Two Dimensional Linear PDE's in a Convex Polygon

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Abstract

A method is introduced for solving boundary value problems for linear PDE's in convex polygons. It consists of three algorithmic steps: (1) *Given a PDE*, construct two compatible eigenvalue equations. (2) *Given a polygon*, perform the simultaneous spectral analysis of these two equations. This yields an integral representation in the complex k-plane of the solution $q(x_1, x_2)$ in terms of a function $\hat{q}(k)$, and an integral representation in the (x_1, x_2)-plane of $\hat{q}(k)$ in terms of the values of q and of its derivatives on the boundary of the polygon. These boundary values are in general related, thus only some of them can be prescribed. (3) *Given appropriate boundary conditions*, express the part of $\hat{q}(k)$ involving the unknown boundary values in terms of the boundary conditions. This is based on the existence of a simple global relation formulated in the complex k-plane, and on the invariant properties of this relation.

As an illustration, the following integral representations are obtained: (a) $q(x,t)$ for a general dispersive evolution equation of order n in a domain bounded by a linearly moving boundary; (b) $q(x,y)$ for the Laplace, modified Helmholtz and Helmholtz equations in a convex polygon. These general formulae and the analysis of the associated global relations are used to discuss typical boundary value problems for evolution equations and for elliptic equations.

1 Introduction

As soon as linear two dimensional PDE's made their appearance, d'Alembert and Euler discovered a general approach for constructing large classes of their solutions. This approach involved separating variables and superimposing solutions of the resulting ODE's. The method of separation of variables naturally led to the solution of PDE's by a transform pair. The prototypical such pair is the direct and the inverse Fourier tranforms; variations of this fundamental transform include the Laplace, Mellin, sine, cosine transforms, and their discrete analogues. The proper transform for a given boundary value problem is specified by the PDE, by the domain, and by the given boundary conditions. For example, the heat equation on the half line with $q(0,t)$ given is solved by the sine transform, while the same equation but with $q_x(0,t)$ given is solved by the cosine transform. For some simple boundary value problems, there exists an algorithmic procedure for deriving the associated transform, see for example [1], [2]. This procedure is based on the spectral analysis of a *single* eigenvalue equation. The transform method has been enormously successful for solving a great variety of initial and boundary value problems. However, for sufficiently complicated problems the classical transform method fails. For example, there does not exist a proper analogue of the sine transform for solving third order evolution equations on the half line. Similarly, there do not exist proper transforms for solving many boundary value problems for elliptic equations even of second order and in simple domains. The failure of the transform method led to the development of several ingenious but ad hoc techniques, which include: conformal mappings for the Laplace and the biharmonic equations; the Jones method [3] and the formulation of the Wiener-Hopf factorization problem [4]; the use of some integral representation, such as that of Sommerfeld; the formulation of a difference equation, such as the Malyuzhinet's equation.

A new general approach for solving boundary value problems for two dimensional linear and integrable nonlinear PDE's was announced in [5]. For linear PDE's, this involves three steps: (1) *Given a PDE, construct two compatible linear eigenvalue equations* [6] which, in analogy with the theory of nonlinear integrable PDE's, we call a *Lax pair* [7]. (2) *Perform the simultaneous spectral analysis of the Lax pair.* This yields an integral representation of $q(x_1, x_2)$ in terms of a function $\hat{q}(k)$ which we call the *spectral function*, and an integral representation of $\hat{q}(k)$ in terms of the values of q and of its derivatives on the boundary of the domain. The implementation of this step for some simple evolution equations and for the Laplace equation in some simple domains is discussed in [5], [8], [9]. However, the implementation presented in these papers is *not* the optimal one. Although it can be used for simple domains (like those in [5], [8], [9]), it becomes prohibitively complicated for arbitrary polygonal domains. A new implementation was found in [10] for the particular case of the Laplace equation in a polygon. Here we will show that this new implementation, appropriately extended, can be effectively applied to other linear two dimensional PDE's. (3) *Given appropriate boundary conditions, analyze the global relation satisfied by the boundary values of q and of its derivatives.* This step is necessary, since $\hat{q}(k)$ involves some unknown boundary values. For the simple problems considered in [5], [8], [9], the analysis of the global relation was successfully used to express $\hat{q}(k)$ in terms of the given boundary conditions. However, *no* general method for analyzing the global relation was established. In this paper, we will introduce such a method.

1.1 Lax pairs

The algorithmic derivation of Lax pairs is discussed in §2. Here we only note that for elliptic equations, it is sometimes convenient to write the Lax pair in $(z, \bar{z})$ as opposed to (x, y) coordinates, where $z = x + iy$, $\bar{z} = x - iy$. For example a Lax pair of the equation

$$q_{xx} + q_{yy} + 4\alpha q = 0, \quad \alpha \ \text{ constant}, \tag{1.1}$$

is given by

$$\mu_z - ik\mu = q_z, \quad \mu_{\bar{z}} - \frac{i\alpha}{k}\mu = -\frac{i\alpha}{k}q, \tag{1.2}$$

where $\mu(z, \bar{z}, k)$ is a scalar function of the arguments indicated and $k \in \mathbb{C}$ is the spectral parameter.

1.2 The simultaneous spectral analysis

The new implementation in the case that the domain is a convex polygon is discussed in §3. The general results obtained in §3 are used in §4 to derive the integral representations of q and of $\hat{q}$ for the following problems: (a) A general dispersive evolution equation of order n involving a linearly moving boundary,

$$q_t + i\sum_0^n \alpha_j(-i\partial_x)^j q = 0, \quad \beta t < x < \infty, \quad 0 < t < T, \tag{1.3}$$

where $\beta = \tan\theta$, $0 \leq \theta < \frac{\pi}{2}$, T is a positive constant, and all α_j are real. (b) Equation (1.1) in a convex polygon Ω. These formulae are given in Propositions 1.1 and 1.2 below.

Proposition 1.1 *Assume that equation (1.3) is supplemented with appropriate initial and boundary conditions, so that there exists a solution $q(x,t)$ which is sufficiently smooth up to the boundary, and which has sufficient decay as $x \to \infty$, uniformly in $0 < t < T$. Then $q(x,t)$ can be expressed in the form*

$$q(x,t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \exp\left[ikx - iw(k)t\right]\hat{q}_0(k)dk + \int_{\partial D_+} \exp\left[ikx - iw(k)t\right]\hat{Q}(k)dk\right], \qquad \omega(k) = \sum_0^n \alpha_j k^j, \tag{1.4}$$

where the spectral functions $\hat{q}_0(k)$, $\hat{Q}(k)$, and the curve ∂D_+ are defined as follows:

$$\hat{q}_0(k) = \int_0^\infty \exp\left[-ikx\right] q(x,0)dx, \quad Im\, k \le 0, \tag{1.5a}$$

$$\hat{Q}(k) = \sum_{j=1}^{n} a_{n-j}(k)\hat{Q}_j(k) - \beta\hat{Q}_0(k), \quad a_j(k) = \alpha_n k^j + \alpha_{n-1}k^{j-1} + \ldots + \alpha_{n-j}, \tag{1.5b}$$

$$\hat{Q}_j(k) = \int_0^T \exp\left[i(w(k) - k\beta)t\right](-i\partial_x)^j q(\beta t, t)dt, \quad j = 0,\ldots,n-1, \quad k \in \mathbb{C}; \tag{1.5c}$$

the curve ∂D_+ is the boundary of the domain D_+ defined by

$$D_+ = \left\{k \in \mathbb{C} : Im\,(w(k) - \beta k) > 0, \quad Im\, k > 0\right\}, \tag{1.6}$$

and oriented in such a way that D_+ is on the left-hand side of the increasing direction of ∂D_+.

Furthermore, the following global relation is valid in $D_- = \{k \in \mathbb{C}$: $Im\,(w(k) - \beta k) > 0$, $Im\, k < 0\}$,

$$\hat{Q}(k) = -\hat{q}_0(k) + \exp\left[iw(k)T\right] \int_{\beta T}^\infty \exp\left[-ikx\right] q(x,T)dx, \quad k \in D_-. \tag{1.7}$$

Example 1.1 Let q solve

$$q_t + q_{xxx} = 0, \quad \beta t < x < \infty, \;\; 0 < t < T. \tag{1.8}$$

Then $w(k) = -k^3$, ∂D^+ is the curve $-k_I^3 + 3k_I k_R^2 + \beta k_I = 0$, and

$$\hat{Q}(k) = \hat{Q}_2(k) + ik\hat{Q}_1(k) - (k^2 + \beta)\hat{Q}_0(k), \quad \hat{Q}_l(k) = \int_0^T \exp\left[-ik\beta t - ik^3 t\right] \partial_x^l q(\beta t, t)dt, l = 0,1,2. \tag{1.9}$$

Proposition 1.2 *Let Ω be a convex closed polygon in the complex z-plane, $z = x + iy$, with corners $z_1, \cdots, z_n$, see figure 1. Let the real-valued function $q(x,y)$ satisfy equation (1.1) in Ω. Assume that appropriate boundary conditions are prescribed on the boundary of Ω such that there exists a solution $q(x,y)$ which is sufficiently smooth up to the boundary of Ω. Then $q(x,y)$ can be expressed as follows: Define the functions $\rho_{i,j}(k)$ by*

$$\rho_{i,j}(k) = \int_{z_i}^{z_j} \exp\left[-ikz - \frac{i\alpha}{k}\bar{z}\right]\left(q_z(z,\bar{z})dz - \frac{i\alpha}{k}q(z,\bar{z})d\bar{z}\right), \qquad z_{n+1} = z_1, \tag{1.10}$$

where $\partial_z = (\partial_x - i\partial_y)/2$.

(i) The solution of the Laplace equation, i.e. of equation (1.1) with $\alpha = 0$, can be expressed by

$$\frac{\partial q}{\partial z} = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{l_j} \exp\left[ikz\right] \rho_{j+1,j}(k)dk, \tag{1.11}$$

where $\rho_{j+1,j}(k)$ are given by (1.10) with $\alpha = 0$, and l_j are the rays in the complex k-plane

$$l_j = \{k \in \mathbb{C} : \arg k = -\arg(z_j - z_{j+1})\}, \quad j = 1, \cdots, n, \tag{1.12}$$

oriented from zero to infinity.

(ii) The solution of the modified Helmholtz equation, i.e. of equation (1.1) with $\alpha = -\beta^2$, can be expressed by

$$q = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{l_j} \exp\left[ikz - \frac{i\beta^2}{k}\bar{z}\right] \rho_{j+1,j}(k)\frac{dk}{k}, \tag{1.13}$$

where $\rho_{j+1,j}(k)$ are given by equation (1.10) with $\alpha = -\beta^2$, and l_j are as in (i).

(iii) The solution of the Helmholtz equation, i.e. of equation (1.1) with $\alpha = \beta^2$, can be expressed by

$$q = \frac{1}{2\pi i}\left[\sum_{j=1}^{n}\int_{\tilde{l}_j}\exp\left[ikz + \frac{i\beta^2}{k}\bar{z}\right]\rho_{j+1,j}(k)\frac{dk}{k} + \sum_{j=1}^{2n}\int_{L_j}\exp\left[ikz + \frac{i\beta^2}{k}\bar{z}\right]\rho^{(j)}(k)\frac{dk}{k}\right], \tag{1.14}$$

where $\rho_{j+1,j}(k)$ are given by equation (1.10) with $\alpha = -\beta^2$, and $\tilde{l}_j$, L_j, $\rho^{(j)}(k)$ are defined as follows: $\tilde{l}_j$ is the union of the two rays, oriented away from the origin, given by

$$\tilde{l}_j = \{k \in \mathbb{C} : [\arg k = -\arg(z_j - z_{j+1}),\ |k| > \beta] \cup [\arg k = \pi - \arg(z_j - z_{j+1}),\ |k| < \beta]\}; \tag{1.15}$$

L_j are the circular arcs formed by the intersection of the ray $\tilde{l}_j$ with the circle $|k| = \beta$; if α_{j+1}, A_i, α_j are the points of intersection of the circle $|k| = \beta$ with the rays $\{\tilde{l}_{j+1},\ |k| < \beta\}$, $\{\tilde{l}_i,\ |k| > \beta\}$, $\{\tilde{l}_j,\ |k| < \beta\}$, where A_i is between α_{j+1} and α_j, then $\rho^{(j)}$ on $L_j = (A_i, \alpha_j)$ is $\rho_{i,j+1}$, see figure 2.

The spectral functions $\rho_{j+1,j}(k)$ satisfy the global relation

$$\sum_{j=1}^{n}\rho_{j+1,j}(k) = 0, \quad k \in \mathbb{C}. \tag{1.16}$$

If Ω is open the following modifications are made: The corners z_1, z_n are moved to infinity and $q(x,y)$ is assumed to have sufficient decay as $z \to \infty$. The spectral function $\rho_{1,n}$ is zero thus the summations in equations (1.11), (1.13), (1.14) are only up to $n-1$. The functions $\rho_{2,1}(k)$ and $\rho_{n,n-1}(k)$ are not defined for all k but for k in S_1 and S_n respectively, where S_1 and S_n are defined as follows: For the Laplace and the modified Helmholtz equations, S_1 and S_n are the half planes

$$\begin{aligned} S_1 &= \{k \in \mathbb{C}, \arg k \in [-\arg(z_2 - z_1), \pi - \arg(z_2 - z_1)]\} \\ S_n &= \{k \in \mathbb{C}, \arg k \in [-\arg(z_{n-1} - z_n), \pi - \arg(z_{n-1} - z_n)\}; \end{aligned} \tag{1.17}$$

for the Helmholtz equation S_1 and S_n are the domains in the k -plane obtained from the map $\lambda = (1-\beta^2/|k|^2)k$ of the sectors $\tilde{S}_1$, $\tilde{S}_n$ in the λ plane, where $\tilde{S}_1$, $\tilde{S}_n$ are as in (1.17), but with k replaced by λ. Finally, the global relation (1.16) is replaced by

$$\sum_{j=1}^{n-1}\rho_{j+1,j}(k) = 0, \quad k \in S_1 \cap S_n. \tag{1.18}$$

Example 1.2 Let Ω be the equilateral triangle with corners $z_j = R\exp\left[-i\frac{2\pi}{3}(j-1)\right]$, $j = 1, 2, 3$. Using, $\arg(z_1 - z_2) = \arg(z_2 - z_1) - \pi = \pi/6$, $\arg(z_2 - z_3) = \arg(z_3 - z_2) - \pi = -\pi/2$, $\arg(z_3 - z_1) = 5\pi/6$, it follows that the rays l_j, $j = 1, 2, 3$, are defined by $\arg k = -\pi/6$, $\pi/2$, $5\pi/6$, respectively. The contours and the spectral functions for the Helmholtz equations are indicated in figure 3.

Example 1.3 Let Ω be the first quadrant of the complex z-plane, $z_1 = \infty$, $z_2 = 0$, $z_3 = i\infty$. Using $\arg(z_1 - z_2) = 0$ and $\arg(z_2 - z_3) = \arg(z_3 - z_2) - \pi = -\pi/2$, it follows that the rays l_j, $j = 1, 2$, are defined by $\arg k = 0$, $\pi/2$, respectively. For the Laplace and for the modified Helmholtz equations, if $k \in S_1$, S_n, then $\arg k \in [\pi, 0]$, $[\pi/2, 3\pi/2]$, respectively; thus the global relation is valid for any $k \in [\pi, 3\pi/2]$. The contours and the spectral functions for the Helmholtz equation are indicated in figure 4; in this case the global relation is valid for $k \in S$, where

$$S = \left\{k \in \mathbb{C}, \quad \left(\arg k \in [\pi, \frac{3\pi}{2}], |k| > \beta\right) \cup \left(\arg k \in [0, \frac{\pi}{2}], |k| < \beta\right)\right\}.$$

1.3 The analysis of the global relation

The formulae presented in propositions 1.1 and 1.2 are derived starting from a given PDE and a given domain, thus they are valid for *any* boundary conditions, provided that these boundary conditions yield a well posed problem. A fundamental limitation of these formulae is that they are derived under the a-priori assumption of existence of solutions. Furthermore, for a given boundary value problem, the spectral functions involve some unknown boundary values. For example, the spectral functions (1.5) depend on the initial condition $q(x,0)$ and on the n boundary conditions $\{\partial_x^l q(\beta t,t)\}_0^{n-1}$; however a boundary value problem for equation (1.3) is well posed provided that $q(x,0)$ and either $n/2$ (if n is even) or $(n\pm 1)/2$ (if n is odd) boundary conditions are given [12], thus $\hat{Q}(k)$ involves either $n/2$ or $(n\mp 1)/2$ unknown functions. Similarly, a necessary condition for equation (1.1) to be well posed is that one relation is prescribed among q, q_x and q_y on the boundary, thus each $\rho_{j+1,j}(k)$ involves one unknown function.

In this paper we introduce a method for expressing the part of $\hat{q}(k)$ involving the unknown boundary values in terms of the given boundary conditions. This method consists of three algorithmic steps: (i) Use the definition of $\hat{q}(k)$ to express it in terms of the given boundary conditions and some unknown functions denoted by $\Psi_j(k)$. Substitute these expressions in the global relation to obtain an equation for the unknown functions $\Psi_j(k)$. (ii) Use certain invariant transformations in the complex k-plane to construct from this equation a set of additional equations. (iii) Use this set of equations to obtain $\Psi_j(k)$ either through the solution of a system of algebraic equations, or through the solution of a matrix Riemann-Hilbert problem [11]. For the implementation of the third step we note: 1. For typical boundary value problems for evolution equations and for a large class of boundary value problems for elliptic equations, the functions $\Psi_j(k)$ can be obtained using only *algebraic manipulations*. This is based on the fact that $q(x_1,x_2)$ remains invariant if $\hat{q}(k)$ is replaced by $\hat{q}(k)+\tilde{q}(k)$, where $\tilde{q}(k)$ is a suitable analytic function. This function can be used to eliminate some of the unknown functions $\Psi_j(k)$, thus allowing one to determine the remaining unknown functions by solving a system of algebraic equations. 2. For general boundary value problems for elliptic equations, the functions $\Psi_j(k)$ satisfy a Riemann-Hilbert problem. This problem can be obtained as follows: (a) Determine the domains in the complex k-plane where each unknown function is bounded and analytic. These domains are separated by certain curves. (b) For each of these curves, use the equations obtained in step (ii) to compute the "jumps" of these functions.

The above method is used in §5 to indicate how the function $\hat{q}(k)$ can be constructed for: (a) Equation (1.3) with

$$q(x,0)=q_0(x),\ \ 0<x<\infty;\quad (-i\partial_x)^j q(\beta t,t)=f_j(t),\ \ 0<t<T,\ \ j=0,\dots,N-1, \tag{1.19}$$

where $q_0(x)$ is a decaying function, $f_j(t)$ is a smooth function and

$$N=n/2,\ \ n \text{ even};\quad N=(n+1)/2,\ \ n \text{ odd and } \alpha_n>0;\quad N=(n+1)/2,\ \ n \text{ odd and } \alpha_n>0. \tag{1.20}$$

(b) The Laplace equation and the modified Helmholtz equation in a wedge and a right isosceles triangle, where on each side

$$\alpha_j q_x+\beta_j q_y=f_j,\quad \sqrt{\alpha_j^2+\beta_j^2}=1, \tag{1.21}$$

with α_j and β_j real constants and f_j a real smooth function.

Regarding these problems we note: 1. For problem (a) the functions $\Psi_j(k)$ satisfy a system of algebraic equations. 2. For problem (b) with arbitrary α_j, β_j, the functions $\Psi_j(k)$ satisfy a Riemann-Hilbert problem. However, for a particular choice of α_j, β_j, the functions $\Psi_j(k)$ satisfy purely algebraic equations. The classical Dirichlet and Neumann boundary conditions are subcases of this particular class. 3. The determination of $\hat{q}(k)$ for other elliptic equations is similar to (b). Actually, using a certain transformation in the complex k-plane, see equation (3.4), it is possible to generalize the results valid for the Laplace equation to other elliptic equations. In this sense, the Laplace equation provides the simplest but still a generic example of the application of this new method to linear elliptic PDE's. 4. The method for determining $\hat{q}(k)$ can be applied to an arbitrary convex polygon, but in the general case the details are cumbersome, so for brevity of presentation we will only discuss here two simple examples. Boundary value problems for a general regular polygon will be discussed elsewhere.

2 Lax Pairs

Let $L(\partial_{x_1}, \partial_{x_2})$ be a linear differential operator in ∂_{x_1} and ∂_{x_2} with constant coefficients. A Lax pair of the equation $Lq = 0$ is given by [9]

$$\mu_{x_1} - ik\mu = q, \quad L\mu = 0. \tag{2.1}$$

In order to simplify the spectral analysis of equations (2.1) we rewrite these equations in a more convenient form. Using equation (2.1a) to eliminate the ∂_{x_1}-derivatives in equation (2.1b), the latter equation becomes an ODE. For evolution equations, this ODE is of first order. For example, if q satisfies equation (1.3) then equations (2.1) with $x_1 = x$, $x_2 = t$ become

$$\mu_x - ik\mu = q, \quad \mu_t + iw(k)\mu = -\tilde{q}(x,t,k), \tag{2.2}$$

where $\tilde{q}(x,t,k)$ is defined by

$$\tilde{q}(x,t,k) = \sum_{j=1}^{n} \alpha_j \left[(-i\partial_x)^{j-1} + k(-i\partial_x)^{j-2} + k^2(-i\partial_x)^{j-3} + \cdots + k^{j-1}\right] q(x,t).$$

If the highest ∂_{x_2}-derivative in the operator L is of order m, the resulting ODE is of order m. However, by rewriting this equation in a matrix form we can reduce the order of the relevant equation. In particular, if the associated matrix can be diagonalized, then the resulting equation is of first order. For example, if q satisfies equation (1.1), equation (2.1b) yields the ODE

$$\mu_{yy} - (k^2 - 4\alpha)\mu = -(q_x + ikq).$$

Rewriting this equation as a first order matrix equation for the vector $N = (\mu_y, \mu)^T$, and diagonalizing the relevant matrix, equations (2.1) yield

$$\nu_x - ik\nu = -\frac{i}{2}\left(q_y - \sqrt{k^2 - 4\alpha}\, q\right), \quad \nu_y + \sqrt{k^2 - 4\alpha}\,\nu = \frac{i}{2}(q_x + ikq). \tag{2.3}$$

In order to simplify the dependence on the spectral parameter, we let $k = \lambda + \alpha/\lambda$. Equations (2.3) indicate that $\nu \sim -q/2 + O(1/\lambda)$, $\lambda \to \infty$; in order to have a solution which decays as $\lambda \to \infty$, we let $\tilde{\nu} = \nu + q/2$. Renaming λ and $\tilde{\nu}$ as k and μ, and letting $z = x + iy$, equations (2.3) become equations (1.2). We note that in this simple example, equations (1.2) can be derived directly by viewing equation (1.1) as an evolution equation for q_z,

$$(q_z)_{\bar{z}} + \alpha q = 0. \tag{2.4}$$

The Lax pair of equation (2.4) is given by equation (1.2a) together with $\mu_{z\bar{z}} + \alpha\mu = 0$. Using (1.2a) and (2.4) this equation becomes (1.2b).

The explicit form of a Lax pair associated with a general linear differential operator in two independent variables is given in [21].

3 The Simultaneous Spectral Analysis

We assume that the equation $Lq = 0$ can be written as the compatibility condition of two first order equations,

$$\mu_{x_1} - if_1(k)\mu = q_1(x_1, x_2, k), \quad \mu_{x_2} - if_2(k)\mu = q_2(x_1, x_2, k), \tag{3.1}$$

where $f_1(k)$, $f_2(k)$ are given analytic functions of k, and q_1, q_2 are analytic functions of q and of its derivatives which depend analytically on k. The essential step in the spectral analysis of equations (3.1) is the construction of a solution $\mu(x_1, x_2, k)$ which is bounded in k for all complex k. We will show that in the case of a polygonal domain it is actually possible to construct a function μ which is sectionally analytic; this means that μ has different representations in different domains of the complex k-plane, $\mu = \mu_j$ for $k \in D_j$, where $\bigcup_j D_j = \mathbb{C}$, and each μ_j is analytic.

Proposition 3.1 Let Ω be the convex polygon defined in §1 with z replaced by $\zeta = x_1 + ix_2$. Let $f_1(k)$, $f_2(k)$ be analytic functions of k. Let $q_1(x_1,x_2,k)$, $q_2(x_1,x_2,k)$ be analytic functions of k, which are sufficiently smooth functions of x_1, x_2, and which if Ω is unbounded have sufficient decay as $\zeta \to \infty$. Assume that

$$(eq_1)_{x_2} = (eq_2)_{x_1}, \quad e = \exp[-if_1(k)x_1 - if_2(k)x_2]. \tag{3.2}$$

Define $\mu_j(x_1,x_2,k)$ by

$$\mu_j(x_1,x_2,k) = \int_{\Gamma(\zeta_j,\zeta)} \exp\left[if_1(k)(x_1-x_1') + if_2(k)(x_2-x_2')\right](q_1(x_1',x_2',k)dx_1' + q_2(x_1',x_2',k)dx_2'), \tag{3.3}$$

while $\Gamma(\zeta_j,\zeta)$ is a ray in the (x_1,x_2) plane defined as follows: If Ω is bounded, $\Gamma(\zeta_j,\zeta)$, $1\le j\le n$, is the ray from ζ_j to ζ; if Ω is unbounded $\Gamma(\zeta_j,\zeta)$, $2\le j\le n-1$, is the ray from ζ_j to ζ, $\Gamma(\zeta_1,\zeta)$ is a ray parallel to the ray $[\zeta_1=\infty,\zeta_2]$, and $\Gamma(\zeta_n,\zeta)$ is a ray parallel to the ray $(\zeta_n=\infty,\zeta_{n-1})$.

Then,

(i) μ_j satisfies the compatible equations (3.1).

(ii) If $\zeta' \in \Gamma(\zeta_j,\zeta)$, where $1\le j\le n$ when Ω is bounded and $2\le j\le n-1$ when Ω is unbounded, then $\arg(\zeta-\zeta')$ satisfies

$$\arg(\zeta-\zeta') \in [\arg(\zeta_{j-1}-\zeta_j), \arg(\zeta_{j+1}-\zeta_j)];$$

when Ω is unbounded, if $\zeta'\in\Gamma(\zeta_1,\zeta)$ or $\zeta'\in\Gamma(\zeta_n,\zeta)$, then $\arg(\zeta'-\zeta)$ satisfies, respectively

$$\arg(\zeta-\zeta') = \arg(\zeta_2-\zeta_1), \quad \arg(\zeta-\zeta') = \arg(\zeta_{n-1}-\zeta_n).$$

(iii) Let

$$\lambda = (f_2(k))_I + i(f_1(k))_I. \tag{3.4}$$

The exponential appearing in μ_j is bounded for $\lambda \in \tilde{S}_j$, where $\tilde{S}_j$ is a wedge in the complex λ-plane defined as follows. If Ω is bounded, $\tilde{S}_j$ is defined by

$$\tilde{S}_j = \{\lambda\in\mathbb{C}:\ \arg\lambda \in [-\arg(\zeta_{j-1}-\zeta_j), \pi - \arg(\zeta_{j+1}-\zeta_j)]\}, \quad 1\le j\le n; \tag{3.5a}$$

if Ω is unbounded, $\tilde{S}_j$ is defined by equations (3.5a) for $2\le j\le n$, $\tilde{S}_1$ is the half plane defined by

$$\tilde{S}_1 = \{\lambda\in\mathbb{C}:\ \arg\lambda\in[-\arg(\zeta_2-\zeta_1), \quad \pi-\arg(\zeta_2-\zeta_1)]\}, \tag{3.5b}$$

and $\tilde{S}_n$ is the half plane defined by

$$\tilde{S}_n = \{\lambda\in\mathbb{C}:\ \arg\lambda\in[-\arg(\zeta_{n-1}-\zeta_n), \quad \pi-\arg(\zeta_{n-1}-\zeta_n)]\}. \tag{3.5c}$$

(iv) The union of the sectors $\tilde{S}_j$ is the entire complex λ-plane.

(v) The sector $\tilde{S}_j$ lies to the right of $\tilde{S}_{j+1}$. For $1\le j\le n-1$ these sectors have as a common boundary the ray l_j defined by equation (1.12) with z and k replaced by ζ and λ respectively. If Ω is bounded, $\tilde{S}_n$ and $\tilde{S}_{n+1}=\tilde{S}_1$ intersect along the ray l_n; if Ω is unbounded $\tilde{S}_n$ and $\tilde{S}_1$ have a two dimensional intersection.

(vi) In a domain where both μ_i and μ_j are defined, they are related by

$$\mu_i - \mu_j = \exp\left[if_1(k)x_1 + if_2(k)x_2\right]\rho_{i,j}(k), \tag{3.6}$$

where

$$\rho_{i,j}(k) = \int_{\zeta_i}^{\zeta_j} \exp\left[-if_1(k)x_1 - if_2(k)x_2\right](q_1dx_1 + q_2dx_2). \tag{3.7}$$

In particular, equation (3.6) is valid for $i=j+1$ on $\lambda\in l_j$, $j=i,\ldots,n$.

(vii) The spectral functions $\rho_{j+1,j}(k)$ satisfy the cyclic relation (1.16). If Ω is unbounded,$\rho_{1,n}(k)=0$ and $\rho_{j+1,j}(k)$ satisfy the relation

$$\rho_{2,1}(k) + \rho_{3,2}(k) + \cdots + \rho_{n,n-1}(k) = 0, \quad \lambda\in\tilde{S}_1\cap\tilde{S}_n. \tag{3.8}$$

Proof (i) The function μ_j is a solution of equations (3.1) even if $\Gamma(\zeta_j,\zeta)$ is replaced by any smooth curve from ζ_j to ζ; furthermore, μ_j is independent of the particular choice of this curve. This is a consequence of equation (3.2), i.e. of the compatibility of equations (3.1). Indeed, this equation implies that the 1-form

$$W(x_1,x_2,k)=\exp[if_1(k)x_1+if_2(k)x_2](q_1dx_1+q_2dx_2) \tag{3.9}$$

is closed. Thus $\oint W=0$.

(ii)-(vii) The particular choice of $\Gamma(\zeta_j,\zeta)$ was discovered in [10] in connection with the Laplace equation. In this case the exponential appearing in μ_j is given by $\exp[ik(z-z')]$. The exponential appearing in (3.3) is bounded provided that

$$f_{1_I}(x_1-x_1')+f_{2_I}(x_2-x_2')>0 \quad \text{or} \quad Im[\lambda(\zeta-\zeta')]>0.$$

Thus the introduction of the function λ reduces the proof of (ii)-(vii) to the analogous proof for the Laplace equation. **QED**

This result shows that it is always possible to construct a sectionally analytic function μ. Indeed, let

$$\mu(x_1,x_2,k)=\mu_j(x_1,x_2,k), \quad \lambda\in\tilde{S}_j, \quad j=1,\cdots,n.$$

Under the transformation defined by equation (3.4), the sectors $\tilde{S}_j$, $j=1,\ldots,n$, are mapped to the domains S_j, $j=1,\ldots,N$. Assume that: (i) The domains S_j cover the entire k-plane. (ii) $\mu=O(1/k)$, $k\to\infty$. Suppose that L_{ij} is a curve in the intersection of S_i and S_j, then on this curve equation (3.6) is valid. These facts make it possible to reconstruct $\mu(x_1,x_2,k)$ as the unique solution of a Riemann-Hilbert problem [11], [20],

$$\mu(x_1,x_2,k)=\frac{1}{2i\pi}\sum_{i,j}\int_{L_{ij}}\exp[if_1(k')x_1+if_2(k')x_2]\,\frac{\rho_{i,j}(k')}{k'-k}dk'. \tag{3.10}$$

Equation (3.10) expresses μ in terms of the spectral function $\hat{q}(k)$, where $\hat{q}(k)$ is the union of all the $\rho_{i,j}(k)$'s. The spectral function defined by equation (3.7) involves integrals of q and of its derivatives along the boundary of the polygon. Then, either of equations (3.1) yields $q(x_1,x_2)$ as an integral of $\hat{q}(k)$ along the curve L, which is the union of the curves L_{ij}.

Remark 3.1. Suppose that instead of equations (3.1) we have

$$\mu_z-if_1(k)\mu=q_1, \quad \mu_{\bar{z}}-if_2(k)\mu=q_2. \tag{3.11}$$

Then the particular solution

$$\mu_j(z,\bar{z},k)=\int_{\Gamma(z_j,z)}\exp[if_1(k)(z-z')+if_2(k)(\bar{z}-\bar{z}')]\,(q_1(z',\bar{z}',k)dz'+q_2(z',\bar{z}',k)d\bar{z}'), \tag{3.12}$$

is also well defined. In analogy with the results of proposition 3.1, we now find

$$\rho_{i,j}(k)=\int_{z_i}^{z_j}\exp[-if_1(k)z-if_2(k)\bar{z}]\,(q_1dz+q_2d\bar{z}), \tag{3.13}$$

$$\mu(z,\bar{z},k)=\frac{1}{2i\pi}\sum_{i,j}\int_{L_{ij}}\exp[if_1(k')z+if_2(k')\bar{z}]\,\frac{\rho_{i,j}(k')}{k'-k}dk', \tag{3.14}$$

$$\lambda=(f_1-f_2)_R+i(f_1+f_2)_I. \tag{3.15}$$

For some problems it is more convenient to use equations (3.11)-(3.14), instead of equations (3.1), (3.3), (3.7), (3.10).

4 Derivation of propositions 1.1 and 1.2

(i) *A dispersive equation.*

Let $q(x,t)$ satisfy equation (1.3). Comparing equations (2.2) and (3.1) we find $x_1 = t$, $x_2 = x$, $f_1 = -w(k)$, $f_2 = k$, $q_1 = -\tilde{q}$, $q_2 = q$. Using $\arg(\zeta_1 - \zeta_2) = \frac{\pi}{2}$, $\arg(\zeta_2 - \zeta_3) = \theta$, $\arg(\zeta_4 - \zeta_3) = \frac{\pi}{2} - \theta$ (see figure 5), $\arg\varphi = \arg(-\varphi) - \pi$, equations (3.5) imply that the functions $\mu_1 = \mu_4$, μ_2, μ_3, defined by equation (3.3), are bounded for $\arg\lambda$ in the following intervals,

$$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right], \quad \left[-\frac{\pi}{2}, -\theta\right], \quad \left[-\theta, \frac{\pi}{2}\right]. \tag{4.1}$$

Equation (3.7) yields

$$\rho_{2,1} = \int_{\beta T}^{\infty} \exp\left[-ikx + iw(k)T\right] q(x,T)dx, \quad \rho_{4,3} = \int_{\infty}^{0} \exp\left[-ikx\right] q(x,0)dx,$$

$$\rho_{3,2} = \int_0^T \exp\left[-ik\beta t + iw(k)t\right] (\beta q(\beta t, t) - \tilde{q}(\beta t, t, k))dt. \tag{4.2}$$

Using the definitions (1.5) it follows that $\rho_{4,3} = -\hat{q}_0(k)$, $\rho_{3,2} = -\hat{Q}(k)$. The equation $\rho_{4,3} + \rho_{3,2} + \rho_{2,1} = 0$, is equation (1.7). The jumps defined by (4.2) split into the jump $\hat{q}_0(k)$ along $\lambda_I = 0$ and into the jump $\hat{Q}(k)$ along the boundary of the sector $-\frac{\pi}{2} \leq \arg\lambda \leq -\theta$, see figure 6. The transformation $\lambda = k_I - iw_I$ maps the sectors defined by (4.1) into the following domains in the k-plane: the lower half k-plane, the region D_+ defined by equation (1.7), and the region $\{k \in \mathbb{C},\ \mathrm{Im}(w(k) - \beta k) < 0\}$. Equation (3.10) together with $q = -i\lim_{k\to\infty}(k\mu)$, yield equation (1.4).

(ii) *The Laplace equation.*

Let q satisfy equation (1.1) with $\alpha = 0$. Comparing equations (1.2) evaluated at $\alpha = 0$, with equations (3.11) it follows that $f_1 = k$, $f_2 = 0$, $q_1 = q_z$, $q_2 = 0$. Equation (3.15) yields $\lambda = k$, hence L_{ij} is l_j. Equations (3.13) and (3.14), together with $q_z = -i\lim_{k\to\infty}(k\mu)$, yield equations (1.10) with $\alpha = 0$, and (1.11).

(iii) *The modified Helmholtz equation.*

Let q satisfy equation (1.1) with $\alpha = -\beta^2$. Comparing equations (1.2) evaluated at $\alpha = -\beta^2$, with equations (3.11) it follows that $f_1 = k$, $f_2 = -\beta^2/k$, $q_1 = q_z$, $q_2 = i\beta^2 q/k$. Equation (3.15) yields $\lambda = (1 + \beta^2/|k|^2)k$, hence there exist two singular points in the k-plane, $k = 0$ and $k = \infty$. However, the sectors S_j in the k-plane coincide with the sectors $\tilde{S}_j$ in the λ plane, hence again L_{ij} is l_j. Equations (3.13) and (3.14), together with $q = \lim_{k\to 0}\mu$, yield equations (1.10) with $\alpha = -\beta^2$, and (1.13).

(iv) *The Helmholtz equation.*

Let q satisfy equation (1.1) with $\alpha = \beta^2$. In this case equation (3.15) yields $\lambda = (1 - \beta^2/|k|^2)k$, hence there exist two singular points in the k-plane, $k = 0$ and $k = \infty$. The sector $\tilde{S}_j$ in the λ plane is mapped to the domains $S_j^{(1)}$ and $S_j^{(2)}$ in the k-plane depicted in figure 7. The union of these domains is the k-plane. The determination of the spectral function on L_j (see figure 2) follows from the fact that L_j separates regions where μ_i and μ_{j+1} are defined. We note that it is sufficient to determine the spectral function on one circular arc only; the others follow from the condition that at any corner the sum of the spectral functions is zero (provided that all orientations point towards this corner).

5 The Solution of Boundary Value Problems

Using the methodology introduced in §1.3, we now discuss some typical boundary value problems for evolution equations and for elliptic equations.

5.1 Evolution equations

Let $q(x,t)$ satisfy equations (1.3) and (1.19). In this case the spectral function $\hat{q}(k)$ consists of $\hat{q}_0(k)$ and of $\hat{Q}(k)$, see equation (1.5). $\hat{q}_0(k)$ is the Fourier transform of the known function $q_0(x)$, see equation (1.5a).

$\hat{Q}(k)$ involves the known functions $\hat{Q}_l(k)$, $0 \le l \le N-1$, and the unknown functions $\psi_l(k)$, $N \le l \le n-1$, where

$$\hat{Q}_l(k) = \int_0^T \exp\left[i(\omega(k) - \beta k)t\right] f_l(t)\, dt, \quad \psi_l(k) = \int_0^T \exp\left[i(\omega(k) - \beta k)t\right](-i\partial_x)^l q(\beta t, t)\, dt. \tag{5.1}$$

Substituting equations (5.1) in equation (1.5b), and then substituting the resulting equation in the global relation (1.7), we find 1 equation involving $n - N$ unknown functions $\psi_l(k)$ and the unknown integral term appearing in the rhs of equation (1.7). However, this integral term does *not* contribute to $q(x_1, x_2)$ and can be neglected. This is a consequence of the fact that $q(x,t)$ remains invariant if $\hat{Q}(k)$ is replaced by $\hat{Q}(k) + \exp\big(i\omega(k)T\big)\tilde{q}(k)$, where $\tilde{q}(k)$ is a function analytic for $\mathrm{Im}k \ge 0$.

Each unknown function $\psi_l(k)$ remains invariant under the transformation $k \mapsto \lambda(k)$, where $\lambda(k)$ are the roots of the equation

$$\omega(k) - \beta k = \omega\big(\lambda(k)\big) - \beta\lambda(k). \tag{5.2}$$

It is shown in [12] that there exist $n - N$ roots in D_-, where N is defined by equation (1.20). Thus the above transformations yield the $n - N$ algebraic equations

$$\hat{Q}\big(\lambda(k)\big) = -\hat{q}_0\big(\lambda(k)\big), \tag{5.3}$$

for the $n - N$ functions $\psi_l(k)$. Then equations (5.1) and (1.5b) yield $\hat{Q}(k)$, and equation (1.4) yields $q(x,t)$.

Example 5.1 Consider equation (1.8). In this case $n = 3$ and $\alpha_n = -1$, thus $N = 1$. Equation (5.2) becomes $k^3 + \beta k = \lambda^3 + \beta\lambda$, thus

$$\lambda_1 = \frac{-k + i\sqrt{3k^2 + 4\beta}}{2}, \quad \lambda_2 = \frac{-k - i\sqrt{3k^2 + 4\beta}}{2}.$$

Using the definition of $\hat{Q}(k)$, i.e. equation (1.5b), equations (5.3) become

$$\psi_2(k) + i\lambda_j\psi_1(k) - (\lambda_j^2 + \beta)\hat{Q}_0(k) = -\hat{q}_0(\lambda_j), \quad j = 1, 2, \tag{5.4}$$

where

$$\hat{q}_0(k) = \int_0^\infty \exp\left[ikx\right] q_0(x) dx, \qquad \hat{Q}_0(k) = \int_0^T \exp\left[-ik\beta t - ik^3 t\right] f_0(t) dt.$$

Equations (5.4) are 2 equations for the 2 unknown functions ψ_l, $l = 1, 2$; solving these equations for ψ_l and substituting these expressions in $\hat{Q}(k)$, we find $\hat{Q}(k)$ in terms of $q_0(x)$ and $f_0(t)$,

$$\hat{Q}(k) = -(3k^2 + \beta)\hat{Q}_0(k) + \frac{(\lambda_2 - k)\hat{q}_0(\lambda_1) - (\lambda_1 - k)\hat{q}_0(\lambda_2)}{\lambda_1 - \lambda_2}.$$

5.2 Elliptic equations

For concreteness we consider a closed polygon and we assume that on every side of the polygon, $q(x,y)$ satisfies the boundary condition specified by equation (1.21). Unbounded polygons and other boundary conditions can be treated in a similar way. For brevity of presentation we give the details for the Laplace equation and indicate the modifications needed for other elliptic equations.

The Laplace equation

The first step of the method introduced in §1.3 involves substituting the boundary condition (1.21) in the definition of $\rho_{j+1,j}(k)$. For the Laplace equation, using the local coordinates $z = z_j + s\exp[i\theta_j]$, this yields

$$\rho_{j+1,j}(k) = (\beta_j + i\alpha_j)\exp\left[i\theta_j - ikz_j\right] \int_0^{|z_{j+1} - z_j|} \exp\left[-ie^{i\theta_j} ks\right] q_j(s)\, ds - F_j(k), \tag{5.6}$$

where $q_j = -\beta_j q_x + \alpha_j q_y$, and F_j is a known function. Substituting equation (5.6) into the global relation (1.16) we find

$$\sum_{j=1}^n \exp\left[i\omega_j - ikz_j\right]\psi_j(-ie^{i\theta_j}k) = F(k), \quad k \in \mathbb{C}, \tag{5.7}$$

where

$$\omega_j = \theta_j + \arg(\beta_j + i\alpha_j), \quad \psi_j = \int_0^{|z_{j+1}-z_j|} \exp\left[-ie^{i\theta_j}ks\right] q_j(s)ds, \tag{5.8}$$

and $F(k)$ is a known function. The complex conjugate of equation (5.7) yields

$$\sum_{j=1}^{n} \exp\left[-i\omega_j + ik\bar{z}_j\right] \psi_j(ie^{-i\theta_j}k) = \overline{F(\bar{k})}, \quad k \in \mathbb{C}. \tag{5.9}$$

Equations (5.7) and (5.9) are 2 equations for the n pairs of unknown functions $\{\psi_j(-ie^{i\theta_j}k), \psi_j(ie^{-i\theta_j}k)\}$.

The second step involves finding the transformations in the complex k-plane which leave each of the above pairs invariant. For the pair corresponding to the fixed subscript l, this transformation is $\{k \mapsto -\exp(-2i\theta_l)k; k \mapsto -\exp(2i\theta_l)k\}$. In general, the transformations corresponding to different l's are different, thus this procedure can be continued. This leads to a proliferation of the unknown functions, and thus for an arbitrary polygon the general construction becomes cumbersome. This step will be illustrated below for two simple examples.

The third step involves using the equations obtained in the second step to determine the unknown functions either through the solution of a system of algebraic equations or through the solution of a Riemann-Hilbert (RH) problem.

Example 5.2. Let $q(x, y)$ satisfy the Laplace equation in a wedge of angle $\pi/4$, with the boundary condition (1.21) on each side of the wedge, see figure 8a. In this case,

$$q_x - iq_y = \int_{\ell_1} \exp\left[ikz\right] \rho_{2,1}(k)\, dk + \int_{\ell_2} \exp\left[ikz\right] \rho_{3,2}(k)\, dk, \tag{5.10}$$

where the rays ℓ_1, ℓ_2 are depicted in figure 8b, and the spectral functions are defined by

$$\rho_{2,1}(k) = e^{i\omega_1} \int_0^\infty \exp\left[-ikx\right] q_1(x)\, dx - F_1(k), \quad \rho_{3,2}(k) = e^{i\omega_2} \int_0^\infty \exp\left[-ie^{i\frac{\pi}{4}}ks\right] q_2(s)\, ds - F_2(k), \tag{5.11}$$

where F_1, F_2 are known functions and $\omega_1 = \arg(-\beta_1 - i\alpha_1)$, $\omega_2 = \arg(\beta_2 + i\alpha_2) + \pi/4$. Let ψ_1 and ψ_2 be the unknown functions in equation (5.11). The global relation (1.18) yields

$$\psi_1(-ik) + e^{i\omega}\psi_2(-ie^{i\pi/4}k) = F(k), \quad \arg k \in [\pi, 2\pi - \pi/4], \tag{5.12a}$$

where $\omega = \omega_2 - \omega_1$ and $F(k)$ is a known function. The complex conjugate of equation (5.12a) is

$$\psi_1(ik) + e^{-i\omega}\psi_2(ie^{-i\pi/4}k) = \overline{F(\bar{k})}, \quad \arg k \in \left[\frac{\pi}{4}, \pi\right]. \tag{5.12b}$$

The set $\{\psi_1(-ik); \psi_1(ik)\}$ is invariant under $k \mapsto -k$, while the set involving ψ_2 is invariant under $\{k \mapsto ik; k \mapsto -ik\}$. Thus equations (5.12) are supplemented by the equations

$$\psi_1(k) + e^{i\omega}\psi_2(ie^{-i\pi/4}k) = F(ik), \quad \arg k \in \left[\frac{\pi}{2}, \frac{5\pi}{4}\right], \tag{5.13a}$$

$$\psi_1(k) + e^{-i\omega}\psi_2(-ie^{i\pi/4}k) = \overline{F(-i\bar{k})}, \quad \arg k \in \left[\frac{3\pi}{4}, \frac{3\pi}{2}\right], \tag{5.13b}$$

and the equations obtained by replacing k with $-k$ in (5.12) and (5.13); we refer to the latter equations as $(5.12)'$ and $(5.13)'$.

Equations (5.12), (5.13), $(5.12)'$, $(5.13)'$ are the basic equations needed for the determination of ψ_1 and ψ_2: equations (5.12) and (5.13) are valid for $k \in \mathbb{R}^-$ and the function $\{\psi_1(-ik), \psi_1(ik)\}$ has a jump across this ray; thus we must use these equations to compute this jump. Using (5.12a), (5.13b), (5.13a), (5.12b), we find

$$\psi_1(-ik) - e^{4i\omega}\psi_1(ik) = f(k), \quad k \in \mathbb{R}^-, \tag{5.14a}$$

where $f(k)$ is a known function. Letting $k \mapsto -k$ in this equation (which is equivalent to using equations (5.12)$'$, (5.13)$'$) we find

$$\psi_1(-ik) - e^{-4i\omega}\psi_1(ik) = e^{-4i\omega} f(-k), \quad k \in \mathbb{R}^+. \tag{5.14b}$$

Equations (5.14) together with the estimate $\psi_1(k) = \mathcal{O}(1/k)$, $k \to \infty$, define a scalar RH problem, which can be solved in closed form by standard methods. We note that this RH problem is discontinuous at the origin, unless $\exp(8i\omega) = 1$.

We will now show that if the condition $\exp(8i\omega) = 1$ is satisfied, it is possible to avoid solving the above RH problem and to determine the spectral functions using only algebraic manipulations. Indeed, we first note that $\exp(ikz)$ is bounded in the wedge formed by the rays ℓ_1 and ℓ_2: $0 < \arg k < 3\pi/4$, $0 < \arg z < \pi/4$, thus $0 < \arg(kz) < \pi$. Hence the rhs of equation (5.10) is invariant if $\rho_{2,1} \to \rho_{2,1} + \tilde{q}(k)$, $\rho_{3,2} \to \rho_{3,2} - \tilde{q}(k)$, where $\tilde{q}(k)$ is analytic and bounded for $0 < \arg k < 3\pi/4$. Using equations (5.12b)$'$, (5.13a)$'$, (5.13b)$'$, (5.12a)$'$, and equations (5.13b), (5.13a), (5.12b), respectively, we find

$$\psi_1(-ik) = e^{-4i\omega}\psi_1(ik) + f_1(k), \quad \psi_2(-ie^{i\pi/4}k) = -e^{3i\omega}\psi_1(ik) + f_2(k),$$

where $f_1(k)$ and $f_2(k)$ are known functions. Thus

$$q_x - iq_y = e^{i\omega_1 - 4i\omega}\int_{\ell_1} \exp[ikz]\,\psi_1(ik)\,dk - e^{i\omega_2+3i\omega}\int_{\ell_2} \exp[ikz]\,\psi_1(ik)\,dk + G(k), \tag{5.15}$$

where $G(k)$ is a known function. Hence if $\exp(i\omega_1 - 4i\omega) = \exp(i\omega_2 + 3i\omega)$, i.e. $\exp(8i\omega) = 1$, the unknown function $\psi_1(ik)$ gives zero contribution. For Neumann boundary conditions q_y is given on (z_2, z_1) and $q_y - q_x$ is given on (z_2, z_3), thus $\alpha_1 = 0$, $\beta_1 = 1$, $\alpha_2 = -\beta_2 = 1/\sqrt{2}$. For Dirichlet boundary conditions q_x is given on (z_2, z_1) and $q_y + q_x$ is given on (z_2, z_3), thus $\alpha_1 = 1$, $\beta_1 = 0$, $\alpha_2 = \beta_2 = 1/\sqrt{2}$. In both cases $\exp(8i\omega) = 1$; thus the spectral functions for the Neumann and Dirichlet problems can be obtained purely through algebra.

The case of a wedge of an arbitrary angle can be solved through a discontinuous RH problem, see [10].

Example 5.3. Let $q(x, y)$ satisfy the Laplace equation in a right isosceles triangle, with the boundary condition (1.21) on each side of the triangle, see figure 9a. In this case,

$$q_x - iq_y = \int_{\ell_1} \exp[ikz]\,\rho_{2,1}(k)\,dk + \int_{\ell_2} \exp[ikz]\,\rho_{3,2}(k)\,dk + \int_{\ell_3} \exp[ikz]\,\rho_{1,3}(k)\,dk, \tag{5.16}$$

where the rays ℓ_1, ℓ_2, ℓ_3 are depicted in figure 9b, and the spectral functions are given by

$$\rho_{2,1}(k) = e^{i\omega_1}\int_0^l \exp[-ikx]\,q_1(x)\,dx - F_1(k), \quad \rho_{3,2}(k) = e^{i\omega_2}\int_0^l \exp[ky]\,q_2(y)\,dy - F_2(k),$$

$$\rho_{1,3}(k) = e^{i\omega_3}e^{kl}\int_0^l \exp[-(1+i)kx]\,q_3(x)\,dx - F_3(k), \tag{5.17}$$

where F_j, are known functions and $\omega_1 = \arg(-\beta_1 - i\alpha_1)$, $\omega_2 = \arg(-\alpha_2 + i\beta_2)$, $\omega_3 = \arg(\beta_3 + i\alpha_3) - \pi/4$. If ψ_j's denote the unknown functions in equations (5.17), the global relation (1.16) and its complex conjugate yield

$$e^{i\omega_1}\psi_1(-ik) + e^{i\omega_2}\psi_2(k) + e^{i\omega_3}e^{kl}\psi_3\big((-1-i)k\big) = F(k), \tag{5.18a}$$

$$e^{-i\omega_1}\psi_1(ik) + e^{-i\omega_2}\psi_2(k) + e^{-i\omega_3}e^{kl}\psi_3\big((-1+i)k\big) = \overline{F(\bar{k})}, \tag{5.18b}$$

where $F(k) = \sum_{j=1}^{3} F_j(k)$. The set involving ψ_3 remains invariant under $\{k \mapsto -ik; k \mapsto ik\}$. Thus we supplement equations (5.18) with the equations

$$e^{i\omega_1}\psi_1(-k) + e^{i\omega_2}\psi_2(-ik) + e^{i\omega_3}e^{-ikl}\psi_3\big((-1+i)k\big) = F(-ik), \tag{5.19a}$$

$$e^{-i\omega_1}\psi_1(-k) + e^{-i\omega_2}\psi_2(ik) + e^{-i\omega_3}e^{ikl}\psi_3\big((-1-i)k\big) = \overline{F(-i\bar{k})}. \tag{5.19b}$$

The terms involving ψ_1 in equations (5.18), and ψ_2 in equations (5.19), remain invariant under $k \mapsto -k$, thus we supplement the above equations with the equations obtained from equations (5.18) and (5.19) by replacing k with $-k$; we refer to the latter equations as (5.18)$'$ and (5.19)$'$.

Equations (5.18), (5.19), (5.18)$'$, (5.19)$'$ can be used to obtain a RH problem for the unknown functions. In the particular case of $\exp[4i\omega_{12}] = 1$, and $\exp[8i\omega_{32}] = 1$, where $\omega_{ij} = \omega_i - \omega_j$, it is possible to avoid solving this RH problem and to obtain the spectral functions using only algebraic manipulations, see [10] for details. Both, the Neumann and Dirichlet boundary conditions belong to this particular case.

The Modified Helmholtz equation.

Using integration by parts it follows that the unknown part of $\rho_{j+1,j}(k)$ is given by

$$\mathrm{e}^{-ikz_j+i\frac{\beta^2}{k}\bar{z}_j}\Big[\frac{(\beta_j+i\alpha_j)\mathrm{e}^{i\theta_j}}{2}+\frac{\beta^2(\alpha_j\sin\theta_j-\beta_j\cos\theta_j)}{\beta^2-k^2e^{2i\theta_j}}\Big]\psi_j\big(-ie^{i\theta_j}k+i\frac{\beta^2}{k}e^{-i\theta_j}\big), \tag{5.20}$$

where ψ_j is defined in equation (5.8b). The complex conjugate of ψ_j yields $\psi_j\big(ie^{-i\theta_j}k - i\beta^2 e^{i\theta_j}/k\big)$, thus the pair of the unknown functions corresponding to $j = l$ remains invariant under the *same* transformation used for the Laplace equation. This implies that the procedure for obtaining the ψ_j's is *identical* with that for the Laplace equation. In particular, one obtains similar RH problems but with more complicate jump conditions, since $\exp[-ikz_j + i\omega_j]$ is now replaced by the coefficient of ψ_j in equation (5.20). It is important to emphasize that, just as in the case of the Laplace equation, for a large class of boundary value problems, the RH problems can be avoided and the spectral functions can be obtained through algebraic manipulations.

Example 5.4. Let $q(x, y)$ satisfy the modified Helmholtz equation in a wedge of angle $\pi/4$, with the boundary conditions (1.21) on every side of the wedge. In this case

$$\rho_{2,1}(k) = \frac{e^{i\omega_1}k^2+\beta^2e^{-i\omega_1}}{2(\beta^2-k^2)}\Psi_1(-ik+\frac{i\beta^2}{k}) - F_1(k),$$

$$\rho_{3,2}(k) = \frac{e^{i\pi/4}(e^{i\omega_2}k^2+\beta^2e^{-i\omega_2})}{2(\beta^2-ik^2)}\Psi_2(-ie^{i\pi/4}k+\frac{i\beta^2e^{-i\pi/4}}{k}) - F_2(k),$$

where $\omega_1 = \arg(\beta_1 + i\alpha_1)$, $\omega_2 = \arg(\alpha_2 - i\beta_2)$ and F_1, F_2 are known functions. Following steps identical to those used in example 5.2, we find a RH problem for the function Ψ_1 with a jump along the real axis given by $J(k)$ for $k \in \mathbb{R}^-$ and $J^{-1}(k)$ for $k \in \mathbb{R}^+$, where

$$J(k) = -e^{4i\omega}\frac{g_1(k)\overline{g_2(\bar{k})}}{\overline{g_1(\bar{k})}g_2(k)}, \qquad \omega = \omega_2 - \omega_1, \qquad g_j(k) = k^4 - \beta^4e^{4i\omega_j}, \quad j = 1, 2.$$

This problem is continuous at $k = 0$ if and only if $e^{8i\omega} = 1$. If the condition $e^{4i\omega} = 1$ is valid, then $J(k) = J^{-1}(k)$. In this case it is possible to avoid solving this RH problem and to determine the spectral functions using only algebraic manipulations.

Remark 5.2. Other elliptic equations can be treated similarly. For example, the unknown functions for Helmholtz equation remain invariant under the transformation used for the Laplace equation and the transformation $k \mapsto -\beta^2/k$.

6 Conclusions

Let $q(x_1, x_2)$ satisfy a linear two dimensional PDE, $Lq = 0$, in a polygon Ω. We propose the following steps for identifying and solving well-posed boundary value problems for $Lq = 0$, $(x_1, x_2) \in \Omega$. (1) Construct its Lax pair, see §2. (2) Under the *assumption of existence of a solution* $q(x_1, x_2)$, construct the spectral functions $\rho_{i,j}(k)$ and express $q(x_1, x_2)$ in terms of $\rho_{i,j}(k)$, see §3. (3) Investigate the global relation $\sum_1^n \rho_{j+1,j}(k) = 0$, $k \in \mathbb{C}$, for a closed polygon, or the analogous equation for an open polygon; this relation can be used to identify well-posed problems, and to express $\rho_{i,j}(k)$ in terms of the given boundary conditions, see §5. (4) Rigorously justify the formulae constructed in (2) and (3) *without the a-priori assumption of existence*: Define $\rho_{i,j}(k)$ in terms of the given boundary conditions through the formulae obtained in (3) and define $q(x_1, x_2)$ in terms of $\rho_{i,j}(k)$ through the formulae obtained in (2). Then, by choosing the given boundary data in an appropriate Sobolev space, prove that $q(x_1, x_2)$ is well defined, it solves the given PDE and it satisfies the given boundary conditions.

In this paper we have implemented (1) and (2) for equations (1.1) and (1.3), and have illustrated the implementation of (3) for some typical boundary value problems. The rigorous investigation of equation (1.3), i.e. step (4), is presented in [12] where existence and uniqueness is proved for initial and boundary data in a certain fractional Sobolev space. Two-point boundary value problems for equation (1.3) are solved in [13]. Several other boundary value problems for equation (1.1) are discussed in [10], [14]. Boundary value problems for the bi-harmonic equation are under investigation.

We conclude with some remarks: 1. Since the differential 1-form $W(x_1, x_2, k)$ defined by equation (3.9) is closed, it follows that

$$\oint_{\partial\Omega} W = 0,$$

where $\partial\Omega$ denotes the boundary of the polygonal domain. This is precisely the global relation. An important consequence of this fact is that it is possible to analyse the global relation *before* finding an integral representation for the solution. 2. Equations (1.10)–(1.14) provide the proper generalization of the Fourier transform for the Laplace, the modified Helmholtz and the Helmholtz equations in an arbitrary convex polygon. 3. The simultaneous spectral analysis of the Lax pair of a linear PDE in a polygon yields a scalar Riemann-Hilbert (RH) problem. The corresponding analysis of a nonlinear integrable PDE yields a matrix RH problem, see for example [5], [9], [15], [16]. We note that RH problems have appeared recently in a variety of other important applications in mathematical physics, including matrix models in quantum gravity, statistical mechanics, orthogonal polynomials and random matrices [17]. 4. For elliptic equations with general boundary conditions, the investigation of the global relation also yields a RH problem. We note that both the Wiener-Hopf and the Malyuzhinet's problems can be formulated as RH problems. This, in addition to emphasizing the fundamental role played by RH problems, suggests that: The extensive armament of ingenious techniques developed by many investigators for the solution of particular boundary value problems for elliptic equations can be used for the solution of the global relation satisfied by the spectral function. 5. An important advantage of the formulae obtained by the new method is their explicit x_1, x_2 dependence. This is particularly useful for the study of certain asymptotic properties of $q(x_1, x_2)$. For example, for linear evolution PDE's it is straightforward to obtain the long time asymptotic behavior of $q(x, t)$ using the steepest descent method. For nonlinear integrable PDE's this can be carried out using the elegant nonlinearization of the steepest descent method of Deift and Zhou [18]. 6. It is possible to extend the new method to a domain containing curved parts [19]. In this case the role of the RH problem is played by the d-bar problem [11]. 7. The expression for $q(x_1, x_2)$ is *spectrally decomposed*, i.e. the x_1, x_2 dependence is separated from the spectral functions $\rho_{i,j}(k)$. This is consistent with the celebrated Ehrenpreis principle, see [20] for further discussion. 8. Recently, progress has been made for the extension of this method to multidimensions, see [20].

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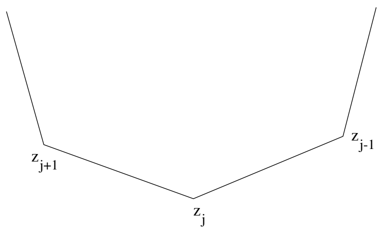

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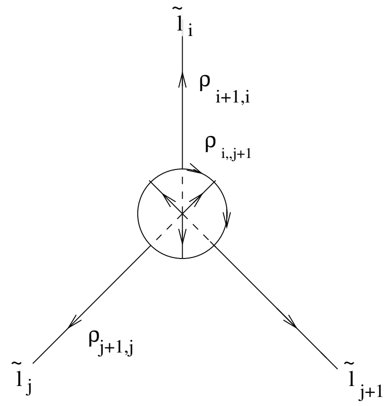

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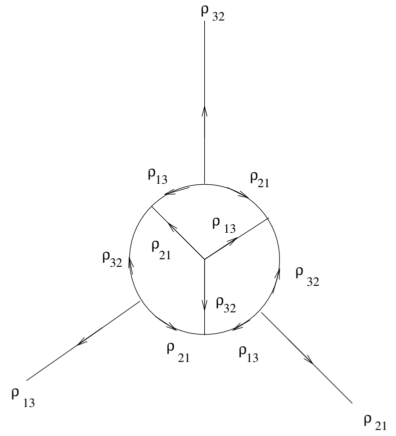

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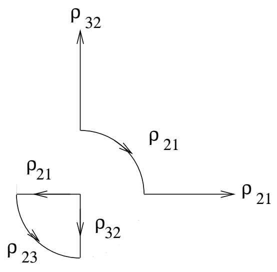

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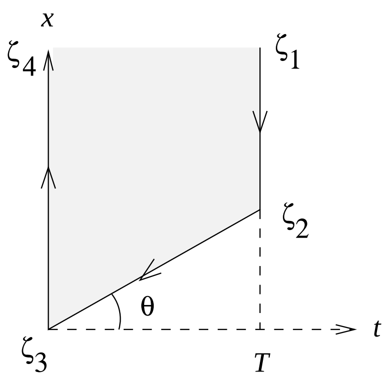

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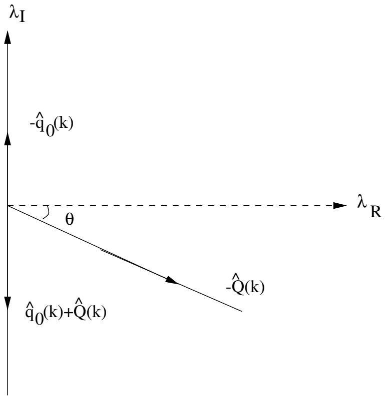

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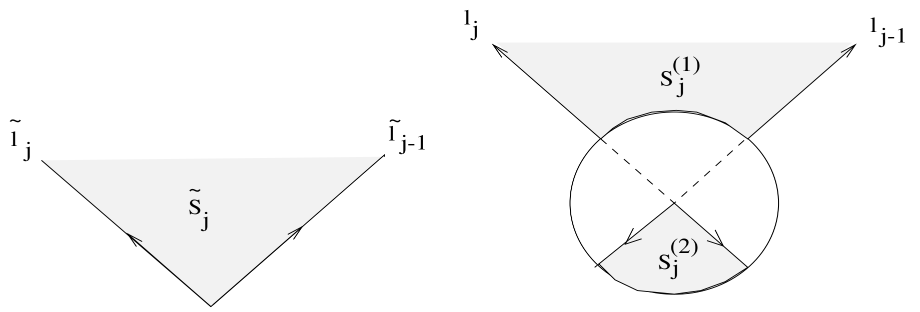

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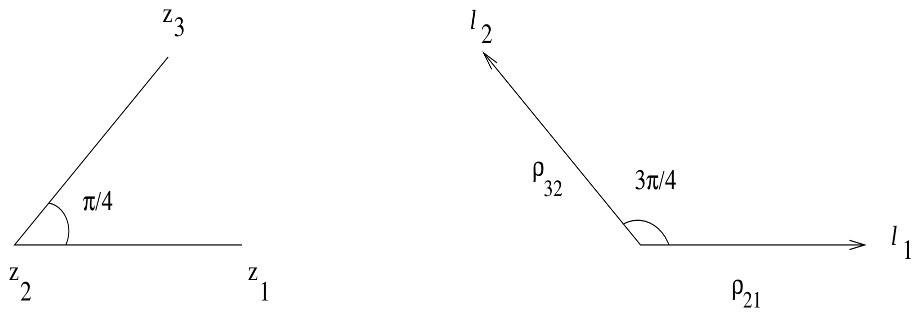

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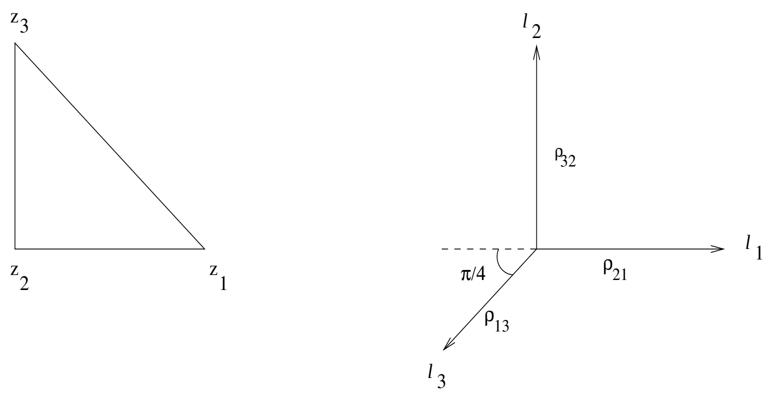

Figure 9: