Steady Inviscid Hydrodynamics with Constant Speed

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Abstract

A long–standing problem in hydrodynamics posed by Gilbarg and investigated by several authors, including R. Prim, L.N. Howard, R.H. Wasserman, A.W. Marris, C. Rogers and W.K. Schief, is the problem of determining all steady inviscid flows which satisfy the constraint that the speed is constant along streamlines. Here it is shown that the general solution of this problem can be expressed either in terms of elliptic functions or in terms of the solutions of a second order ODE introduced by Chazy in 1911. This ODE can be mapped to the classical fourth Painlevé equation.

1 Introduction

Steady inviscid hydrodynamics is characterized by the equations

$$\mathrm{div}\mathbf{q} = 0, \quad \rho_0(\mathbf{q}\cdot\nabla)\mathbf{q} + \nabla p = 0, \tag{1.1}$$

where $\mathbf{q}$ is the fluid velocity and p, ρ_0 are the pressure and constant density, respectively. A long–standing problem in hydrodynamics posed by Gilbarg [1] and subsequently investigated by several authors including Prim [2], Howard [3], Wasserman [4], Marris [5], Rogers and Schief [6]–[10], is the problem of solving equations (1.1) under the constraint that the speed $|\mathbf{q}|$ is constant along streamlines.

Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ denote the orthogonal basis of the tangent $\mathbf{t}$, principal normal $\mathbf{n}$, and binormal $\mathbf{b}$ vectors to the streamlines. It is known [11]–[13] that the associated directional derivatives of these vectors are determined in terms of eight geometric quantities which satisfy a set of nine equations. These equations are reviewed in Section 2. Included among these geometric quantities are the *torsion* τ and the *curvature* κ of the streamlines, as well as the so called *abnormality* Ω defined by $\Omega = \mathbf{t}\cdot\mathrm{curl}\mathbf{t}$.

We will first show that equations (1.1) supplemented with the constraint of constant speed, i.e.

$$\mathbf{q} = q\mathbf{t}, \quad q = |\mathbf{q}|, \quad q = \text{constant}, \tag{1.2}$$

imply

$$\mathbf{t}\cdot\nabla p = 0, \quad \mathbf{b}\cdot\nabla p = 0,$$

thus the trajectories of the $\mathbf{t}$ and $\mathbf{b}$ vector fields lie on the surface p=constant. We will then show that equations (1.1) and (1.2) are *equivalent* to the following equations valid on surfaces of constant pressure: The torsion τ and the function ρ which is simply related to the curvature κ by $\kappa^2 = \rho$, satisfy the equations

$$\begin{aligned} \frac{\partial \rho}{\partial b} &= -2\frac{\partial}{\partial s}(\rho\tau), \\ \frac{\partial \tau}{\partial b} &= \frac{\partial}{\partial s}\left(\frac{1}{2\rho}\frac{\partial^2 \rho}{\partial s^2} - \frac{1}{4\rho^2}\left(\frac{\partial \rho}{\partial s}\right)^2 + \frac{\rho}{2} - \tau^2\right), \end{aligned} \tag{1.3}$$

as well as the *constraint*

$$\tau = 2f(b) + 4g(b)s + \frac{X}{\rho}, \tag{1.4}$$

where X is defined by

$$X^2 \doteqdot \left[2f^2(b) + h(b) + \left(\dot{f}(b) + 8f(b)g(b)\right)s + \left(8g(b)^2 + \dot{g}(b)\right)s^2\right]\rho^2 - \frac{\rho^3}{4} - \frac{1}{4}\left(\frac{\partial \rho}{\partial s}\right)^2; \tag{1.5}$$

throughout this paper dot denotes ordinary differentiation, and $f(b)$, $h(b)$, $g(b)$ must be determined by the requirement that the constraint (1.4) is compatible with equations (1.3).

It will also be shown that the abnormality Ω is given by

$$\Omega = 4f(b) + 8g(b)s. \tag{1.6}$$

Finally we will find the general set of solutions satisfying equations (1.3) and (1.4):
(i)

$$f(b) = \frac{1}{4}\frac{b_1a_2 - b_2a_1}{a_1b + a_2}, \quad g(b) = \frac{1}{8}\frac{a_1}{a_1b + a_2}, \quad h(b) = \frac{c_1}{a_1b + a_2} - \frac{1}{8}\left(\frac{b_1a_2 - b_2a_1}{a_1b + a_2}\right)^2, \tag{1.7}$$

where a_1, a_2, b_1, b_2, c_1 are constants and $a_1 > 0$.

If $c_1 \neq 0$, define the variables $\hat{b}$, $\hat{s}$, ζ, by

$$\hat{b} = b + \frac{a_2}{a_1}, \quad \hat{s} = s + b_1\frac{a_2}{a_1} - b_2, \quad \zeta = \frac{\hat{s}}{\sqrt{\hat{b}}} + \left(\frac{c_2}{c_1} - b_1\right)\sqrt{\hat{b}}. \tag{1.8}$$

The abnormality Ω is given by

$$\Omega = \frac{\hat{s}}{\hat{b}}, \tag{1.9}$$

while the torsion τ and the curvature κ, $\kappa^2 = \rho$, are given by the expressions

$$\rho = \frac{w(\zeta)}{\hat{b}}, \quad \tau = \frac{\hat{s}}{2\hat{b}} + \frac{1}{4\sqrt{\hat{b}}}\left[-\zeta + \frac{\int^{\zeta} w(\zeta')\mathrm{d}\zeta'}{w(\zeta)}\right], \tag{1.10}$$

where $w(\zeta)$ satisfies the ODE

$$w_{\zeta}{}^2 + w^3 - \frac{4c_1}{a_1}w^2 + \frac{1}{4}\left(\zeta w - \int^{\zeta} w(\zeta')\mathrm{d}\zeta'\right)^2 = 0. \tag{1.11}$$

If $c_1 = 0$, the constant c_2/c_1 in the definition of ζ (equation $(1.8)_3$) is replaced by the arbitrary constant c.

For the functions f, g, h defined by equations (1.7), a particular solution for ρ, τ, Ω is

$$\rho = \frac{c}{a_1 b + a_2}, \quad \tau = \frac{a_1 s + b_1 a_2 - b_2 a_1}{2(a_1 b + a_2)}, \quad \Omega = \frac{a_1 s + b_1 a_2 - b_2 a_1}{a_1 b + a_2}, \tag{1.12}$$

where c is a constant.

The ODE (1.11) was studied by Chazy in 1911 [26] and is related to the classical fourth Painlevé equation [27], [28].

(ii)

$$f = \frac{b_1}{4}, \quad g = 0, \quad h = c_1 - \frac{{b_1}^2}{8}, \tag{1.13}$$

where b_1, c_1 are constants. The torsion τ and the curvature κ, $\kappa^2 = \rho$, are given by

$$\rho = w(\zeta), \quad \tau = \frac{1}{2}(b_1 - 2c_3) + \frac{c_2}{w(\zeta)}, \tag{1.14}$$

where c_2 and c_3 are constants and $w(\zeta)$ satisfies the ODE

$${w_\zeta}^2 + {w_\zeta}^3 - 4c_1 w^3 + 4(c_2 - c_3 w)^2 = 0, \quad \zeta = s + (2c_3 - b_1)b - b_2. \tag{1.15}$$

In this case the abnormality is constant,

$$\Omega = b_1. \tag{1.16}$$

For the functions f, g, h defined by equations (1.13), a particular solution for ρ, τ, Ω is

$$\rho = \frac{1}{a_2 \varphi^2(y)}, \quad \tau = \frac{b_1}{2}, \quad \Omega = b_1, \tag{1.17}$$

where

$$\varphi(y) = A_1 e^{a_2\sqrt{c_1}y} + A_2 e^{-a_2\sqrt{c_1}y}, \quad A_1 A_2 = \frac{1}{16 a_1 c_1}, \quad y = \frac{s - b_1 b - b_2}{a_2}. \tag{1.18}$$

Historical Remarks

Hamel studied in [14] *irrotational* constant speed flows; in this case curl$\mathbf{q} = 0$, $\Omega = 0$, and there exist minimal surfaces [15] which are orthogonal to streamlines. Hamel employed the classical Weierstrass [16] representation of minimal surfaces to show that the relevant streamlines can only be straight lines or helices mounted on concentric circular cylinders. In [17] Marris presented an alternative demonstration of Hamel's theorem (using the geometric formulation reviewed in Section 2) showing that $\Omega = 0$ iff $\kappa = 0$ or $\kappa_s = 0$ or $\tau_s = 0$. In the context of equations (1.1) the Gilbarg problem [1] is a generalization of Hamel's problem and may be formulated as follows: Is a flow uniquely determined by its streamlines? In [2] Prim showed that the answer is affirmative unless the flow has constant speed along each individual streamline. Gilbarg [1] and Prim [2] solved the Gilbarg problem completely for planar and axisymmetric flows respectively, where the streamlines are necessarily concentric circles or straight lines. Wasserman [4] reduced the rotationally symmetric case to a pair of ODEs which has been recently solved in terms of elliptic integrals [8], [9]. Necessary constraints on the geometry of admissible constant speed flows are given in [5]. Precisely such flows was the subject of the PhD thesis of Howard

[3]. A remarkable connection between equations (1.1), (1.2) and integrable systems was established in [6] and [7] where it was shown that these equations are equivalent to the Heisenberg spin equation

$$\frac{\partial \mathbf{t}}{\partial b} = \mathbf{t} \wedge \frac{\partial^2 \mathbf{t}}{\partial s^2},$$

subject to the geometrical constrain $\mathrm{div}\mathbf{t} = 0$ (this equation is a direct consequence of (1.1) and (1.2)). However, the question of finding all solutions of Heisenberg equation which are compatible with this constraint remained open. We note that the Heisenberg spin equation is equivalent to the celebrated nonlinear Schrödinger equation which, in turns encodes the classical Da Rios system [18] which was originally derived in 1906 in connection with the spatial evolution of an isolated vortex filament in an unbounded inviscid liquid. Equations (1.3) constitute precisely the Da Rios system.

The concept of constant abnormality played an important role in the advances made by Marris [19], [20] in his investigation of Ericksen's [21] problem of determining all deformations that can be sustained by a perfectly elastic, isotropic body subject only to surface tractions [22], see the review in [23]. Other connections on similar problems in elasticity and in hydrodynamics have been discussed in [24].

2 Geometric Preliminaries

Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ denote the orthogonal basis of the tangent $\mathbf{t}$, principal normal $\mathbf{n}$, and binormal vector $\mathbf{b}$ to the streamlines. Let

$$\frac{\delta}{\delta s}, \quad \frac{\delta}{\delta n}, \quad \frac{\delta}{\delta b}$$

denote the directional derivatives along these three vectors, i.e.

$$\frac{\delta}{\delta s} = \mathbf{t} \cdot \nabla, \quad \frac{\delta}{\delta n} = \mathbf{n} \cdot \nabla, \quad \frac{\delta}{\delta b} = \mathbf{b} \cdot \nabla. \tag{2.1}$$

Following [12], we introduce the notations

$$\theta_{ns} = \mathbf{n} \cdot \frac{\delta \mathbf{t}}{\delta n}, \quad \theta_{bs} = \mathbf{b} \cdot \frac{\delta \mathbf{t}}{\delta b}$$

and

$$\Omega = \mathbf{t} \cdot \mathrm{curl}\mathbf{t}, \quad \Omega_n = \mathbf{n} \cdot \mathrm{curl}\mathbf{n}, \quad \Omega_b = \mathbf{b} \cdot \mathrm{curl}\mathbf{b}.$$

It can be shown (see [6] for details) that[1]

$$\begin{aligned} &\mathrm{div}\mathbf{t} = \theta_{ns} + \theta_{bs}, \\ &\Omega_b = \Omega - \Omega_n - 2\tau, \\ &\mathrm{curl}\mathbf{t} = \Omega \mathbf{t} + \kappa \mathbf{b}. \end{aligned} \tag{2.2}$$

[1] Equation $(2.2)_3$ was first obtained by Masotti [11] and was later rediscovered by Emde [13] and Bjørgum [12].

The identity curlgrad$\phi = \mathbf{0}$, yields the following commutator relations

$$\begin{aligned}
\frac{\delta^2}{\delta n \delta b} - \frac{\delta^2}{\delta b \delta n} &= -\Omega \frac{\delta}{\delta s} + \mathrm{div}\mathbf{b} \frac{\delta}{\delta n} - (\kappa + \mathrm{div}\mathbf{n}) \frac{\delta}{\delta b}, \\
\frac{\delta^2}{\delta b \delta s} - \frac{\delta^2}{\delta s \delta b} &= -\Omega_n \frac{\delta}{\delta n} + \theta_{bs} \frac{\delta}{\delta b}, \\
\frac{\delta^2}{\delta s \delta n} - \frac{\delta^2}{\delta n \delta s} &= -\kappa \frac{\delta}{\delta s} - \theta_{ns} \frac{\delta}{\delta n} - \Omega_b \frac{\delta}{\delta b}.
\end{aligned} \tag{2.3}$$

It can be shown [24] that the directional derivatives of the orthogonal triad $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ are given by

$$\begin{aligned}
\frac{\delta}{\delta s} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} &= \begin{pmatrix} 0 & \kappa & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \\
\frac{\delta}{\delta n} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} &= \begin{pmatrix} 0 & \theta_{ns} & \Omega_b + \tau \\ -\theta_{ns} & 0 & -\mathrm{div}\mathbf{b} \\ -(\Omega_b + \tau) & \mathrm{div}\mathbf{b} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \\
\frac{\delta}{\delta b} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} &= \begin{pmatrix} 0 & -(\Omega_n + \tau) & \theta_{bs} \\ \Omega_n + \tau & 0 & \kappa + \mathrm{div}\mathbf{n} \\ -\theta_{bs} & -(\kappa + \mathrm{div}\mathbf{n}) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}.
\end{aligned} \tag{2.4}$$

Using the commutator relations (2.3) it can be shown that the compatibility of the above system yields the following set of nine conditions on the eight geometric quantities κ, τ, Ω, Ω_n, div$\mathbf{n}$, div$\mathbf{b}$, θ_{ns} and θ_{bs} [24]:

$$\begin{aligned}
&\frac{\delta \theta_{ns}}{\delta b} + \frac{\delta}{\delta n}(\tau + \Omega_n) = (\kappa + \mathrm{div}\mathbf{n})(\Omega - 2\Omega_n - 2\tau) + \mathrm{div}\mathbf{b}(\theta_{bs} - \theta_{ns}) + \Omega\kappa \\
&\frac{\delta}{\delta b}(\tau + \Omega_n - \Omega) + \frac{\delta \theta_{bs}}{\delta n} = (\kappa + \mathrm{div}\mathbf{n})(\theta_{ns} - \theta_{bs}) + \mathrm{div}\mathbf{b}(\Omega - 2\Omega_n - 2\tau) \\
&\frac{\delta}{\delta b}\mathrm{div}\mathbf{b} + \frac{\delta}{\delta n}(\kappa + \mathrm{div}\mathbf{n}) = (\tau + \Omega_n)(\tau + \Omega_n - \Omega) - \theta_{ns}\theta_{bs} - \tau\Omega - (\mathrm{div}\mathbf{b})^2 - (\kappa + \mathrm{div}\mathbf{n})^2 \\
&\frac{\delta}{\delta s}(\tau + \Omega_n) + \frac{\delta \kappa}{\delta b} = -\Omega_n \theta_{ns} - (2\tau + \Omega_n)\theta_{bs} \\
&\frac{\delta \theta_{bs}}{\delta s} = -{\theta_{bs}}^2 + \kappa(\kappa + \mathrm{div}\mathbf{n}) - \Omega_n(\tau + \Omega_n - \Omega) + \tau(\tau + \Omega_n) \\
&\frac{\delta}{\delta s}(\kappa + \mathrm{div}\mathbf{n}) - \frac{\delta \tau}{\delta b} = -\Omega_n \mathrm{div}\mathbf{b} - \theta_{bs}(2\kappa + \mathrm{div}\mathbf{n}) \\
&\frac{\delta \kappa}{\delta n} - \frac{\delta \theta_{ns}}{\delta s} = \kappa^2 + {\theta_{ns}}^2 + (\tau + \Omega_n)(3\tau + \Omega_n) - \Omega(2\tau + \Omega_n) \\
&\frac{\delta}{\delta s}(\tau + \Omega_n - \Omega) = -\theta_{ns}(\Omega_n - \Omega) + \kappa \mathrm{div}\mathbf{b} + \theta_{bs}(-2\tau - \Omega_n + \Omega) \\
&\frac{\delta \tau}{\delta n} + \frac{\delta}{\delta s}\mathrm{div}\mathbf{b} = -\kappa(\Omega_n - \Omega) - \theta_{ns}\mathrm{div}\mathbf{b} + (\kappa + \mathrm{div}\mathbf{n})(-2\tau - \Omega_n + \Omega).
\end{aligned} \tag{2.5}$$

3 Intrinsic Decomposition of the Hydrodynamic System

Equations (2.1) imply

$$\nabla = \mathbf{t}\frac{\delta}{\delta s} + \mathbf{n}\frac{\delta}{\delta n} + \mathbf{b}\frac{\delta}{\delta b}.$$

This equation and the definition $\mathbf{q} = q\mathbf{t}$ imply

$$\mathrm{div}\mathbf{q} = \frac{\delta q}{\delta s} + q\mathrm{div}\mathbf{t}, \quad \mathbf{q}\cdot\nabla = q\frac{\delta}{\delta s}.$$

Using these expressions and assuming $\frac{\delta q}{\delta s} = 0$, the hydrodynamic equations (1.1) become

$$\mathrm{div}\mathbf{t} = 0, \quad \rho_0 q^2\frac{\delta\mathbf{t}}{\delta s} + \nabla p = 0.$$

Replacing $\mathrm{div}\mathbf{t}$ by the rhs of equation $(2.2)_1$, and $\frac{\delta\mathbf{t}}{\delta s}$ by $\kappa\mathbf{n}$ (see $(2.4)_1$) we find

$$\theta_{ns} = -\theta_{bs}, \quad \nabla p = -\rho_0 q^2\kappa\mathbf{n}. \tag{3.1}$$

In what follows we scrutinize the consequences of these two equations.

3.1 Constant Pressure Surfaces

Equation $(3.1)_2$ implies that

$$\mathbf{t}\cdot\nabla p = 0, \quad \mathbf{b}\cdot\nabla p = 0.$$

Thus the trajectories of the $\mathbf{t}$ and $\mathbf{b}$ vector fields lie on the surface $p =$ constant. Let the associated s–lines and b–lines be taken as parametric curves on this surface. The first fundamental form can be taken as

$$I = \mathrm{d}s^2 + h^2\mathrm{d}b^2.$$

Then

$$\nabla_{p=\mathrm{const}} = \mathbf{t}\frac{\delta}{\delta s} + \mathbf{b}\frac{\delta}{\delta b} = \mathbf{t}\frac{\partial}{\partial s} + \frac{\mathbf{b}}{h}\frac{\partial}{\partial b}.$$

The Gauss–Weingarten equations associated with I are equations (2.4), with $\frac{\delta}{\delta s} = \frac{\partial}{\partial s}$, as well as the equations

$$\frac{\partial}{\partial b}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}0 & -\tau h & \frac{\partial h}{\partial s}\\ \tau h & 0 & V\\ -\frac{\partial h}{\partial s} & -V & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}, \tag{3.2}$$

where h and V define the second fundamental form of this surface [25].

The compatibility condition of equations $(2.4)_1$ with $\frac{\delta}{\delta s} = \frac{\partial}{\partial s}$ and equations (3.2), yield the Gauss–Mainardi–Codazzi equations for the constant pressure surfaces,

$$\frac{\partial\kappa}{\partial b} = -\frac{1}{h}\frac{\partial}{\partial s}(h^2\tau), \quad \frac{\partial\tau}{\partial b} = \frac{\partial V}{\partial s} + \kappa\frac{\partial h}{\partial s}, \quad \frac{\partial^2 h}{\partial s^2} = \kappa V + \tau^2 h. \tag{3.3}$$

Comparing equations (3.2) with equations $(2.4)_3$ where $\frac{\delta}{\delta b} = \frac{1}{h}\frac{\partial}{\partial b}$, we deduce that

$$V = h(\kappa + \mathrm{div}\mathbf{n}), \quad \frac{1}{h}\frac{\partial h}{\partial s} = \theta_{bs}, \quad \Omega_n = 0. \tag{3.4}$$

3.2 Further Consequences of Equations (3.1)

Equation $(3.1)_2$ is equivalent to

$$\frac{\partial p}{\partial s}=0, \quad \frac{\partial p}{\partial b}=0, \quad \frac{\delta p}{\delta n}=-\rho_0 q^2\kappa. \tag{3.5}$$

The commutator relations (2.3) imply that equations (3.5) are compatible iff

$$\theta_{ns}=-\frac{1}{\kappa}\frac{\partial \kappa}{\partial s}, \quad \Omega_n=0, \quad \frac{\partial q}{\partial b}=-\frac{h}{2\kappa}q\mathrm{div}(\kappa\mathbf{b}). \tag{3.6}$$

Equations $(3.1)_1$, $(3.4)_2$, $(3.6)_1$, yield

$$\frac{1}{h}\frac{\partial h}{\partial s}=\frac{1}{\kappa}\frac{\partial \kappa}{\partial s}, \quad \text{i.e.} \quad h=F\kappa, \quad \frac{\partial F}{\partial s}=0.$$

Using $\frac{h}{\kappa}=F$ in $(3.6)_3$ it follows that this equation is compatible with $\frac{\partial q}{\partial s}=0$ iff $\frac{\partial(\mathrm{div}\kappa\mathbf{b})}{\partial s}=0$. Using equation $(2.2)_3$ to replace $\kappa\mathbf{b}$ we can compute the divergence of this expression,

$$\mathrm{div}\kappa\mathbf{b}=\mathrm{div}(\mathrm{curl}\mathbf{t}-\Omega\mathbf{t})=-\frac{\partial\Omega}{\partial s}. \tag{3.7}$$

Hence $\frac{\partial q}{\partial s}=0$ is compatible with $(3.6)_3$ iff $\frac{\partial^2\Omega}{\partial s^2}=0$.
In summary:
1. p satisfies the compatible set of equations

$$\frac{\partial p}{\partial s}=\frac{\partial p}{\partial b}=0, \quad \frac{\delta p}{\delta n}=-\rho_0 q^2\kappa,$$

while q satisfies the compatible set of equations

$$\frac{\partial q}{\partial s}=0, \quad \frac{\partial q}{\partial b}=\frac{1}{2}F\frac{\partial\Omega}{\partial s},$$

where F is an arbitrary function such that $\frac{\partial F}{\partial s}=0$.
2. The eight geometric quantities κ, τ, Ω, Ω_n, θ_{nb}, θ_{bs}, $\mathrm{div}\mathbf{b}$, $\mathrm{div}\mathbf{n}$, satisfy the following set of equations:
(i) $(3.1)_1$, $(3.6)_1$, $(3.6)_2$, $\frac{\partial^2\Omega}{\partial s^2}=0$.
(ii) Equations (3.3) with $V=h(\kappa+\mathrm{div}\mathbf{n})$, $h=F\kappa$, i.e.

$$\begin{gathered}\frac{1}{F}\frac{\partial \kappa}{\partial b}=-\frac{1}{\kappa}\frac{\partial}{\partial s}(\kappa^2\tau),\\ \frac{1}{F}\frac{\partial \tau}{\partial b}=\frac{\partial}{\partial s}(\kappa(\kappa+\mathrm{div}\mathbf{n}))+\kappa\frac{\partial\kappa}{\partial s}, \quad \frac{\partial^2\kappa}{\partial s^2}=\kappa^2(\kappa+\mathrm{div}\mathbf{n})+\tau^2\kappa.\end{gathered} \tag{3.8}$$

(iii) Equations (2.5).

In what follows we analyze the set of equations satisfied by the eight geometrical quantities. By redefining b,

$$F\mathrm{d}b\to\mathrm{d}b,$$

we assume without loss of generality that $F=1$.

Proposition 3.1. Let $\mathbf{t}$ denote the unit vector tangent to the streamlines of a steady ideal hydrodynamic flow for which the speed of the fluid flow is constant along streamlines. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the associated orthogonal basis described in Section 2. The directional derivatives of this basis are determined by the eight geometrical quantities (see equations (2.4) with Ω_b given by equation $(2.2)_2$)

$$\kappa, \tau, \Omega, \Omega_n, \theta_{ns}, \theta_{bs}, \mathrm{div}\mathbf{n}, \mathrm{div}\mathbf{b}.$$

These quantities satisfy the following equations:

$$\theta_{ns} = -\theta_{bs} = -\frac{1}{\kappa}\frac{\partial \kappa}{\partial s}, \quad \Omega_n = 0, \tag{3.9}$$

$$\mathrm{div}\mathbf{n} = \frac{1}{\kappa^2}\frac{\partial^2 \kappa}{\partial s^2} - \frac{\tau^2}{\kappa} - \kappa, \quad \mathrm{div}\mathbf{b} = -\frac{1}{\kappa}\frac{\partial \Omega}{\partial s} + \frac{1}{\kappa}\frac{\partial \tau}{\partial s} + \frac{2\tau}{\kappa^2}\frac{\partial \kappa}{\partial s}, \tag{3.10}$$

$$\frac{\partial \kappa}{\partial b} = -\kappa\frac{\partial \tau}{\partial s} - 2\tau\frac{\partial \kappa}{\partial s}, \quad \frac{\partial \tau}{\partial b} = \frac{\partial}{\partial s}\left(\frac{1}{\kappa}\frac{\partial^2 \kappa}{\partial s^2} + \frac{1}{2}\kappa^2 - \tau^2\right), \tag{3.11}$$

$$\frac{\partial^2 \Omega}{\partial s^2} = 0, \quad \frac{\partial \Omega}{\partial b} = \frac{\partial}{\partial s}\left(4\tau^2 + 4\left(\frac{1}{\kappa}\frac{\partial \kappa}{\partial s}\right)^2 + \kappa^2 - 4\tau\Omega + \frac{1}{2}\Omega^2\right), \tag{3.12}$$

$$\frac{\delta \kappa}{\delta n} = -\frac{1}{\kappa}\frac{\partial^2 \kappa}{\partial s^2} + 2\left(\frac{1}{\kappa}\frac{\partial \kappa}{\partial s}\right)^2 + \kappa^2 + 3\tau^2 - 2\tau\Omega, \tag{3.13}$$

$$\frac{\delta \tau}{\delta n} = \left(\kappa - \frac{\tau^2}{\kappa} + \frac{1}{\kappa^2}\frac{\partial^2 \kappa}{\partial s^2}\right)\Omega + \frac{2\tau^3}{\kappa} + \frac{6\tau}{\kappa^3}\left(\frac{\partial \kappa}{\partial s}\right)^2 - \frac{1}{\kappa}\frac{\partial^2 \tau}{\partial s^2} - \frac{4\tau}{\kappa^2}\frac{\partial^2 \kappa}{\partial s^2} - \frac{2}{\kappa^2}\frac{\partial \kappa}{\partial s}\frac{\partial \Omega}{\partial s}. \tag{3.14}$$

Proof. Throughout the derivation we use

$$\frac{\delta}{\delta s} = \frac{\partial}{\partial s}, \quad \frac{\delta}{\delta b} = \frac{1}{\kappa}\frac{\partial}{\partial b}.$$

Equations (3.9) are equations $(3.1)_1$, $(3.6)_1$, $(3.6)_2$. Equation $(3.10)_1$ is equation $(3.8)_3$, equation $(3.11)_1$ is equation $(3.8)_1$ and equation $(3.11)_2$ is equation $(3.8)_2$ where $\kappa + \mathrm{div}\mathbf{n}$ has been replaced by equation $(3.10)_1$.

We will show that the nine equations (2.5) yield only the three additional equations $(3.12)_2$, (3.13), (3.14): Using (3.9), equation $(2.5)_4$ becomes equation $(3.11)_1$. Using (3.9), equations $(2.5)_5$ and $(2.5)_6$ become equations $(3.8)_3$ and $(3.8)_2$ respectively. The identity

$$\mathrm{div}\kappa\mathbf{b} = \kappa\mathrm{div}\mathbf{b} + \frac{\delta \kappa}{\delta b}$$

together with equation (3.7) imply

$$\kappa\mathrm{div}\mathbf{b} = -\frac{\partial \Omega}{\partial s} - \frac{1}{\kappa}\frac{\partial \kappa}{\partial b}.$$

Replacing in this equation $\frac{\partial \kappa}{\partial b}$ by $(3.11)_1$, equation (3.10) follows. Subtracting equations $(2.5)_1$ and $(2.5)_4$ and using $\Omega_n = 0$ we find

$$\frac{\delta}{\delta b}\theta_{ns} - \frac{\delta}{\delta s}\mathrm{div}\mathbf{b} = \theta_{bs}\mathrm{div}\mathbf{b}.$$

Using the expression for θ_{ns}, div$\mathbf{b}$, θ_{bs}, this equation becomes an identity. Similarly, using these expressions as well as equation $(3.11)_1$, equations $(2.5)_1$ and $(2.5)_3$ become equations (3.14) and (3.13). Finally, using the commutator relation (see $(2.3)_3$)

$$\frac{\delta}{\delta n}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\delta}{\delta n} + \kappa\frac{\partial}{\partial s} - \frac{1}{\kappa}\frac{\partial \kappa}{\partial s}\frac{\delta}{\delta n} + \frac{\Omega - 2\tau}{\kappa}\frac{\partial}{\partial b} \tag{3.15}$$

equation $(2.5)_2$ becomes equation $(3.12)_2$.

We note that, using the commutator relation (3.15), as well as the commutator relation $(3.1)_1$, it can be verified that equations $(3.11)_1$ and (3.13), as well as equations $(3.11)_2$ and (3.14) are compatible. **QED**

Proposition 3.2. Let $\mathbf{t}$ denote the unit vector tangent to the streamlines of steady ideal hydrodynamic flow for which the speed of the fluid flow is constant along streamlines. Then the abnormality $\Omega = \mathbf{t} \cdot \mathrm{curl}\mathbf{t}$ is given by (1.6) where the functions $f(b)$, $g(b)$ are restricted by the condition that the torsion τ and function ρ, which is related to the commutator κ by $\kappa^2 = \rho$, satisfy equations (1.3) as well as the constraint (1.4).

Proof. According to Proposition 3.1, the curvature κ, the torsion τ, and the abnormality Ω satisfy equations (3.11) and (3.12). Letting $\kappa^2 = \rho$, equations (3.11) become equations (1.3). The integration of $(3.12)_1$ yields (1.6), where the factors 4 and 8 have been included for convenience. We now show that equation $(3.12)_2$ is equivalent to equation (1.4) where X is defined by equation (1.5): The bracket in the rhs of equation $(3.12)_2$ can be written as

$$4\left(\tau - \frac{\Omega}{2}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial s}\right)^2 + \rho - \frac{\Omega^2}{2};$$

integrating the lhs of equation $(3.12)_1$ with respect to s and differentiating with respect to b we find

$$4\dot{f}(b)s + 4\dot{g}(b)s^2 + \dot{h}(b),$$

where $h(s)$ is the integration constant. Equating the above two expressions and replacing Ω by equation (1.6) we find (1.4). **QED**

4 A First Integral

As it was noted in the introduction the functions $f(b)$ and $g(b)$ appearing in equation (1.4) must be chosen by the requirement that equation (1.4), which defines a relation between $\{s, b, \tau, \rho, \rho_s\}$, is compatible with equations (1.3). If one attempts to analyze this compatibility directly, the relevant computations are prohibitively complicated. In what follows we introduce a change of coordinates, which will allow as to make progress. We use subscripts to denote partial differentiation.

Let the functions $f(b)$, $g(b)$, $h(b)$, be expressed in terms of the new functions $F(b)$, $G(b)$, $H(b)$, by the equations

$$4f = \dot{F} - \frac{\dot{G}}{G}F, \quad 8g = \frac{\dot{G}}{G}, \quad G \neq 0, \tag{4.1}$$

$$h = H - F\frac{\ddot{F}}{4} + \frac{F^2}{8}\left(\frac{\ddot{G}}{G} - \frac{\dot{G}^2}{G^2}\right) - \frac{\dot{F}^2}{8} + \frac{F\dot{F}\dot{G}}{4G}.$$

Let the independent variables b and s be defined in terms of the new variables x and y by the equations

$$b = x, \quad s = yG(x) + F(x). \tag{4.2}$$

Define the quadratic and cubic in y polynomials by

$$Q(x,y) = \frac{G\ddot{G}}{8}y^2 + \frac{G\ddot{F}}{4}y + H,$$

$$C(x,y) = \frac{G^2\dddot{G} + 3G\dot{G}\ddot{G}}{24}y^3 + \frac{G^2\dddot{F} + 3G\dot{G}\ddot{F}}{8}y^2 + (G\dot{H} + 2H\dot{G})y + c(x). \tag{4.3}$$

In the new coordinates, equations (1.3)–(1.5) become

$$\begin{aligned}
&\tau = \frac{\dot{F}}{2} + \frac{\dot{G}}{2}y + \frac{X}{\rho},\\
&X^2 = Q\rho^2 - \frac{\rho^3}{4} - \frac{{\rho_y}^2}{4G^2},\\
&(G\rho)_x + 2X_y = 0,\\
&\tau_x G - (\dot{G}y + \dot{F})\tau_y = \left(\frac{\rho_{yy}}{2\rho G^2} - \frac{{\rho_y}^2}{4\rho^2 G^2} + \frac{\rho}{2} - \tau^2\right)_y.
\end{aligned} \tag{4.4}$$

Indeed, writing τ from equation (1.4) in the new coordinates we find equation $(4.4)_1$. The change of variables (4.2) implies

$$\partial_s = \frac{1}{G}\partial_y, \quad \partial_b = \partial_x - \left(y\frac{\dot{G}}{G} + \frac{\dot{F}}{G}\right)\partial_y.$$

Writing X^2 from equation (1.5) in the new coordinates we find equation $(4.4)_2$. Using the expression $(4.4)_1$ in equations (1.3) we find equations $(4.4)_3$ and $(4.4)_4$.

Proposition 4.1. Let I denote the expression

$$I = 4Q - \rho. \tag{4.5}$$

Equations (4.4) are compatible iff the following two equations for $X(x,y)$ are compatible:

$$X_y = \frac{1}{I}\left(2Q_y X - \rho_y\left(X + C - \frac{\dot{G}}{4}\int \rho(x,y')\mathrm{d}y'\right)\right), \tag{4.6}$$

$$\begin{aligned}
X_x \;=\;& \left(\frac{2Q_x}{I} + \frac{\dot{G}}{G}\left(\frac{\rho}{2I} - 1\right)\right)X + \left(\frac{Q_{yy}}{I} - \frac{2{Q_y}^2}{I^2}\right)\frac{\rho_y}{G^3}\\
+\;& \left(\frac{\rho^2}{2I} - \rho\right)\frac{Q_y}{G} - \frac{2\rho_y}{GI^2}\left(X + C - \frac{\dot{G}}{4}\int \rho(x,y')\mathrm{d}y'\right)^2.
\end{aligned} \tag{4.7}$$

These equations are compatible iff

$$
\begin{aligned}
&\frac{16Q_y}{I}\left(X+C-\frac{\dot{G}}{4}\int\rho(x,y')\mathrm{d}y'\right)\left(C-\frac{\dot{G}}{4}\int\rho(x,y')\mathrm{d}y'\right)\\
&+\left(2Q_{yy}-\frac{8{Q_y}^2}{I}\right)\frac{\rho_y}{G^2}+I\left(GC_x+\frac{1}{4}(\dot{G}^2-G\ddot{G})\int\rho(x,y')\mathrm{d}y'\right)\\
&-\left(C-\frac{\dot{G}}{4}\int\rho(x,y')\mathrm{d}y'\right)(-I\dot{G}+8\dot{G}Q+8GQ_x)\\
&+Q_y\left(\frac{16{Q_y}^2}{IG^2}-\frac{8Q_{yy}}{G^2}+8Q\rho-16Q^2\right)-4(\dot{G}Q+GQ_x)X=0. \qquad (4.8)
\end{aligned}
$$

For the derivation of this result we have assumed that I, ρ_y, X do not vanish. The cases that this assumption is violated are studied in the appendix A and in [29].

Proof. This result is based on the fact that equation $(4.4)_4$ can be integrated. Indeed, differentiating $(4.4)_2$ with respect to y we find

$$
\frac{2X}{\rho}\frac{X_y}{\rho_y}=Q_y\frac{\rho}{\rho_y}+2Q-\frac{3}{4}\rho-\frac{1}{2G^2}\frac{\rho_{yy}}{\rho}. \qquad (4.9)
$$

Replacing τ by equation $(4.4)_1$ and ρ_x by equation $(4.4)_3$, we find that the lhs of equation $(4.4)_4$ becomes

$$
G\left(\frac{\ddot{F}}{2}+\frac{\ddot{G}}{2}y+\frac{X_x}{\rho}\right)+\frac{\dot{G}X}{\rho}+\frac{2XX_y}{\rho^2}-(\dot{G}y+\dot{F})\left(\frac{\dot{G}}{2}+\frac{X_y}{\rho}-\frac{X\rho_y}{\rho^2}\right). \qquad (4.10)
$$

Using equation (4.9) to replace ρ_{yy} in the parenthesis appearing in the rhs of equation $(4.4)_4$, we find

$$
-2\frac{X}{\rho}\frac{X_y}{\rho_y}-(\dot{F}+\dot{G}y)\frac{X}{\rho}+Q_y\frac{\rho}{\rho_y}+\left(H-\frac{\dot{F}^2}{4}\right)+\left(\frac{G\ddot{F}}{4}-\frac{\dot{F}\dot{G}}{2}\right)y+\left(\frac{G\ddot{G}}{8}-\frac{\dot{G}^2}{4}\right)y^2.
$$

According to $(4.4)_4$ we differentiate the above expression by ∂_y and we equate the resulting expression with (4.10); this yields

$$
\frac{X_x}{\rho}=-\frac{2X}{G\rho}\left(\frac{X_y}{\rho_y}\right)_y-\frac{2{X_y}^2}{G\rho\rho_y}-\frac{Q_y}{G}\frac{\rho\rho_{yy}}{{\rho_y}^2}+\frac{\ddot{G}}{4}\frac{\rho}{\rho_y}-\frac{2\dot{G}}{G}\frac{X}{\rho}. \qquad (4.11)
$$

Using the derivative of equation $(4.4)_2$ to eliminate X_x, equation (4.11) can be written in the form

$$
\begin{aligned}
&\frac{I}{G}\left(\frac{X_y}{\rho_y}\right)_y+\frac{I_y}{G}\frac{X_y}{\rho_y}+\left(-\frac{\ddot{G}}{2}\frac{X_y}{\rho_y}y-\frac{\ddot{G}}{2}\frac{X}{\rho_y}+\frac{\ddot{G}}{2}\frac{\rho_{yy}}{{\rho_y}^2}Xy\right)\\
&+\left(-\frac{\ddot{F}}{2}\frac{X_y}{\rho_y}+\frac{\ddot{F}X}{2}\frac{\rho_{yy}}{{\rho_y}^2}\right)+\frac{X_y}{G}+\frac{2\dot{G}}{G}\left(H+\frac{G\ddot{F}}{4}y+\frac{G\ddot{G}}{8}y^2\right)\\
&+\dot{H}+\left(\frac{G\ddot{H}}{4}\right)_x y+\left(\frac{G\ddot{G}}{8}\right)_x y^2-\frac{\dot{G}}{G}\frac{\rho}{4}=0. \qquad (4.12)
\end{aligned}
$$

For the derivation of equation (4.12) we have also used equation (4.9).

It is remarkable that equation (4.12) can be integrated to equation (4.6), where $c(x)$ appearing in $C(x,y)$ is the constant (with respect to y) of integration.

Equations $(4.4)_2$ and (4.6) yield the following equations for ρ_{yy} and for X_{yy},

$$\frac{\rho_{yy}}{G^2} = I\rho - \frac{\rho^2}{2} + \frac{2Q_y\rho_y}{G^2 I} + \frac{4X}{I}\left(X + C - \frac{\dot{G}}{4}\int \rho(x,y')\mathrm{d}y'\right), \tag{4.13}$$

$$\begin{aligned} X_{yy} &= \left(\frac{2Q_{yy}}{I} - \frac{4{Q_y}^2}{I^2} - G^2\rho + \frac{G^2\rho^2}{2I}\right)X + \frac{\rho_y}{I}\left(\frac{\dot{G}}{4}\rho - C_y\right) \\ &- \frac{4G^2X}{I^2}\left(X + C - \frac{\dot{G}}{4}\int \rho(x,y')\mathrm{d}y'\right)^2 \\ &+ G^2\rho\left(\frac{\rho}{2I} - 1\right)\left(C - \frac{\dot{G}}{4}\int \rho(x,y')\mathrm{d}y'\right). \end{aligned} \tag{4.14}$$

Indeed, differentiating equation $(4.4)_2$ with respect to y, and then replacing in the resulting equation X_y by the rhs of equation (4.6) and X^2 by the rhs of equation $(4.4)_2$, we find equation (4.13). Furthermore, differentiating equation (4.6) with respect to y and then replacing in the resulting equation X_y and ρ_{yy} by equations (4.6) and (4.13), we find equation (4.14).

Using equations (4.6), (4.13), (4.14), equation (4.11) becomes equation (4.7).

A tedious calculation yields that the compatibility of equations (4.6) and (4.7) is equation (4.8). **QED**

Remark 4.1. In equations (4.6) and (4.7), the integral with respect to y of the function $\rho(x,y)$ is defined to within an arbitrary function of x. However, this is without loss of generality since the arbitrary function of x can be absorbed in the function $c(x)$ appearing in $C(x,y)$.

5 The Set of Solutions

In order to find ρ and τ we adopt the following strategy. We find *all* possible functions F, G, H such that the compatibility condition (4.8) is *identically* satisfied. It then follows that equations (4.6) and (4.7), where X is defined in terms of ρ by equation $(4.4)_2$, are *two compatible* equations for the *single* function ρ. In order to determine ρ it is more convenient to use equations (4.6) and $(4.4)_3$, instead of equations (4.6) and (4.7). Having determined ρ, the function τ follows from equation $(4.4)_1$.

It turns out that in the most general case, equation (4.8) is identically satisfied provided that the functions F, G, H and the function $c(x)$ appearing in the definition of $C(x,y)$ (equation (4.3)) satisfy the following equations

$$\ddot{F} = \ddot{G} = 0, \quad G\dot{H} + \dot{G}H = c\dot{G} + G\dot{c} = 0. \tag{5.1}$$

Thus

$$G(x) = a_1x + a_2, \quad F(x) = b_1x + b_2, \quad H(x) = \frac{c_1}{a_1x + a_2}, \quad c(x) = \frac{c_2}{a_1x + a_2}, \quad {a_1}^2 + {a_2}^2 \neq 0. \tag{5.2}$$

Indeed, the coefficient of X in equation (4.8) is given by

$$\frac{16Q_y\left(C-\frac{\dot{G}}{4}\int\rho(x,y')\mathrm{d}y'\right)}{I}-4(\dot{G}Q+GQ_x).$$

Thus this coefficient vanishes identically iff

$$Q_y=0,\quad \dot{G}Q+GQ_x=0.$$

These equations imply the first three of equations (5.1). With this choice of Q, equation (4.8) simplifies to $a_1C+GC_x=0$, or

$$\left(a_1(G\dot{H}+2H\dot{G})+G\frac{\mathrm{d}}{\mathrm{d}x}(G\dot{H}+2H\dot{G})\right)y+c\dot{G}+G\dot{c}=0.$$

Hence, if c is chosen according to the fourth of equations (5.1) then the second term above vanishes, while the coefficient of y also vanishes:

$$a_1H\dot{G}+G\frac{\mathrm{d}}{\mathrm{d}x}(Ha_1)=a_1(G\dot{H}+\dot{G}H)=0.$$

We will now show that equation (4.6) can be integrated: Using $\rho=4H(x)-I$ and $C=a_1H(x)y+c(x)$, equation (4.6) becomes

$$X_y-\frac{I_y}{I}X=c(x)\frac{I_y}{I}+\frac{a_1}{4}\frac{I_y}{I}\int I(x,y')\mathrm{d}y'.$$

After dividing by I this equation can be rewritten as

$$\left(\frac{X}{I}\right)_y=-\left(\frac{c(x)}{I}\right)_y+\frac{a_1}{4}\left(y-\frac{\int I(x,y')\mathrm{d}y'}{I}\right)_y.$$

Thus

$$X=-c(x)+a(x)I+\frac{a_1}{4}\left(yI-\int I(x,y')\mathrm{d}y'\right),\tag{5.3}$$

where $a(x)$ is the constant (with respect to y) of integration.

Since the rhs of equation (5.3) provides the general solution of equation (4.6), we must choose the arbitrary function $a(x)$ in such a way that this solution is compatible with the general solution of equation $(4.4)_3$. The fact that such a choice exists is the consequence of the fact that the compatibility condition (4.8) is identically satisfied. In order to find the general solution of equation $(4.4)_3$ we use equation (5.3) to compute X_y,

$$X_y=-\left(a(x)+\frac{a_1}{4}y\right)\rho_y.$$

Thus equation $(4.4)_3$ becomes

$$(G\rho)_x-\frac{2a(x)+\frac{a_1}{2}y}{G}(G\rho)_y=0.$$

Hence,

$$\rho=\frac{W(z)}{G},\quad z=y\sqrt{a_1x+a_2}+2\int^x\frac{a(x')}{\sqrt{a_1x'+a_2}}\mathrm{d}x'.\tag{5.4}$$

Substituting the expression $I = (4c_1 - W)/G$ in the rhs of equation (5.3) we find

$$X = \frac{1}{G}\left[4a(x)c_1 - c_2 - \left(a(x) + \frac{a_1}{4}y\right)W(z) + \frac{a_1}{4\sqrt{G}}\int^z W(z')\mathrm{d}z'\right]. \tag{5.5}$$

Replacing in the lhs of equation $(4.4)_2$ X by the above expression and using $\rho = W/G$, $Q = c_1/G$, equation $(4.4)_2$ becomes

$$W_z{}^2 + W^3 - 4c_1W^2 + 4\left[(4a(x)c_1 - c_2)\sqrt{G} - \left(a(x) + \frac{a_1}{4}y\right)\sqrt{G}W(z) + \frac{a_1}{4}\int^z W(z')\mathrm{d}z'\right]^2 = 0. \tag{5.6}$$

(i) $a_1c_1 \neq 0$**.**

The function

$$\left(a(x) + \frac{a_1}{4}y\right)\sqrt{G}$$

is proportional to z iff $a(x)$ is a constant. This constant must equal $c_2/4c_1$ in order for the first term in the above bracket to be independent of x.

In summary,

$$\begin{aligned} &\rho = \frac{W(z)}{a_1x + a_2}, \quad \tau = \frac{b_1}{2} + \frac{a_1}{2}y + \frac{a_1}{4\sqrt{a_1x + a_2}}\left(-z + \frac{\int^z W(z')\mathrm{d}z'}{W(z)}\right), \\ &z = \left(y + \frac{c_2}{c_1a_1}\right)\sqrt{a_1x + a_2}, \end{aligned} \tag{5.7}$$

where $W(z)$ solves the ODE

$$W_z{}^2 + W^3 - 4c_1W^2 + \frac{a_1{}^2}{4}\left(-z + \int^z W(z')\mathrm{d}z'\right)^2 = 0. \tag{5.8}$$

Using the definitions (4.1) and (4.2) to express F, G, x, y in terms of f, g, b, s we find the following expressions:

$$\rho = \frac{W(z)}{a_1b + a_2}, \quad \tau = \frac{a_1s + b_1a_2 - b_2a_1}{2(a_1b + a_2)} + \frac{a_1}{4\sqrt{a_1b + a_2}}\left(-z + \frac{\int^z W(z')\mathrm{d}z'}{W(z)}\right), \tag{5.9}$$

where $W(z)$ satisfies the ODE

$$W_z{}^2 + W^3 - 4c_1W^2 + \frac{a_1{}^2}{4}\left(zW - \int^z W(z')\mathrm{d}z'\right)^2 = 0, \quad z = \frac{s - b_1b - b_2}{\sqrt{a_1b + a_2}} + \frac{c_2}{a_1c_1}\sqrt{a_1b + a_2}. \tag{5.10}$$

Furthermore, the definition (1.6) of Ω yields

$$\Omega = \frac{a_1s + b_1a_2 - b_2a_1}{a_1b + a_2}. \tag{5.11}$$

Introducing the variables $\hat{b}$ and $\hat{s}$ by equations (1.8), as well as the variables $\zeta = z\sqrt{a_1}$, $w = W/a_1$, equations (5.9)–(5.11) become equations (1.9)–(1.11).

(ii) $a_1 \neq 0$, $c_1 = 0$.

Again we must take $a(x)$ to be a constant, which for convenience we choose $a_1 c/4$. In order for the bracket in equation (5.6) to be independent of x, we must choose $c_2 = 0$, thus

$$a(x) = \frac{a_1 c}{4}, \quad c_2 = 0, \quad z = (y + c)\sqrt{a_1 x + a_2}. \tag{5.12}$$

Hence

$$\rho = \frac{W(z)}{a_1 x + a_2}, \quad \tau = \frac{b_1}{2} + \frac{a_1}{2} y + \frac{a_1}{4\sqrt{a_1 x + a_2}} \left(-z + \frac{\int^z W(z')\mathrm{d}z'}{W(z)} \right), \tag{5.13}$$

where $W(z)$ satisfies

$$W_z{}^2 + W^3 + \frac{a_1{}^2}{4} \left(zW - \int^z W(z')\mathrm{d}z' \right)^2 = 0. \tag{5.14}$$

Rewriting the above in terms of s and b we find

$$\rho = \frac{W(z)}{a_1 b + a_2}, \quad \tau = \frac{a_1 s + b_1 a_2 - b_2 a_1}{2(a_1 b + a_2)} + \frac{a_1}{4\sqrt{a_1 b + a_2}} \left(-z + \frac{\int^z W(z')\mathrm{d}z'}{W(z)} \right),$$
$$z = \frac{s - b_1 b - b_2}{\sqrt{a_1 b + a_2}} + c\sqrt{a_1 b + a_2}.$$

These equations are similar with equations (5.9) and (5.10), where $c_1 \to 0$ and $c_2/(a_1 c_1)$ is replaced by c. Thus in the limit $c_1 \to 0$, ζ is given by equation $(1.8)_3$ with c_2/c_1 replaced by c.

(iii) $a_1 = 0$.

The bracket in (5.6) is independent of x iff $a(x) = c_3$. Thus

$$\rho = \frac{W(z)}{a_2}, \quad \tau = \frac{b_1}{2} - c_3 + \frac{4c_1 c_3 - c_2}{W(z)}, \quad z = y\sqrt{a_2} + \frac{2c_3 x}{\sqrt{a_2}},$$

where $W(z)$ satisfies

$$W_z{}^2 + W^3 - 4c_1 W^2 + 4a_2 \left[(4c_1 c_3 - c_2) - c_3 W \right]^2 = 0.$$

Expressing x and y in terms of s and b we find

$$z = \frac{s + (2c_3 - b_1)b - b_2}{\sqrt{a_2}}.$$

Letting $w = W/a_2$, $\zeta = z\sqrt{a_2}$ and renaming $(4c_1 c_3 - c_2)/a_2$, c_1/a_2, as c_2 and c_1, respectively, we find equations (1.13)–(1.16).

6 Conclusions

It appears that, following the contributions of several investigators, the long–standing problem of determining the steady inviscid hydrodynamic flows which are subject to the constraint of constant speed, has been solved.

In this paper we have presented the general solution assuming that: (a) $\rho_y \neq 0$, (b) the compatibility condition (4.8) is identically satisfied. Regarding the latter assuption, we note that if equation (4.8) is *not* identically satisfied, then there exist additional restrictions on $\rho(x, y)$. In spite of this fact, there exist particular solutions for which these additional restrictions are satisfied. Both cases (a) and (b) are investigated in [29].

The general solution of the Gilbarg problem can be expressed either in terms of elliptic functions or in terms of solutions of a certain ODE introduced by J. Chazy in 1911. This ODE, which was also rederived by F.J. Bureau, can be solved in terms of the fourth Painlevé transcendent. Indeed consider for siplicity the case of equation (1.11) with $c_1 = 0$. Letting $w = u_\zeta$, and then replacing u and ζ by $iu/4$ and $i\zeta/4$ respectively, we find

$$u_{\zeta\zeta}{}^2 + 4u_\zeta{}^3 - 4(\zeta u_\zeta - u)^2 = 0.$$

Letting

$$u = \frac{v_\zeta{}^2}{8v} - \frac{v^3}{8} - \frac{1}{2}\zeta v(\zeta + v),$$

we find that $v(\zeta)$ solves the fourth Painlevé equation

$$v_{\zeta\zeta} = \frac{1}{2}\frac{v_\zeta{}^2}{v} + \frac{3}{2}v^3 + 4\zeta v^2 + 2\zeta^2 v.$$

The general solution of the fourth Painlevé equation cannot be written explicitely. However, in applications one is usually interested in the asymptotic characterisation of the solution of an ODE and in particular in the associated connection formulas. It is remarkable that for the Painlevé equations there exist *explicit* connection formulas [30]. These formulas should provide the necessary information needed for the investigation of certain concrete physical problems, which will be analyzed elsewhere.

Appendix A

We will show that if either of the functions I or X vanish, then equations (4.4) yield for ρ and τ expressions which are *particular* cases of the expressions (5.9). The case of $\rho_y = 0$ is analyzed in [29].

(i) $I = 0$**.**

Equation $(4.4)_2$ becomes $X^2 + 4Q_y{}^2/G^2 = 0$, thus $X = Q_y = 0$. Hence G, F are given by the first two expressions in (5.2). Equations $(4.4)_1$ and $(4.4)_3$ yield

$$\rho = \frac{w(y)}{G}, \quad \tau = \frac{a_1 y + b_1}{2}.$$

Substituting these expressions in equation $(4.4)_4$ we find $w(y) = c$. Rewriting the above equations in terms of s and b we find

$$\rho = \frac{c}{a_1 b + a_2}, \quad \tau = \frac{a_1 s + b_1 a_2 - b_2 a_1}{2(a_1 b + a_2)}. \tag{A.1}$$

The functions f, g, h are given by equations (1.7) and Ω is given by equation (5.11). Comparing with equations (5.9) and (5.10), we see that this is a particular case corresponding to $W = \text{const}$.

(ii) $X = 0$**.**

Equations $(4.4)_1$ and $(4.4)_3$ yield

$$\rho = \frac{w(y)}{G}, \quad \tau = \frac{\dot{F} + \dot{G}y}{2}. \tag{A.2}$$

Then equations $(4.4)_2$ and $(4.4)_4$ imply

$$\left(\frac{w'(y)}{w(y)}\right)^2 + Gw(y) = G^3\left(\ddot{G}\frac{y^2}{2} + \ddot{F}y\right) + 4G^2H, \tag{A.3}$$

$$\frac{w''(y)}{w(y)} - \frac{1}{2}\left(\frac{w'(y)}{w(y)}\right)^2 + Gw(y) = G^3\left(\ddot{G}\frac{y^2}{2} + \ddot{F}y\right) + a(x). \tag{A.4}$$

Subtracting these equations we find

$$\frac{w''(y)}{w(y)} - \frac{3}{2}\left(\frac{w'(y)}{w(y)}\right)^2 = a(x) - 4G^2H.$$

Hence,

$$\frac{w''(y)}{w(y)} - \frac{3}{2}\left(\frac{w'(y)}{w(y)}\right)^2 = -2c_1a_2, \quad a(x) = 4G^2H - 2c_1a_2, \tag{A.5}$$

where we chose the constant in the form $-2c_1a_2$ for later convenience. Letting $w = \varphi^{-2}$, the first of equations (A.5) becomes the linear equation

$$\varphi'' - c_1a_2\varphi = 0. \tag{A.6}$$

Then equation (A.3) becomes

$$4\varphi'^2 + G = \left[G^3\left(\ddot{G}\frac{y^2}{2} + \ddot{F}y\right) + 4G^2H\right]\varphi^2. \tag{A.7}$$

Multiplying equation (A.6) by φ' and integrating we find

$$\varphi'^2 - c_1a_2\varphi^2 = -\frac{a_2}{4}, \tag{A.8}$$

where we chose the constant in the form $-a_2/4$ for later convenience. Equations (A.7) and (A.8) imply

$$G = a_2, \quad F = b_1x + b_2, \quad H = \frac{c_1}{a_2}. \tag{A.9}$$

These equations are the particular case of the first three equations (5.2) corresponding to $a_1 = 0$. Equations (A.2) become

$$\tau = \frac{b_1}{2}, \quad \rho = \frac{1}{a_2\varphi^2(y)}, \quad \varphi(y) = A_1e^{\sqrt{c_1a_2}y} + A_2e^{-\sqrt{c_1a_2}y}, \quad A_1A_2 = \frac{1}{16c_1}, \quad y = \frac{s - b_1b - b_2}{a_2}.$$

Also

$$f = \frac{b_1}{4}, \quad g = 0, \quad h = \frac{c_1}{a_2} - \frac{{b_1}^2}{8}$$

and $\Omega = b_1$. Renaming c_1/a_2 as c_1 we find equations (1.17), (1.18).

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References

[1] D. Gilbarg, On the Flow Patterns Common to Cenrtain Classes of Plane Fluid Motions. J. Math. and Phys. **26**, 137–142 (1947).

[2] R. Prim, On the Uniqueness of Flows with Given Streamlines. J. Math. and Phys. **28**, 50–53 (1949).

[3] L.N. Howard, Constant Speed Flows. PhD Thesis, Princeton University (1953).

[4] R.H. Wasserman, On a Class of Three–Dimensional Compressible Fluid Flows. J. Math. Anal. Appl. **5**, 119–135 (1962).

[5] A.W. Marris, On Motions with Constant Speed and Streamline Parameters. Arch. Rat. Mech. Anal. **90**, 1–14 (1985).

[6] C. Rogers, On the Heisenberg Spin Equation in Hydrodynamics. Informes de Matématica, Série B **127**, Instituto de Matématica Pura a Aplicada, Rio de Janeiro, Brazil (2000).

[7] C. Rogers and W.K. Schief, On Geodesic Hydrodynamic Motions. Heisenberg Spin Connections. J. Math. Anal. Appl. **251**, 855–870 (2000).

[8] W.K. Schief, Nested Toroidal Flux Surfaces in Magnetohydrostatics. Generation via Soliton Theory. J. Plasma Phys. (to appear).

[9] W.K. Schief, Hidden Integrability in Ideal Magnetohydrodynamics. The Pohlmeyer–Lund–Regge Model. Phys. Plasma **10**, 2677–2685 (2003).

[10] G. Grice and W.K. Schief, Constant Speed Flows and the Nonlinear Schrödinger Equation (in preparation).

[11] A. Masotti, Decomposizione Intrinseca del Vortice a sue Applicazioni. Instituto Lombardo di Scienze a Lettere Rendiconti (2) **60**, 869–874 (1927).

[12] O. Bjørgum, On Beltrami Vector Fields and Flows, Part I. Universitet I. Bergen, Arbok Naturvitenskapelig rekke n–1 (1951).

[13] F. Emde, Der Einfluß der Feldlinien auf Divergenz und Rotor. Archiv für Elektrotechnik **39**, 2–8 (1948).

[14] G. Hamel, Potentialströmungen mit Konstanter Geschwindigkeit. Sitzgber. preuss. Akad. Wiss. phys.–math. Kl. 5–20 (1937).

[15] R.K. Ricca, Rediscovery of Da Rios Equation. Nature **352**, 561–562 (1991).

[16] C.E. Weatherburn, Differential Geometry of Three Dimensions, Vol. I; II. Cambridge University Press (1927); (1930).

[17] A.W. Marris, Hamel's Theorem. Arch. Rational Mech. Anal. **51**, 85–105 (1973); corrigendum, ibid. **51**, 388 (1973).

[18] L.S. Da Rios, Sul Moto d'um Liquido Indefinito con un Filetto Vorticoso. Rend. Circ. Mat. Palermo **22**, 117–135 (1906).

[19] A.W. Marris, Universal Deformations in Incompressible Isotropic Elastic Materials. J. Elasticity **5**, 111–128 (1975).

[20] A.W. Marris, Two New Theorems on Ericksen's Problem. Arch. Rat. Mech. Anal. **79**, 131–173 (1982).

[21] J. Ericksen, Deformations Possible in Every Isotropic, Incompressible, Perfectly Elastic Body. Z. angew. Math. Phys. **5**, 466–489 (1954).

[22] G. Saccomandi, Universal Results in Finite Elasticity, in Y.B. Fu and R.W. Ogden, eds. Nonlinear Elasticity: Theory and Applications, London Mathematical Society Lecture Notes Series **283**, Cambridge University Press, 97–130 (2001).

[23] J.K. Knowles, Universal States of Finite Anti–plane Shear: Ericksen's Problem in Miniature. Amer. Math. Monthly **86**, 109–113 (1979).

[24] A.W. Marris, A.W. Wang and C.C. Wang, Solenoidal Screw Fields of Constant Magnitude, Arch. Rat. Mech. Anal. **39**, 227–244 (1970).

[25] A.S. Fokas, I.M. Gel'fand, Surfaces on Lie Groups, on Lie Algebras, and their Integrability, Comm. Math. Phys. **177**, 203–220 (1996).

[26] J. Chazy, Sur les équations différentielles du troisième ordre et d'order supérieur dont l'intégrale générale a ses points critiques fixes, Acta Math. **41**, 317–385 (1911).

[27] F.J. Bureau, Integration of some Nonlinear Systems of Ordinary Differential Equations, Ann. Mat. Pura Appl. **94**, 345–360 (1972).

[28] C.M. Cosgrove, G. Scoufis, Painlevé Classifications of a Class of Differential Equations of the Second Order and Second Degree, Stud. Appl. Math. **88**, 25–87 (1993).

[29] V. Marinakis, Particular Solutions of the Gilbarg Problem (preprint).

[30] A.S. Fokas, A.R. Its, A.A. Kapaev, V. Novokshenov, *The Isomonodromy Approach to Painlevé Equations*, AMS (in press).