, $j=1,\cdots,n-1$, denote the boundary values at $x=0$. A subset of these functions is prescribed as boundary conditions. We assume that these latter functions are sufficiently smooth and are compatible with $q_0(x)$ (and $q_1(x)$) at $x=t=0$.
- A_{11}, A_{12}, A_{21}, A_{22} denote the 11, 12, 21, 22, entries of the 2×2 matrix A.
- mKdVI and mKdVII refer to equations $(1.3)^+$ and $(1.3)^-$ respectively.
- $\bar{D}$ denotes the closure of the domain D of the complex k-plane where the function $\phi(x,k)$ to be defined later is analytic. For the NLS, sG, mKdV I, D is $\mathbb{C}^+$, i.e. the upper half complex k-plane, for the mKdV II, D is $\mathbb{C}^-$.
- $\dot{g}_0(t)=\dfrac{dg_0(t)}{dt}, \quad \dot{g}_1(t)=\dfrac{dg_1(t)}{dt}, \quad \dot{q}_0(x)=\dfrac{dq_0(x)}{dx}.$
- ∂D_1, ∂D_2, ∂D_3 denote the oriented boundaries of the domains $0<\arg k<\frac{\pi}{2}$, $\{\mathrm{Im}\, k>0\cap|k|>1\}$, $\frac{4\pi}{3}<\arg k<\frac{5\pi}{3}$, respectively, see Figure 1.1.

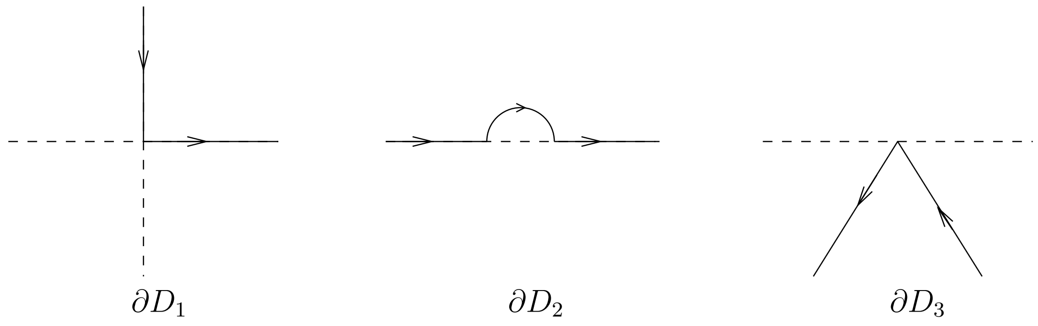

Fig. 1.1 The oriented boundaries $\partial D_j, j = 1, 2, 3$.

Lax Pair Formulation
Equations (1.1) - (1.3), the KdV equation, as well as several other nonlinear PDEs of physical significance, admit a Lax pair of the form

$$\Psi_x + if_1(k)\sigma_3\Psi = Q(x,t,k)\Psi, \tag{1.4a}$$

$$\Psi_t + if_2(k)\sigma_3\Psi = \tilde{Q}(x,t,k)\Psi, \quad k \in \mathbb{C}, \tag{1.4b}$$

where the eigenfunction $\Psi(x,t,k)$ is a 2×2 matrix of the arguments indicated, $f_1(k)$, $f_2(k)$ are analytic functions of k, and Q, $\tilde{Q}$ are analytic functions of k, of $q(x,t)$, of $\bar{q}(x,t)$, and of derivatives of these functions. For example for the NLS, $f_1 = k$, $f_2 = 2k^2$, $Q(x,t)$ is an off-diagonal 2×2 matrix with $Q_{12} = q$, $Q_{21} = \rho\bar{q}$, $\rho = \pm 1$, and

$$\tilde{Q}(x,t,k) = 2kQ - iQ_x\sigma_3 - i\rho|q|^2 q\sigma_3. \tag{1.5}$$

Formulation of the Problem
Let $q(x,t)$ satisfy an integrable evolution PDE on the half-line, $0 < x < \infty$, $t > 0$, and let n be the highest derivative of this PDE. In this situation, an initial-boundary value problem is at least linearly well-posed if one prescribes initial conditions as well as N boundary conditions, where for n even N equals $n/2$ and for n odd, depending on the sign of the highest derivative, N equals either $(n-1)/2$ or $(n+1)/2$ [2]. For the NLS, the sG, the mKdV I, and the mKdV II, the problems for which the following functions are prescribed are linearly well-posed:

$$\{g_0(t), q_0(x)\}, \quad \{g_0(t), q_0(x), q_1(x)\}, \quad \{g_0(t), q_0(x)\}, \quad \{g_0(t), g_1(t), q_0(x)\}, \tag{1.6}$$

where $q_0(x)^1, q_1(x) \in S(\mathbb{R}^+)$, and $g_0(t)$, $g_1(t)$ have sufficient smoothness. We define the generalized Dirichlet to Neumann map as follows: *Determine the boundary values which are not prescribed as boundary conditions, in terms of the initial and boundary conditions.* For the NLS, the sG, the mKdVI, and the mKdVII, with the initial and boundary conditions (1.6), constructing this map means determining $\{g_1(t)\}$ for the NLS, the sG, the mKdVII, and $\{g_1(t), g_2(t)\}$ for the mKdVI.

[1]For the sG $q_0(x) - 2\pi m \in S(\mathbb{R}^+)$, where m is an integer.

The Global Relation

It has been emphasized by the author that for a given initial-boundary value problem the relation coupling the initial conditions and the boundary values takes a simple form in the space of spectral functions. These functions, which will be denoted by $\{a(k), b(k), A(t,k), B(t,k)\}$, are constructed as follows:

The Spectral Functions Associated with the Initial Condition

These functions are defined in terms of the initial condition $q_0(x)$ (and $q_1(x)$) by the formulae

$$a(k) = \phi_2(0,k), \quad b(k) = \phi_1(0,k), \tag{1.7}$$

where the vector $\phi(x,k)$ with components $\phi_1(x,k)$ and $\phi_2(x,k)$ is the following solution of the x-part of the associated Lax pair evaluated at $t=0$:

$$\begin{aligned} &\phi_x + if_1(k)\sigma_3\phi = Q(x,0,k)\phi, \quad 0 < x < \infty, \quad k \in \bar{D}, \\ &\phi(x,k) = e^{if_1(k)x}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1)\right) \quad \text{as} \quad x \to \infty, \end{aligned} \tag{1.8}$$

where $\bar{D}$ denotes the closure of the domain D of the complex k-plane defined in the Notations.

The Spectral Functions Associated with the Boundary Values

These functions are defined in terms of the boundary values $\{g_j(t)\}_0^{n-1}$ by the formulae

$$A(t,k) = e^{if_2(k)t}\overline{\Phi_2(t,\bar{k})}, \quad B(t,k) = -e^{if_2(k)t}\Phi_1(t,k), \tag{1.9}$$

where the vector $\Phi(t,k)$ with components $\Phi_1(t,k)$ and $\Phi_2(t,k)$ is the following solution of the t-part of the associated Lax pair evaluated at $x=0$:

$$\begin{aligned} &\Phi_t + if_2(k)\sigma_3\Phi = \tilde{Q}(0,t,k)\Phi, \quad t > 0, \quad k \in \mathbb{C}, \\ &\Phi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{1.10}$$

For the NLS, the sG, the mKdVI, and the mKdVII, the functions $f_2(k)$ and $\tilde{Q}(0,t,k)$

are given by the following expressions:

NLS $\quad f_2 = 2k^2, \quad \tilde{Q}(0,t,k) = 2kQ_1(t) + Q_0(t),$

$$Q_1(t) = \begin{pmatrix} 0 & g_0(t) \\ \rho\bar{g}_0(t) & 0 \end{pmatrix}, \quad Q_0(t) = i\rho \begin{pmatrix} -|g_0(t)|^2 & \rho g_1(t) \\ -\bar{g}_1(t) & |g_0(t)|^2 \end{pmatrix}. \tag{1.11}$$

sG $\quad f_2 = \frac{1}{4}\left(k + \frac{1}{k}\right), \quad \tilde{Q}(0,t,k) = iQ_0(t) - \frac{i}{k}Q_{-1}(t),$

$$Q_0(t) = -\frac{1}{4}(\dot{g}_0(t) + g_1(t)) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$Q_{-1}(t) = \frac{1}{4} \begin{pmatrix} \cos g_0(t) - 1 & i\sin g_0(t) \\ -i\sin g_0(t) & 1 - \cos g_0(t) \end{pmatrix}. \tag{1.12}$$

mKdVI $\quad f_2 = 4k^3, \quad \tilde{Q}(0,t,k) = Q_0(t) - 2ikQ_1(t) + 4k^2Q_2(t),$

$$Q_0(t) = (2\rho g_0^3(t) - g_2(t)) \begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix}, \quad Q_1(t) = \rho g_0(t)^2\sigma_3 + g_1(t) \begin{pmatrix} 0 & -1 \\ \rho & 0 \end{pmatrix},$$

$$Q_2(t) = g_0(t) \begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix}. \tag{1.13}$$

mKdVII $\quad$ as *mKdVI* but with

$$g_0 \to -g_0, \quad g_1 \to g_1, \quad g_2 \to -g_2. \tag{1.14}$$

The Explicit Solution of the Global Relation

The spectral functions satisfy the global relation [3]

$$B(t,k) - R(k)A(t,k) = e^{2if_2(k)t}\frac{c(t,k)}{a(k)}, k \in \bar{D}, \tag{1.15a}$$

where

$$R(k) = \frac{b(k)}{a(k)}, \quad k \in \bar{D}, \tag{1.15b}$$

and $c(t,k)$ as a function of k is analytic for $k \in D$, and is of $O(1/k)$ as $k \to \infty$. Using the analyticity properties of $c(t,k)$ it is possible to eliminate this function, by integrating equation (1.15a) in an appropriate domain of the complex k-plane and by using Jordan's lemma. The resulting equation is a *single* equation coupling the known spectral functions $\{a(k), b(k)\}$ with the unknown spectral functions $\{A(t,k), B(t,k)\}$. It is remarkable that this equation, together with the equations obtained from this equation by using certain invariant transformations, can be solved *explicitly* for the missing boundary values in terms of $\Phi(t,k)$ and the given initial and boundary conditions. Indeed, the following formulae will be derived in Section 4.

NLS

$$g_1(t) = \frac{g_0(t)}{\pi}\int_{\partial D_1} e^{-2ik^2t}\left(\Phi_2(t,k) - \Phi_2(t,-k)\right)dk$$

$$+\frac{2i}{\pi}\int_{\partial D_1}\left[e^{-2ik^2t}k\left(\Phi_1(t,k) - \Phi_1(t,-k)\right) + ig_0(t)\right)dk + \frac{4i}{\pi}\int_{\partial D_1} e^{-2ik^2t}kR(k)\overline{\Phi_2(t,\bar{k})}dk. \tag{1.16}$$

sG

$$g_1(t) = -\dot{g}_0(t) + \frac{i}{\pi}\int_{\partial D_2}\frac{e^{-\frac{i}{4}(k+\frac{1}{k})}}{k^2}\left(\Phi_1(t,k) - \Phi_1(t,\frac{1}{k}\right)dk$$

$$+\frac{i}{\pi}\int_{\partial D_2} e^{-\frac{i}{4}(k+\frac{1}{k})}\left(\frac{1}{k^2} - 1\right)R(k)\overline{\Phi_2(t,\bar{k})}dk. \tag{1.17}$$

mKdVII

$$g_2(t) = 2\rho g_0^3(t) + \frac{\rho {g_0}^2(t)}{\pi}\int_{\partial D_3} e^{-4ik^3t}\chi_1(t,k)dk - \frac{2i}{\pi}g_0(t)\int_{\partial D_3} e^{-4ik^3t}k\chi_2(t,k)dk$$

$$-\frac{g_1(t)}{\pi}\int_{\partial D_3} e^{-4ik^3t}\widehat{\chi}_2(t,k)dk - \frac{12}{\pi}\int_{\partial D_3}\left[\frac{1}{3}e^{-4ik^3t}k2(\chi_1(t,k) + \hat{\chi}_1(t,k)) - \frac{g_1(t)}{4} - \frac{ikg_0(t)}{2} - \xi(t)\right]dk$$

$$+\frac{3g_0(t)}{\pi^2}\int_{\partial D_3} e^{-12ik^3t}\widehat{\chi}_2(t,k)dk - \frac{12}{\pi}\int_{\partial D_3} e^{-4ik^3t}k^2R(k)\overline{\Phi_2(t,\bar{k})}dk, \tag{1.18a}$$

where

$$\chi_1(t,k) = \Phi_1(t,k) + E\Phi_1(t,Ek) + E^2\Phi_1(t,E^2k), \quad E = e^{\frac{2i\pi}{3}}, \tag{1.18b}$$

$$\chi_2(t,k) = \Phi_2(t,k) + E^2\Phi_2(t,Ek) + E\Phi_2(t,E^2k), \tag{1.18c}$$

$$\widehat{\chi}_2(t,k) = \Phi_2(t,k) + E\Phi_2(t,Ek) + E^2\Phi_2(t,E^2k). \tag{1.18d}$$

$$\hat{\chi}_1(t,k) = \Phi_1(t,k) + E2\Phi_1(t,Ek) + E\Phi_1(t,E^2k) \tag{1.18e}$$

$$\xi(t) = \frac{g_0(t)}{4\pi}\int_{\partial D_3} e^{-12ik^3t}\hat{\chi}_2(t,k)dk. \tag{1.18f}$$

mKdVI

$$g_1(t) = \frac{g_0}{\pi}\int_{\partial D_3} e^{-4ik^3t}\widehat{\chi}_2(t,k)dk + \frac{6}{i\pi}\int_{\partial D_3}\left[\frac{e^{-4ik^3t}k}{3}\chi_1(t,k) - \frac{g_0(t)}{2i}\right]dk \tag{1.19a}$$

$$-\frac{6}{i\pi}\int_{\partial D_3} e^{-8ik^3t}kG_1(t,k)dk,$$

$$g_2(t) = 2\rho g_0^3(t) - \frac{2g_0(t)}{\pi}\int_{\partial D_3} e^{-4ik^3t}\left[\frac{\rho g_0(t)}{2}\chi_1(t,k) + ik\chi_2(t,k)\right]dk + \frac{12}{\pi}\int_{\partial D_3} e^{-8ik^3t}k^2 G_2(t,k)dk \tag{1.19b}$$

$$+\frac{g_1(t)}{\pi}\int_{\partial D_3} e^{-4ik^3t}\widehat{\chi}_2(t,k)dk - \frac{12}{\pi}\int_{\partial D_3}\left[\frac{e^{-4ik^3t}}{3}k^2\chi_1(t,k) - \frac{k}{2i}g_0(t)\right]dk,$$

where $\chi_1, \chi_2, \widehat{\chi}_2$, are defined by (1.18b)- (1.18d), and G_1, G_2 are defined by

$$G_1 = \frac{(RA)(E^2k,t) - (RA)(Ek,t)}{(E^2 - E)k}, \quad G_2 = \frac{(RA)(E^2k,t) - E(RA)(Ek,t)}{E-1}, \tag{1.19c}$$

with $A(t,k) = e^{4ik^3t}\overline{\Phi_2(t,\overline{k})}$.
In the above formulae if $a(k) \neq 0$ for $k \in \bar{D}$, then the contours ∂D_j, $j = 1,2,3$, are defined in the Notations, while if $a(k)$ has zeros in D then these contours are deformed to pass above all these zeros.

Global Existence
For the NLS, the functions $\Phi_1(t,k)$ and $\Phi_2(t,k)$ satisfy the two ODEs (1.10), where $\tilde{Q}(0,t,k)$ is given by equation (1.11) and $g_1(t)$ is given by equation (1.16). Thus Φ_1 and Φ_2 satisfy two nonlinear ODEs, which are explicitly defined in terms of $g_0(t)$ and $b(k)/a(k)$ (and this latter function can be computed in terms of $q_0(x)$). The situation for the sG, mKdVI, and mKdVII, is similar.
For the focusing NLS, the sG, and the two focusing versions of the mKdV, equations (1.10) yield

$$|\Phi_1(t,k)|^2 + |\Phi_2(t,k)|^2 = 1, \tag{1.20}$$

for k such that $Imk^2 = 0$, $Im(k + 1/k) = 0$, $Imk^3 = 0$, respectively. This relation, together with the fact that the nonlinear ODEs satisfied by Φ_1 and Φ_2 are equivalent to nonlinear Volterra integral equations, imply global existence.

The Main Result
Combining equations (1.10) (with $\tilde{Q}(0,t,k)$ given by equations (1.11)-(1.14)) with equations (1.16)-(1.19), as well as using equation (1.20), we obtain the main result of this paper.
Theorem 1.1 Let $q(x,t)$ satisfy either the NLS equation (1.1), or the sG equation (1.2), or the mKdVII equation $(1.3)^-$, or the mKdVI equation $(1.3)^+$, on $0 < x < \infty$, $t > 0$. Let $q(x,t)$ satisfy the following initial and boundary conditions:

NLS

$$q(x,0) = q_0(x), \quad 0 < x < \infty; \quad q(0,t) = g_0(t), \quad t > 0.$$

sG

$$q(x,0) = q_0(x), q_t(x,0) = q_1(x), \quad 0 < x < \infty; \quad q(0,t) = g_0(t), \quad t > 0.$$

mKdVII

$$q(x,0) = q_0(x), \quad 0 < x < \infty; \quad q(0,t) = g_0(t), q_x(0,t) = g_1(t), \quad t > 0,$$

mKdVI

$$q(x,0) = q_0(x), \quad 0 < x < \infty; \quad q(0,t) = g_0(t), \quad t > 0.$$

The functions $q_0(x)$ [2] and $q_1(x)$ belong to the Schwartz class, the functions $g_0(t)$ and $g_1(t)$ are sufficiently smooth, and furthermore these functions are compatible at $x = t = 0$, i.e

NLS

$$q_0(0) = g_0(0)$$

sG

$$q_0(0) = g_0(0), \quad q_1(0) = \frac{dg_0(0)}{dt};$$

mKdVII

$$q_0(0) = g_0(0), \quad \frac{dq_0(0)}{dx} = g_1(0);$$

mKdVI

$$q_0(0) = g_0(0).$$

For the NLS and the sG, $g_1(t)$ is given by equations (1.16) and (1.17) respectively, while for the mKdVII and the mKdVI, $g_2(t)$ and $\{g_1(t),\, g_2(t)\}$ are given by equations (1.18) and (1.19) respectively. The functions $\Phi_1(t,k)$ and $\Phi_2(t,k)$ satisfy the ODEs (1.10), where for the NLS, the sG, the mKdVI and the mKdVII, $f_2(k)$ and $\tilde{Q}(0,t,k)$ are defined in equations (1.11)-(1.14). For the sG, the focusing NLS (equation (1.1) with $\rho = -1$), the focusing mKdVI (equation $(1.3)^+$ with $\rho = -1$) and the focusing mKdVII (equation $(1.3)^-$ with $\rho = -1$) the above functions exist globally. For the defocusing cases of the NLS, mKdVI and mKdVII, the above functions exist at least if t is sufficiently small or if an appropriate norm of the given boundary conditions is sufficiently small.

Organization of the Paper

The Lax pair formulation of equations (1.1)-(1.3) is presented in section 2. The global relation (1.15) is also derived in section 2. In order to solve the global relation it is convenient to use the so-called Gel'fand-Levitan-Marchenko representation of the function $\Phi(t,k)$. The representations associated with the NLS, the sG, and two versions of the mKdV are derived in Section 3. The explicit solution of the global relation, i.e. equations (1.16)-(1.19), is presented in section 4. The linearization of the ODEs associated with the NLS for the particular case $g_0(t) = 0$ is presented in section 5. These results are discussed further in section 6.

[2] For the sG $q_0(x) - 2\pi m \in S(\mathbb{R}^+)$, where m is an integer.

2 The Lax Pair Formulation and the Global Relation

The NLS, sG, mKdVI, and mKdVII, admit the Lax pair formulation (1.4) where $f_1, f_2, Q, \tilde{Q}$ are given by the following formulas:

$$\textit{NLS} \qquad f_1 = k, \quad f_2 = 2k^2, \quad Q(x,t) = \begin{pmatrix} 0 & q \\ \rho\bar{q} & 0 \end{pmatrix}, \quad \tilde{Q} \text{ by equation (1.5).} \tag{2.1}$$

$$\textit{sG} \qquad f_1 = \frac{1}{4}\left(k - \frac{1}{k}\right), \quad f_2 = \frac{1}{4}\left(k + \frac{1}{k}\right),$$

$$Q(x,t,k) = \begin{pmatrix} \frac{i}{4k}(\cos q - 1) & -\frac{i}{4}(q_x + q_t) - \frac{\sin q}{4k} \\ \frac{-i}{4}(q_x + q_t) + \frac{\sin q}{4k} & -\frac{i}{4k}(\cos q - 1) \end{pmatrix},$$

$$\tilde{Q}(x,t,k) = Q(x,t,-k). \tag{2.2}$$

$$\textit{mKdVI} \quad f_1 = k, \quad f_2 = 4k^3, \quad Q(x,t) = \begin{pmatrix} 0 & q \\ \rho q & 0 \end{pmatrix},$$

$$\tilde{Q}(x,t) = 2Q^3 - Q_{xx} - 2ik(Q^2 + Q_x)\sigma_3 + 4k^2 Q. \tag{2.3}$$

mKdVII : as with *mKdVI* but with

$$f_1 = -k, \tag{2.4}$$

and all the terms of $\widetilde{Q}$ change sign except the term involving k.

We now derive the global relation (1.15).

Proposition 2.1. Assume that there exists a function $\Psi(x,t,k)$ satisfying equations (1.4). Furthermore assume that this function satisfies the symmetry relations $\Psi_{22}(x,t,k) = \overline{\Psi_{11}(x,t,\bar{k})}$ and $\Psi_{12}(x,t,k) = \sigma\overline{\Psi_{21}(x,t,\bar{k})}$, where $\sigma^2 = 1$. Define $\{a(k), b(k), A(t,k), B(t,k)\}$ by equations (1.7)-(1.10). Then the global relation (1.15) is valid, where

$$c(t,k) = \int_0^\infty e^{2if_1(k)\xi}(Q_{11}\mu_1 + Q_{12}\mu_2)(\xi,t,k)d\xi. \tag{2.5}$$

The functions $\mu_1(\xi,t,k)$, $\mu_2(\xi,t,k)$ are analytic for k such that $\operatorname{Im} f_1(k) > 0$. Furthermore, $\mu_1 = O(1/k)$, $\mu_2 = O(1)$ as $k \to \infty$.

Proof. Let $\Psi_j(x,t,k)$, $j = 2,3$, be the two solutions of equations (1.4) which satisfy the boundary conditions

$$\Psi_2(0,0,k) = \Psi_3(\infty,0,k). \tag{2.6}$$

The explicit integral representations of Ψ_j are given in [3]. These solutions satisfy the relation

$$\Psi_3(0,t,k) = \Psi_2(0,t,k)\Psi_3(0,0,k). \tag{2.7}$$

Indeed, since Ψ_j, $j = 2,3$ satisfy both equations (1.4) it follows that $\Psi_3(x,t,k) = \Psi_2(x,t,k)J(k)$. Evaluating this equation at $x = t = 0$, we find

$$J(k) = \Psi_3(0,0,k).$$

Letting $x = 0$, we find equation (2.7).
Let the vectors $\phi(x,k)$ and $\Phi(t,k)$ denote the second columns vectors of the matrices $\Psi_3(x,0,k)$ and $\Psi_2(0,t,k)$. Then these vectors satisfy equations (1.8) and (1.10) respectively. Assume that $Q(x,0,k)$ and $\tilde{Q}(0,t,k)$ are such that $\Psi_3(x,0,k)$ and $\Psi_2(0,t,k)$ satisfy the following symmetry relations

$$\Psi_3(x,0,k) = \begin{pmatrix} \overline{\phi_2(x,\bar{k})} & \phi_1(x,k) \\ \sigma\overline{\phi_1(x,\bar{k})} & \phi_2(x,k) \end{pmatrix}, \quad \Psi_2(0,t,k) = \begin{pmatrix} \overline{\Phi_2(t,\bar{k})} & \Phi_1(t,k) \\ \sigma\overline{\Phi_1(t,\bar{k})} & \Phi_2(t,k) \end{pmatrix}, \tag{2.8}$$

where $\sigma^2 = 1$. For the case of the NLS, the sG, and the mKdV, σ equals ρ, -1, and ρ, respectively.
The 12 element of equation (2.7) is

$$\overline{\Phi_2(t,\bar{k})}\phi_1(0,k) + \Phi_1(t,k)\phi_2(0,k) = (\Psi_3(0,t,k))_{12}.$$

Using the definitions (1.7) and (1.9), and multiplying the lhs of the above equation by $\exp[if_2t]$ we find

$$A(t,k)b(k) - B(t,k)a(k) = e^{if_2t}(\Psi_3(0,t,k))_{12}. \tag{2.9}$$

The definition of $\Psi_3(x,t,k)$ implies

$$\Psi_3(x,t,k) = e^{-i(f_1x+f_2t)\sigma_3} - \int_x^\infty e^{if_1\xi\sigma_3}(Q\Psi_3)(\xi,t,k)d\xi.$$

Hence the rhs of equation (2.9) becomes

$$-e^{if_2t}\int_0^\infty e^{2if_1\xi}(Q\Psi_3)_{12}(\xi,t,k)d\xi.$$

Letting

$$\Psi_3(x,t,k) = \begin{pmatrix} \overline{\mu_2(x,t,\bar{k})} & \mu_1(x,t,k) \\ \sigma\overline{\mu_1(x,t,\bar{k})} & \mu_2(x,t,k) \end{pmatrix} e^{-i(f_1x+f_2t)\sigma_3}$$

equation (2.9) becomes equation (1.15), where $c(t,k)$ is given by equation (2.5) and the vector μ satisfies the equations

$$\begin{aligned} \mu_1(x,t,k) &= -\textstyle\int_x^\infty e^{2if_1(\xi-x)}(Q_{11}\mu_1 + Q_{12}\mu_2)(\xi,t,k)d\xi, \\ \mu_2(x,t,k) &= 1 - \textstyle\int_x^\infty (Q_{21}\mu_1 + Q_{22}\mu_2)(\xi,t,k)d\xi. \end{aligned}$$

Since $\xi - x \geq 0$, the vector μ is bounded and analytic in the domain of the complex k-plane where $\text{Im } f_1(k) > 0$. **QED**

3 The Gel'fand-Levitan-Marchenko Representations

We first express $\Phi(t,k)$ in terms of the auxiliary functions $\{L_j(t,s), M_j(t,s), N_j(t,s)\}_1{}^2$.

Proposition 3.1 Let the vector $\Phi(t,k)$ satisfy equations (1.10), where $f_2(k)$ and $\tilde{Q}(0,t,k)$ are defined by (1.11)-(1.14).
Then the vector Φ can be represented in the following form:

$$NLS \qquad \Phi(t,k) = \begin{pmatrix} 0 \\ e^{if_2 t} \end{pmatrix} + \int_{-t}^{t} \begin{pmatrix} L_1(t,s) - \frac{i}{2} g_0(t) M_2(t,s) + k M_1(t,s) \\ L_2(t,s) + \frac{i\rho}{2} \bar{g}_0(t) M_1(t,s) + k M_2(t,s) \end{pmatrix} e^{if_2 s} ds, \tag{3.1}$$

$$sG \qquad \Phi(t,k) = \begin{pmatrix} 0 \\ e^{if_2 t} \end{pmatrix} + \int_{-t}^{t} \begin{pmatrix} iL_1(t,s) - \frac{1}{2k}\left[M_1(t,s) + \tan\left(\frac{g_0(t)}{2}\right) M_2(t,s)\right] \\ L_2(t,s) + \frac{i}{2k}\left[-M_2(t,s) + \tan\left(\frac{g_0(t)}{2}\right) M_1(t,s)\right] \end{pmatrix} e^{if_2 s} ds, \tag{3.2}$$

$$mKdVI \quad \Phi(t,k) = \begin{pmatrix} 0 \\ e^{if_2 t} \end{pmatrix} + \int_{-t}^{t} \cdot$$

$$\begin{pmatrix} L_1(t,s) + \frac{1}{2} g_0(t) M_2(t,s) + \frac{1}{4} g_1(t) N_2(t,s) + ik\left(M_1(t,s) - \frac{1}{2} g_0(t) N_2(t,s)\right) + k^2 N(t,s) \\ L_2(t,s) - \frac{1}{2}\rho g_0(t) M_1(t,s) + \frac{1}{4}\rho g_1(t) N_1(t,s) + ik\left(M_2(t,s) + \frac{1}{2}\rho g_0(t) N_1(t,s)\right) + k^2 N_2(t,s) \end{pmatrix} e^{if_2 s} ds. \tag{3.3}$$

The functions $\{L_j, M_j, N_j\}_1^2$ satisfy the following equations.

NLS

$$L_1(t,t) = \frac{i}{2} g_1(t), \quad M_1(t,t) = g_0(t), \quad L_2(t,-t) = M_2(t,-t) = 0, \tag{3.4a}$$

$$\begin{aligned}
&L_{1_t} - L_{1_s} = i g_1(t) L_2 + \alpha(t) M_1 + \beta(t) M_2, \\
&L_{2_t} + L_{2_s} = -i\rho \bar{g}_1(t) L_1 - \alpha(t) M_2 + \rho \bar{\beta}(t) M_1, \\
&M_{1_t} - M_{1_s} = 2 g_0(t) L_2 + i g_1(t) M_2, \\
&M_{2_t} + M_{2_s} = 2\rho \bar{g}_0(t) L_1 - i\rho \bar{g}_1(t) M_1,
\end{aligned} \tag{3.4b}$$

where $\alpha(t)$ and $\beta(t)$ are defined by the equations

$$\alpha(t) = \frac{\rho}{2}(g_0 \bar{g}_1 - \bar{g}_0 g_1), \quad \beta(t) = \frac{1}{2}(\dot{g}_0 - \rho |g_0|^2 g_0). \tag{3.4c}$$

sG

$$L_1(t,t) = -\frac{1}{8}(\dot{g}_0(t) + g_1(t)), \quad M_1(t,t) = -\frac{1}{4}\sin g_0(t), \quad L_2(t,-t) = M_2(t,-t) = 0, \tag{3.5a}$$

$$L_{1_t} - L_{1_s} = -\tfrac{1}{4}(\dot{g}_0(t) + g_1(t))L_2 + \tfrac{1}{4}\tan\left(\tfrac{g_0(t)}{2}\right)M_2,$$

$$L_{2_t} + L_{2_s} = \tfrac{1}{4}(\dot{g}_0(t) + g_1(t))L_1 - \tfrac{1}{4}\tan\left(\tfrac{g_0(t)}{2}\right)M_1,$$

$$M_{1_t} - M_{1_s} = -\tfrac{\dot{g}_0(t)}{2(1+\cos g_0(t))}\left[(1+\cos g_0(t))M_2 + \sin g_0(t)M_1\right]$$

$$+\tfrac{1}{4}(\dot{g}_0(t) + g_1(t))M_2 - \tfrac{1}{2}\sin g_0(t)L_2,$$

$$M_{2_t} + M_{2_s} = \tfrac{\dot{g}_0(t)}{2(1+\cos g_0(t)}\left[-\sin g_0(t)M_2 + (1+\cos g_0(t))M_1\right]$$

$$-\tfrac{1}{4}(\dot{g}_0(t) + g_1(t))M_1 + \tfrac{1}{2}\sin g_0(t)L_1. \tag{3.5b}$$

mKdVI

$$N_1(t,t) = 2g_0(t), \quad M_1(t,t) = g_1(t), \quad L_1(t,t) = \rho g_0^3(t) - \frac{1}{2}g_2(t),$$
$$N_2(t,-t) = M_2(t,-t) = L_2(t,-t) = 0, \tag{3.6a}$$

$$N_{1_t} - N_{1_s} = (\rho g_0^3(t) - g_2(t))N_2 - 2g_1(t)M_2 + 4g_0(t)L_2,$$

$$N_{2_t} + N_{2_s} = (g_0^3(t) - \rho g_2(t))N_1 + 2\rho g_1(t)M_1 + 4\rho g_0(t)L_1,$$

$$M_{1_t} - M_{1_s} = (\rho g_0^3(t) - g_2(t))M_2 + 2g_1(t)L_2 + \alpha(t)N_1 - \beta(t)N_2,$$

$$M_{2_t} + M_{2_s} = (g_0^3(t) - \rho g_2(t))M_1 - 2\rho g_1(t)L_1 - \alpha(t)N_2 + \beta(t)N_1,$$

$$L_{1_t} - L_{1_s} = (2\rho g_0^3(t) - g_2(t))L_1 + \gamma(t)M_1 - \tfrac{1}{2}\dot{g}_0(t)M_2 + \tfrac{1}{4}\rho g_0(t)\dot{g}_0(t)N_1 + \delta N_2,$$

$$L_{2_t} + L_{2_s} = (2g_0^3(t) - \rho g_2(t))L_1 - \gamma(t)M_2 + \tfrac{1}{2}\rho\dot{g}_0(t)M_1 + \tfrac{1}{4}\rho g_0(t)\dot{g}_0(t)N_2 + \lambda\delta N_1, \tag{3.6b}$$

where $\alpha, \beta, \gamma, \delta$ are defined by equations

$$\alpha(t) = \frac{3}{2}g_0^4 - \rho g_0 g_2 + \frac{1}{2}\rho g_1^2, \quad \beta(t) = \frac{1}{2}(\rho g_0^2 g_1 - \dot{g}_0),$$

$$\gamma(t) = \rho g_0 g_2 - \frac{1}{2}\rho g_1^2 - \frac{3}{2}g_0^4, \quad \delta(t) = \frac{1}{4}\rho g_0 g_1^2 - \frac{1}{2}\rho g_0^2 g_2 - \frac{1}{4}\dot{g}_1 + \frac{3}{4}g_0^5. \tag{3.6c}$$

Proof. For the derivation of this result we will use the following identity, which can be derived using integration by parts,

$$(-i)f_2\int_{-t}^{t} F(t,s)e^{-if_2 s\sigma_3}ds = \left[F(t,t)e^{-if_2 t\sigma_3} - F(t,-t)e^{if_2 t\sigma_3} - \int_{-t}^{t} F_s(t,s)e^{-if_2 s\sigma_3}ds\right]\sigma_3. \tag{3.7}$$

We will also use

$$\sigma_3 F \sigma_3 = \begin{pmatrix} F_{11} & -F_{12} \\ -F_{21} & F_{22} \end{pmatrix}.$$

The vector $\Phi(t,k)$ is the second column vector of the matrix $\Psi(t,k)$; the latter matrix satisfies equation (1.10a) as well as the boundary condition $\Psi(0,k) = I$.

NLS

Substituting the representation

$$\Psi(t,k) = e^{-if_2 t\sigma_3} + \int_{-t}^{t} (L(t,s) + kM(t,s)) e^{-if_2 s\sigma_3} ds, \tag{3.8}$$

(where L and M are 2×2 matrices) in equation (1.10a), and using equation (3.7) we find the following equations

$$M(t,-t) + \sigma_3 M(t,-t)\sigma_3 = 0 \tag{3.9a}$$

$$L(t,-t) + \sigma_3 L(t,-t)\sigma_3 + iQ_1(t)M(t,-t)\sigma_3 = 0, \tag{3.9b}$$

$$M(t,t) - \sigma_3 M(t,t)\sigma_3 = 2Q_1(t), \tag{3.10a}$$

$$L(t,t) - \sigma_3 L(t,t)\sigma_3 - iQ_1(t)M(t,t)\sigma_3 = Q_0(t), \tag{3.10b}$$

$$M_t(t,s) + \sigma_3 M_s(t,s)\sigma_3 = 2Q_1(t)L(t,s) + Q_0(t)M(t,s), \tag{3.11a}$$

$$L_t(t,s) + \sigma_3 L_s(t,s)\sigma_3 = -iQ_1(t)M_s(t,s)\sigma_3 + Q_0(t)L(t,s). \tag{3.11b}$$

Equations (3.9b) and (3.10b) suggest replacing the function L by $\tilde{L}$, where

$$L(t,s) = \tilde{L}(t,s) + \frac{i}{2}Q_1(t)\sigma_3 M(t,s). \tag{3.12}$$

Using this equation in equations (3.9)-(3.11) we find the following equations:

$$M(t,-t) + \sigma_3 M(t,-t)\sigma_3 = 0, \tag{3.13a}$$

$$\tilde{L}(t,-t) + \sigma_3 \tilde{L}(t,-t)\sigma_3 = 0, \tag{3.13b}$$

$$M(t,t) - \sigma_3 M(t,t)\sigma_3 = 2Q_1(t), \tag{3.14a}$$

$$\tilde{L}(t,t) - \sigma_3 \tilde{L}(t,t)\sigma_3 = Q_0(t), \tag{3.14b}$$

$$M_t + \sigma_3 M_s \sigma_3 = 2Q_1(t)\tilde{L} + (iQ_1^2(t)\sigma_3 + Q_0(t))M, \tag{3.15a}$$

$$\tilde{L}_t + \sigma_3\tilde{L}_s\sigma_3 = (Q_0(t) - iQ_1(t)\sigma_3 Q_1(t))\tilde{L} + \frac{i}{2}(Q_0(t)Q_1(t)\sigma_3 - iQ_1^3(t) - Q_1(t)\sigma_3 Q_0(t) - \dot{Q}_1(t)\sigma_3)M. \tag{3.15b}$$

For the derivation of equation (3.13b) we have used the identity

$$Q_1(t)\sigma_3 M(t,-t) - Q_1(t)M(t,-t)\sigma_3 + 2Q_1(t)M(t,-t)\sigma_3 = 0, \tag{3.16}$$

which follows from the fact that $M(t,-t)$ is off diagonal (see (3.9a)). Similarly for the derivation of equation (3.14b) we have used an equation similar to (3.16) where $M(t,-t)$ is replaced by the matrix $M(t,t)$ which is also off-diagonal (see (3.10a)).

Equations (3.13)-(3.15) imply that the 2×2 matrices $\tilde{L}$ and M have the following structure,

$$\tilde{L} = \begin{pmatrix} \bar{L}_2 & L_1 \\ \rho\bar{L}_1 & L_2 \end{pmatrix}, \quad M = \begin{pmatrix} \bar{M}_2 & M_1 \\ \rho\bar{M}_1 & M_2 \end{pmatrix}. \tag{3.17}$$

Using these representations, equations (3.13)-(3.15) imply equations (3.4b). Furthermore the second column vector of equation (3.8) yields equation (3.1).

sG
Substituting the representation

$$\Psi(t,k) = e^{-if_2 t\sigma_3} + \int_{-t}^{t} (L(t,s) + \frac{1}{k} M(t,s)) e^{-if_2 s\sigma_3} ds, \tag{3.18}$$

into equation (1.10a) and using equation (3.7) we find the following equations:

$$L(t,-t) + \sigma_3 L(t,-t)\sigma_3 = 0, \tag{3.19a}$$

$$M(t,-t) + (\sigma_3 + 4Q_{-1})M(t,-t)\sigma_3 = 0, \tag{3.19b}$$

$$L(t,t) - \sigma_3 L(t,t)\sigma_3 = iQ_0(t), \tag{3.20a}$$

$$M(t,t) - (\sigma_3 + 4Q_{-1})M(t,t)\sigma_3 = -iQ_{-1}(t), \tag{3.20b}$$

$$L_t + \sigma_3 L_s \sigma_3 = iQ_0(t)L + iQ_{-1}(t)M, \tag{3.21a}$$

$$M_t + (\sigma_3 + 4Q_{-1})M_s\sigma_3 = iQ_0(t)M - iQ_{-1}(t)L. \tag{3.21b}$$

Let $R(t)$ denote the matrix

$$R(t) = \begin{pmatrix} \cos g_0(t) & i\sin g_0(t) \\ -i\sin g_0(t) & -\cos g_0(t) \end{pmatrix}.$$

This matrix can be written in the form

$$R(t) = G^{-1}(t)\sigma_3 G(t), \quad G(t) = \begin{pmatrix} 1+\cos g_0(t) & i\sin g_0(t) \\ i\sin g_0(t) & 1+\cos g_0(t) \end{pmatrix}. \tag{3.22}$$

Equations (3.19b) and (3.20b) suggest replacing $M(t,s)$ by $\tilde{M}(t,s)$, where

$$M(t,s) = G^{-1}(t)\tilde{M}(t,s). \tag{3.23}$$

Then equations (3.19)-(3.21) become

$$\begin{aligned} & L(t,-t) + \sigma_3 L(t,-t)\sigma_3 = 0, \quad \tilde{M}(t,-t) + \sigma_3\tilde{M}(t,-t)\sigma_3 = 0, \\ & L(t,t) - \sigma_3 L(t,t)\sigma_3 = iQ_0(t), \quad \tilde{M}(t,t) - \sigma_3\tilde{M}(t,t)\sigma_3 = -iG(t)Q_{-1}(t), \\ & L_t + \sigma_3 L_s\sigma_3 = iQ_0(t)L + iQ_{-1}(t)G^{-1}(t)\tilde{M}, \\ & \tilde{M}_t + \sigma_3\tilde{M}_s\sigma_3 = (\dot{G}(t)G^{-1}(t) + iG(t)Q_0(t)G^{-1}(t))\tilde{M} - iG(t)Q_{-1}(t)L. \end{aligned} \tag{3.24}$$

Using the identities

$$iQ_{-1}(t)G^{-1}(t) = \frac{\sin g_0}{4(1+\cos g_0)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, iG(t)Q_{-1}(t) = \frac{\sin g_0}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\dot{G}(t)G^{-1}(t)+iG(t)Q_0(t)G^{-1}(t) = \frac{\dot{g}_0}{2(1+\cos g_0)} \begin{pmatrix} -\sin g_0 & i(1+\cos g_0) \\ i(1+\cos g_0) & -\sin g_0 \end{pmatrix} - \frac{i}{4}(g_1+\dot{g}_0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

equations (3.24) imply that the 2×2 matrices L and M have the following structure,

$$L = \begin{pmatrix} L_2 & iL_1 \\ iL_1 & L_2 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} iM_2 & -M_1 \\ M_1 & -iM_2 \end{pmatrix}, \quad L_j, M_j \in \mathbb{R}. \tag{3.25}$$

Using these representations, equations (3.24) imply equations (3.5b), and the second column vector of equation (3.18) yields equation (3.2).

mKdVI
Substituting the representation

$$\Psi(t,k) = e^{-if_2t\sigma_3} + \int_{-t}^{t} (L(t,s) + ikM(t,s) + k^2N(t,s))e^{-if_2s\sigma_3}ds, \tag{3.26}$$

into equation (1.10a) and using equation (3.7) we find the following equations:

$$N(t,-t) + \sigma_3 N(t,-t)\sigma_3 = 0, \tag{3.27a}$$

$$M(t,-t) + \sigma_3 M(t,-t)\sigma_3 + Q_2(t)N(t,-t)\sigma_3 = 0, \tag{3.27b}$$

$$L(t,-t) + \sigma_3 L(t,-t)\sigma_3 + \frac{1}{2}Q_1(t)N(t,-t)\sigma_3 - Q_2(t)M(t,-t)\sigma_3 = 0. \tag{3.27c}$$

$$N(t,t) - \sigma_3 N(t,t)\sigma_3 = 4Q_2(t), \tag{3.28a}$$

$$M(t,t) - \sigma_3 M(t,t)\sigma_3 = -2Q_1(t) + Q_2(t)N(t,t)\sigma_3, \tag{3.28b}$$

$$L(t,t) - \sigma_3 L(t,t)\sigma_3 = Q_0(t) + \frac{1}{2}Q_1(t)N(t,t)\sigma_3 - Q_2(t)M(t,t)\sigma_3, \tag{3.28c}$$

$$N_t + \sigma_3 N_s \sigma_3 = Q_0(t)N + 2Q_1(t)M + 4Q_2(t)L \tag{3.29a}$$

$$M_t + \sigma_3 M_s \sigma_3 = Q_0(t)M - 2Q_1(t)L - Q_2(t)N_s\sigma_3 \tag{3.29b}$$

$$L_t + \sigma_3 L_s \sigma_3 = Q_0(t)L - \frac{1}{2}Q_1(t)N_s\sigma_3 + Q_2(t)M_s\sigma_3. \tag{3.29c}$$

Equations (3.27c), (3.28b) and (3.28c) suggest replacing M and L by $\tilde{M}$ and $\tilde{L}$, where

$$M = \tilde{M} + \frac{1}{2}Q_2(t)\sigma_3 N$$

$$L = \tilde{L} - \frac{1}{2}Q_2(t)\sigma_3\tilde{M} + \frac{1}{4}g_1(t), \tag{3.30}$$

and Λ denotes

$$\Lambda = \begin{pmatrix} 0 & 1 \\ \rho & 0 \end{pmatrix}. \tag{3.31}$$

Using (3.30), equations (3.27)-(3.29) become

$$F(t,-t) + \sigma_3 F(t,-t)\sigma_3 = 0, \quad F = N \text{ or } \tilde{M} \text{ or } \tilde{L},$$

$$N(t,t) - \sigma_3 N(t,t)\sigma_3 = 4g_0(t)\Lambda,$$

$$\tilde{M}(t,t) - \sigma_3 \tilde{M}(t,t)\sigma_3 = -2g_1(t)\Lambda\sigma_3,$$

$$\tilde{L}(t,t) - \sigma_3 \tilde{L}(t,t)\sigma_3 = (2\rho g_0^3(t) - g_2(t))\Lambda,$$

$$\begin{aligned}
&N_t + \sigma_3 N_s \sigma_3 = (\rho g_0^3(t) - g_2(t))\Lambda N + 2g_1(t)\Lambda\sigma_3\tilde{M} + 4g_0(t)\Lambda\tilde{L},\\
&\tilde{M}_t + \sigma_3 \tilde{M}_s \sigma_3 = (\rho g_0^3(t) - g_2(t))\Lambda\tilde{M} + (\alpha(t) + \beta(t)\Lambda)\sigma_3 N - 2g_1\Lambda\sigma_3\tilde{L},\\
&\tilde{L}_t + \sigma_3 \tilde{L}_s \sigma_3 = (2\rho g_0^3(t) - g_2(t))\Lambda\tilde{L} + (\gamma(t) + \tfrac{1}{2}\dot{g}_0(t)\Lambda)\sigma_3\tilde{M}\\
&\qquad + (\delta(t)\Lambda + \tfrac{1}{4}\rho g_0(t)\dot{g}_0(t))N,
\end{aligned} \tag{3.32}$$

where $\alpha, \beta, \gamma, \delta$ are defined by equations (3.6c).
These equations imply that $N, \tilde{M}, \tilde{L}$ have the following structure:

$$N = \begin{pmatrix} N_2 & N_1 \\ \rho N_1 & N_2 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} -M_2 & M_1 \\ -\rho M_1 & M_2 \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} L_2 & L_1 \\ \rho L_1 & L_2 \end{pmatrix}. \tag{3.33}$$

Using these representations, equations (3.32) imply equations (3.6b), and the second column vector of equation (3.26) yields equation (3.3) **QED**

The Expressions of $\Phi(t,k)$ in terms of $\{\lambda_j, \mu_j, \nu_j\}_1^2$
We now express $\Phi(t,k)$ in terms of the auxiliary functions $\{\lambda_j, \mu_j, \nu_j\}_1^2$.

Proposition 3.2
Let the vector $\Phi(t,k)$ be expressed in terms of the functions $\{L_j(t,s),\, M_j(t,s),\, N_j(t,s)\}_1^2$ by equations (3.1)-(3.3). Define the functions $\{\lambda_j, \mu_j, \nu_2\}_1^2$ by the following equations:

$$\lambda_j(t,k) = \int_{-t}^{t} e^{if_2(s-t)} L_j(t,s)ds, \quad \mu_j(t,k) = \int_{-t}^{t} e^{if_2(s-t)} M_j(t,s)ds,$$

$$\nu_j(t,k) = \int_{-t}^{t} e^{if_2(s-t)} N_j(t,s)ds, \quad j = 1,2, \quad t > 0, \quad k \in \mathbb{C}. \tag{3.34}$$

Then $\Phi(t,k)$ is given by the following expressions:

NLS

$$\begin{aligned}\Phi_1(t,k) &= e^{4ik^2t}\left(\lambda_1 - \frac{i}{2}g_0(t)\mu_2 + k\mu_1\right),\\ \Phi_2(t,k) &= e^{4ik^2t}\left(1+\lambda_2 + \frac{i\rho}{2}\bar{g}_0(t)\mu_1 + k\mu_2\right).\end{aligned} \tag{3.35}$$

sG

$$\begin{aligned}\Phi_1(t,k) &= e^{\frac{i}{4}(k+\frac{1}{k})t}\left(i\lambda_1 - \frac{1}{2k}\mu_1 - \frac{1}{2k}\tan\left(\frac{g_0(t)}{2}\right)\mu_2\right),\\ \Phi_2(t,k) &= e^{\frac{i}{4}(k+\frac{1}{k})t}\left(1+\lambda_2 - \frac{i}{2k}\mu_2 + \frac{i}{2k}\tan\left(\frac{g_0(t)}{2}\right)\mu_1\right).\end{aligned} \tag{3.36}$$

mKdVI

$$\begin{aligned}\Phi_1(t,k) &= e^{4ik^3t}\left(\lambda_1 + \frac{1}{2}g_0(t)\mu_2 + \frac{1}{4}g_1(t)\nu_2 + ik\left(\mu_1 - \frac{1}{2}g_0(t)\nu_2\right) + k^2\nu_1\right),\\ \Phi_2(t,k) &= e^{4ik^3t}\left(1+\lambda_2 - \frac{1}{2}\rho g_0(t)\mu_1 + \frac{\rho}{4}g_1(t)\nu_1 + ik\left(\mu_2 + \frac{1}{2}\rho g_0(t)\nu_1\right) + k^2\nu_2\right).\end{aligned} \tag{3.37}$$

Proof
Equations (3.1)-(3.3) express $\Phi(t,k)$ in terms of $\{L_j, M_j, N_j\}_1^2$. Using these equations as well as the definitions (3.34), equations (3.1)-(3.3) become equations (3.35)-(3.37). Indeed, if $\Phi(t,k)$ is represented in the form

$$\Phi(t,k) = \begin{pmatrix} 0 \\ e^{if_2t} \end{pmatrix} + \int_{-t}^{t} \begin{pmatrix} F_1(t,s) \\ F_2(t,s) \end{pmatrix} e^{if_2s}ds, \tag{3.38}$$

then Φ_1 and Φ_2 are given by the expressions

$$\Phi_1(t,k) = e^{if_2t}f_1(t,k), \quad \Phi_2(t,k) = e^{if_2t}(1+f_2(t,k)), \tag{3.39}$$

where f_1 and f_2 are obtained from F_1 and F_2 by replacing $\{L_j, M_j, N_j\}_1^2$ with $\{\lambda_j, \mu_j, \nu_j\}_1^2$.

4 The Solution of the Global Relation

Equations (1.9) express $A(t,k)$ and $B(t,k)$ in terms of $\overline{\Phi_2(t,\bar{k})}$ and $\Phi_1(t,k)$. Thus, if Φ_1 and Φ_2 are represented in the form (3.39), then $A(t,k)$ and $B(t,k)$ are given by the equations

$$A(t,k) = 1 + \overline{f_2(t,\bar{k})}, \quad B(t,k) = -e^{2if_2t}f_1(t,k).$$

These equations and equations (3.35)-(3.37) yield the following expressions for $A(t,k)$ and $B(t,k)$

NLS

$$
\begin{aligned}
A(t,k) &= 1+\overline{\lambda_2(t,\bar{k})}-\tfrac{i\rho}{2}g_0(t)\overline{\mu_1(t,\bar{k})}+k\overline{\mu_2(t,\bar{k})},\\
B(t,k) &= -e^{2if_2t}\left[\lambda_1(t,k)-\tfrac{i}{2}g_0(t)\mu_2(t,k)+k\mu_1(t,k)\right],
\end{aligned}
\tag{4.1}
$$

sG

$$
\begin{aligned}
A(t,k) &= 1+\overline{\lambda_2(t,\bar{k})}+\tfrac{i}{2k}\overline{\mu_2(t,\bar{k})}-\tfrac{i}{2k}\tan\left(\tfrac{g_0(t)}{2}\right)\overline{\mu_1(t,\bar{k})},\\
B(t,k) &= -e^{2if_2t}\left[i\lambda_1(t,k)-\tfrac{1}{2k}\mu_1(t,k)-\tfrac{1}{2k}\tan\left(\tfrac{g_0(t)}{2}\right)\mu_2(t,k)\right],
\end{aligned}
\tag{4.2}
$$

mKdVI

$$
\begin{aligned}
A(t,k) &= 1+\overline{\lambda_2(t,\bar{k})}-\tfrac{1}{2}\rho g_0(t)\overline{\mu_1(t,\bar{k})}+\tfrac{\rho}{4}g_1(t)\overline{\nu_1(t,\bar{k})}\\
&\quad -ik\left(\overline{\mu_2(t,\bar{k})}+\tfrac{1}{2}\rho g_0(t)\overline{\nu_1(t,\bar{k})}\right)+k^2\overline{\nu_2(t,\bar{k})},\\
B(t,k) &= -e^{2if_2t}\left[\lambda_1(t,k)+\tfrac{1}{2}g_0(t)\mu_2(t,k)+\tfrac{1}{4}g_1(t)\nu_2(t,k)\right.\\
&\quad \left.+ik\left(\mu_1(t,k)-\tfrac{1}{2}\rho g_0(t)\nu_2(t,k)\right)+k^2\nu_1(t,k)\right].
\end{aligned}
\tag{4.3}
$$

If Φ_1 and Φ_2 are represented in the form (3.38), then the identities

$$
\begin{aligned}
&\textstyle\int_{-t}^{t} F(t,s)e^{-if_2s}ds = 2e^{-if_2t}\int_0^t F(t,t-2\tau)e^{2if_2\tau}d\tau,\\
&\textstyle\int_{-t}^{t} F(t,s)e^{if_2s}ds = 2e^{-if_2t}\int_0^t F(t,2\tau-t)e^{2if_2\tau}d\tau,
\end{aligned}
$$

imply that $A(t,k)$ and $B(t,k)$ are given by the expressions

$$
A(t,k)=1+2\int_0^t \bar{F}_2(t,t-2\tau)e^{2if_2\tau}d\tau,\quad B(t,k)=-2\int_0^t F_1(t,2\tau-t)e^{2if_2\tau}d\tau.
$$

Thus, we find the following alternative representations for $A(t,k)$ and $B(t,k)$.

NLS

$$
\begin{aligned}
A(t,k) &= 1+2\textstyle\int_0^t e^{2if_2\tau}\left[\bar{L}_2(t,t-2\tau)-\tfrac{i\rho}{2}g_0(t)\bar{M}_1(t,t-2\tau)+k\bar{M}_2(t,t-2\tau)\right]d\tau,\\
B(t,k) &= -2\textstyle\int_0^t e^{2if_2\tau}\left[L_1(t,2\tau-t)-\tfrac{i}{2}g_0(t)M_2(t,2\tau-t)+kM_1(t,2\tau-t)\right]d\tau.
\end{aligned}
\tag{4.4}
$$

sG

$$
\begin{aligned}
A(t,k) &= 1+2\textstyle\int_0^t e^{2if_2\tau}\left[L_2(t,t-2\tau)+\tfrac{i}{2k}\left(M_2(t,t-2\tau)-\tan\left(\tfrac{g_0(t)}{2}\right)M_1(t,t-2\tau)\right)\right]d\tau,\\
B(t,k) &= -2\textstyle\int_0^t e^{2if_2\tau}\left[iL_1(t,2\tau-t)-\tfrac{1}{2k}\left(M_1(t,2\tau-t)+\tan\left(\tfrac{g_0(t)}{2}\right)M_2(t,2\tau-t)\right)\right]d\tau.
\end{aligned}
\tag{4.5}
$$

mKdVI

$$\begin{aligned} A(t,k) &= 1 + 2\int_0^t e^{2if_2\tau}\left[L_2(t,t-2\tau) - \tfrac{1}{2}\rho g_0(t)M_1(t,t-2\tau) + \tfrac{1}{4}\rho g_1(t)N_1(t,t-2\tau)\right. \\ &\qquad \left. -ik\left(M_2(t,t-2\tau) + \tfrac{1}{2}\rho g_0(t)N_1(t,t-2\tau)\right) + k^2 N_2(t,t-2\tau)\right]d\tau, \\ B(t,k) &= -2\int_0^t e^{2if_2\tau}\left[L_1(t,2\tau-t) + \tfrac{1}{2}g_0(t)M_2(t,2\tau-t) + \tfrac{1}{4}g_1(t)N_2(t,2\tau-t)\right. \\ &\qquad \left. +ik\left(M_1(t,2\tau-t) - \tfrac{1}{2}g_0(t)N_2(t,2\tau-t)\right) + k^2 N_1(t,2\tau-t)\right]d\tau. \end{aligned} \tag{4.6}$$

The Solution of the Global Relation in Terms of $\{\lambda_j,\mu_j,\nu_j\}_1^2$.

The NLS

For the solution of global relation (1.15) in the case of the NLS we will use the following two identities:

$$\int_{\partial D_1} k\left[\int_0^t e^{4ik^2(\tau-t')}K(\tau,t)d\tau\right]dk = \frac{\pi}{4}K(t',t), \tag{4.7}$$

$$\int_{\partial D_1} k^2\left[\int_0^t e^{4ik^2(\tau-t')}K(\tau,t)d\tau\right]dk = \int_{\partial D_1} k^2\left[\int_0^{t'} e^{4ik^2(\tau-t')}K(\tau,t)d\tau - \frac{K(t',t)}{4ik^2}\right]dk, \tag{4.8}$$

where

$$t>0, \quad t'>0, \quad t'<t,$$

the contour ∂D_1 denotes the union of the contour $(i\infty,0]$ and of the contour $[0,\infty)$ (i.e. the oriented boundary of the first quadrant) and $K(\tau,t)$ is a smooth function of the arguments indicated.

Equation (4.7) follows from the usual Fourier transform identity by mapping ∂D_1 to the real axis.

In order to derive equation (4.8) we rewrite the lhs of this equation as the rhs plus the term

$$\int_{\partial D_1} k^2\left[\int_{t'}^t e^{4ik^2(\tau-t')}K(\tau,t)d\tau + \frac{K(t',t)}{4ik^2}\right]dk.$$

The integrand of the above integral is analytic and bounded in the domain of the complex-k plane enclosed by ∂D_1. Also its zero order term (with respect to $(k^2)^{-1}$) is given by $K(t,t)e^{4ik^2(t-t')}/2i$, thus Jordan's lemma implies that this term vanishes.

We will now show that $g_1(t)$ is given by the expression

$$\begin{aligned} g_1(t) = \frac{4i}{\pi}\int_{\partial D_1} e^{-4ik^2t}kR(k)dk + \frac{2g_0(t)}{\pi}\int_{\partial D_1} k\left[\mu_2 + \rho e^{-4ik^2t}R(k)\bar{\mu}_1\right]dk \\ + \frac{4i}{\pi}\int_{\partial D_1}\left[k^2\mu_1 - \frac{g_0(t)}{2i} + e^{-4ik^2t}kR(k)(\bar{\lambda}_2 + k\bar{\mu}_2)\right]dk. \end{aligned} \tag{4.9}$$

We first assume that $a(k)\neq 0$ for $k\in D_1$. We let $f=2ik^2$ in the global equation (1.15a), we multiply the resulting equation by $k\exp[-4ik^2t']$, and then integrate along ∂D_1. The rhs vanishes because $kc(k,t)/a(k)$ is analytic and of $0(1)$ for $\mathrm{Im}k \geq 0$, and the oscillatory term $\exp[4ik^2(t-t')]$ is bounded in the first quadrant. The term $kR(k)A(t,k)$ is of $0(1)$ as $k \to \infty$, thus this term times $\exp[-4ik^2t']$ is well defined. Hence, replacing in the global relation $B(t,k)$ by the rhs of equation (4.4b) we find

$$-2\int_{\partial D_1}\left[\int_0^t e^{4ik^2(\tau-t')}\left(kL_1(t,2\tau-t)-\frac{i}{2}g_0(t)kM_2(t,2\tau-t)+k^2M_1(t,2\tau-t)d\tau\right)\right]dk$$
$$-\int_{\partial D_1} e^{-4ik^2t'}kR(k)A(t,k)dk=0. \tag{4.10}$$

We recall that $L_1(t,t)=ig_1(t)/2$ and $M_1(t,z)=g_0(t)$, but $M_2(t,t)$ is unknown. Hence we will manipulate the first and the third terms of the lhs of equation (4.10) but will not manipulate the second term of the lhs of equation (4.10). Using equations (4.7) and (4.8) to simplify the former two terms, equation (4.10) yields

$$-\frac{\pi}{2}L_1(t,2t'-t)+ig_0(t)\int_{\partial D_1}k\left[\int_0^t e^{4ik^2(\tau-t')}M_2(t,2\tau-t)d\tau\right]dk$$
$$-2\int_{\partial D_1}\left[k^2\int_0^{t'} e^{4ik^2(\tau-t')}M_1(t,2\tau-t)d\tau-\frac{M_1(t,2t'-t)}{4i}\right]dk-\int_{\partial D_1} e^{-4ik^2t'}kR(k)A(t,k)dk=0. \tag{4.11}$$

The second integral in the rhs of this equation equals $ig_0(t)\pi M_2(t,2t'-t)/4$, thus this integral is well defined.
Letting $t' \to t$ in equation (4.11) and using the relations between $L_1(t,t)$ and $g_1(t)$, and $M_1(t,t)$ and $g_0(t)$, we find

$$\frac{\pi}{4i}g_1(t)+ig_0(t)\int_{\partial D_1}k\left[\int_0^t e^{4ik^2(\tau-t)}M_2(t,2\tau-t)d\tau\right]dk$$
$$-\int_{\partial D_1}\left[2k^2\int_0^t e^{4ik^2(\tau-t)}M_1(t,2\tau-t)d\tau-\frac{g_0(t)}{2i}\right]dk-\int_{\partial D_1} e^{-4ik^2t}kR(k)A(t,k)dk=0.$$

Writing the expressions involving M_1 and M_2 in terms of μ_1 and μ_2, as well as replacing $A(t,k)$ by the rhs of equation (4.4a), we find equation (4.9).
If $a(k)$ has zeros for $k \in D_1$, then the integral involving $kc(t,k)\exp[4ik^2(t-t')]/a(k)$ does not vanish but has a contribution due to the relevant residues. The simplest way to compute this contribution is to deform the contour ∂D_1 to include the relevant zeros.

sG
For the solution of the global relation (1.15) in the case of the sG equation we will use the identity

$$\int_{\partial D_2}\left(1-\frac{1}{k^2}\right)\left[\int_0^t e^{\frac{i}{2}(k+\frac{1}{k})(\tau-t')}K(\tau,t)d\tau\right]dk=4\pi K(t',t), \tag{4.12}$$

where

$$t>0, \quad t'>0, \quad t'<t,$$

the contour ∂D_2 is defined in the Notations and $K(\tau,t)$ is a smooth function of the arguments indicated.
This equation follows from the usual Fourier transform identity by mapping the contour ∂D_2 to the real axis.

We will now show that $g_1(t)$ is given by the following expression

$$\begin{aligned} g_1(t) = -\dot{g}_0(t) - \frac{i}{\pi}\int_{\partial D_2}\left(1-\frac{1}{k^2}\right)e^{-\frac{i}{2}(k+\frac{1}{k})t}R(k)dk \\ +\frac{i}{\pi}\tan\left(\frac{g_0(t)}{2}\right)\int_{\partial D_2}\left(1-\frac{1}{k^2}\right)\left(\mu_2 + ie^{-\frac{i}{2}(k+\frac{1}{k})t}R(k)\bar{\mu}_1\right)\frac{dk}{2k} \\ +\frac{i}{\pi}\int_{\partial D_2}\left(1-\frac{1}{k^2}\right)\left[\frac{\mu_1}{2k} - e^{-\frac{i}{2}(k+\frac{1}{k})t}R(k)\left(\bar{\lambda}_2 + \frac{i}{2k}\bar{\mu}_2\right)\right]dk. \end{aligned} \tag{4.13}$$

Indeed, for the sG the expression for $B(t,k)$ is given by equation (4.5b). After multiplying the global relation by

$$\left(1-\frac{1}{k^2}\right)e^{-\frac{i}{2}(k+\frac{1}{k})t'},$$

and then integrating along ∂D_2, the rhs vanishes, while the first term of $B(t,k)$ (with the aid of (4.12)) becomes $-8i\pi L_1(t, 2t'-t)$. Taking the limit $t' \to t$ and using equation (3.5a) relating $L_1(t,t)$ with $\dot{g}_0(t) + g_1(t)$, we find equation (4.13).

mKdVII

For the solution of the global relation (1.15) in the case of the mKdVII equation we will use the following identities:

$$\int_{\partial D_3} k^2\left[\int_0^t e^{8ik^3(\tau-t')}K(\tau,t)d\tau\right]dk = \frac{\pi}{12}K(t',t), \tag{4.14}$$

$$\int_{\partial D_3} k^m\left[\int_0^t e^{8ik^3(\tau-t')}K(\tau,t)d\tau\right]dk = \int_{\partial D_3} k^m\left[\int_0^{t'} e^{8ik^3(\tau-t')}K(\tau,t)d\tau - \frac{K(t',t)}{8ik^3}\right]dk, \tag{4.15}$$

where m equals 3 or 4,

$$t>0, \quad t'>0, \quad t'<t,$$

∂D_3 is defined in the Notations and $K(\tau,t)$ is a smooth function of the arguments indicated.
The derivation of these equations is similar with the derivation of equations (4.7) and (4.8). In particular, regarding equation (4.15) we note that the lhs equals the rhs plus an integral involving the integrant

$$k^m\left[\int_{t'}^t e^{8ik^3(\tau-t')}K(\tau,t)d\tau + \frac{K(t',\tau)}{8ik^3}\right].$$

The leading order of this integral equals $\exp[8ik^3(\tau-t)]O(k^{m-3})$ and if $k \in D_3$, i.e. if $4\pi \le \arg k^3 \le 5\pi$, the exponential is bounded $(\tau - t' \ge 0)$, thus Jordan's lemma implies that the above integral vanishes.

We will now show that $g_2(t)$ is given by the expression

$$
\begin{aligned}
g_2(t) = 2\rho g_0^3(t) &- \frac{12i}{\pi}\int_{\partial D_3}\left[k^3\mu_1 - \frac{g_1(t)}{4i}\right]dk - \frac{6ig_0(t)}{\pi}\int_{\partial D_3}\left[k^3\nu_2 - \frac{N_2(t,t)}{4i}\right]dk \\
&- \frac{12}{\pi}\int_{\partial D_3}\left[k^4\nu_1 + \frac{k}{2i}g_0(t)\right]dk \\
&- \frac{12}{\pi}\int_{\partial D_3} k^2R(k)e^{-8ik^3t}\left[1+\bar{\lambda_2} - ik\bar{\mu_2} + k^2\bar{\nu_2}\right]dk \qquad (4.16a) \\
&+ \frac{6}{\pi}g_0(t)\int_{\partial D_3}\left[k^2\mu_2 - \rho k^2R(k)e^{-8ik^3t}(\bar{\mu_1} + ik\bar{\nu_1})\right]dk \\
&- \frac{3}{\pi}g_1(t)\int_{\partial D_3} k^2\left[\nu_2 + \rho R(k)e^{-8ik^3t}\bar{\nu_1}\right]dk,
\end{aligned}
$$

where

$$
N_2(t,t) = \frac{6}{\pi}\int_{\partial D_3} e^{-8ik^3t}k^2\nu_2 dk. \qquad (4.16b)
$$

Indeed, for the mKdVII equation, $B(t,k)$ is given by an equation similar to equation (4.6b) where g_0 and g_2 are replaced by $-g_0$ and $-g_2$. Thus proceeding as with the NLS equation, in analogy with equation (4.10) we now find

$$
\begin{aligned}
-2\int_{\partial D_3}\Bigg[\int_0^t e^{8ik^3(\tau-t')}\Bigg(& k^2L_1(t,2\tau-t) - \frac{1}{2}g_0(t)k^2M_2(t,2\tau-t) + \frac{1}{4}g_1(t)k^2N_2(t,2\tau-t) \\
& ik^3M_1(t,2\tau-t) + \frac{i}{2}g_0(t)k^3N_2(t,2\tau-t) + k^4N_1(t,2\tau-t)\Bigg)dk \\
& - \int_{\partial D_3} e^{-8ik^3t'}k^2R(k)A(t,k)dk = 0. \qquad (4.17)
\end{aligned}
$$

Using equations (4.14) and (4.15) the terms involving L_1, M_1, N_2, N_1 can be rewritten as

$$
\begin{aligned}
-\frac{\pi}{6}L_1(t,2t'-t) &- 2i\int_{\partial D_3} k^3\left[\int_0^t e^{8ik^3(\tau-t')}M_1(t,2\tau-t) - \frac{M_1(t,2t'-\tau)}{8ik^3}\right]dk \\
&-ig_0(t)\int_{\partial D_3} k^3\left[\int_0^t e^{8ik^3(\tau-t')}N_2(t,2\tau-t) - \frac{N_2(t,2t'-\tau)}{8ik^3}\right]dk \\
&-2\int_{\partial D_3} k^4\left[\int_0^t e^{8ik^3(\tau-t')}N_1(t,2\tau-t) - \frac{N_1(t,2t'-\tau)}{8ik^3}\right]dk. \qquad (4.18)
\end{aligned}
$$

We recall that

$$
L_1(t,t) = -\rho g_0^3(t) + \frac{1}{2}g_2(t), \quad M_1(t,t) = g_1(t), \quad N_1(t,t) = -2g_0(t). \qquad (4.19)
$$

Hence substituting the expression (4.18) into (4.17), taking the limit $t' \to t$, using equations (4.19), and recalling that $A(t,k)$ is given by equation (4.6a) but with g_0 and g_2

replaced by $-g_0$ and $-g_2$, we find equation (4.16a). The expression for $N_2(t,t)$ can be obtained using equation (4.14):

$$\frac{\pi}{2}N_2(t,t) = \int_{\partial D_3} k^2 \left[\int_0^t e^{8ik^3(\tau-t)} N_2(t, 2\tau - t) d\tau\right] dk = \frac{1}{2}\int_{\partial D_3} e^{-8ik^3 t} k^2 \nu_2 dk.$$

mKdVI

We will show that $g_1(t)$ and $g_2(t)$ are given by the following expressions:

$$g_1(t) = \frac{3}{\pi} g_0 \int\limits_{\partial D_3} k^2 \nu_2 dk + \frac{6}{i\pi} \int\limits_{\partial D_3} \left(k^3 \nu_1 - \frac{g_0(t)}{2i}\right) dk - \frac{6}{i\pi}\int_{\partial D_3} e^{-8ik^3 t} k G_1 dk, \tag{4.20a}$$

$$\begin{aligned} g_2(t) = 2\rho g_0^3(t) + \frac{6g_0(t)}{\pi} \int\limits_{\partial D_3} k^2 \mu_2 dk + \frac{3}{\pi} g_1(t) \int\limits_{\partial D_3} k^2 \nu_2 dk - \frac{12}{\pi} \int\limits_{\partial D_3} (k^4 \nu_1 - \frac{k}{2i} g_0(t)) dk \\ + \frac{12}{\pi}\int_{\partial D_3} e^{-8ik^3 t} k^2 G_2 dk, \end{aligned} \tag{4.20b}$$

where $E = e^{2\pi i/3}$, G_1, G_2 are defined by (1.19c), and $A(k,t)$ is defined by

$$A = 1 + \overline{\lambda_2} - \frac{1}{2}\rho g_0(t)\overline{\mu_1} + \frac{\rho}{4} g_1(t)\overline{\nu_1} - ik(\overline{\mu_2} + \frac{1}{2}\rho g_0(t)\overline{\nu_1}) + k^2\overline{\nu_2}. \tag{4.20c}$$

Indeed, the function $B(t,k)$ is defined by equation (4.6b). Replacing in this definition k by Ek and by $E^2 k$ and then solving the two resulting equations for the terms containing

$$M_1 - \frac{1}{2}g_0(t)N_2, \quad L_1 + \frac{1}{2}g_0(t)M_2 + \frac{1}{4}g_1(t)N_2,$$

we find

$$\begin{aligned} -2i\int_0^t e^{8ik^3\tau} k\left(M_1 - \frac{1}{2}g_0 N_2\right) d\tau = -2\int_0^t e^{8ik^3\tau} k^2 N_1 d\tau + \frac{B(t,E^2k) - B(t,Ek)}{E^2 - E}, \\ -2\int_0^t e^{8ik^3\tau}\left(L_1 + \frac{1}{2}g_0 M_2 + \frac{1}{4}g_1 N_2\right) d\tau = -2\int_0^t e^{8ik^3\tau} k^2 N_1 d\tau + \frac{EB(t,Ek) - B(t,E^2k)}{E - 1}. \end{aligned} \tag{4.21}$$

The global relation for the mKdVI is valid for $\mathrm{Im}k \geq 0$, thus replacing in this relation k by Ek and by $E^2 k$ we find two equations, which are both valid for $k \in D_3$,

$$\begin{aligned} B(t,Ek) - R(Ek)A(t,Ek) = \frac{e^{8ik^3 t} c(t,E,k)}{a(Ek)}, \\ B(t,E^2k) - R(E^2k)A(t,E^2k) = \frac{e^{8ik^3 t} c(t,E^2k)}{a(E^2k)}, \quad k \in D_3. \end{aligned} \tag{4.22}$$

Replacing in equations (4.21) $B(t,Ek)$ and $B(t,E^2k)$ by the solution of equations (4.22), and then multiplying equations (4.21a) and (4.21b) by $k\exp[-8ik^3t']$ and by $k^2\exp[-8ik^3t']$ respectively, we obtain the following equations which are also valid for $k\in D_3$:

$$
\begin{aligned}
-2i\int_0^t e^{8ik^3(\tau-t')}k^2\left(M_1-\frac{1}{2}g_0N_2\right)d\tau &= -2\int_0^t e^{8ik^3\tau}k^3N_1d\tau \\
&+k\left[\frac{R(E^2k)A(t,E^2k)-R(Ek)A(t,Ek)}{E^2-E}\right]+\frac{e^{8ik^3t}kC_1}{E^2-E}, \\
-2\int_0^t e^{8ik^3(\tau-t')}k^2\left(L_1+\frac{1}{2}g_0M_2+\frac{1}{4}g_1N_2\right)d\tau &= -2\int_0^t e^{8ik^3\tau}k^4N_1d\tau \\
&+\frac{k^2\left[ER(Ek)A(t,Ek)-R(E^2k)A(t,E^2k)\right]}{E-1}+\frac{e^{8ik^3t}k^2C_2}{E-1},
\end{aligned}
\tag{4.23}
$$

where C_1 and C_2 involve combinations of c/a.
Integrating equations (4.23) around ∂D_3, using the identity (4.14) to compute the first term in the lhs of each of the equations (4.23a) and (4.23b), using the identity (4.15) to evaluate the first term in the rhs of each of the equations (4.23a) and (4.23b), taking the limit $t'\to t$, and recalling equations (3.6a), we find equations (4.20).

Remark 4.1
It is straightforward to verify that all the integrals appearing in equations (4.9), (4.13), (4.16), (4.20a) and (4.20b) are well defined. In this respect it is useful to know the large k behavior of the functions λ_j, μ_j, ν_j. Let $f(t,k)$ denote anyone of these functions, and let $F(t,s)$ denote the corresponding functions L_j, M_j, N_j. Then

$$
f(t,k)=\frac{F(t,t)-e^{-2if_2(k)t}F(t,-t)}{if_2(k)}+O\left(\frac{1}{(f_2(k)^2}\right),\quad k\to\infty. \tag{4.24}
$$

Indeed, by definition (see equation (3.34))

$$
f(t,k)=e^{-if_2t}\int_{-t}^{t}e^{-if_2s}F(t,s)ds,
$$

which implies

$$
f(t,k)=2e^{-2if_2t}\int_0^t e^{2if_2\tau}F(t,2\tau-t)d\tau.
$$

This equation and integration by parts yield equation (4.24).

The Expressions of $\{\lambda_j,\mu_j,\nu_j\}_1^j$ in terms of Φ
Equations (4.9), (4.13), (4.16), (4.20), express the unknown boundary values in terms of the given boundary conditions, of $b(k)/a(k)$ (which can be computed in terms of the given initial conditions), and of the functions $\{\lambda_j,\mu_j,\nu_j\}_1^j$. We will now show that these latter functions can be expressed in terms of $\Phi(t,k)$. For this purpose we will use equations (3.35)-(3.37), which express $\Phi(t,k)$ in terms of $\{\lambda_j,\mu_j,\nu_j\}_1^2$, as well as the fact that these

latter functions depend on k only through $f_2(k)$, see equations (3.34) (thus they remain invariant under those transformations which leave $f_2(k)$ invariant).

NLS

Supplementing the two equations (3.35) with the two equations obtained from equations (3.35) by replacing k with $-k$, we obtain four equations relating $\{\Phi_j(t,k), \Phi_j(t,-k)\}_{j=1}^2$ with $\{\lambda_j(t,k^2), \mu_j(t,k^2)\}_{j=1}^2$. Solving these four equations we find the following expressions.

$$\begin{aligned}
k\mu_1(t,k^2) &= \frac{e^{-2ik^2t}}{2}\left(\Phi_1(t,k) - \Phi_1(t,-k)\right), \\
k\mu_2(t,k^2) &= \frac{e^{-2ik^2t}}{2}\left(\Phi_2(t,k) - \Phi_2(t,-k)\right), \\
k\lambda_1(t,k^2) &= \frac{e^{-2ik^2t}}{2}\left[k\left(\Phi_1(t,k) + \Phi_1(t,-k)\right) + \frac{i}{2}g_0(t)\left(\Phi_2(t,k) - \Phi_2(t,-k)\right)\right], \\
k\left(1 + \lambda_2(t,k^2)\right) &= \frac{e^{-2ik^2t}}{2}\left[k\left(\Phi_2(t,k) + \Phi_2(t,-k)\right) - \frac{i\rho}{2}\bar{g}_0(t)\left(\Phi_1(t,k) - \Phi_1(t,-k)\right)\right].
\end{aligned} \tag{4.25}$$

sG

Supplementing the two equations (3.36) with the two equations obtained from equations (3.36) by replacing k with $1/k$, we obtain a system of four equations. Solving these four equations we find the following expressions

$$\begin{aligned}
(k^2-1)\lambda_1(t,k+\frac{1}{k}) &= -ie^{-\frac{i}{4}(k+\frac{1}{k})}\left(k^2\Phi_1(t,k) - \Phi_1(t,\frac{1}{k})\right), \\
(k^2-1)\left(1 + \lambda_2(t,k+\frac{1}{k})\right) &= e^{-\frac{i}{4}(k+\frac{1}{k})}\left(k^2\Phi_2(t,k) - \Phi_2(t,\frac{1}{k})\right), \\
\mu_1(t,k+\frac{1}{k}) &= \frac{e^{-\frac{i}{4}(k+\frac{1}{k})}}{k-\frac{1}{k}}\left[2\cos^2\left(\frac{g_0(t)}{2}\right)\left(\Phi_1(t,k) - \Phi_1(t,\frac{1}{k})\right)\right. \\
&\qquad \left. + i\sin(g_0(t))\left(\Phi_2(t,k) - \Phi_2(t,\frac{1}{k})\right)\right], \\
\mu_2(t,k+\frac{1}{k}) &= \frac{e^{-\frac{i}{4}(k+\frac{1}{k})}}{k-\frac{1}{k}}\left[\sin(g_0(t))\left(\Phi_1(t,k) - \Phi_1(t,\frac{1}{k})\right)\right. \\
&\qquad \left. -2i\cos^2\left(\frac{g_0(t)}{2}\right)\left(\Phi_2(t,k) - \Phi_2(t,\frac{1}{k})\right)\right].
\end{aligned} \tag{4.26}$$

mKdVI

Supplementing the two equations (3.37) with the four equations obtained from equation (3.37) by replacing k with Ek and with E^2k we obtain a system of six equations. Solving these six equations, and making use of the identity

$$1 + E + E^2 = 0,$$

we find the following expressions:

$$k^2\nu_1(t,k^3)=\frac{e^{-4ik^3t}}{3}\left(\Phi_1(t,k)+E\Phi_1(t,Ek)+E^2\Phi_1(t,E^2k)\right),$$

$$k^2\nu_2(t,k^3)=\frac{e^{-4ik^3t}}{3}\left(\Phi_2(t,k)+E\Phi_2(t,Ek)+E^2\Phi_2(t,E^2k)\right),$$

$$\begin{aligned}k^2\mu_1(t,k^3)=\frac{e^{-4ik^3t}}{3}&\left[\frac{g_0(t)}{2}\left(\Phi_2(t,k)+E\Phi_2(t,Ek)+E^2\Phi_2(t,E^2k)\right)\right.\\&\left.-ik\left(\Phi_1(t,k)+E^2\Phi_1(t,Ek)+E\Phi_1(t,E^2k)\right)\right],\end{aligned}$$

$$\begin{aligned}k^2\mu_2(t,k^3)=-\frac{e^{-4ik^3t}}{3}&\left[\frac{\rho g_0(t)}{2}\left(\Phi_1(t,k)+E\Phi_1(t,Ek)+E^2\Phi_1(t,E^2k)\right)\right.\\&\left.+ik\left(\Phi_2(t,k)+E^2\Phi_2(t,Ek)+E\Phi_2(t,E^2k)\right)\right],\end{aligned}$$

$$\begin{aligned}k^2\lambda_1(t,k^3)=\frac{e^{-4ik^3t}}{3}&\left[\frac{\rho}{4}g_0^2(t)\left(\Phi_1(t,k)+E\Phi_1(t,Ek)+E^2\Phi_1(t,E^2k)\right)\right.\\&\left.+\frac{ig_0(t)}{2}k\left(\Phi_2(t,k)+E^2\Phi_2(t,Ek)+E\Phi_2(t,E^2k)\right)\right]\\&-\frac{g_1(t)}{4}\left(\Phi_2(t,k)+E\Phi_2(t,Ek)+E^2\Phi_2(t,E^2k)\right)\\&+k^2\left(\Phi_1(t,k)+\Phi_1(t,Ek)+\Phi_1(t,E^2k)\right)\Big],\end{aligned}$$

$$\begin{aligned}k^2(1+\lambda_2(t,k^3))=\frac{e^{-4ik^3t}}{3}&\left[\frac{\rho}{4}g_0^2(t)\left(\Phi_2(t,k)+E\Phi_2(t,Ek)+E^2\Phi_2(t,E^2k)\right)\right.\\&-\frac{i\rho g_0(t)}{2}k\left(\Phi_1(t,k)+E^2\Phi_1(t,Ek)+E\Phi_1(t,E^2k)\right)\\-\frac{\rho}{4}g_1(t)\left(\Phi_1(t,k)+E\Phi_1(t,Ek)+E^2\Phi_1(t,E^2k)\right)&+k^2\left(\Phi_2(t,k)+\Phi_2(t,Ek)+\Phi_2(t,E^2k)\right).\end{aligned}\tag{4.27}$$

Using equations (4.25)-(4.27) in equations (4.9), (4.13), (4.16), (4.20), we find equations (1.16)-(1.18). We note that before we use the expressions (4.27) in equations (4.16), we first replace in equations (4.27) g_0 and g_2 by $-g_0$ and $-g_2$, respectively.

Global Existence

Equations (1.10) for the sG, as well as for the focusing versions of the NLS, the mKdVI, and the mKdVII, i.e. equations (1.2) as well s equations (1.1), $(1.3)^+$, and $(1.3)^-$ with $\rho=-1$, imply

$$\frac{d}{dt}\left(|\Phi_1(t,k)|^2+|\Phi_2(t,k)|^2\right)=\text{constant},\quad Imf_2(k)=0.$$

This equation, together with the boundary conditions $\Phi_1(0,k)=0,\Phi_2(0,k)=1$, imply the important relation (1.20).

5 Linearizable Boundary Conditions

It was shown in [1], [3] that for certain particular boundary conditions, it is possible to bypass the nonlinear equations satisfied by the missing boundary values and to obtain the spectral functions $\{A(k), B(k)\}$ using only algebraic manipulations. In what follows we show that for such boundary conditions the nonlinear ODEs satisfied by $\{\Phi_1, \Phi_2\}$ can be linearized. To achieve this we use the so called direct linearizing transform [10]. For brevity of presentation we only consider the particular case of the NLS with $g_0(t) = 0$. Other linearizable boundary conditions can be analyzed in a simpler manner.
If $g_0 = 0$, equations (1.10) with f_2 and $\tilde{Q}(0,t,k)$ defined by (1.11), become

$$\begin{aligned} &\Phi_{1_t} + 2ik^2\Phi_1 - ig_1(t)\Phi_2 = 0 \\ &\Phi_{2_t} - 2ik^2\Phi_2 + i\rho\bar{g}_1(t)\Phi_1 = 0 \\ &\Phi_1(0,k) = 0, \quad \Phi_2(0,k) = 1. \end{aligned} \tag{5.1}$$

These equations imply $\Phi_1(t,-k) = \Phi_1(t,k), \Phi_2(t,-k) = \Phi_2(t,k)$. Thus equation (1.16) yields

$$g_1(t) = \frac{4i}{\pi}\int_{\partial D_1} e^{-2ik^2t} kR(k)\overline{\Phi_2(t,\bar{k})}dk. \tag{5.2}$$

It turns out that equations (5.1), with $g_1(t)$ given by equation (5.2), can be linearized.

Theorem 5.1 Let $q(x,t)$ satisfy the NLS equation (1.1) on the half line $0 < x < \infty$, $t > 0$, with

$$q(x,0) = q_0(x), \quad 0 < x < \infty, \quad q(0,t) \equiv 0, \quad t > 0, \tag{5.3}$$

where $q_0(x) \in S(\mathbb{R}^+)$ and $q_0(0) = 0$. Let the contour ∂D_1 be as in theorem 1.1, and let the function $R(k) = b(k)/a(k)$ be defined in terms of the function $q_0(x)$ as in theorem 1.1. Then the function $g_1(t) = g_x(0,t)$ is given by equation (5.2), where the functions $\Phi_1(t,k)$ and $\Phi_2(t,k)$ satisfy the linear equations

$$\begin{aligned} &\Phi_1(t,k) + \frac{i}{\pi}\int_{\partial D_1} e^{2i(k^2-l^2)t}\frac{lR(l)}{l^2-k^2}\overline{\Phi_2(t,\bar{l})}dl = \frac{ie^{-2ik^2t}}{\pi}\int_{\partial D_1}\frac{lR(l)}{l^2-k^2}dl, \\ &\Phi_2(t,k) + \frac{i\rho}{\pi}\int_{\partial D_1} e^{2i(k^2-l^2)t}\frac{lR(l)}{l^2-k^2}\overline{\Phi_1(t,\bar{l})}dl = e^{2ik^2t}, \quad k \in \mathbb{C}/\partial D_1. \end{aligned} \tag{5.4}$$

Proof In what follows we prove this result under the *assumption* that the homogeneous version of equations (5.4) has only the trivial solution. This assumption is relaxed in the remark 5.1.
If k is off the contour ∂D_1, then the integrals appearing in equations (5.4) are well defined. We denote the rhs of equation (5.4a) by $F(t,k)$ and we denote by $Jf(t,l)$ the integral

$$\Big(Jf(t,l)\Big)(t,k) = \frac{i}{\pi}\int_{\partial D_1} e^{2i(k^2-l^2)t}\frac{lR(l)}{l^2-k^2}f(t,l)dl.$$

Differentiating equation (5.4a) with respect to t, multiplying equation (5.4a) by $2ik^2$, and multiplying equation (5.4b) by $-ig_1(t)$, we find the following equations

$$\Phi_{1_t} + J\overline{\Phi_{2_t}(t,\bar{l})} + \frac{2e^{2ik^2t}}{\pi}\int_{\partial D_1} e^{-2il^2t} lR(l)\overline{\Phi_2(t,\bar{l})}dl = -2ik^2F,$$

$$2ik^2\Phi_1 + J(2ik^2\overline{\Phi_2(t,\bar{l})}) = 2ik^2F,$$

$$-ig_1\Phi_2 + J(-i\rho g_1\overline{\Phi_1(t,\bar{l})}) = -ig_1(t)e^{2ik^2t}.$$

Adding these equations we find

$$L+J\left(\overline{\Phi_{2_t}(t,\bar{l})} + 2ik^2\overline{\Phi_2(t,\bar{l})} - i\rho g_1\overline{\Phi_1(t,\bar{l})}\right) = e^{2ik^2t}\left[-ig_1(t) - \frac{2}{\pi}\int_{\partial D_1} e^{-2il^2t} lR(l)\overline{\Phi_2(t,\bar{l})}dl\right], \tag{5.5}$$

where L denotes the lhs of equation (5.1a). Using the identity

$$J\left(2ik^2\overline{\Phi_2(t,\bar{l})}\right) = J\left(2il^2\overline{\Phi_2(t,\bar{l})}\right) + \frac{2e^{2ik^2t}}{\pi}\int_{\partial D_1} e^{-2il^2t} lR(l)\overline{\Phi_2(t,\bar{l})}dl,$$

as well as equation (5.2), equation (5.5) becomes

$$L + J\bar{M} = 0 \tag{5.6a}$$

where M denotes the lhs of equation (5.1b) with k replaced by l.
Following similar steps, one finds

$$M + \rho J\bar{L} = 0. \tag{5.6b}$$

Hence the unique solvability of equations (5.4) implies that the first two of equations (5.1) are valid. Also evaluating equations (5.4) at $t = 0$, we find $\Phi_1(0,k) = 0$, $\Phi_2(0,k) = 1$. **QED**

Remark 5.1 Let D_1^+ denote the first quadrant of the complex k-plane, and let $D_1^- = \mathbb{C}/\{D_1^+\}$. By taking the limit as k approaches ∂D_1 either from D_1^+ or D_1^-, equations (5.5) yield a system of singular integral equations. This system is equivalent to a matrix 2×2 Riemann-Hilbert problem. Using the “symmetry” properties of the jump matrix of this Riemann-Hilbert problem, it can be shown that its homogeneous version has only the trivial solution (the analysis of the relevant Riemann-Hilbert problem is similar to the one presented in [11]).

6 Conclusion

A general approach for analyzing initial-boundary value problems for integrable nonlinear evolution PDEs on the half line, $0 < x < \infty$, $0 < t < T$, was introduced in [5]. The rigorous implementation of this approach, which for the NLS, sG, KdV, and mKdV, is presented in [3], [6], [7], involves the following steps:

(i) Given $q_0(x)$ construct the functions $\{a(k), b(k)\}$, see (1.7).

(ii) Given boundary conditions (say $g_0(t)$ for the NLS), characterize the unknown boundary values (say $g_1(t)$ for the NLS), by the requirement that the functions $\{A(k,t), B(k,t)\}$, which are defined by (1.9), satisfy the global relation (1.15a).
(iii) Given $\{a(k), b(k), A(k) = A(k,T), B(k) = B(k,T)\}$, define $q(x,t)$ through the solution of a 2×2 matrix Riemann-Hilbert problem. This problem, which has the distinctive feature that its jump matrix has explicit (x,t) dependence in the exponential form $\exp[if_1 x + if_2 t]$, has a unique, global solution. It can be shown that the function $q(x,t)$ satisfies the given PDE as well as the given initial and boundary conditions.
The most complicated step in the above construction is the characterization of the unknown boundary values. For the NLS with given $g_0(t)$ and with $q_0(x) \equiv 0$, it can be shown [6] that the global relation is equivalent to the integral relation

$$\int_{\partial D_1} k \int_0^t e^{4ik^2(t'-t)+i\rho\int_0^{t'}|g_0|^2 d\tau}(2kg_0(t') + ig_1(t'))\overline{\Phi_2(t',\bar{k})}dt'dk = 0. \tag{6.1}$$

This equation as well as the two equations satisfied by $\Phi_1(t,k)$ and $\Phi_2(t,k)$ constitute a *coupled* system of three equations for the three unknown functions $g_1(t)$, $\Phi_1(t,k)$, $\Phi_2(t,k)$. We have shown in this paper that it is possible to solve the global relation *explicitly*. For example, we have solved equation (6.1) for $g_1(t)$ in terms of $g_0(t)$, $\Phi_1(t,k)$, $\Phi_2(t,k)$. This is a dramatic simplification; indeed, although the new system is still a system of *nonlinear* Volterra integral equations, the nonlinearity is now in a simple explicit form, while in the original approach of [6] one had to deal with a transcendental nonlinearity.

In order to obtain this explicit solution of the global relation, we found it necessary to use the Gel'fand-Levitan-Marchenko representation of $\Phi(t,k)$, namely to express $\Phi_1(t,k)$ and $\Phi_2(t,k)$ in terms of the auxiliary functions $\{L_j(t,s),\ M_j(t,s),\ N_j(t,s)\}_{j=1}^2$. However, we emphasize that the use of this representation is only a mathematical tool for solving the global relation. Indeed, after we obtain the unknown boundary values in terms of the given data and the functions $\{L_j, M_j, N_j\}_{j=1}^2$, we eliminate these functions by using the fact that they can be expressed in terms of Φ_1 and Φ_2. As a result of this construction we obtain two nonlinear ODEs for Φ_1 and Φ_2. These two equations are equivalent to two nonlinear integral equations of the Volterra type, thus it is straightforward to establish existence of solution for small t, or for "small" boundary conditions. Furthermore, using the identity (1.20) it is possible to establish *global* existence for the sG and for the focusing versions of NLS, mKdVI and mKdVII.

It was shown in [3], [6], that there exist particular choices of boundary conditions for which the global relation can be solved using algebraic manipulations without the need to formulate a nonlinear Volterra integral equation. Thus for these "linearizable" boundary conditions, initial-boundary value problems are solved as effectively as initial value problems. We have shown in this paper that for such linearizable boundary conditions, the generalized Dirichlet to Neumann map can be linearized, see section 5.
We conclude with some historical remarks:

1. The first attempt to characterize the spectral functions was made in [8] and led to a formal *nonlinear* RH problem. A similar formulation was presented in [9]. In [9] a different formulation was also presented based on an attempt to express g_1 explicitly in terms of Φ

using certain analyticity arguments. However, all these formal attempts yield a system of nonlinear *Fredholm* integral equations for the spectral functions. This is to be contrasted with the formulation of [6] as well as with the simplified formulation presented here, which yield a system of nonlinear *Volterra* integral equations. The insurmountable problem with the former formulations is that, since the spectral functions are characterized to within an equivalent class ($c(t,k)$ in (1.15a) is arbitrary), one *cannot* rigorously establish solvability for the associated Fredholm equations.

2. In the case of the NLS, sG and mKdVI, for economy of presentation we have concentrated on the case that $g_0(t)$ is given. The analysis applies mutatis-mutandis for the case that $g_1(t)$ is given. Similarly, for the mKdVII one can prescribe $g_0(t)$ and $g_2(t)$ or $g_1(t)$ and $g_2(t)$ instead of $g_0(t)$ and $g_1(t)$.

3. The analysis of the mKdVI is more complicated than the analysis of NLS, sG, and mKdVII, since in the former case the *single* global relation (1.11) must be solved for the *two* unknown boundary values $g_1(t)$ and $g_2(t)$. We have been able to solve this problem by supplementing the global relation with an equation constructed from the global relation via the use of certain invariant transformations. The importance of such invariant transformations was first understood in [2].

4. The first explicit solution of the global relation in the case of the NLS was obtained in [12]. However, it was not realized in [12] that the functions $\{\lambda_j(t,k^2), \mu_j(t,k^2)\}_{j=1}^2$ can be expressed in terms of $\{\Phi_j(t,k), \Phi_j(t,-k)\}_{j=1}^2$. Hence in order to obtain a "closed" system of equations, it was necessary to use $\{\lambda_j, \mu_j\}_{j=1}^2$ instead of Φ_1 and Φ_2. Thus the Dirichlet to Neumann map is characterized in [12] in terms of *four* ODEs as opposed to *two* ODEs used here (similarly the analysis of the mKdV would involve *six* ODEs). Also the global existence for the focusing NLS was *not* obtained in [12].

5. The Gel'fand-Levitan-Marchenko representation associated with the eigenfunctions of the NLS is presented in [13].

6. An important advantage of the approach of [5] is that it yields the solution in terms of a Riemann-Hilbert problem with exponential (x,t) dependence. Thus it is possible to study both the long time asymptotic (using the Deift-Zhou method [14], [15], [11]) and the small dispersion limit (using the Deift-Zhou-Venakides method) [16], [17], [18].

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