GENERALIZED FOURIER TRANSFORMS, THEIR NONLINEARIZATION AND THE IMAGING OF THE BRAIN

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1. Introduction

Among the most important applications of the Fourier transform are (a) the solution of the Cauchy problem of linear evolution PDEs and (b) the solution of certain inverse problems such as the one appearing in computerized tomography (the inversion of the Radon transform).

The first goal of this article is to show that Fourier transforms (FT) can be (a) *nonlinearized* and (b) *generalized*. Nonlinear FTs can be used for the solution of the Cauchy problem of certain nonlinear evolution PDEs, while generalized FTs can be used for the solution of certain important inverse problems such as the one arising in single particle emission computerized tomography (the inversion of the so-called attenuated Radon transform).

The second goal of this article is to establish that certain *abstract* integral representations, sometimes called the Ehrenpreis-Palamodov representations, which have been shown to represent the general solution of linear PDEs in convex domains, can be (a) *made effective* and (b) *nonlinearized*. As illustrations of (a) and (b) we will solve initial-boundary value problems on the half-line for linear and for certain nonlinear evolution PDEs, respectively.

The third goal of this article is to emphasize the emergence of a *mathematical unification*. This encompasses:

(i) The derivation of the classical transforms and their application to the solution of linear PDEs.
(ii) The use of several specialized techniques for the solution of linear PDEs, such as the method of images, and the so-called Wiener-Hopf technique.
(iii) The Green's function approach.
(iv) The solution of certain inverse problems.
(v) The Ehrenpreis-Palamodov integral representations.
(vi) The inverse scattering method for solving the Cauchy problem of integrable nonlinear PDEs.
(vii) A new method for analyzing boundary value problems for linear and certain nonlinear PDEs.

This unification is based on a novel approach to the spectral analysis of a single eigenvalue equation, as well as on the *simultaneous* spectral analysis of a compatible pair of eigenvalue equations. This spectral analysis is based on the formulation of two fundamental mathematical problems of complex analysis, namely the Riemann-Hilbert [1] and the ∂-bar [2] problems.

A schematic summary of the above discussion is given below:

I. Fourier Transforms can be

<u>*nonlinearized*</u> and <u>*generalized*</u>

↓ ↓

Cauchy problem for nonlinear PDEs — single particle emission computerized tomography

II. The Ehrenpreis-Palomodov integral representations can be

<u>*made effective*</u> and <u>*nonlinearized*</u>

↓ ↓

boundary value problems for linear PDEs — boundary value problems for nonlinear PDEs

III. Mathematical Unification

Parts I and II will be discussed in sections 2–7 and sections 8–9 respectively. Part III and further generalizations will be discussed in section 10.

2. The Cauchy Problem for Linear Evolution PDEs on the Line

Let $\omega(k)$ be a real polynomial of degree n. Let $q(x,t)$ satisfy the Cauchy problem for the linear PDE[1]

$$\partial_t q + i\omega(-i\partial_x)q = 0 \qquad -\infty < x < \infty,\ t > 0, \tag{2.1a}$$

$$q(x,0) = q_0(x) \qquad -\infty < x < \infty, \tag{2.1b}$$

where $q_0(x) \in S(\mathbb{R})$.[2] The solution of this problem involves the construction of two maps: The *direct map* is defined by

$$q_0(x) \to \hat{q}_0(k) = \int_{-\infty}^{\infty} e^{-ikx} q_0(x)dx, \quad k \in \mathbb{R}, \tag{2.2}$$

while the *inverse map* is defined by

$$e^{-i\omega(k)t}\hat{q}_0(k) \to q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-i\omega(k)t}\hat{q}_0(k)dk, \quad -\infty < x < \infty, \quad t > 0. \tag{2.3}$$

The solution $q(x,t)$ defined by equation (2.3) has several remarkable features: (i) It is general, i.e. it is valid for any dispersive PDE with symbol $\omega(k)$. (ii) It is "spectrally decomposable", namely the spectral function $\hat{q}_0(k)$ is separated from the (x,t) dependence, and the latter dependence appears in an exponential form. (iii) It is both elegant and

[1] For simplicity we assume that $\omega(k)$ is real, i.e. we study only dispersive PDEs.

[2] For simplicity we assume throughout this paper, except for §8, that the initial conditions $q_0(x)$ or $q_0(x_1,x_2)$ belong in the Schwartz space denoted by $S(\mathbb{R})$ or $S(\mathbb{R}^2)$. It is of course possible to derive similar results on less restrictive function spaces.

useful. For example, it is straightforward to compute the large t asymptotics. (iv) It is straightforward to make it rigorous. In this respect we note that an Applied Mathematician starts with equation (2.1) and then by applying the FT, derives equation (2.3); at this stage he/she considers the problem solved. In contrast, for an Analyst, the derivation of equation (2.3) is just the *starting point* of solving the Cauchy problem (2.1). Indeed, an Analyst *defines* $\hat{q}_0(k)$ in terms of the given function $q_0(x)$ by equation (2.2), and also *defines* $q(x,t)$ in terms of $\hat{q}_0(k)$ by equation (2.3); then he/she proves that this function $q(x,t)$ satisfies the PDE (2.1a) and also the initial condition (2.1b).

An Alternative Type of Separability

The solution expressed by equation (2.3) is based on the separability of equation (2.1a). Indeed, separation of variables gives rise to the eigenvalue problem

$$k^n\psi(x,k) + i\omega\left(-i\frac{d}{dx}\right)\psi(x,k) = \delta(x), \quad -\infty < x < \infty, \quad k \in \mathbb{C},$$
$$\psi(x,k) \to 0 \qquad \text{as } |x| \to \infty.$$

The spectral analysis of this problem gives rise to the classical FT, which is the appropriate integral transform for the associated Cauchy problem.

We claim that there exists an alternative, perhaps deeper kind of separability: Equation (2.1a) is the *compatibility* condition of the following equations satisfied by the scalar function $\mu(x,t,k)$:

$$\mu_x - ik\mu = q(x,t), \tag{2.4a}$$

$$\mu_t + i\omega(k)\mu = \sum_{j=0}^{n-1} c_j(k)(-i\partial_x)^j q(x,t), \quad k \in \mathbb{C}, \tag{2.4b}$$

where the polynomials $\{c_j(k)\}_0^{n-1}$ are determined by

$$\frac{\omega(\ell) - \omega(k)}{\ell - k} = -\sum_{j=0}^{n-1} c_j(k)\ell^j. \tag{2.5}$$

Equations (2.4), which are *two* equations for the *single* function μ, are compatible provided that q satisfies the PDE (2.1a). Indeed, if (2.1a) is valid, then equation (2.4a) is compatible with the equation $(\partial_t + i\omega(-i\partial_x))\mu = 0$. Replacing in the latter equation the x-derivatives of μ using equation (2.4a), this equation becomes equation (2.4b).

Our claim regarding the importance of the formulation (2.4) is based on the following considerations:

(i) There exist certain nonlinear PDEs, called *integrable*, which are amenable to exact analysis. Since, these equations are nonlinear they are *not* separable, but they *do* admit a formulation which is a natural generalization of equations (2.4). The simplest example of an integrable nonlinear PDE is the celebrated nonlinear Schrödinger equation (NLS)

$$iq_t + q_{xx} - 2|q|^2 q = 0. \tag{2.6}$$

It can be verified [3] that this equation is the compatibility condition of the following pair of eigenvalue equations for the 2×2 matrix-valued function $\Psi(x,t,k)$:

$$\Psi_x + ik\sigma_3\Psi = Q\Psi, \quad k \in \mathbb{C}, \tag{2.7a}$$

$$\Psi_t + 2ik^2\sigma_3\Psi = (2kQ - iQ_x\sigma_3 - i|q|^2\sigma_3)\Psi, \tag{2.7b}$$

where σ_3 is the third Pauli matrix and Q is defined in terms of q,

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q(x,t) = \begin{bmatrix} 0 & q(x,t) \\ \bar{q}(x,t) & 0 \end{bmatrix}. \tag{2.8}$$

The compatible pair of linear eigenvalue equations associated with a given integrable PDE is called a Lax pair [4]. We note that for the linearized version of the NLS, $\omega(k) = k^2$, thus equations (2.4) become

$$\mu_x - ik\mu = q, \tag{2.9a}$$

$$\mu_t + ik^2\mu = iq_x - kq. \tag{2.9b}$$

Equations (2.7) are a non-Abelian version of equations (2.9).

(ii) Although the formulation (2.4) provides *no* advantage for the solution of the Cauchy problem for equation (2.1a), it *does* provide a novel approach for the solution of boundary value problems for equation (2.1a), see section 8.

The solution of the Cauchy problem on the infinite line for the NLS equation (2.6) is based on the analysis of the Lax pair (2.7). Thus, for pedagogical reasons we conclude this section by using equations (2.4) to rederive the solution (2.3).

We first *assume* that the solution of the Cauchy problem on the infinite line exists and has sufficient smoothness and decay . In order to construct the direct map, we treat equation (2.4a) as an ODE for the unknown function $\mu(x,t,k)$ in terms of $q(x,t)$, where $t > 0$ is fixed and k is an arbitrary *complex* parameter. By integrating with respect to x from either $-\infty$ or $+\infty$, it follows that $\mu = \mu^+$, $k \in \mathbb{C}^+$ and $\mu = \mu^-$, $k \in \mathbb{C}^-$, where $\mathbb{C}^{\pm} = \{k \in \mathbb{C}, \, \mathrm{Im} k \gtrless 0\}$, and

$$\mu^+(x,t,k) = \int_{-\infty}^{x} e^{ik(x-\xi)} q(\xi,t)d\xi, \quad \mu^-(x,t,k) = \int_{\infty}^{x} e^{ik(x-\xi)} q(\xi,t)d\xi. \tag{2.10}$$

For k real, both functions μ^+ and μ^- are well defined, and since they both satisfy the same ODE (2.4a), it follows that for $k \in \mathbb{R}$ these functions are simply related:

$$\mu^+(x,t,k) - \mu^-(x,t,k) = e^{ikx}\hat{q}(k,t), \quad k \in \mathbb{R}, \tag{2.11}$$

where $\hat{q}(k,t)$ is the FT of $q(x,t)$. We note that $\hat{q}(k,t)$ can be thought of as a "scattering function",

$$\hat{q}(k,t) = \lim_{x\to\infty} e^{-ikx}\mu^+(x,t,k). \tag{2.12}$$

Equations (2.10) express $\mu(x,t,k)$ in terms of $q(x,t)$. In order to construct the inverse map, we use an alternative representation for μ, namely we express μ in terms of the function $\hat{q}(k,t)$: The functions μ^+ and μ^- defined by equations (2.10) are analytic in k for Im $k > 0$ and Im $k < 0$ respectively. Furthermore, equations (2.10) imply that $\mu = O(1/k)$ as $k \to \infty$. These facts together with the "jump condition" (2.11) define a scalar Riemann-Hilbert problem for the sectionally analytic function $\mu(x,t,k)$. The unique solution of this problem is

$$\mu(x,t,k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ilx}\hat{q}(l,t)}{l-k}dl, \quad k \in \mathbb{C}, \quad \mathrm{Im}k \neq 0. \tag{2.13}$$

Substituting μ in equation (2.4a) we find the classical inverse FT formula.

The above analysis of equation (2.4a) will be referred to as the *spectral analysis*. In order to derive equation (2.3), all that remains is to show that $\hat{q}_t + i\omega(k)\hat{q} = 0$. This equation is a direct consequence of equation (2.4b). Indeed, it follows by substituting equation (2.13) into equation (2.4b) and using the fact that $\{\partial_x^j q(x,t)\}_0^{n-1}$ vanishes as $x \to \infty$. In summary, the spectral analysis of equation (2.4a) yields the FT, while the time evolution of the Fourier transform is a direct consequence of equation (2.4b). Although we have constructed $q(x,t)$ under the assumption of existence, we can rigorously justify *a posteriori* this formula *without* this assumption. Indeed, the above construction motivates the *definitions* of both the direct and the inverse maps (2.2) and (2.3) respectively. In order to prove that $q(x,0) = q_0(x)$ we must derive the inverse FT of $q_0(x)$ in terms of $\hat{q}_0(k)$. This can be achieved by performing the spectral analysis of equation (2.4a) evaluated at $t = 0$. In this case, instead of $q(x,t)$ we have the *known* function $q_0(x)$, thus every step of the spectral analysis can be rigorously justified.

3. The Cauchy Problem for the NLS on the Line

The analysis is conceptually similar with the analysis of equations (2.4) presented in Section 2. Indeed, we first assume that $q(x,t)$ exists and perform the spectral analysis of equation (2.7a). This yields a "nonlinear", or more precisely a non-Abelian FT. Furthermore, the time evolution of this nonlinear FT, which turns out to be linear, is a direct consequence of equation (2.7b). Having obtained the correct formula for $q(x,t)$, this formula can be rigorously justified following steps conceptually similar with those of the analogous treatment of equations (2.4). In what follows we present the nonlinear versions of equations (2.2) and (2.3) [5].

Direct map $\quad q_0(x) \mapsto \{a(k), b(k)\}$

$$a(k) = 1 - \int_{-\infty}^{\infty} \bar{q}_0(x)\phi_1(x,k)\,dx, \quad b(k) = -\int_{-\infty}^{\infty} e^{2ikx}q_0(x)\phi_2(x,k)\,dx, \tag{3.1}$$

where the vector $(\phi_1(x,k), \phi_2(x,k))^T$ is defined in terms of $q_0(x)$ through the solution of the linear Volterra integral equation

$$\begin{aligned} \phi_1(x,k) &= \int_{\infty}^{x} e^{-2ik(x-x')}q_0(x')\phi_2(x',k)\,dx' \\ \phi_2(x,k) &= 1 + \int_{\infty}^{x} \bar{q}_0(x')\phi_1(x',k)\,dx' \end{aligned} \tag{3.2}$$

for $-\infty < x < \infty$ and $\mathrm{Im}\,k \geq 0$.

Inverse map $\{a(k), e^{-4ik^2t}b(k)\} \mapsto q(x,t)$

$$q(x,t) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\Big[e^{-2ikx}\frac{e^{-4ik^2t}b(k)}{\overline{a(k)}}M_{11}^{+}(x,t,k) + \frac{|b(k)|^2}{|a(k)|^2}M_{12}^{+}(x,t,k)\Big]dk, \tag{3.3}$$

where $M(x,t,k)$ is defined in terms of $b(k)/a(k)$ through the solution of the following matrix RH problem:

(i) M is analytic in $k \in \mathbb{C}\setminus\mathbb{R}$.
(ii) $M = \operatorname{diag}(1,1) + O\big(1/k\big), \quad k \to \infty$.
(iii) Let $M = M^{+}$ for $\operatorname{Im} k \geq 0$, $M = M^{-}$ for $\operatorname{Im} k \leq 0$; then M satisfies the "jump condition",

$$M^{-}(x,t,k) = M^{+}(x,t,k)\begin{bmatrix} 1 & -\dfrac{e^{-4ik^2t}b(k)}{\overline{a(k)}}e^{-2ikx} \\ \dfrac{e^{4ik^2t}\overline{b(k)}}{a(k)}e^{2ikx} & 1-\dfrac{|b(k)|^2}{|a(k)|^2} \end{bmatrix}, \qquad k \in \mathbb{R}. \tag{3.4}$$

Suppose that $q_0(x) \in S(\mathbb{R})$. The rigorous justification of the solution of the Cauchy problem of the NLS equation (2.6) involves the following steps. The vector $(\phi_1,\phi_2)^T$ is defined in terms of $q_0(x)$ through the solution of a linear Volterra integral equation, thus the direct map is well defined. The inverse map is also well defined provided that the RH problem which defines M in terms of $a(k)$ and $b(k)$ has a unique global solution. This is indeed the case: If the coefficients of the "jump matrix" appearing in equation (3.4) are in $H_1(\mathbb{R})$, then the RH problem is equivalent to a Fredholm integral equation of index zero. Using the fact that the jump matrix is Hermitian positive definite, it can be shown that the Fredholm integral equation has a unique global solution. After establishing that the direct and the inverse maps are well defined, it remains to show that (a) $q(x,t)$ satisfies the NLS equation, and (b) $q(x,0) = q_0(x)$. For (a), one uses ideas of the so-called dressing method introduced by Zakharov and Shabat, namely one shows that if M satisfies a RH problem with the jump condition (3.4), and if $q(x,t)$ is defined in terms of M by equation (3.3), then M satisfies the Lax pair (2.7), hence q satisfies the NLS. For (b), one evaluates equation (3.4) at $t=0$, and then one uses the fact that if $\mathbf{S}$ denotes the direct map (3.1) and if $\mathbf{Q}$ denotes the inverse map (3.3) evaluated at $t=0$, then $\mathbf{Q}^{-1} = \mathbf{S}$.

If $\bar{q}$ in the definition of Q (equation (2.8)) is replaced by $-\bar{q}$, then the resulting RH problem is singular, namely $a(k)$ can have zeros. These zeros are important because they give rise to *solitons*. A singular RH problem can be mapped to a regular RH problem supplemented with a set of algebraic equations.

Equations (3.1) and (3.3) provide a nonlinear analogue of the direct and inverse Fourier transform formulae (2.2) and (2.3) respectively (with $\omega(k) = k^2$). Indeed, if $q_0(x)$ is small, then equations (3.2) imply $\phi_1 \sim 0$ and $\phi_2 \sim 1$, thus $a(k) \sim 1$ and $b(k)$ tends to the FT of $q_0(x)$. Furthermore, if $a \sim 1$ and b is small, then the RH defining M implies that $M \sim \operatorname{diag}(1,1)$, thus equation (3.3) implies that $q(x,t)$ tends to the formula (2.3) (with k replaced by $2k$ and $\omega(k) = k^2$).

4. Fourier Transform on the Plane

We saw in Section 2 that the Fourier transform on the line can be derived through the spectral analysis of the differential operator d/dx. In this section we show that the Fourier transform and inverse Fourier transform on the plane can be derived as the direct and inverse maps in the spectral analysis of the ∂-bar operator.

Consider the following spectral problem:

$$\frac{\partial \mu}{\partial \bar{x}} - \frac{ik}{2}\mu = q(x_1, x_2), \tag{4.1}$$

where $\partial/\partial\bar{x} = (1/2)(\partial/\partial x_1 + i\partial/\partial x_2)$ is the ∂-bar operator for the complex variable $x = x_1 + ix_2$, $k = k_1 + ik_2$, and $q \in S(\mathbb{R}^2)$. Using the inversion of the ∂-bar operator, it follows that the solution of (4.1) which vanishes at ∞ is given by

$$\mu(x,k) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{e^{i[k(\bar{x}-\bar{x}')+\bar{k}(x-x')]/2} q(x_1', x_2')}{x - x'}\, dx_1' dx_2' \qquad (x' = x_1' + ix_2'). \tag{4.2}$$

It follows from (4.2) that

$$\lim_{k\to\infty} \mu(x,k) = 0, \tag{4.3}$$

$$\frac{\partial \mu}{\partial \bar{k}} = e^{i(k_1 x_1 + k_2 x_2)} \hat{q}(k_1, k_2), \tag{4.4}$$

where

$$\hat{q}(k_1, k_2) = \frac{i}{2\pi}\int_{\mathbb{R}^2} e^{-i(k_1 x_1' + k_2 x_2')} q(x_1', x_2')\, dx_1' dx_2' \tag{4.5}$$

is the Fourier transform of q.

Equation (4.2) expresses μ in terms of $q(x_1, x_2)$. Using equations (4.3) and (4.4), it follows that it is possible to find an alternative representation of μ, namely we can express μ in terms of $\hat{q}(k_1, k_2)$. Indeed, the solution of (4.4) satisfying the boundary condition (4.3) is given by

$$\mu(x,k) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{e^{i(k_1' x_1 + k_2' x_2)} \hat{q}(k_1', k_2')}{k - k'}\, dk_1' dk_2' \qquad (k' = k_1' + ik_2'). \tag{4.6}$$

Substituting (4.6) into (4.1) yields the Fourier inversion formula

$$q(x_1, x_2) = \frac{-i}{2\pi}\int_{\mathbb{R}^2} e^{i(k_1 x_1 + k_2 x_2)} \hat{q}(k_1, k_2)\, dk_1 dk_2. \tag{4.7}$$

5. The Cauchy Problem for DSII on the Plane

The Fourier transform on the plane can be *nonlinearized* and the resulting transform can be used to solve certain nonlinear evolution equations in two space dimensions. The starting point of the nonlinearization is the following non-Abelian analog of (4.1):

$$\frac{\partial \mu}{\partial \bar{x}} - \frac{ik}{4}\mu = Q\bar{\mu}, \tag{5.1}$$

where μ is a 2×2 matrix and

$$Q = \begin{bmatrix} 0 & q_{12}(x_1, x_2) \\ q_{21}(x_1, x_2) & 0 \end{bmatrix} \tag{5.2}$$

is an off-diagonal 2×2 matrix whose components belong to $S(\mathbb{R}^2)$. We look for solutions of (5.1) of the form

$$\mu = e^{i(k_1x_1+k_2x_2)/2}\psi = e^{i(k\bar{x}+\bar{k}x)/4}\psi, \tag{5.3}$$

$$\lim_{x\to\infty} \psi = I, \tag{5.4}$$

where I denotes the 2×2 identity matrix.

Equation (5.1) implies that ψ satisfies the equation

$$\frac{\partial \psi}{\partial \bar{x}} = e^{-i(k\bar{x}+\bar{k}x)/2} Q\bar{\psi}. \tag{5.5}$$

The solution of (5.5) satisfying (5.4) is defined by the integral equation

$$\psi(x_1, x_2, k_1, k_2) = I + \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{e^{-i(k\bar{x}'+\bar{k}x')/2}}{x - x'} Q(x_1', x_2')\overline{\psi(x_1', x_2', k_1, k_2)}\, dx'. \tag{5.6}$$

This equation implies

$$\lim_{k\to\infty} \psi_d = I, \tag{5.7}$$

$$\lim_{k\to\infty} \frac{-ik}{2} e^{i(k\bar{x}+\bar{k}x)/2}\psi_o(x_1, x_2, k_1, k_2) = Q(x_1, x_2), \tag{5.8}$$

where ψ_d (resp. ψ_o) denotes the diagonal (resp. off-diagonal) part of μ.

Let $\nu = \bar{\psi}_d + e^{i(k\bar{x}+\bar{k}x)/2}\psi_o$. It follows from (5.7) and (5.8) that

$$\lim_{k\to\infty} \nu = I, \tag{5.9}$$

$$\lim_{k\to\infty} \frac{-ik}{2}\nu_o(x_1, x_2, k_1, k_2) = Q(x_1, x_2). \tag{5.10}$$

Furthermore, equation (5.6) implies that ν satisfies

$$\frac{\partial \nu}{\partial \bar{k}} = e^{i(k\bar{x}+\bar{k}x)/2}\bar{\nu}\hat{Q}, \tag{5.11}$$

where $\hat{Q}$ is the 2×2 off-diagonal matrix defined by

$$\hat{Q}(k_1, k_2) = \frac{i}{2\pi}\int_{\mathbb{R}^2} e^{-i(k_1x_1+k_2x_2)} Q(x_1, x_2)\overline{\psi_d(x_1, x_2, k_1, k_2)}\, dx_1 dx_2. \tag{5.12}$$

Given $\hat{Q}$, the solution of (5.11) with the boundary condition (5.10), is defined by the integral equation

$$\nu(x_1, x_2, k_1, k_2) = I + \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{e^{i(k'\bar{x}+\bar{k}'x)/2}}{k - k'}\overline{\nu(x_1, x_2, k_1', k_2')}\,\hat{Q}(k_1', k_2')\, dk_1' dk_2'. \tag{5.13}$$

The asymptotic relation (5.10), and equation (5.13) imply

$$Q(x_1, x_2) = \frac{-i}{2\pi} \int_{\mathbb{R}^2} e^{2i(k_1 x_1, k_2 x_2)} \overline{\nu_o(x_1, x_2, k_1, k_2)}\, \hat{Q}(k_1, k_2)\, dk_1 dk_2. \tag{5.14}$$

The *direct map* $Q \mapsto \hat{Q}$ defined by equation (5.12), and the *inverse map* $\hat{Q} \mapsto Q$ defined by equation (5.14), provide a nonlinear generalization of the two dimensional FT. Indeed, if $Q(x_1, x_2)$ is small, then equation (5.6) implies that $\psi \sim \operatorname{diag}(1,1)$, and equation (5.12) implies that $\hat{Q}(k_1, k_2)$ tends to the FT of $Q(x_1, x_2)$. Furthermore, if $\hat{Q}$ is small, equation (5.13)implies that $\nu \sim \operatorname{diag}(1,1)$, thus equation (5.14) implies that $Q(x_1, x_2)$ tends to the inverse FT of $\hat{Q}(k_1, k_2)$.

If we now let $\hat{Q}$ evolve according to the differential equation

$$\frac{d\hat{Q}}{dt} = i(k_1^2 - k_2^2)\hat{Q} \tag{5.15}$$

and denote the function obtained from $\hat{Q}(k_1, k_2, t)$ through the inverse map by $Q(x_1, x_2, t)$, then Q satisfies the partial differential equation

$$\frac{\partial Q}{\partial t} = i\left[\frac{\partial^2 Q}{\partial x_2^2} - \frac{\partial^2 Q}{\partial x_1^2}\right] - 8i \begin{bmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{bmatrix} Q, \tag{5.16}$$

where ϕ is defined by

$$\begin{gathered} \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = \left[\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}\right](Q_{12}\bar{Q}_{21}) \\ \lim_{x\to\infty} \phi = 0. \end{gathered} \tag{5.17}$$

In the particular case where $Q_{12} = q = \pm Q_{21}$, equations (5.16) and (5.17) reduced to the Davey-Stewartson-II equations:

$$\begin{gathered} \frac{\partial q}{\partial t} = i\Big(\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}\Big) q - 8i\phi q, \\ \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = \pm\Big(\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}\Big)|q|^2. \end{gathered} \tag{5.18}$$

The rigorous justification of the solution of the Davey-Stewartson-II equations by the nonlinear Fourier transforms associated with the spectral problem (5.1), depends on the solvability of the integral equations (5.6) and (5.13) [6], [7]. These integral equations are Fredholm equations with index zero on the space of bounded continuous functions on $\mathbb{R}^2$. They are uniquely solvable if Q and $\hat{Q}$ are small. In the case where $Q_{12} = Q_{21}$ (resp. $\hat{Q}_{12} = \hat{Q}_{21}$), any solution of the homogeneous equation is a *generalized analytic* function that vanishes at ∞, and the only such solution is the zero solution by the *generalized Liouville* theorem. Therefore the integral equations are uniquely solvable without assuming Q or $\hat{Q}$ to be small. This means that, when the sign in the equation for ϕ is positive, the Cauchy problem of (5.18) is uniquely solvable for any initial data in $S(\mathbb{R}^2)$.

6. Generalized Fourier Transforms and the Imaging of the Brain

One of the authors and I.M. Gel'fand emphasized in [8] that the approach illustrated in the spectral analysis of equations (2.4a) and (4.1), provides a novel approach for deriving linear transforms. As an application of this approach, one of the authors and R. Novikov rederived the celebrated Radon transform by performing the spectral analysis of the following eigenvalue equation for the scalar function $\mu(x_1, x_2, k)$,

$$\frac{1}{2}(k + \frac{1}{k})\mu_{x_1} + \frac{1}{2i}(k - \frac{1}{k})\mu_{x_2} = f(x_1, x_2), \quad -\infty < x_1, x_2 < \infty, \quad k \in \mathbb{C}, \tag{6.1}$$

where $f \in S(\mathbb{R}^2)$. Although the Radon transform can be derived in a simpler way by using the two-dimensional FT, the advantage of the derivation of [9] was demonstrated recently by R. Novikov who showed that a similar analysis applied to a slight generalization of equation (6.1), yields the inversion of the so-called attenuated Radon transform [10]. Implementing this inversion was one of the most important open problems in the field of medical imaging. We recall that the Radon transform provides the mathematical basis of computerized tomography (CT). Similarly, the attenuated Radon transform provides the mathematical basis of a new imaging technique of great significance, namely of the so-called Single Photon Emission Computerized Tomography (SPECT). Before discussing the mathematics of CT and of SPECT we first present a brief introduction of these remarkable imaging techniques.

CT

In brain imaging, computerized tomography is the computer aided reconstruction of a mathematical function that represents the x-ray attenuation coefficient of the brain tissue (and is therefore related to its density). Let $f(x_1, x_2)$ denote the x-ray attenuation coefficient at the point (x_1, x_2). This means that x-rays transversing a small distance $\Delta\tau$ at (x_1, x_2) suffer a relative intensity loss $\Delta I/I = -f\Delta\tau$. Taking the limit and solving the resulting ODE we find $I_1/I_0 = \exp[-\int_L f d\tau]$, where L denotes the part of the line that transverses the tissue. Since I_1/I_0 is known from the measurements, the basic mathematical problem of CT is to reconstruct a function from the knowledge of its line integrals. The line integral of a function is called its Radon transform. In order to define this transform we introduce local coordinates: Let the line L make an angle θ with the positive x_1-axis. A point (x_1, x_2) on this line can be specified by the variables (ρ, τ), where ρ is the distance from the origin and τ is a parameter along the line, see Figure 6.1.

The variables (x_1, x_2) and (τ, ρ), for fixed θ, are related by the equations

$$x_1 = \tau\cos\theta - \rho\sin\theta, \quad x_2 = \tau\sin\theta + \rho\cos\theta. \tag{6.2}$$

We will denote a function $f(x_1, x_2)$ written in the local coordinates (θ, τ, ρ) by $F(\tau, \rho, \theta)$, i.e.

$$F(\tau, \rho, \theta) = f(\tau\cos\theta - \rho\sin\theta, \tau\sin\theta + \rho\cos\theta). \tag{6.3}$$

Thus the Radon transform of the function $f(x_1, x_2)$, which we will denote by $\hat{f}(\rho, \theta)$, is defined by

$$\hat{f}(\rho, \theta) = \int_{-\infty}^{\infty} F(\tau, \rho, \theta) d\tau. \tag{6.4}$$

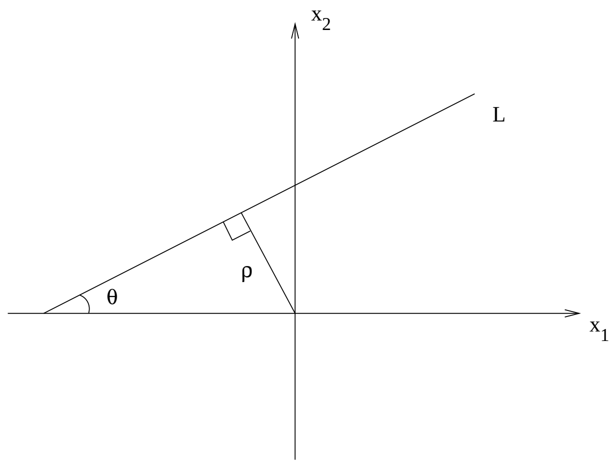

FIGURE 6.1. Local coordinates for the mathematical formulation of PET and SPECT.

In summary, the basic mathematical problem in CT is to reconstruct a function $f(x_1, x_2)$ from the knowledge of its Radon transform $\hat{f}(p, \theta)$.

The advent of CT made possible for the first time direct images of the brain tissue. Furthermore, the subsequent development of Magnetic Resonance Imaging (MRI) allowed striking discrimination between grey and white matter. This has had a tremendous impact on the entire field of medical imaging. Although the first applications of CT and MRI were in brain imaging, later these techniques were applied to many other areas of medicine. Indeed, it is impossible to think of medicine today without CT and MRI. However, in-spite of their enormous impact these techniques are capable of imaging only *structures* as opposed to *functional* characteristics.

PET and SPECT

The study of functional characteristics became possible only in the late 80s with the development of functional MRI, of Positron Emission Tomography (PET), and of SPECT. Using these new techniques it is now possible to observe neural activity in living humans with ever-increasing precision. For example, we now know that understanding of the context of language is associated mainly with the left prefrontal part of the brain, while understanding of the emotional tone of speech is associated mainly with the right. Another example, perhaps of interest to mathematicians, is that during the performance of mathematical calculations there exists bilateral activation of the frontal cortex and of the posterial parietal cortex. There exist a vast number of clinical applications from epilepsy and migrains, to differential diagnosis of schizophrenia and of Alzheimer's disease. Furthermore, just like with CT and MRI, the above new techniques are now used beyond neuroscience in a wide range of medical areas. These include pharmacology, oncology and cardiology.

In PET, the patient is injected with a dose of FGD which is a normal molecule of glucose attached to an atom of radioactive fluorine. The more active cells absorb more FDG. The fluorine atom in FDG suffers a radioactive decay emitting a positron, which when colliding with an electron liberates energy in the form of *two* beams of gamma rays which are picked

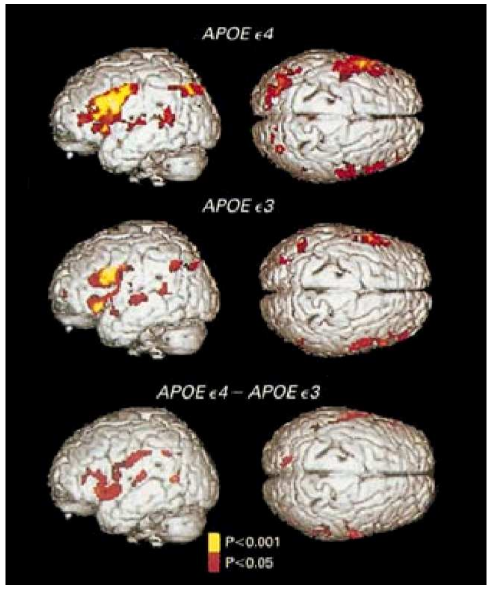

FIGURE 6.2. Images showing the difference in brain activation during a memory test between two groups of subjects. One group carries the apolipoprotein Eε4 allele which is a known risk factor for Alzheimer's disease. The increase in activation may reflect a need to compensate for minor defects in memory – even though both groups appear normal. (Image reproduced with permission from: Bookheimer S Y, Strojwas B S, Cohen M S, et al. 2000. *Patterns of Brain Activation in People at Risk for Alzeheimer's Disease*, New England Journal of Medicine **343** (7), 450–456. Copyright ©2000 Massachusetts Medical Society)

by the PET scanner simultaneously. In SPECT the situation is similar by instead of FDG one uses Xenon-133 which emits a *single* photon.

Let $f, g, L(x)$, denote the x-ray attenuation coefficient, the distribution of the radioactive material, and the part of the ray from the tissue to the detector. Then in SPECT the following integral I is known from the measurements.

$$I = \int_L e^{-\int_{L(x)} f ds} g d\tau.$$

This integral is called the attenuated (with respect to f) Radon transform of g and will be denoted by $\hat{g}_f$. Writing f and g in local coordinates we find

$$\hat{g}_f(\rho, \theta) = \int_{-\infty}^{\infty} e^{-\int_\tau^\infty F(s,\rho,\theta)ds} G(\tau, \rho, \theta) d\tau. \tag{6.5}$$

Thus the basic mathematical problem of SPECT is to reconstruct the function $g(x_1, x_2)$ from the knowledge of its attenuated Radon transform $\hat{g}_f$ and of the associated x-ray attenuation coefficient $f(x_1, x_2)$.

7. The Mathematics of PET and SPECT

It has been recently shown [11] that by scrutinizing the analysis of [9], it is possible to derive the attenuated Radon transform almost immediately.[3] In this respect, we first review the main steps in the spectral analysis of equation (6.1): The lhs of this equation motivates the introduction of the complex variable z,

$$z = \frac{1}{2i}(k - \frac{1}{k})x_1 - \frac{1}{2}(k + \frac{1}{k})x_2. \tag{7.1}$$

Equation (6.1), after changing variables from (x_1, x_2) to $(z, \bar{z})$, becomes

$$\frac{1}{2i}\left(\frac{1}{|k|^2} - |k|^2\right)\frac{\partial \mu}{\partial \bar{z}} = f, \quad |k| \neq 1. \tag{7.2}$$

Using the inverse ∂-bar formula, it follows that the unique solution of this equation satisfying the boundary condition $\mu = 0(1/z)$ as $z \to \infty$, is

$$\mu(x_1, x_2, k) = \frac{1}{2\pi i} sgn\left(\frac{1}{|k|^2} - |k|^2\right)\int_{\mathbb{R}^2}\int \frac{f(x_1', x_2')dx_1'dx_2'}{z' - z}, \quad |k| \neq 1. \tag{7.3}$$

This equation provides the solution of the direct problem, i.e. it expresses μ in terms of $f(x_1, x_2)$. In order to solve the inverse problem, i.e. in order to find an alternative representation of μ (in terms of an appropriate spectral function) we note that μ is an analytic function of k in the entire complex k-plane (including infinity) except for the unit circle. Thus in order to reconstruct μ, it is sufficient to compute the "jump" $\mu^+ - \mu^-$, where μ^+ and μ^- denote the limits of μ as k approaches the unit circle from inside and outside the unit disk. A simple computation yields

$$\mu^{\pm} = \lim_{\varepsilon \to 0} \mu(x_1, x_2, (1 \mp \varepsilon)e^{i\theta}) = \mp P^{\mp}\hat{f}(\rho, \theta) - \int_{\tau}^{\infty} F(s, \rho, \theta)ds, \tag{7.4}$$

where $P^{\mp}$ denote the usual projectors in the variable ρ, i.e.

$$(P^{\mp}f)(\rho) = \mp\frac{f}{2} + \frac{1}{2i\pi}Hf,$$

and H denotes the Hilbert transform.

Equations (7.4) imply $\mu^+ - \mu^- = -H\hat{f}(\rho, \theta)/i\pi$. Substituting this expression in the equation

$$\mu = \frac{1}{2i\pi}\int_0^{2\pi} \frac{(\mu^+ - \mu^-)(e^{i\theta'})ie^{i\theta'}d\theta'}{e^{i\theta'} - k}, \quad |k| \neq 1, \tag{7.5}$$

we find μ in terms of $\hat{f}$,

$$\mu(x_1, x_2, k) = -\frac{1}{2i\pi^2}\int_0^{2\pi} \frac{e^{i\theta'}[H\hat{f}(x_2\cos\theta' - x_1\sin\theta', \theta')]d\theta'}{e^{i\theta'} - k}.$$

Substituting this expression in equation (6.1) we find the inverse Radon transform formula,

$$f(x_1, x_2) = -\frac{i}{4\pi^2}\left(\partial_{x_1} - i\partial_{x_2}\right)\int_0^{2\pi} e^{i\theta}[H\hat{f}(x_2\cos\theta - x_1\sin\theta, \theta)]d\theta. \tag{7.6}$$

[3]A numerical implementation of the inverse attenuated Radon transform is presented in [11].

We now present the derivation of the inverse attenuated Radon transform. Instead of starting with equation (7.2), we start with the equation

$$\nu(k)\frac{\partial\mu}{\partial\bar{z}} + f(x_1,x_2)\mu = g(x_1,x_2), \quad \nu(k) := \frac{1}{2i}\left(\frac{1}{|k|^2} - |k|^2\right), \tag{7.7}$$

where $f \in S(\mathbb{R}^2)$, $g \in S(\mathbb{R}^2)$, $k \in \mathbb{C}$, and $|k| \neq 1$. Equation (7.7) implies

$$\mu \exp\left[\partial_{\bar{z}}^{-1}\frac{f}{\nu}\right] = \partial_{\bar{z}}^{-1}\left(\frac{g}{\nu}\exp\left[\partial_{\bar{z}}^{-1}\frac{f}{\nu}\right]\right). \tag{7.8}$$

This equation provides the solution of the direct problem, ie. it expresses μ in terms of f and g. Since μ is an analytic function of k in the entire complex k-plane except for the unit circle, it follows that μ satisfies the alternative representation (7.5). Thus, in order to express μ in terms of an appropriate spectral function, all that remains is to compute $\mu^+ - \mu^-$. But this can be easily derived using equation (7.4). Indeed, equation (7.4) can be rewritten in the form

$$\lim_{k\to k^\pm}\left\{\partial_{\bar{z}}^{-1}\left(\frac{f(x_1,x_2)}{\nu(k)}\right)\right\} = \mp P^{\mp}\hat{f}(\rho,\theta) - \int_\tau^\infty F(s,\rho,\theta)ds.$$

Using this equation, equation (7.8) can be used to compute $\mu^\pm$, and then equation (7.5) proveds an alternative representation of μ in terms of $\hat{g}_f(\rho,\theta)$ and of F. Substituting this representation in equation (7.7) we find the inverse attenuated Radon transform

$$g(x_1,x_2) = -\frac{1}{4\pi}(\partial_{x_1} - i\partial_{x_2})\int_0^{2\pi} J(x_1,x_2,\theta)e^{i\theta}\mathrm{d}\theta,$$

where

$$J(x_1,x_2,\theta) = -e^{\int_\tau^\infty F(s,\rho,\theta)\mathrm{d}s}\left(e^{P^-\hat{f}(\rho,\theta)}P^-e^{-P^-\hat{f}(\rho,\theta)} + e^{-P^+\hat{f}(\rho,\theta)}P^+e^{P^+\hat{f}(\rho,\theta)}\right)\hat{g}_f(\rho,\theta).$$

8. Linear Evolution PDEs on the Half-Line

After solving the Cauchy problem for integrable evolution PDEs on the line and on the plane (see sections 3 and 5), the most important open problem in the analysis of integrable PDEs becomes the solution of boundary value problems, such as the solution of the NLS on the half-line. This problem remained essentially open for almost thirty years, until the recent emergence of a uniform transform method introduced by one of the authors. The difficulty of this problem can be illustrated by considering the Korteweg-deVries equation on the half-line, $0 < x < \infty$. This equation, which is another well known integrable PDE, is

$$q_t + q_x + q_{xxx} + 6qq_x = 0. \tag{8.1}$$

It was noted in sections 2 and 3 that the solution of the Cauchy problem on the infinite line for an integrable evolution PDE can be constructed by using a nonlinear FT. Thus, a natural strategy for solving equation (8.1) is to solve the linearized version of equation (8.1) by the appropriate x-transform, and then to nonlinearize this transform. However, this strategy fails: Let us consider an initial-boundary value problem for the equation

$$q_t + q_x + q_{xxx} = 0, \quad 0 < x < \infty, \quad 0 < t < T, \tag{8.2}$$

where T is a positive fixed constant, and q decays for large x. Although it is *not* immediately obvious how many boundary conditions must be prescribed at $x = 0$, it turns out that this problem is well-posed with *one* boundary condition. Thus, we consider equation (8.2) with $q(x, 0) = q_0(x) \in H^2(\mathbb{R}^+)$ and $q(0, t) = g_0(t) \in H^2(0, T)$. In order to derive the appropriate x-transform we analyze the eigenvalue equation obtained by separation of variables. It turns out that the associated ordinary differential operator is *not* self-adjoint, and furthermore it does *not* admit a self-adjoint extension. This implies that there does *not* exist an appropriate x-transform for third order evolution PDEs defined in the half-line (ie. there do *not* exist proper analogs of the sine and of the cosine transforms which are the appropriate transforms for the Dirichlet and the Neumann problems of second order evolution PDEs on the half-line). Of course, one can use the appropriate t-transform, which is actually the Laplace transform. However, since in this transform $0 < t < \infty$, the Laplace transform is problematic if the growth of $g_0(t)$ as $t \to \infty$ is faster than linearly exponential functions. Furthermore, as it was mentioned earlier, the solution of the Cauchy problem of the associate nonlinear PDE (8.1) was based on the nonlinearization of the $x-$ and not of the $t-$transform.

The above failure of the classical transforms should be contrasted with the existence of the abstract Ehrenpreis-Palomodov integral representations [12]. For equation (2.1a) this fundamental representation implies that if a well-posed problem is defined in a convex domain,[4] then there exists a measure $\rho(k)$ such that the solution $q(x, t)$ is given by

$$q(x,t) = \int_L e^{ikx - i\omega(k)t} d\rho(k). \tag{8.3}$$

The limitation of this result is that it does *not* provide an algorithm for constructing $\rho(k)$ in terms of the given initial and boundary conditions. However, it does imply that there must exist a representation for the solution of (8.2) which can be expressed in the form (8.3). This is indeed the case. Actually, the following general result has been proved by the authors: Consider the linear evolution PDE (2.1a) in the domain $0 < x < \infty$, $0 < t < T$, with the initial condition $q(x, 0) = q_0(x) \in H^{\tilde{n}/2}(\mathbb{R}^+)$, and the N boundary conditions $\partial_x^l q(0, t) = g_l(t) \in H^{\frac{1}{2}+\frac{2\tilde{n}-2l-1}{2n}}(0, T)$, $l = 0, 1, \dots, N - 1$, where

$$N = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } \alpha_n > 0 \\ \frac{n-1}{2} & \text{if } n \text{ is odd and } \alpha_n < 0 \end{cases}, \qquad \tilde{n} = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases},$$

and $\alpha_n \neq 0$ is the coefficient of k^n in $\omega(k)$. Assume that $q_0(x)$ and $g_l(t)$ are compatible at $x = t = 0$, ie. $g_l(0) = \partial_x q_0(0)$, $0 \leq l \leq N - 1$. Then the unique solution of (2.1) is given by the Ehrenpreis-Palomodov representation

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - i\omega(k)t} \hat{q}_0(k)\, dk + \frac{1}{2\pi} \int_{\partial D_+} e^{ikx - i\omega(k)t} \tilde{g}(k)\, dk, \tag{8.4}$$

where ∂D_+ is the oriented boundary of the domain $D_+ = \{k \in \mathbb{C} : \operatorname{Im} \omega(k) > 0, \operatorname{Im} k > 0\}$ oriented so that D_+ is to the left of the increasing direction, $\hat{q}_0(k)$ is the Fourier transform of

[4] In the abstract formulation it is required that the domain is bounded and smooth, so this result does not apply immediately to our domain.

$q_0(x)$, and $\tilde{g}(k)$ can be written explicitly in terms of $\{\tilde{g}_l(k)\}_0^{N-1}$ which are the t-transforms of $\{g_l(t)\}_0^{N-1}$,

$$\tilde{g}_l(k) = \int_0^T e^{i\omega(k)t} g_l(t)\,dt, \quad k \in \mathbb{C},\ 0 \le l \le N-1, \tag{8.5}$$

and in terms of $\hat{q}_0(\nu_i(k))$, $i = 1, \ldots, n-1$, where $\nu_i(k)$ are determined by solving $\omega(\nu_i(k)) - \omega(k) = 0$.

For equation (8.2), $\omega(k) = k - k^3$, ∂D_+ consists of part of the real k-axis and of parts of the curve $k_I^2 - 3k_R^2 + 1 = 0$ $(k = k_R + ik_I)$, see Figure 8.1, and $\tilde{g}(k)$ is given by the following expression:

$$\tilde{g}(k) = \frac{\nu_1 - k}{\nu_2 - \nu_1}\hat{q}_0(\nu_2) + \frac{k - \nu_2}{\nu_2 - \nu_1}\hat{q}_0(\nu_1) + (1 - 3k^2)\tilde{g}_0(k), \tag{8.6}$$

where $\tilde{g}_0(k)$ is the t-transform of $g_0(t)$ (see equation (8.5) with $l = 0$ and $\omega = k - k^3$), and ν_1, ν_2 are the nontrivial solutions of the equation $\nu - \nu^3 = k - k^3$, ie. they are the two solutions of the quadratic equation $\nu^2 + \nu k + k^2 - 1 = 0$.

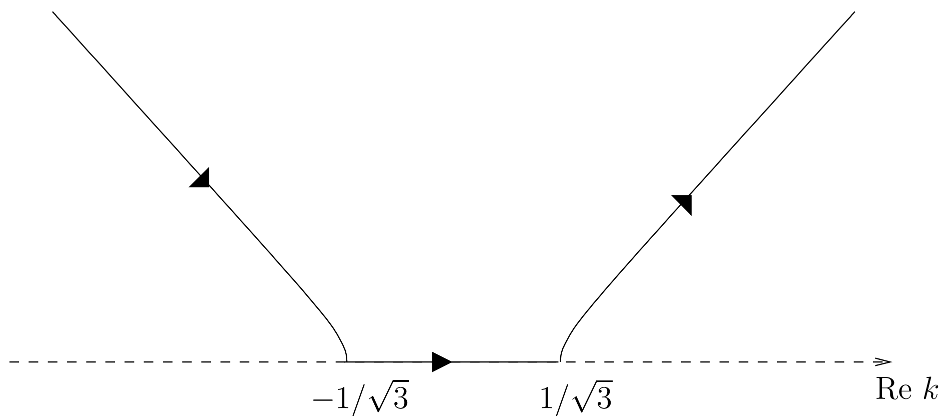

FIGURE 8.1. The contour ∂D_+ for equation (8.2)

This result is the implementation to the initial-boundary value problem of (8.2) of a general approach initiated by one of the authors [13]. In this approach, the PDE, the domain, and the given initial and boundary conditions are treated as follows:

(i) The PDE is rewritten in the form of a Lax pair. For equation (2.1a) the associated Lax pair is given by equations (2.4).

(ii) For a given domain the simultaneous spectral analysis of the Lax pair gives rise to two maps, a direct and an inverse map. The direct map constructs, from the *initial* and *boundary values*, appropriate x and t transforms, which we call *spectral functions*. The inverse map constructs, from the spectral functions, the representation $q(x,t)$ of the solution. For the initial-boundary value problem of (2.1a) on the half-line, these maps are defined below:

Direct Map

$$\{q_0(x), q(0,t), \partial_x q(0,t), \ldots, \partial_x^{n-1} q(0,t)\} \mapsto \hat{q}_0(k), \tilde{g}(k),$$

where

$$\hat{q}_0(k) = \int_0^\infty e^{-ikx} q_0(x)\,dx, \quad \operatorname{Im} k \le 0, \tag{8.7}$$

$$\tilde{g}(k) = \sum_{j=0}^{n-1} c_j(k)\tilde{g}_j(k), \quad \tilde{g}_j(k) := \int_0^T e^{i\omega(k)t}\partial_x^j q(0,t)\,dt, \quad k \in \mathbb{C}, \tag{8.8}$$

Inverse Map

$$e^{-i\omega(k)t}\hat{q}_0(k), e^{-i\omega(k)t}\tilde{g}(k) \mapsto q(x,t),$$

with $q(x,t)$ given by (8.4).

(iii) For given boundary conditions (ie. for a given subset of the boundary values) using the fact that the spectral functions satisfy a simple algebraic equation, which we have called the *global relation*, the unknown part of $\tilde{g}(k)$ can be determined. For equation (2.1a) on the half-line the relevant global relation is

$$\hat{q}_0(k) + \tilde{g}(k) = e^{i\omega(k)T}\int_0^\infty e^{-ikx} q(x,T)\,dx, \quad \operatorname{Im} k \le 0. \tag{8.9}$$

For the IBV problem of (2.1a) formulated earlier, N of the functions $\{\partial_x^i q(0,t)\}_0^{n-1}$ are known, then the lhs of (8.9) involves $n-N$ unknowns. Furthermore, the rhs of (8.9) is also unknown. However, in spite of this ominous looking situation, it is actually possible to compute $\tilde{g}(k)$ by solving a system of $n-N$ linear algebraic equations. This is based on the crucial observation that the functions $\tilde{g}_l(k)$ defined by (8.8), remain invariant under the transforms in the complex k plane which leaves $\omega(k)$ invariant. As an illustrative example we consider equation (8.2). In this case equation (8.9) becomes

$$\tilde{g}_2(k) + ik\tilde{g}_1(k) = (k^2-1)\tilde{g}_0(k) - \hat{q}_0(k) + e^{i\omega(k)}\hat{q}_T(k), \quad \operatorname{Im} k \le 0, \tag{8.10}$$

where $\omega = k^3 - k$ and $\hat{q}_T(k)$ denotes the integral appearing on the rhs of (8.9). Replacing in (8.10) k by $\nu_1(k) \in D_1$ and $\nu_2(k) \in D_2$ (cf. Figure 8.2), we find the two equations

$$\begin{aligned} \tilde{g}_2(k) + i\nu_1(k) &= (\nu_1^2(k)-1)\tilde{g}_0(k) - \hat{q}_0(\nu_1(k)) + e^{i\omega(\nu_1(k))}\hat{q}_T(\nu_1(k)), \qquad k \in D_+, \\ \tilde{g}_2(k) + i\nu_2(k) &= (\nu_2^2(k)-1)\tilde{g}_0(k) - \hat{q}_0(\nu_2(k)) + e^{i\omega(\nu_2(k))}\hat{q}_T(\nu_2(k)), \qquad k \in D_+. \end{aligned} \tag{8.11}$$

If $\nu_1(k) \in D_1$ and $\nu_2(k) \in D_2$, then $k \in D_+$, thus the equations (8.11) are valid in D_+. These equations can be considered as two equations for the two unknown functions $\tilde{g}_1$ and $\tilde{g}_2$. Solving for these unknown functions and substituting the resulting expressions in $\tilde{g}(k)$ we find that $\tilde{g}(k)$ equals the rhs of (8.6) plus an expression involving $\hat{q}_T(\nu_1(k))$ and $\hat{q}_T(\nu_2(k))$. However, the product of this latter expression with $\exp[ikx - i\omega(k)t]$ gives rise to an expression which is bounded and analytic in D_+. Thus its contribution vanishes.

9. The NLS on the Half-Line

We first present the nonlinear version of the direct map, ie. the nonlinear version of equations (8.7) and (8.8) with $n=2$, $c_0=-1$ and $c_1=i$.

Direct Map

$$\{q_0(x), g_0(t), g_1(t)\} \mapsto \{a(k), b(k), A(k), B(k)\}, \quad g_0(t) := q(0,t),\ g_1(t) := \partial_x q(0,t).$$

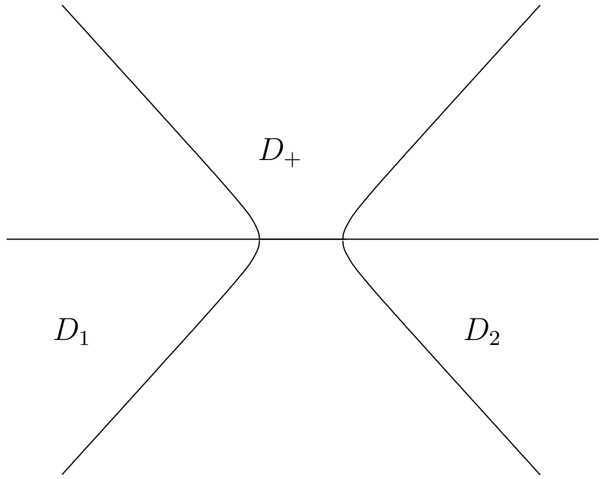

FIGURE 8.2. Domains D_+, D_1 and D_2 for (8.2)

The spectral functions $a(k)$ and $b(k)$ are defined by

(9.1) $$a(k) = \phi_2(0,k), \qquad b(k) = \phi_1(0,k), \qquad \operatorname{Im} k \geq 0,$$

where $(\phi_1(x,k), \phi_2(x,k))^T$ is the solution of the linear Volterra integral equation

(9.2) $$\begin{aligned} \phi_1(x,k) &= \int_\infty^x e^{-2ik(x-x')} q_0(x')\phi_2(x',k)\,dx', \quad \operatorname{Im} k \geq 0 \\ \phi_2(x,k) &= 1 + \int_\infty^x \bar{q}_0(x')\phi_1(x',k)\,dx', \qquad\qquad \operatorname{Im} k \geq 0. \end{aligned}$$

The spectral functions $A(k)$ and $B(k)$ are defined by

(9.3) $$A(k) = \overline{\Phi_2(T,\bar{k})}, \qquad B(k) = -e^{4ik^2T}\Phi_1(T,k), \qquad k \in \mathbb{C},$$

where $(\Phi_1(t,k), \Phi_2(t,k)^T$ is the solution of the linear Volterra integral equation

(9.4) $$\begin{aligned} \Phi_1(t,k) &= \int_0^t e^{-4ik^2(t-t')}\big[-i|g_0(t')|^2\Phi_1(t',k) + (2kg_0(t') + ig_1(t'))\Phi_2(t',k)\big]\,dt' \\ \Phi_2(t,k) &= 1 + \int_0^t \big[(2k\overline{g_0(t')} - i\,\overline{g_1(t')})\Phi_1(t',k) + i|g_0(t')|^2\Phi_2(t',k)\big]\,dt'. \end{aligned}$$

Inverse Map

$$\{a(k), e^{-4ik^2t}b(k), A(k), e^{-4ik^2t}B(k)\} \mapsto q(x,t).$$

(9.5) $$\begin{aligned} q(x,t) = -\frac{1}{\pi}\Big\{ &\int_{\partial D_2} \overline{\Gamma(\bar{k})} e^{-2ikx-4ik^2t} M_{11}^+\,dk + \int_{-\infty}^\infty \gamma(k)e^{-2ikx-4ik^2t}M_{11}dk \\ &+ \int_0^\infty |\gamma(k)|^2 M_{12}^+ dk\Big\}, \end{aligned}$$

where ∂D_2 denotes the boundary of the third quadrant of the complex k-plane and $M(x,t,k)$ satisfies the following 2×2 matrix RH problem:

(i) M is analytic in $k \in \mathbb{C} \setminus L$ where L denotes the union of the real and the imaginary axes.
(ii) $M = \operatorname{diag}(1,1) + O(1/k)$ as $k \to \infty$.
(iii) $M^- = M^+ J$, $k \in L$ (cf. Figure 9.1) where

$$J_1 = \begin{bmatrix} 1 & 0 \\ \Gamma(k) e^{2ikx+4ik^2t} & 1 \end{bmatrix}, \qquad J_2 = J_3 J_4^{-1} J_1, \qquad J_3 = \begin{bmatrix} 1 & \overline{\Gamma(k)} e^{-2ikx-4ik^2t} \\ 0 & 1 \end{bmatrix}$$

$$J_4 = \begin{bmatrix} 1 & -\gamma(k) e^{-2ikx-4ik^2t} \\ \overline{\gamma(k)} e^{i2kx+4ik^2t} & 1 - |\gamma|^2 \end{bmatrix},$$

and

$$\gamma(k) = \frac{b(k)}{\overline{a(k)}}, \quad \Gamma(k) = \frac{1}{a(k)} \left[\frac{\overline{A(\bar{k})}}{\overline{B(\bar{k})}} a(k) - b(k) \right]^{-1}.$$

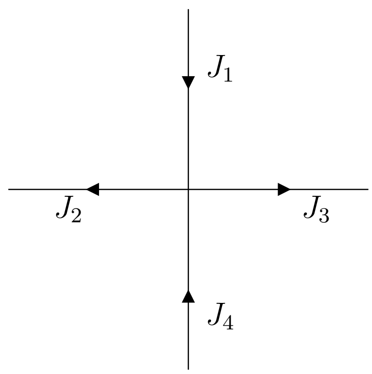

FIGURE 9.1. The contour L with jumps J_1, J_2, J_3 and J_4

The spectral functions satisfy the following nonlinear analog of (8.9).

$$a(k)B(k) - b(k)A(k) = e^{4ik^2t} c_T(k), \quad \operatorname{Im} k \geq 0, \tag{9.6}$$

where $c_T(k)$ is a function analytic for $\operatorname{Im} k \geq 0$ and of $O(1/k)$ as $k \to \infty$.

Regarding the Dirichlet problem of the NLS the following result is valid: Given the initial condition $q(x,0) = q_0(x) \in S(\mathbb{R}^+)$, define $a(k)$ and $b(k)$ by (9.1). Given $a(k)$ and $b(k)$ as well as the Dirichlet boundary condition $q(0,t) = g_0(t)$, use the global relation (9.6) and the equation (9.4) defining $(\Phi_1(t,k), \Phi_2(t,k))^T$ to determine the function $g_1(t)$ through the solution of a system of nonlinear ODEs [14]. Given $g_0(t)$ and $g_1(t)$ define $A(k)$ and $B(k)$ by (9.3) and then define $q(x,t)$ by (9.5). This function $q(x,t)$ satisfies NLS, and also $q(x,0) = q_0(x)$, $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t)$.

From the above result it follows that although the method for solving IBV problems for integrable nonlinear PDEs is conceptually similar to the method for solving linear PDEs, there exists the technical difficulty of characterizing the unknown boundary values through the solution of a system of nonlinear ODEs. However, there exist certain particular boundary conditions for which the nonlinear ODEs can be avoided and $A(k)$, $B(k)$ can be explicitly expressed in terms of $a(k)$, $b(k)$ and the given boundary condition. An example of such a boundary condition for the NLS is the Robin problem $q_x(0,t) - c\,q(0,t) = 0$, where c is a real constant [15]. In this case

$$\frac{B(k)}{A(k)} = -\frac{2k + ic\,b(-k)}{2k - ic\,a(-k)}. \tag{9.7}$$

The direct map defined by equations (9.1) and (9.3) is the nonlinearization of the corresponding direct map of the linear version of the NLS. Indeed, if the initial condition $q_0(x)$ is small, equations (9.2) imply that $\phi_1 \sim 0$ and $\phi_2 \sim 1$, thus equations (9.1) imply that $a \sim 1$ and $b(k)$ tends to the FT of $q_0(x)$. Similarly, if the boundary values $g_0(t) = q(0,t)$ and $g_1(t) = q_x(0,t)$ are small, equations (9.4) imply $\Phi_1 \sim 0$ and $\Phi_2 \sim 1$, thus equations (9.3) imply that $A \sim 1$ and that $B(k)$ tends to the time-transform of $2kg_0(t) + ig_1(t)$ (which is the spectral function associated with the linearized NLS). Similarly, the inverse map expressed by equation (9.5) is the nonlinearization of the inverse map expressed by equation (8.4). Indeed, if the spectral functions are small, the RH problem defining M implies that $M \sim \text{diag}(1,1)$, and then equation (9.5) implies that $q(x,t)$ tends to the expression defined by equation (8.4) (for the case of $\omega(k) = k^2$).

10. Conclusions

We have shown that Fourier transforms in one and two space dimensions can be nonlinearized. Nonlinear FTs can be used for the solution of the Cauchy problem of integrable nonlinear evolution PDEs in one and two spatial dimensions. Examples of nonlinear FTs in one and two spatial dimensions are presented in §3 and §5.

The derivation of nonlinear FTs is based on the spectral analysis of matrix linear eigenvalue problems. For example, the derivation of the nonlinear FTs presented in sections 3 and 5 is based on the spectral analysis of the matrix eigenvalue equations (2.7a) and (5.1). This analysis is based on formulating a RH problem and a ∂-bar problem, respectively. These two problems can also be used for the derivation of the classical FTs: The spectral analysis of the scalar equation (2.9a) and the formulation of a RH problem give rise to the FT in one dimension, while the spectral analysis of the scalar eigenvalue equation (4.1) and the formulation of a ∂-bar problem gives rise to the FT in two dimensions (see §2 and §4 respectively).

Integrable nonlinear PDEs have the distinctive feature that they can be written as the compatibility condition of two eigenvalue equations. The solution of their Cauchy problem is based on using one of these eigenvalue equations (the t-independent part) to derive a nonlinear FT. The other eigenvalue equation (the t-dependent part) determines the time evolution of the nonlinear Fourier data. Linear PDEs can also be written as the compatibility condition of two eigenvalue equations. For example, equation (2.1a) is the compatibility condition of the eigenvalue equations (2.4).

This provides the first unification: Both linear and integrable nonlinear evolution PDEs in one and two spatial dimensions can be written as the compatibility condition of two linear eigenvalue equations, called a Lax pair. Furthermore, the Cauchy problem of these equations can be analyzed by performing the spectral analysis of the t-independent part of the Lax pair.

For linear equations, this analysis yields the FT in one and two spatial dimensions, while for nonlinear equations it yields nonlinear generalizations of the FTs.

The formulation of linear PDEs in terms of Lax pairs, appears to express a deeper kind of separability than the classical separation of variables. This claim is based on the following considerations: (i) It is the Lax pair formulation that generalizes to nonlinear integrable PDEs. The Lax pairs of nonlinear PDEs are non-Abelian versions of the corresponding Lax pairs of linear PDEs. (ii) The Lax pair formulation of linear PDEs provides a spectral approach to the solution of boundary value problems for linear PDEs. It provides the effective implementation, as well as the spectral meaning, of the Ehrenpreis-Palamodov type integral representation. For example, using this approach it is shown in §8, that for linear evolution PDEs on the half-line, the abstract Ehrenpreis-Palamodov type integral representation (8.3), takes the concrete form given by equation (8.4). In comparison with initial value problems the analysis of IBV problems or linear PDEs uses two novel ideas: (a) In order to determine the direct and inverse maps, one uses the *simultaneous* spectral analysis of both parts of the Lax pair, as opposed of the case of the Cauchy problem where one performs the spectral analysis of only the t-independent part of the Lax pair. (b) One eliminates the unknown boundary values by using the fact that the spectral functions satisfy a simple algebraic equation, called the *global relation*.

Our approach to IBV problems of integrable nonlinear evolution PDEs is conceptually similar with that of linear PDEs: (a) The simultaneous spectral analysis of the Lax pair yields a direct and an inverse map. (b) The spectral functions $a(k)$ and $b(k)$ are defined in terms of the given function $q_0(x)$. However, for the Dirichlet problem, the spectral functions $A(k)$ and $B(k)$ depend on the known function $g_0(t)$ and on the *unknown* function $g_1(t)$. It is again possible to eliminate $g_1(t)$ using the global relation. But for nonlinear problems this step is in general *nonlinear*. Indeed, although for some particular boundary conditions it is possible to express A and B *explicitly* in terms of the known spectral functions and the known boundary conditions (see equation (9.7)), for general boundary conditions, one must solve a system of two *nonlinear* ODEs in order to compute $g_1(t)$.

This provides a further unification: IBV problems for both linear and integrable nonlinear PDEs can be analyzed by (a) performing the simultaneous spectral analysis of the Lax pair, and (b) using the global relation to eliminate the unknown boundary values. For linear PDEs this yields the effective implementation of the Ehrenpreis-Palamodov integral representation. For nonlinear equations, it provides the extension of this fundamental representation to integrable nonlinear PDEs.

For linear evolution PDEs in our spatial dimension, the solution of both the Cauchy and IBV problems can be expressed in terms of integrals involving explicit exponential (x, t) dependence. This makes it possible to study effectively the asymptotic properties of the solution. For example, one can use the steepest descent method to study the long time behavior of the solution. For nonlinear evolution PDEs, in the case of the Cauchy problem the solution can be expressed in terms of an integral involving explicit exponential (x, t) dependence as well as a function M (see equation (3.3)). Although M has a complicated (x, t) dependence it is defined in terms of a RH problem which has a jump matrix with explicit exponential (x, t) dependence. This allows one to obtain rigorous asymptotic results using the powerful and elegant nonlinearization of the steepest descent method of Deift and Zhou [16]. In this way, both the long time asymptotics and the zero-dispersive limit [17] can be computed. An advantage of the new method for solving IBV problems is that it provides a similar representation for $q(x,t)$, where the associated integrals and RH problems, although now more complicated, they still have explicit exponential (x,t) dependence. This again allows one to compute effectively the asymptotic behaviour of the solution using the Deift-Zhou method.

In this article we have concentrated on IBV problems on the half-line. Similar results can be obtained for both linear and integrable nonlinear PDEs of the finite interval [19].

It was emphasized by I.M. Gel'fand and one of the authors [8] that the RH and ∂-bar formalism presented in sections 2 and 4, provides a new approach to inverting linear integral transforms. This idea was used in [9] where it was shown that the spectral analysis of equation (6.1) (or equivalently of equation (7.2)) and the RH and ∂-bar formalisms yield the Radon transform pair. It is remarkable that the analysis of a slight generalization of this equation, namely of equation (7.7), gives rise to the attenuated Radon transform. This transform provides the mathematical basis of a new powerful technique for the functional imaging of the brain called SPECT, see section 6 and 7.

The methodology for solving IBV problems for linear evolution PDEs can be also applied to linear elliptic PDEs. The analogues of the direct and inverse maps presented in §8, for the Laplace and Helmholtz equations in a convex polygon are given in [20]. However, elliptic PDEs present a new difficulty: For a general polygon and for general boundary conditions the global relation *cannot* be solved in closed form, but it yields an *auxiliary* RH problem. For simple polygons, this RH problem has a jump on the infinite line, thus it is equivalent to a Wiener-Hopf problem. This explains the central role played by the Wiener-Hopf technique in many earlier work.

This article is dedicated to Peter Lax, whose decisive discovery of Lax pairs is the basis of *all* developments presented here.

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