A New Transform Method for Evolution PDEs

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Abstract

We introduce a new transform method for solving initial-boundary value problems for linear evolution PDEs with spatial derivatives of arbitrary order. This method is illustrated by solving several such problems on the half line $\{t > 0, 0 < x < \infty\}$, and on the quarter plane, $\{t > 0, 0 < x_j < \infty,\ j = 1, 2\}$. For equations in one space dimension this method constructs $q(x, t)$ as an integral in the complex k-plane involving an x-transform of the initial condition and a t-transform of the boundary conditions. For equations in two space dimensions it constructs $q(x_1, x_2, t)$ as an integral in the complex (k_1, k_2)-planes involving an (x_1, x_2)-transform of the initial condition, an (x_2, t)-transform of the boundary conditions at $x_1 = 0$, and an (x_1, t)-transform of the boundary conditions at $x_2 = 0$. This method is simple to implement and it yields integral representations which are particularly convenient for computing the long time asymptotics of the solution.

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1 Introduction

Let the scalar function $q(x,t)$ satisfy a linear evolution equation on the half line,

$$(\partial_t + w(-i\partial_x))(q(x,t)) = 0, \quad 0 < x < \infty, \quad t > 0, \tag{1.1}$$

where $w(k)$ is an arbitrary polynomial. Physically significant examples of such problems are: the Schrödinger equation with zero potential

$$iq_t + q_{xx} = 0, \quad w(k) = ik^2; \tag{1.2}$$

the heat equation

$$q_t = q_{xx} + q_x, \quad w(k) = k^2 - ik; \tag{1.3}$$

the Airy equation

$$q_t + q_{xxx} + q_x = 0, \quad w(k) = i(k - k^3). \tag{1.4}$$

Equation (1.2) contains the elastic wave equation

$$u_{tt} + u_{xxxx} = 0, \tag{1.5}$$

under the reduction $q = u + iv$, u, v real.

A new transform method for solving initial-boundary value (IBV) problems for (1.1) was introduced in [1]. Here we first illustrate this method by solving some IBV problems for equations (1.2)-(1.5), and then extend this method to linear separable evolution PDEs on the quarter plane,

$$(\partial_t + w_1(-i\partial_{x_1}) + w_2(-i\partial_{x_2}))\, q(x_1, x_2, t) = 0, \quad 0 < x_j < \infty, \quad j = 1,2, \quad t > 0, \tag{1.6}$$

where $w_1(k)$ and $w_2(k)$ are arbitrary polynomials. Examples of such problems are:

$$iq_t + q_{x_1x_1} + q_{x_2x_2} = 0, \quad w_j(k) = ik^2, \quad j = 1,2,; \tag{1.7}$$

$$q_t = q_{x_1x_1} + q_{x_1} + q_{x_2x_2} + q_{x_2}, \quad w_j(k) = k^2 - ik, \quad j = 1,2; \tag{1.8}$$

$$q_t + q_{x_1x_1x_1} + q_{x_2x_2x_2}, \quad w_j(k) = -ik^3, \quad j = 1,2. \tag{1.9}$$

Equation (1.7) contains the elastic wave equation

$$\partial_t^2 u + \left(\partial_{x_1}^2 + \partial_{x_2^2}\right)^2 u = 0, \tag{1.10}$$

under the reduction $q = iv$, u, v real.

One may attempt to solve equation (1.1) using a Laplace transform in t. This yields the ODE

$$w(-i\partial_x)\hat{q}(x,s) + s\hat{q} = q(x,0), \quad 0 < x < \infty. \tag{1.11}$$

However, this approach has several disadvantages: (i) Since t is finite, the Laplace transform is not an appropriate transform. Although one can justify this approach by appealing to "causality" arguments, one must be particularly careful in the case that the boundary values at $x = 0$ are periodic functions of t. (ii) The solution of the ODE (1.11) involves the investigation of the algebraic equation $w(\lambda) + s = 0$. This is cumbersome, even for the simple equation (1.4), where λ satisfies the cubic equation $\lambda^3 - \lambda + is = 0$.

Alternatively one may attempt to solve equation (1.1) using an appropriate x-transform. Indeed *if such a transform exists*, this is a convenient method of solution. For example, equation (1.2) where either $q(0,t)$ or $q_x(0,t)$ is given, can be solved by either the sine or the cosine transform. Similarly equation (1.2) where the combination $q_x(0,t) + \alpha q(0,t)$ is given, α constant, can also solved by an appropriate x-transform [2], [3]. However, there does *not* exist a proper x-transform for neither equation (1.3) nor for equation (1.4) [4].

The applicability of either the t-Laplace transform or an x_j-transform, $j = 1, 2$, is even more problematic for equation (1.6). Indeed, the Laplace transform yields a PDE for which in general there does *not* exist an appropriate $_j$-transform. This is indeed the case for both equations (1.8) and (1.9).

A. The New Method for Equations in One Spatial Variable

For equation (1.1) the number of boundary conditions that one needs to prescribe at $x = 0$ depends on the order of the highest derivative, and if this order is odd it also depends on the coefficient of the highest derivative. Suppose that the highest order term of $w(k)$ is $\alpha_n k^n$. It is shown in §2 that the number of boundary conditions u is given by

$$N = \begin{cases} \frac{n}{2}, & n \quad \text{even} \\ \frac{n+1}{2}, & n \quad \text{odd}, \quad \alpha_n \neq -i \\ \frac{n-1}{2}, & n \quad \text{odd}, \quad \alpha_n = -i. \end{cases} \tag{1.12}$$

For example for equations (1.2)-(1.4), $N = 1$.

Associated with equation (1.1) we prescribe initial and N boundary conditions

$$q(x,0) = q_0(x), 0 < x < \infty; \partial_x^j q(0,t) = g_j(t), t > 0, j = 0, 1, \cdots, N-1, \tag{1.13}$$

where N is given by (1.12), the function $q_0(x)$ has sufficient smoothness and decay, the functions $\{g_j(t)\}_0^{N-1}$ have sufficient smoothness and are compatible with $q_0(x)$ at $x = t = 0$, i.e. $\{\partial_x^j q_0(0) = g_j(0)\}_{j=0}^{N-1}$. Furthermore, we assume that for real k, $Re w(k) \geq 0$. The solution of this IBV problem can be obtained as follows: Given q_0 and $\{g_j\}_0^{N-1}$ compute the functions $\hat{q}_0$ and $\{\hat{g}_s\}_0^{N-1}$ by

$$\hat{q}_0(k) = \int_0^\infty dx e^{-ikx} q_0(0), \quad \hat{g}_j(k,t) = \int_0^t d\tau e^{w(k)\tau} g_j(\tau), \quad j = 0, 1, \cdots, N-1. \tag{1.14}$$

We note that $\hat{q}_0$ is well defined for $Im k \leq 0$, while $\hat{g}_j$ is well defined for all complex k. Given $\hat{q}_0$ and $\{\hat{g}_j\{_0^{N-1}$, compute the function $\hat{g}$ by solving a system of $n - N$ linear algebraic equations; this system is derived in section 2. Denote by D_+ the domain in the upper half of the complex k-plane specified by $Re\ w(k) < 0$. Denote by ∂D_+ the boundary of this domain oriented so that D_+ is on the left of ∂D_+. Then $q(x,t)$ is given by

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty dk e^{ikx - w(k)t} \hat{q}_0(k) + \frac{1}{2\pi} \int_{\partial D_+} dk e^{ikx - w(k)t} \hat{g}(k,t). \tag{1.15}$$

A simple derivation of this result is given in section 2, where also the detailed derivation of $\hat{g}(k,t)$ for equations (1.2)-(1.4) is presented.

Examples 1.1

For equations (1.2)-(1.4) the contours ∂D_+ are the boundary of the first quadrant of the complex k-plane, the curve $\{k_R^2 - k_I^2 + k_I = 0,\ k_I > 0\}$, and the curves $\{k_I^2 - 3k_R^2 + 1 = 0,\ k_I \geq 0\}$ respectively. These contours are depicted in Figures 1.1

Figure 1.1

For equations (1.2)-(1.4) with prescribed values of $q(x,0) = q_0(x)$ and $q(0,t) = g_0(t)$, $\hat{g}_0(k,t)$ is real by equation (1.14b) with $j = 0$, and $\hat{g}(k,t)$ is given respectively by each of the following expressions by

$$\hat{g} = -\hat{q}_0(-k) + 2k\hat{g}_0; \tag{1.16}$$

$$\hat{g} = -\hat{q}_0(i-k) - (2ik+1)\hat{g}_0; \tag{1.17}$$

$$\hat{g} = \frac{\nu_1 - k}{\nu_2 - \nu_1}\hat{q}_0(\nu_2) + \frac{k - \nu_2}{\nu_2 - \nu_1}\hat{q}_0(\nu_1) + (1 - 3k^2)\hat{g}_0, \tag{1.18}$$

where ν_1, ν_2 are the two nontrivial roots (i.e. $\nu \neq k$) of the equation $\nu^3 - \nu = k^3 - k$.

B. The New Method for Equations in Two Space Variables

For equation (1.6) the number of boundary conditions at $x_1 = 0$ and $x_2 = 0$ is determined by w_1 and w_2 respectively. If the highest term in $w_1(k_1)$ is $\alpha_{n_1}^{(1)} k_1^{n_1}$ then the number of the boundary conditions at $x_1 = 0$ is given by N_1 which is defined by equation (1.12) (with α_n and n replaced with $\alpha_{n_1}^{(1)}$ and n_1). Similarly N_2, the number of the boundary conditions at $x_2 = 0$ is determined by $\alpha_{n_2}^{(2)} k_2^{n_2}$, the highest term in $w_2(k_2)$.

Associated with equation (1.6) we prescribe an initial condition, N_1 boundary conditions at $x_1 = 0$, and N_2 boundary conditions at $x_2 = 0$,

$$\begin{aligned}
&q(x_1,x_2,0) = q_0(x_1,x_2), \quad 0 < x_1 < \infty, \quad 0 < x_2 < \infty, \\
&\partial_{x_1}^j q(0,x_2,t) = g_j^{(1)}(x_2,t), \quad 0 < x_2 < \infty, \quad t > 0, \quad j = 0,1,\cdots,N_1 - 1, \\
&\partial_{x_2}^j q(x_1,0,t) = g_j^{(2)}(x_1,t), \quad 0 < x_1 < \infty, \quad t > 0, \quad j = 0,1,\cdots,N_2 - 1,
\end{aligned} \tag{1.19}$$

where the function $q_0(x_1,x_2)$ has sufficient smoothness and decay, the functions $\{g_j^{(1)}(x_2,t)\}_0^{N_1-1}$ have sufficient smoothness are compatible with $q_0(x_1,x_2)$ at $x_1=t=0$, and the functions $\{g_j^{(2)}(x_1,t)\}_0^{N_2-1}$ have sufficient smoothness and are compatible with $q_0(x_1,x_2)$ at $x_2=t=0$. Furthermore, we assume that for real k $Rew_j(k)\geq 0$, $j=1,2$. The solution of this IBV can be obtained as follows:

Denote

$$k=(k_1,k_2), x=(x_1,x_2), kx=k_1x_1+k_2x_2, w(k)=w_1(k_1)+w_2(k_2). \tag{1.20}$$

Given q_0, $\{g_j^{(1)}\}_0^{N_1-1}$, $\{g_j^{(2)}\}_0^{N_2-1}$, compute the functions $\hat{q}_0$, $\{\hat{g}_j^{(1)}\}_0^{N_1-1}$, $\{\hat{g}_j^{(2)}\}_0^{N_2-1}$ by

$$\hat{q}_0(k)=\int_0^\infty dx_1\int_0^\infty dx_2 e^{-ikx}q_0(x),$$

$$\hat{g}_j^{(1)}(k,t)=\int_0^t d\tau\int_0^\infty dx_2 e^{-ik_2x_2+w(k)\tau}g_j^{(1)}(x_2,\tau), \quad j=0,1,\cdots,N_1-1.$$

$$\hat{g}_j^{(2)}(k,t)=\int_0^t d\tau\int_0^\infty dx_1 e^{-ik_1x_1+w(k)\tau}g_j^{(2)}(x_1,\tau), \quad j=0,1,\cdots,N_2-1. \tag{1.21}$$

We note that these functions all well defined for $\{$ $Imk_1\leq 0$, $Imk_2\leq 0\}$, $\{Imk_2\leq 0$, all complex $k_1\}$, $\{Imk_1\leq 0$, all complex $k_2\}$, respectively. Given the above functions compute the functions $\hat{g}(k,t)$, $\hat{g}^{(1)}(k,t)$ and $\hat{g}^{(2)}(k,t)$ by solving a system of linear algebraic equations; this system is discussed in §3. Denote by $D_+^{(1)}$ the domain in the upper half of the complex k_1-plane specified by $Rew_1(k_1)<0$. Denote by $\partial D_+^{(1)}$ the boundary of $D_+^{(1)}$ oriented so that $D_+^{(1)}$ is on the left of $\partial D_+^{(1)}$. Similarly $D_+^{(2)}, \partial D_+^{(2)}$ are associated with $Rew_2(k_2)<0$. Then $q(x,t)$ is given by

$$q(x,t)=\frac{1}{(2\pi)^2}\int_{-\infty}^\infty dk_1\int_{-\infty}^\infty dk_2 e^{ikx-w(k)t}\hat{q}_0(k)+\frac{1}{(2\pi)^2}\int_{\partial D_+^{(1)}}dk_1\int_{\partial D_+^{(2)}}dk_2 e^{ikx-w(k)t}\hat{g}(k,t)$$

$$+\frac{1}{(2\pi)^2}\int_{-\infty}^\infty dk_2\int_{\partial D_+^{(1)}}dk_1 e^{ikx-w(k)t}\hat{g}^{(1)}(k,t)+\frac{1}{(2\pi)^2}\int_{-\infty}^\infty dk_1\int_{\partial D_+^{(2)}}dk_2 e^{ikx-w(k)t}\hat{g}^{(2)}(k,t). \tag{1.22}$$

A simple derivation of this result is given in §3 where also the detailed derivation of $\hat{g}^{(j)}(k,t)$ for equations (1.7)-(1.9) is presented.

C. Generalizations and Long Time Asymptotics

1. The case where instead of prescribing the function and its first $N-1$ derivatives, a different subset of $N-1$ boundary values is prescribed can be solved in a similar way. This is illustrated using equation (1.2) with $q_x(0,t)+\alpha q(0,t)$ given, and equation (1.7) with $q_{x_1}(0,x_2,t)+\alpha_1 q(0,x_1,t)$, $q_{x_2}(x_1,0,t)+\alpha_2 q(x_1,0,t)$ given.

2. It is also possible to solve IBV problems for systems of evolution equations. An example of such a problem is equation (1.5) which is equivalent to a system of two evolution

equations, see equation (4.1). An IBV problem for this equation, where $u(0,t)$ and $u_x(0,t)$ are prescribed, is solved in §4 (this corresponds to solving equation (1.2) with prescribed values of $(\text{Re } q)(0,t)$ and of $(\text{Re}q_x)(0,t)$).

3. The forced versions of equations (1.1) and (1.6) are solved in §4.

4. The representations obtained by this method are very convenient for computing the long time asymptotics of the solution: Suppose that equation (1.1) is valid for $0 < t < T$, where T is a positive constant. It is shown in §4 that the representation (1.15) is equivalent with the representation obtained by replacing $\hat{g}(k,t)$ with $\hat{g}(k) = \hat{g}(k,T)$. In particular if $0 < t < \infty$, then $\hat{g}(k)$ can be computed in terms of $\hat{q}_0(k)$ and of $\{\hat{g}_j(k)\}_0^{N-1}$, where

$$\hat{g}_j(k) = \int_0^\infty d\tau e^{w(k)\tau} g_j(\tau).$$

Thus the only time dependence of $q(x,t)$ appears in the form of $\exp[ikx - w(k)t]$. Hence it is straightforward to obtain the long time asymptotics using the steepest descent method. Similar considerations are valid for equation (1.6).

5. The extension of this method to multidimensional separable PDEs,

$$\left(\partial_t + \sum_1^m w_j(-i\partial_{x_j})\right) q(x_1, \cdots, x_m, t) = 0, \tag{1.23}$$

is straightforward. It is also possible to extend this method to non-separable evolution PDEs. This is briefly discussed in Appendix B and illustrated for the linearized Kadomtsev-Petviashvilli equation.

D. Organization of the Paper

In §2 we present: (i) The derivation of equation (1.15). (ii) The solution of equations (1.2)-(1.4) with prescribed values of $q(x,0)$ and $q(0,t)$. (iii) The solution of equation (1.2) with prescribed values of $q(x,0)$ and of $q_x(0,t) + \alpha q(0,t)$, α constant. In §3 we solve the analogous problems for equations (1.6) and (1.7)-(1.9). In §4 we discuss the elastic wave equation, as well as formal problems, and the long time asymptotics. In §5 we discuss further these results. In Appendix B the main steps needed for the solution of an equation with a fifth order derivative.

2 Equations in One Space Variable

In order to derive equation (1.15) we first note that it is convenient to rewrite equation (1.1) in the form

$$\left(e^{-ikx+w(k)t}q\right)_t + \left(e^{-ikx+w(k)t}X\right)_x = 0, \tag{2.1}$$

where $X(x,t,k)$ is given by the explicit formula

$$X = -i\left(\frac{w(k) - w(l)}{k - l}\right) q, \quad l = -i\partial_x. \tag{2.2}$$

Indeed, replacing in equation (2.1) q_t by $-w(-i\partial_x)q$ we find

$$(w(k)-w(-i\partial_x))q+(\partial_x-ik)X=0,$$

which yields equation (2.2).

Example 2.1

For equation (1.2) $w(k)=ik^2$, thus

$$X=\frac{k^2-l^2}{k-l}q=(k+l)q=-iq_x+kq. \tag{2.3}$$

For equation (1.3) $w(k)=k^2-ik$, thus

$$X=-i\left(\frac{k^2-l^2-i(k-l)}{k-l}\right)q=-i(k+l)q-q=-(q_x+ikq+q). \tag{2.4}$$

For equation (1.4) $w(k)=i(k-k^3)$, thus

$$X=q_{xx}+ikq_x+(1-k^2)q. \tag{2.5}$$

A. The Fourier Transform Representation

The solution of equation (2.1) can be expressed in the form

$$q(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dke^{ikx-w(k)t}\hat{q}_0(k)+\frac{1}{2\pi}\int_{-\infty}^{\infty}dke^{ikx-w(k)t}\hat{g}(k,t), \tag{2.6}$$

where $\hat{q}_0(k)$ is given by equation (1.14a) and $\hat{g}(k,t)$ is defined by

$$\hat{g}(k,t)=\int_0^t d\tau e^{w(k)\tau}X(0,\tau,k). \tag{2.7}$$

Indeed, if we denote the x-Fourier transform of $q(x,t)$ by $\hat{q}(k,t)$

$$\hat{q}(k,t)=\int_0^{\infty}dxe^{-ikx}q(x,t), \tag{2.8}$$

then using equation (2.1) it is straightforward to compute the time evolution of $\hat{q}(k,t)$:

$$\left(e^{w(k)t}\hat{q}(k,t)\right)_t=\int_0^{\infty}dx\left(e^{-ikx+w(k)t}q(x,t)\right)_t=-\int_0^{\infty}dx\left(e^{-ikx+w(k)t}X\right)_x=e^{w(k)t}X(0,t,k).$$

Thus

$$e^{w(k)t}\hat{q}(k,t)=\hat{q}_0(k)+\hat{g}(k,t),\quad \mathrm{Im}k\leq 0. \tag{2.9}$$

We note that $\hat{g}$ is defined for all complex k but $\hat{q}_0$ is defined only for $\mathrm{Im}k\leq 0$, thus equation (2.9) is valid for $\mathrm{Im}k\leq 0$.

Equation (2.9) and the inverse Fourier transform of equation (2.8) yield equation (2.6).

B. An Integral Representation in the Complex k-plane

The first crucial step in the new method is to rewrite equation (2.6) in the form (1.15): The first term in the rhs of equation (2.6) is identical with the first term in the rhs of equation (1.15). The equality of the second terms is a consequence of Cauchy's theorem. Indeed, the term

$$e^{ikx}\int_0^t d\tau e^{-w(k)(t-\tau)}X(0,\tau,k),$$

is analytic and bounded in the region of the complex k-plane where $\mathrm{Im}k>0$ and $\mathrm{Re}w(k)>0$(note that X is analytic in k and $t\geq\tau$). Thus the integral of this term along the real axis can be deformed to an integral along the curve ∂D_+. For example, for equation (1.2), $w(k)=ik^2$, thus the integral along the negative part of the real axis can be deformed to an integral along the positive imaginary axis, see Figure 2.1. For equation (1.3) the integral along the real axis can be deformed along the curve shown in Figure 2.1a; this curve is defined by $\mathrm{Re}(k^2-ik)=0$, $\mathrm{Im}k>0$, i.e. $k_R^2-k_I^2+k_I=0$, $k_I>0$. For equation (1.4) the integral along the rays $(-\infty,-1/\sqrt{3}]$ and $[1/\sqrt{3},\infty)$ can be deformed along the curves shown in Figure 2.1c; these curves are the two braces of the curve $\mathrm{Re}(ik-ik^3)=0$, Im>0, i.e. $k_I^2-3k_R^2+1=0$, $k_I>0$.

Summarizing the above discussion we have:

The solution $q(x,t)$ of equation (1.1) is given by equation (1.15) where $\hat{g}(k,t)$ is defined by equation (2.7) in terms of $X(0,t,k)$.

However, $\hat{g}(k,t)$ involves some unknown boundary values. For example for equations (1.2)-(1.4) substituting the expressions (2.3)-(2.5) into the definition (2.7) we find

$$\hat{g}(k,t)=-i\hat{g}_1(k,t)+k\hat{g}_0(k,t), \tag{2.10}$$

$$\hat{g}(k,t)=-\hat{g}_1(k,t)-(ik+1)\hat{g}_0(k,t), \tag{2.11}$$

$$\hat{g}(k,t)=\hat{g}_2(k,t)+ik\hat{g}_1(k,t)+(1-k^2)\hat{g}_0(k,t). \tag{2.12}$$

If $q(0,t)=g_0(t)$ is given then the term $\hat{g}_0(k,t)$ is known and can be computed in terms of $g_0(t)$ by equation (1.14b) with $j=0$ and $w(k)=ik^2$, k^2-ik, $i(k-k^3)$, respectively. However, $\hat{g}_1$ and $\hat{g}_2$ are unknown functions,

$$\hat{g}_j(k,t)=\int_0^t d\tau e^{w(k)\tau\partial_x^j}q(0,\tau),\quad j=1,2.$$

Thus for equations (1.2) and (1.3) $\hat{g}(k,t)$ involves one unknown function ($\hat{g}_1$), while for equation (1.4) $\hat{g}(k,t)$ involves two unknown functions g_1 and g_2..

C. The Analysis of the Global Condition

The second crucial step of the new method is the determination of the unknown part of $\hat{g}(k,t)$ by using the invariant properties of the algebraic global relation (2.9): For pedagogical reasons we first discuss equations (1.2)-(1.4) before discussing the general case.

For equation (1.2), substituting (2.10) into (2.9) we find

$$-i\hat{g}_1(k,t) = -k\hat{g}_0(k,t) - \hat{q}_0(k) + e^{w(k)t}\hat{q}(k,t), \quad \mathrm{Im}k \le 0. \tag{2.13}$$

We need to compute $\hat{g}(k,t)$ on the curve ∂D_+; since $\hat{g}(k,t)$ is analytic in D_+ this is equivalent to computing $\hat{g}(k,t)$ for $k \in D_+$. Thus our first task is to transform equation (2.13) from the lower half complex k-plane to the domain D_+. In this respect it is important to observe that $\hat{g}_1$ depends on k only through $w(k)$. Thus $\hat{g}_1$ remains invariant by those transformations $k \to \nu(k)$ which preserve $w(k)$. In this example $w(k) = ik^2$, thus $\nu(k) = -k$. Replacing k by $-k$ in equation (2.13) we find

$$-i\hat{g}_1(k,t) = k\hat{g}_0(k,t) - \hat{q}_0(-k) + e^{w(k)t}\hat{q}(-k,t), \quad \mathrm{Im}k \ge 0.$$

Replacing $-i\hat{g}_1$ in the rhs of equation (2.10) by the above expression we find

$$\hat{g}(k,t) = 2k\hat{g}_0(k,t) - \hat{q}_0(-k) + e^{w(k)t}\hat{q}(-k,t).$$

The rhs of this equation involves the unknown function $\hat{q}(-k,t)$. However, this function *does not* contribute to $q(x,t)$. Indeed, the contribution of this term is given by

$$\int_{\partial D_+} dk e^{ikx}\hat{q}(-k,t),$$

and since both $\exp(ikx)$ and $\hat{q}(-k,t)$ are analytic and bounded in D_+ Cauchy's theorem implies that this contribution vanishes. Hence the effective part of $\hat{g}(k,t)$ is given by (1.16).

Similarly for equation (1.3) using (2.11) into (2.4) we find

$$-\hat{g}_1(k,t) = (ik+1)\hat{g}_0(k,t) - \hat{q}_0(k) + e^{w(k)t}\hat{q}(k,t), \quad \mathrm{Im}k \le 0. \tag{2.14}$$

The equation $w(k) = w(\nu(k))$ has one nontrivial root,

$$\nu^2 - i\nu = k^2 - ik, \quad \nu = i - k.$$

Replacing in equation (2.14) k by $i-k$ we find

$$-\hat{g}_1 = -ik\hat{g}_0 - \hat{q}_0(i-k) + e^{w(k)t}\hat{q}(i-k,t)$$

Replacing $-\hat{g}_1$ in equation (2.11) by the rhs of the above expression we find

$$\hat{g}(k,t) = -(2ik+1)\hat{g}_0(k,t) - \hat{q}_0(i-k) + e^{w(k)t}\hat{q}(i-k,t).$$

Again the term $\hat{q}(i-k,t)$ does not contribute to $q(x,t)$: it involves $\exp[(1+ik)x]$ which is bounded in k for $k_R > 1$, thus it is bounded in D_+. Hence the effective part of $\hat{g}$ is given by equation (1.17).

For equation (1.4) using (2.12) into (2.9) we find

$$\hat{g}_2(k,t) + ik\hat{g}_1(k,t) = (k^2-1)\hat{g}_0(k,t) - \hat{q}_0(k) + e^{w(k)t}\hat{q}(k,t), \quad \mathrm{Im}k \le 0. \tag{2.15}$$

The equation $w(k) = w(\nu(k))$, has two nontrivial roots

$$\nu - \nu^3 = k - k^3, \quad \nu_1 = \frac{-k \pm i\sqrt{3k^2-4}}{2}. \tag{2.16}$$

If $nu_1(k) \in D^{(1)}$ and $\nu_2(k) \in D^{(2)}$, then $k \in D_+$. Thus replacing k by $\nu_j(k)$ in equation (2.15) we find two algebraic equations,

$$\hat{g}_2 + i\nu_1\hat{g}_1 = (\nu_1^2-1)\hat{g}_0 - \hat{q}_0(\nu_1) + e^{w(k)t}\hat{q}(\nu_1,t)$$

$$\hat{g}_2 + i\nu_2\hat{g}_1 = (\nu_2^2-1)\hat{g}_0 - \hat{q}_0(\nu_2) + e^{w(k)t}\hat{q}(\nu_2,t).$$

Solving these two equations for $\hat{g}_1$ and $\hat{g}_2$ and then substituting the resulting expressions into equation (2.12) we find that $\hat{g}(k,t)$ is given by the rhs of equation (1.18) plus the term

$$e^{w(k)t}\left[\frac{k-\nu_1}{\nu_2-\nu_1}\hat{q}(\nu_2,t) + \frac{\nu_2-k}{\nu_2-\nu_1}\hat{q}(\nu_1,t)\right];$$

again this term does not contribute to $q(x,t)$ due to analytic considerations.

We now discuss the general case: Suppose that $w(k)$ is a polynomial of degree n and let α_n be the coefficient of k^n. The equation $w(k) = w(\nu(k))$ has $n-1$ nontrivial roots which for large k are given by $\nu_m \sim k\exp[2i\pi m/n]$, $m = 1, \cdots, n-1$. These roots map the domain $\mathrm{Re}w(k) < 0$ onto itself. For large k this domain consists of the n sectors

$$\frac{1}{n}\left(\frac{\pi}{2} - \arg\alpha_n\right) + \frac{2\pi m}{n} < \arg k < \frac{1}{n}\left(\frac{3\pi}{2} - \arg\alpha_n\right) + \frac{2\pi m}{n}, \quad m = 0, 1, \cdots, n-1. \tag{2.17}$$

Suppose that there exist $\tilde{n}$ such sectors in the lower half complex k-plane. Then using ν_m each of these sectors is mapped onto D_+. Thus the global relation gives rise to $\tilde{n}$ algebraic equations. Since $\hat{g}(k,t)$ involves the n unknown functions $\hat{g}_0, \hat{g}_1, \cdots, \hat{g}_{n-1}$, for a well posed problem one must prescribe $n - \tilde{n}$ boundary conditions.

Using that for k real $w(k) \ge 0$, i.e. $-\pi/2 \le \arg\alpha_n \le \pi/2$, equation (2.17) implies that if n is even $\tilde{n} = n/2$, while if n is odd then $\tilde{n} = (n-1)/2$ unless $\alpha_n = -i$ in which case $\tilde{n} = (n+1)/2$.

D. Other Types of Boundary Conditions

For simplicity we only consider equation (1.2), other equations can be treated similarly.

The solution of (1.2) with a combination of Dirichlet and Neumann boundary conditions.

Let $q(x,t)$ satisfy equation (1.2) with the initial condition $q(x,0)=q_0(x)$ and the boundary condition

$$q_x(0,t)+\alpha q(0,t)=f(t), \quad t>0, \tag{2.18}$$

where α is a constant, $q_0(x)$ has sufficient smoothness and decay, $f(t)$ has sufficient smoothness and $f(0)=\partial_x q(0)+\alpha q(0)$. The solution of this IBV problem is given by equation (1.15) where $w(k)=ik^2$, $\hat{q}_0(k)$ is given by equation (1.14a), ∂D_+ is the boundary of the first quadrant of the complex k-plane (see Figure 1.1), and $\hat{g}(k,t)$ can be computed as follows: Given q_0 and f compute $\hat{G}(k,t)$ by

$$\hat{G}(k,t)=(k+i\alpha)\hat{q}_0(-k)-2ki\int_0^t d\tau e^{ik^2\tau}f(\tau). \tag{2.19}$$

Compute $\hat{g}(k,t)$ by

$$\hat{g}(k,t)=\begin{cases}\frac{\hat{G}(k,t)}{k-i\alpha}, & \text{if} \quad 0<\arg\alpha<\frac{3\pi}{2},\\[2ex] \frac{\hat{G}(k,t)-\hat{G}(i\alpha,t)}{k-i\alpha}, & \text{if} \quad -\frac{\pi}{2}\le\arg\alpha\le 0.\end{cases} \tag{2.20}$$

Indeed multiplying equation (2.18) by $\exp(ik^2\tau)$ and integrating we find

$$\hat{g}_1+\alpha\hat{g}_0=\hat{f}, \tag{2.21}$$

where $\hat{f}$ is the integral term in the rhs of equation (2.19). Substituting $\hat{g}_1$ from this equation into the definition of $\hat{g}$ (equation (2.10)) we find

$$\hat{g}=-i\hat{f}+(k_1+i\alpha)\hat{g}_0. \tag{2.22}$$

Substituting this expression into the equation obtained from equation (2.9) by replacing k with $-k$, solving the resulting equation for $\hat{g}_0$ and then substituting the expression for $\hat{g}_0$ into equation (2.22) wee find

$$(k-i\alpha)\hat{g}=\hat{G}-(k+i\alpha)e^{ik^2t}\hat{q}(-k,t), \tag{2.23}$$

where the known function $\hat{G}$ is given by equation (2.19).

If $\pi/2<\arg(i\alpha)<2\pi$, then $k-i\alpha\neq 0$ for k in the first quadrant of the complex k-plane, $\hat{q}(-k,t)/(k-i\alpha)$ is analytic and bounded, and the last term of equation (2.23) does not contribute to $q(x,t)$. However, if $0\le\arg i\alpha\le\pi/2$, then $k-i\alpha=0$ for k in the first quadrant of the complex k-plane and $\hat{q}(-k,t)/(k-i\alpha)$ has a pole. The relevant residue can be computed by evaluating equation (2.23) at $k=i\alpha$:

$$\hat{G}(k_0,t)-2k_0e^{ik_0^2t}\hat{q}(-k_0,t)=0, \quad k_0=i\alpha.$$

Using this equation, equation (2.23) can be rewritten in the form

$$\hat{g}(k,t) = \frac{\hat{G}(k,t) - \hat{G}(k_0,t)}{k - i\alpha} - \left\{ \frac{(k+i\alpha)e^{ik^2t}\hat{q}(-k,t) - 2k_0 e^{ik_0^2 t}\hat{q}(-k_0,t)}{k - i\alpha} \right\}.$$

Since the term in the bracket is now analytic and bounded in the first quadrant of the complex k-plane, it does not contribute to $q(x,t)$.

3 Equations in Two Space Variables

In order to derive equation (1.22) we first note that it is convenient to rewrite equation (1.6) in the form

$$\left(e^{-ikx+w(k)t}q\right)_t + \sum_{m=1}^{2}\left(e^{-ikx+w(k)t}X^{(m)}\right)_{X_m} = 0, \tag{3.1}$$

where the functions $X^{(m)}$ are given by the following explicit formulas,

$$X^{(m)} = -i\left(\frac{w_m(k_m) - w(l_m)}{k_m - l_m}\right)q, \quad l_m = -i\partial_{x_m}, \quad m = 1,2. \tag{3.2}$$

Indeed, using equation (1.6), equation (3.1) becomes

$$\left(w_1(k_1) + w_2(k_2) - w_1(-i\partial_{x_1}) - w_2(-i\partial_{x_2})\right)q + \left(\partial_{x_1} - ik_1\right)X^{(1)} + \left(\partial_{x_2} - ik_2\right)X^{(2)} = 0.$$

If $X^{(m)}$ are given by equations (3.2), then the above equation is identically satisfied.

Examples 3.1

For equation (1.7) $w_1(k_1) = ik_1^2$, $w_2(k_2) = ik_2^2$, thus

$$X^{(m)} = -iq_{x_m} + k_m q, \quad m = 1,2. \tag{3.3}$$

For equation (1.8), $w_m(k_m) = k_m^2 - ik_m$, thus

$$X^{(m)} = -\left(q_{x_m} + ik_m q + q\right), \quad m = 1,2. \tag{3.4}$$

For equation (1.9), $w_m(k_m) = ik_m^3$, thus

$$X^{(m)} = q_{x_m x_m} + ik_m q_{x_m} - k_m^2 q, \quad m = 1,2. \tag{3.5}$$

A. The Fourier Transform Representation

The solution of equation (3.1) can be expressed in the form

$$q(x,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 e^{ikx - w(k)t}\hat{q}_0(k) + \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 e^{ikx - w(k)t}(\hat{g}^{(1)}(k,t) + \hat{g}^{(2)}(k,t)), \tag{3.6}$$

where $\hat{q}_0$ is given by equation (1.21a) and $\hat{g}^{(1)}$, $\hat{g}^{(2)}$ are defined by

$$\hat{g}^{(1)}(k,t) = \int_0^t d\tau \int_0^\infty dx_2 e^{-ik_2x_2+w(k)\tau} X^{(1)}(0,x_2,\tau,k_1),$$

$$\hat{g}^{(2)}(k,t) = \int_0^t d\tau \int_0^\infty dx_1 e^{-ik_1x_1+w(k)\tau} X^{(2)}(x,0_2,\tau,k_2). \tag{3.7}$$

Indeed, if we denote the (x_1,x_2)-Fourier transform of $q(x,t)$ by $\hat{q}(k,t)$,

$$\hat{q}(k,t) = \int_0^\infty dx_1 \int_0^\infty dx_2 e^{-ikx} q(x,t), \tag{3.8}$$

then using equation (3.1) it is straightforward to compare the time evolution of $\hat{q}(k,t)$:

$$\begin{aligned}
\left(e^{w(k)t}\hat{q}(k,t)\right)_t &= \textstyle\int_0^\infty dx_1 \int_0^\infty dx_2 \left(e^{-ikx+w(k)t}q(x,t)\right)_t \\
&= \textstyle\int_0^\infty dx_1 \int_0^\infty dx_2 \left[\left(e^{-ikx+w(k)t}X^{(1)}\right)_{x_1} + \left(e^{-ikx+w(k)t}X^{(2)}\right)_{x_2}\right] \\
&= \textstyle\int_0^\infty dx_2 e^{-ik_2x_2+w(k)t}X^{(1)}(0,x_2,t,k_1) + \int_0^\infty dx_1 e^{-ik_1x_1+w(k)t}X^{(2)}(x_1,0,t,k_2).
\end{aligned}$$

Thus

$$e^{w(k)t}\hat{q}(k,t) = \hat{q}_0(k) + \hat{g}^{(1)}(k,t) + \hat{g}^{(2)}(k,t), \quad \mathrm{Im}k_1 \le 0, \quad \mathrm{Im}k_2 \le 0. \tag{3.9}$$

We note that $\hat{g}^{(m)}$, $m=1,2$, are defined for all complex values of k_1 and k_2 but $\hat{q}_0$ is defined only for $\mathrm{Im}k_1 \le 0$ and $\mathrm{Im}k_2 \le 0$, thus equation (3.9) is valid only for these values of k_1 and k_2. Equation (3.9) and the inverse Fourier transform of equation (3.8) yields equation (3.6).

B. An Integral Representation in the Complex k_1 and k_2 Planes

Equation (3.6) can be rewritten in the form (1.22). This is a direct consequence of the analogous derivation for equation (1.15). For example for the term in equation (3.6) involving $\hat{g}^{91)}(k,t)$, the k_1 integral is given by

$$\int_{-\infty}^\infty dk_1 e^{ik_1x_1-w_1(k_1)t}\left(\int_0^t d\tau \int_0^\infty dx_2 e6{-}ik_2x_2 + (w_1(k_1)+w_2(k_2))\tau X^{(1)}(0,x_2,\tau,k_1)\right);$$

hence k_1 appears exactly in the same form as k appears in equation (2.6). Similarly for the term in equation (3.6) involving $\hat{g}^{(2)}(k,t)$.

Examples 3.2

For equations (1.7)-(1.9) the curves $\partial D_+^{(1)}$ in the complex k_1-plane are identical with the curves $\partial D_+^{((2)}$ in the complex k_2-plane. For equations (1.7) and (1.8) these curves are depicted in Figures 1.a, 1.b. For equation (1.9) the relevant curve is the union of the rays $\arg k = \pi/3$ and $\arg k = 2\pi/3$, see Figure 3.1.

Figure 3.1

Summarizing the above discussion we have:

The solution $q(x,t)$ of equation (1.6) is given by (1.22) where $\hat{g}^{(1)}(k,t)$ and $\hat{g}^{(2)}(k,t)$ are defined by equations (3.7) in terms of $X^{(1)}(0,x_2,t,k)$ and $X^{(2)}(x,0,t,k)$.

However $\hat{g}^{(1)}$ and $\hat{g}^{(2)}$ involve some unknown boundary values. For example for equations (1.7)-(1.9) substituting the expressions for $X^{(j)}$ from equations (3.3)-(3.5) into the definitions (3.7) we find

$$\hat{g}^{(m)} = -i\hat{g}_1^{(m)} + k_j\hat{g}_0^{(m)}, \tag{3.10}$$

$$\hat{g}^{(m)} = -\hat{g}_1^{(m)} - (ik_j+1)\hat{g}_0^{(m)}, \tag{3.11}$$

$$\hat{g}^{(m)} = \hat{g}_2^{(m)} + ik_j\hat{g}_1^{(m)} - k_j^2\hat{g}_0^{(m)}, \quad m=1,2. \tag{3.12}$$

If $q(0,x_2,t) = g_0^{(1)}(x_2,t)$ and $q(x_1,0,t) = g_0^{(2)}(x_1,t)$ are given, then the terms $\hat{g}_0^{(1)}$ and $\hat{g}_0^{(2)}$ are known and can be computed in terms of $g_0^{(1)}$ and $g_0^{(2)}$ by equations (1.21b) and (1.21b) with $j=0$ and $w(k) = i(k_1^2+k_2^2)$, $k_1^2+k_2^2-i(k_1+k_2)$, $-i(k_1^3+k_2^3)$, respectively. However $\hat{g}_1^{(m)}$ and $\hat{g}_2^{(m)}$ are unknown functions:

$$\hat{g}_j^{(1)}(k,t) = \int_0^t d\tau \int_0^\infty dx_2 e^{-ik_2x_2+w(k)\tau}\partial_{x_1}^j q(0,x_2,t),$$

$$\hat{g}_j^{(2)}(k,t) = \int_0^t d\tau \int_0^\infty dx_1 e^{-ik_1x_1+w(k)\tau}\partial_{x_2}^j q(x_1,0,t), \quad j=1,2.$$

C. The Analysis of the Global Relation

The analogue of equation (2.9) is now equation (3.9). Although equation (3.9) involves more unknowns than equation (2.9), it can again be used to determine the unknown parts of both $\hat{g}^{(1)}$ and $\hat{g}^{(2)}$.

The solution of equations (1.7)-(1.9) with Dirichlet boundary conditions

Let $q(x,t)$ satisfy any of the equations (1.7)-(1.9) with the initial condition $q(x_1,x_2,0) = q_0(x_1,x_2)$ and the boundary conditions $q(0,x_2,t) = g_0^{(1)}(x_2,t)$, $q(x_1,0,t) = g_0^{(2)}(x_1,t)$, where q_0, $g_0^{(1)}$, $g_0^{(2)}$ and have sufficient smoothness and decay $g_0^{(1)}(x_2,0) = q_0(0,x_2)$, $g_0^{(2)}(x_1,0) = q_0(x,0)$. The solution of any of these IBV problems can be obtained as follows: Compute $\hat{q}_0$, $\hat{g}_0^{(1)}$, $\hat{g}_0^{(2)}$ using equations (1.21) with $j=0$, $w(k) = w_1(k_1)+w_2(k_2)$, and $w_1(k) = w_2(k) = ik^2$, k^2-ik, $-ik^3$, respectively. Given $\hat{q}_0$, $\hat{g}_0^{(1)}$, $\hat{g}_0^{(2)}$, compare $\hat{g}^{(1)}$, $\hat{g}^{(2)}$, $\hat{g}$, by

$$\begin{aligned}
\hat{g}^{(1)} &= -\hat{q}_0(-k_1,k_2) + 2k_1\hat{g}_0^{(1)},\\
\hat{g}^{(2)} &= -\hat{q}_0(k_1,-k_2) + 2k_2\hat{g}_0^{(2)},\\
\hat{g} &= \hat{q}_0(-k_1,-k_2);
\end{aligned} \tag{3.13}$$

$$\begin{aligned} \hat{g}^{(1)} &= -\hat{q}_0(i-k_1,k_2)-(2ik_1+1)\hat{g}_0^{(1)}, \\ \hat{g}^{(2)} &= -\hat{q}_0(k_1,i-k_2)-(2ik_2+1)\hat{g}_0^{(2)}, \\ \hat{g} &= \hat{q}_0(i-k_1,i-k_2); \end{aligned} \tag{3.14}$$

$$\begin{aligned} \hat{g}^{(1)} &= -3k_1^2\hat{g}_0^{(1)}+E\hat{q}_0(Ek_1,k_2)+E^2\hat{q}_0(E^2k_1,k_2), \quad E=e^{\frac{2i\pi}{3}} \\ \hat{g}^{(2)} &= -3k_2^2\hat{g}_0^{(2)}+E\hat{q}_0(k_1,Ek_2)+E^2\hat{q}_0(k_1,E^2k_2), \\ \hat{g} &= 3k_1^2\hat{g}_0^{(1)}(k_1,Ek_2)+3Ek_1^2\hat{g}_0^{(1)}(k_1,E^2k_2)+3k_2^2\hat{g}_0^{(2)}(Ek_1,k_2)+3Ek_2^2\hat{g}_0^{(2)}(E^2k_1,k_2) \\ &+\hat{q}_0(Ek_1,Ek_2)+\hat{q}_0(E^2k_1,Ek_2)+\hat{q}_0(Ek_1,E^2k_2)+\hat{q}_0(E^2k_1,E^2k_2); \end{aligned} \tag{3.15}$$

The solution of each of these IBV problems is given by

$$\begin{aligned} q(x,t) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_1\int_{-\infty}^{\infty}dk_2 e^{ikx-w(k)t}\hat{q}_0(k)+\frac{1}{(2\pi)^2}\int_{\partial D_+}dk_1\int_{\partial D_+}dk_2 e^{ikx-w(k)t}\hat{g}(k,t) \\ &+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_2\int_{\partial D_+}dk_1 e^{ikx-w(k)t}\hat{g}^{(1)}(k,t)+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_1\int_{\partial D_+}dk_2 e^{ikx-w(k)t}\hat{g}^{(2)}(k,t), \end{aligned} \tag{3.16}$$

where the contour ∂D^+ for equations (1.7), (1.8), (1.9) is depicted in Figures 1.1a, 1.1b, 3.1.

Before deriving these results we observe that $\hat{g}_j^{(1)}$ and $\hat{g}_j^{(2)}$ are invariant under the transformations which leave invariant $w_1(k_1)$ and $w_2(k_2)$ respectively. We now discuss equation (1.7). Since we need to compare $\hat{g}^{(1)}$ for k_1 in the first quadrant of the complex k_1-plane we replace k_1 by $-k_1$ in equation (3.9). Substituting the expression for $\hat{g}^{(1)}$ from equation (3.10) into this equation we find

$$-i\hat{g}_1^{(1)}=k_1\hat{g}_0^{(1)}-\hat{q}_0(-k_1,k_2)+e^{w(k)t}\hat{q}(-k_1,k_2,t)-\hat{g}^{(2)}(-k_1,k_2,t), \quad \mathrm{Im}k_1\geq 0, \quad \mathrm{Im}k_2\leq 0.$$

Replacing $\hat{g}_1^{(1)}$ in the definition of $\hat{g}^{(1)}$ (equation 3.10)) by the above expression we find

$$\hat{g}^{(1)}=2k_1\hat{g}_0^{(1)}-\hat{q}_0(-k_1,k_2)+e^{w(k)t}\hat{q}(-k_1,k_2,t)-\hat{g}^{(2)}(-k_1,k_2,t). \tag{3.17}$$

In a similar way replacing k_2 by $-k_2$ in equation (3.9) and proceeding as above we find

$$\hat{g}^{(2)}=2k_2\hat{g}_0^{(2)}-\hat{q}_0(k_1,-k_2)+e^{w(k)t}\hat{q}(k_1,-k_2,t)-\hat{g}^{(1)}(k_1,-k_2,t). \tag{3.18}$$

The terms involving $\hat{q}(-k_1,k_2,t)$ and $\hat{q}(k_1,-k_2,t)$ do not contribute to $q(x,t)$. Indeed, the associated integrals over ∂D_+ in the complex k_1-plane and over ∂D_+ in the complex k_2-plane vanish for precisely the reasons discussed in §2. Thus equations (3.17) and (3.18) yield the expressions (3.13a), (3.13b) as well as the term

$$-\frac{1}{(2\pi)^2}\left(\int_{-\infty}^{\infty}dk_2\int_{\partial D_+}dk_1 e^{ikx-w(k)t}\hat{g}^{(2)}(-k_1,k_2,t)+\int_{-\infty}^{\infty}dk_1\int_{\partial D_+}dk_2 e^{ikx-w(k)t}\hat{g}^{(1)}(k_1,-k_2,t)\right).$$

We will show that this term equals the second term of the rhs of equation (3.16). Indeed, the definition of $\hat{g}^{(2)}$ (equation (3.7b)) implies that the only exponential k_2 dependence of this term is in $w_2(k_2)$. Thus the integral over the real k_2-axis can be deformed over the integral along the curve ∂D_+ of the complex k_2-plane. Similar considerations are valid for $\hat{g}^{(1)}$, thus the above term equals

$$-\frac{1}{(2\pi)^2}\int_{\partial D_+} dk_1 \int_{\partial D_+} dk_2 e^{ikx-w(k)t}\left(\hat{g}^{(1)}(k_1,-k_2,t)+\hat{g}^{(2)}(-k_1,k_2,t)\right).$$

Letting $k_1 \to k_1$, $k_2 \to k_2$ in the global relation (3.9) we find

$$\hat{g}^{(1)}(k_1,-k_2,t)+\hat{g}^{(2)}(-k_1,k_2,t) = -\hat{q}_0(-k_1,-k_2)+e^{w(k)}\hat{q}(-k_1,-k_2,t).$$

The first term of the rhs of this equation yields the second term in the rhs of equation (3.16), while the second term yields a zero contribution due to analytic considerations.

For equation (1.8) the derivation of equations (3.14), is similar.

For equation (1.9) we first note that the equation $k^3=\nu^3$ has the two nontrivial roots $\nu_1 = k\exp(2i\pi/3)$, $\nu_2 = k\exp(4i\pi/3)$. Furthermore,

Figure 3.2

if ν_1 and ν_2 are in D_1 and D_2 then k is in D_+, where for the regions D_+, D_1, D_2, $\arg k$ is in $[\pi/3, 2\pi/3]$, $[\pi, 4\pi/3]$, $[5\pi/3, 2\pi]$, respectively, see Figure 3.2.

Replacing k by ν_1 and ν_2 in equation (3.9) and using the explicit form of $\hat{g}^{(1)}$ given by equation (3.12) we find

$$\hat{g}_2^{(1)} + iEk_1\hat{g}_1^{(1)} - E^2k_1^2\hat{g}_0^{(1)} = -\hat{q}_0(Ek_1,k_2)+e^{w(k)t}\hat{q}(Ek_1,k_2,t)-\hat{g}^{(2)}(Ek_1,k_2,t),$$

$$\hat{g}_2^{(1)} + iE^2k_1\hat{g}_1^{(1)} - Ek_1^2\hat{g}_0^{(1)} = -\hat{q}_0(E^2k_1,k_2)+e^{w(k)t}\hat{q}(E^2k_1,k_2,t)-\hat{g}^{(2)}(E^2k_1,k_2,t),$$

where $k_1 \in D_+^{(1)}$ and $\mathrm{Im}k_2 \le 0$. Solving these two algebraic equations for $\hat{g}_1^{(1)}$ and $\hat{g}_2^{(1)}$ and substituting the resulting expressions into the $\hat{g}^{(1)}$ (equation (3.12)) we find (using $E^3=1$, $1+E+E^2=0$)

$$\hat{g}_0^{(1)} = -3k_1^2\hat{g}_0^{(1)} + E\hat{q}_0(Ek_1,k_2)+E^2\hat{q}_0(E^2k_1,k_2)-e^{w(k)t}[E\hat{q}(Ek_1,k_2,t)+E^2\hat{q}(E^2k_1,k_2,t)]$$
$$+E\hat{g}^{(2)}(Ek_1,k_2,t)+E^2\hat{g}^{(2)}(E^2k_1,k_2,t), \quad k_1\in D_+, \quad \mathrm{Im}k_2 \le 0. \tag{3.19}$$

Similarly

$$\hat{g}_0^{(2)} = -3k_2^2\hat{g}_0^{(2)} + E\hat{q}_0(k_1, Ek_2) + E^2\hat{q}_0(k_1, Ek_2) - e^{w(k)t}[E\hat{q}(k_1, Ek_2, t) + E^2\hat{q}(k_1, Ek_2, t)]$$
$$+E\hat{g}^{(1)}(k_1, Ek_2, t) + E^2\hat{g}^{(1)}(k_1, E^2k_2, t), \quad k_2 \in D_+, \quad \mathrm{Im}k_1 \le 0. \tag{3.20}$$

As before the terms multiplying $\exp[w(k)t]$ do not contribute. For example $\hat{q}(Ek_1, k_2, t)$ involves $\exp[-ik_1 \exp 2i\pi/3X]$ which is bounded for $\pi/3 \le \arg k_1 \le 4\pi/3$; similarly $\hat{q}(E^2k_1, k_2, t)$ is bounded for $-\pi/3 < \arg k_1 < 2\pi/3$. Thus equations (3.19) and (3.20) yield the expressions (3.14a), (3.14b) as well as the term

$$\frac{E}{(2\pi)^2}\int_{\partial D_+} dk_1 \int_{\partial D_+} dk_2 e^{ikx - w(k)t}\left[\left(\hat{g}^{(1)}(k_1, Ek_2, t) + \hat{g}^{(2)}(Ek_1, k_2, t)\right)\right.$$
$$\left. +E\left(g^{(1)}(k_1, E^2k_2, t) + \hat{g}^{(2)}(E^2k_1, k_2, t)\right)\right]. \tag{3.21}$$

Evaluating the global condition (3.9) at (k_1, k_2) equal (Ek_1, Ek_2), (E^2k_1, Ek_2), (Ek_1, E^2k_2), (E^2k_1, E^2k_2), and dropping for convenience of notation t we find the following tour equations

$$\hat{g}_2^{(1)}(k_1, Ek_2) + iEk_1\hat{g}_1^{(1)}(k_1, Ek_2) + E^2k_1^2\hat{g}_0^{(1)}(k_1, Ek_2) + \hat{g}_2^{(2)}(Ek_1, k_2) + iEk_2\hat{g}_1^{(2)}(Ek_1, k_2)$$
$$+E^2k_2^2\hat{g}_0^{(2)}(Ek_1, k_2) = -\hat{q}_0(Ek_1, Ek_2) + e^{wt}\hat{q}(Ek_1, EK_2),$$
$$\hat{g}_2^{(1)}(k_1, Ek_2) + iE^2k_1\hat{g}_1^{(1)}(k_1, Ek_2) + Ek_1^2\hat{g}_0^{(1)}(k_1, Ek_2) + \hat{g}^{(2)}(E^2k_1, k_2) + iEk_2\hat{g}_1^{(2)}(E^2k_1, k_2)$$
$$+E^2k_2^2\hat{g}_0^{(2)}(E^2k_1, k_2) = -\hat{q}_0(E^2k_1, Ek_2) + e^{wt}\hat{q}(E^2k_1, Ek_2),$$
$$\hat{g}_2^{(1)}(k_1, E^2k_2) + iEk_1\hat{g}_1^{(1)}(k_1, E^2k_2 + E^2k_1^2\hat{g}_0^{(1)}(k_1, E^2k_2) + \hat{g}_2^{(2)}(Ek_1, k_2) + iE^2k_2\hat{g}_1^{(2)}(Ek_1, k_2)$$
$$+Ek_2^2\hat{g}_0^{(2)}(Ek_1, k_2) = -\hat{q}_0(Ek_1, E^2k_2) + e^{wt}\hat{q}(Ek_1, E^2k_2)$$
$$\hat{g}_2^{(1)}(k_1, E^2k_2) + iE^2k_1\hat{g}_1^{(2)}(k_1, E^2k_2) + Ek_1^2\hat{g}_0^{(1)}(k_1, E^2k_2) + \hat{g}_2^{(2)}(E^2k_1, k_2) + iE^2k_2\hat{g}_1^{(2)}(E^2k_1, k_2)$$
$$+Ek_2^2\hat{g}_0^{(2)}(E^2k_1, k_2) = -\hat{q}_0(E^2k_1, E^2k_2) + e^{wt}\hat{q}(E^2k_1, E^2k_2). \tag{3.22}$$

Multiplying equations (3.22a), (3.22b), (3.22c), (3.22d) by $E, E^2, E^2, 1$ respectively adding the resulting equations and using $E + E^2 + 1 = 0$, we find

$$\hat{g}_2^{(1)}(k_1, Ek_2) + E\hat{g}_2^{(1)}(k_1, E^2k_2) + ik_1\hat{g}_1^{(1)}(k_1, Ek_2) + iEk_1\hat{g}_1^{(1)}(k_1, E^2k_2)$$
$$-2k_1^2\hat{g}_0^{(1)}(k_1, Ek_2) - 2Ek_1^2\hat{g}_0^{(1)}(k_1, E^2k_2)$$
$$\hat{g}_2^{(2)}(Ek_1, k_2) + E\hat{g}_2^{(2)}(E^2k_1, k_2) + ik_2\hat{g}_1^{(2)}(Ek_1, k_2) + ik_2E\hat{g}_1^{(2)}(E^2k_1, k_2)$$
$$-2k_2^2\hat{g}_0^{(2)}(Ek_1, k_2) - 2Ek_2^2\hat{g}_0^{(2)}(E^2k_1, k_2)$$
$$= \hat{q}_0(Ek_1, Ek_2) + \hat{q}_0(E^2k_1, Ek_2) + \hat{q}_0(Ek_1, E^2k_2) + \hat{q}_0(E^2k_1, E^2k_2)$$
$$-e^{wt}\left[\hat{q}(Ek_1, Ek_2) + \hat{q}(E^2k_1, Ek_2) + \hat{q}(Ek_1, E^2k_2) + \hat{q}(E^2k_1, E^2k_2)\right].$$

Thus the term appearing in the bracket of (3.21) equals the term appearing in the second term of the rhs of (3.16) plus a term involving $\exp[wt]$ which gives a zero contribution.

D. Other Boundary Conditions

For simplicity we only consider equation (1.7).

The solution of equation (1.7) with a combination of Dirichlet and Neumann boundary conditions.

Let $q(x,t)$ satisfy equation (1.7) with the initial condition $q(x_1,x_2,0)=q_0(x_1,x_2,0)$ and the boundary conditions

$$\begin{aligned} &q_{x_1}(0,x_2,t)\alpha_1 q(0,x_2,t)=f^{(1)}(x_2,t), \quad 0<x_2<\infty, \quad t>0, \\ &q_{x_2}(x_1,0,t)+\alpha_2 q(x_1,0,t)=f^{(2)}(x_1,t), \quad 0<x_1<\infty, \quad t>0, \end{aligned} \tag{3.23}$$

where α_1,α_2 are constants, the functions q_0, f_1,f_2 have sufficient smoothness and decay and $f_1(x_2,0)=\partial_{x_1}q(0,x_2)+\alpha q(0,x_2)$, $f_2(x_1,0)=\partial_{x_2}q(x_1,0)+\alpha_2 q(x_1,0)$. The solution of this IBV problem is given by equation (1.22) where $w(k)=i(k_1^2+k_2^2)$, $\hat{q}_0(k)$ is given by equation (1.21a), $\partial D_+^{(m)}$ is the boundary of the first quadrant of the complex k_m-plane (see Figure 1.1), and $\hat{g}(k,t)$, $\hat{g}^{(1)}(k,t)$, $\hat{g}^{(2)}(k,t)$ can be compared as follows: Given q_0, f_1, and f_2, compute $\hat{G}^{(m)}(k,t)$, $m=1,2$, by

$$\hat{G}^{(1)}(k,t)+(k_1+i\alpha_1)\hat{q}_0(-k_1,k_2)-2ik_1\int_0^t d\tau\int_0^\infty dx_2 e^{-ik_2x_2+w(k)\tau}f_1(x_2,\tau),$$

$$\hat{G}^{(2)}(k,t)=(k_2+i\alpha_2)\hat{q}_0(k_1,-k_2)-2ik_2\int_0^t d\tau\int_0^\infty dx_1 e^{-ik_1x_1+w(k)\tau}f_1(x_1,\tau). \tag{3.24}$$

If $0<\arg\alpha_j<\frac{3\pi}{2}$, $j=1,2$, then compare $\hat{g}^{(1)}$, $\hat{g}^{(2)}$, $\hat{g}$ by

$$\hat{g}^{(1)}=\frac{\hat{G}^{(1)}}{k_1-i\alpha_1}, \quad \hat{g}^{(2)}=\frac{\hat{G}^{(2)}}{k_2-i\alpha_2}, \tag{3.25}$$

$$\hat{g}(k,t)=\frac{k_1+i\alpha_1}{k_1-i\alpha_1}\frac{k_2+i\alpha_2}{k_2-i\alpha_2}\hat{q}_0(-k_1,-k_2)-\frac{2ik_1}{k_1-i\alpha_1}\frac{k_2+i\alpha_2}{k_2-i\alpha_2}\hat{f}_1(k_1,-k_2)-\frac{2ik_2}{k_2-i\alpha_2}\frac{k_1+i\alpha_1}{k_1-i\alpha_1}\hat{f}_2(-k_1,k_2).$$

If $-\frac{\pi}{2}<\arg\alpha_1<0$, $0<\arg\alpha_2<\frac{3\pi}{2}$, then $\hat{g}^{(1)}$ has the additional term

$$-\frac{\hat{G}^{(1)}(i\alpha_1,k_2,t)}{k_1-i\alpha_1}, \tag{3.26a}$$

while $\hat{g}(k,t)$ has the additional term

$$-\frac{1}{k_1-i\alpha_1}\left[2i\alpha_1\frac{k_2+i\alpha_2}{k_2-i\alpha_2}\hat{q}_0(-i\alpha_1,-k_2)+2\alpha_1\frac{k_2+i\alpha_2}{k_2-i\alpha_2}\hat{f}_1(i\alpha_1,-k_2)+\frac{4\alpha_1k_2\hat{f}_2(-i\alpha_1,k_2)}{k_2-i\alpha_2}\right]. \tag{3.26b}$$

Indeed, multiplying equation (3.23a) by $\exp[-ik_2x_2+w(k)t]$ and integrating we find

$$\hat{g}_1^{(1)}=-\alpha_1 g_0^{(1)}+\hat{f}_1,$$

where $\hat{f}_1$ is the integral term in the rhs of equation (3.24a). Substituting $\hat{g}_1^{(1)}$ from this equation into the definition of $\hat{g}^{(1)}$ (equation (3.10)) we find

$$\hat{g}^{(1)} = -i\hat{f}_1 + (k_1 + i\alpha_1)\hat{g}_0^{(1)}. \tag{3.27}$$

Substituting this expression into the equation obtained from equation (3.9) by replacing k_1 with $-k_1$, solving the resulting equation for $\hat{g}_0^{(1)}$ and then substituting the expression for $\hat{g}_0^{(1)}$ into equation (3.27) we find

$$(k_1 - i\alpha_1)\hat{g}^{(1)} = \hat{G}^{(1)} - (k_1 + i\alpha_1)e^{wt}\hat{q}(-k_1, k_2, t) + (k_1 + i\alpha_1)\hat{g}^{(2)}(-k_1, k_2, t). \tag{3.28}$$

Similarly we obtain

$$\hat{g}^{(2)} = -i\hat{f}_2 + (k_2 + i\alpha_2)\hat{g}_0^{(2)}, \tag{3.29}$$

(where $\hat{f}_2$ is the integral term in the rhs of equation (3.24b)) and

$$(k_2 - i\alpha_2)\hat{g}^{(2)} = \hat{G}^{(2)} - (k_2 + i\alpha_2)e^{wt}\hat{q}(k_1, -k_2, t) + (k_2 + i\alpha_2)\hat{g}^{(1)}(k_1, -k_2, t). \tag{3.30}$$

We first assume that $k_1 - i\alpha_1 \neq 0$ for k_1 in the first quadrant of the complex k_1-plane and $k_2 - i\alpha_2 \neq 0$ for k_2 in the first quadrant of the complex k_2-plane. Then proceeding as with the case of Dirichlet boundary conditions we find

$$\hat{g}(k,t) = \frac{(k_1 + i\alpha_1)}{k_1 - i\alpha_1}\hat{g}^{(2)}(-k_1, k_2, t) + \frac{k_2 + i\alpha_2}{k_2 - i\alpha_2}\hat{g}^{(1)}(k_1, -k_2, t). \tag{3.31}$$

Using equations (3.27) and (3.29) to write $\hat{g}^{(1)}$ and $\hat{g}^{(2)}$ in terms of $\{g_0^{(1)}, f_1\}$ and $\{g_0^{(2)}, f_2\}$, the above expression simplifies to

$$\hat{g}(k,t) = -i\left[\frac{k_1 + i\alpha_1}{k_1 - i\alpha_1}\hat{f}_2(-k_1, k_2) + \frac{k_2 + i\alpha_2}{k_2 - i\alpha_2}\hat{f}_1(k_1, -k_2)\right]$$

$$+\frac{k_1 + i\alpha_1}{k_1 - i\alpha_1}\frac{k_2 + i\alpha_2}{k_2 - i\alpha_2}\left[(k_2 - i\alpha_2)\hat{g}_0^{(1)}(-k_1, k_2) + (k_1 - i\alpha_1)\hat{g}_0^{(2)}(k_1, -k_2)\right]. \tag{3.32}$$

Substituting equations (3.27) and (3.29) in equation (3.9) and replacing in the resulting equation k_1, k_2 with $-k_1, -k_2$ we find

$$-i\hat{f}_1(k_1, -k_2) + (-k_1 + i\alpha_1)\hat{g}_0^{(1)}(k_1, -k_2) - i\hat{f}_2(-k_1, k_2) + (-k_2 + i\alpha_2)\hat{g}_0^{(2)}(-k_1, k_2)$$

$$= -\hat{q}_0(-k_1, -k_2) + e^{wt}\hat{q}(-k_1, -k_2, t). \tag{3.33}$$

Thus the bracket in the search term of (3.32) equals

$$\hat{q}_0(-k_1, -k_2) - i\hat{f}_1(k_1, -k_2) - i\hat{f}_2(-k_1, k_2) - e^{wt}\hat{q}(-k_1, -k_2, t).$$

Replacing the bracket in the second term of (3.32) by the above expression we find that $\hat{g}(k,t)$ is given by equation (3.25c) plus a term involving $\exp[wt]$ which does not contribute to $q(x,t)$.

We now assume that $k_1 - i\alpha_1 = 0$ for k_1 in the first quadrant of the complex k_1-plane, while $k_2 - i\alpha_2 \neq 0$ for k_2 in the first quadrant of the complex k_2-plane. Evaluating equation (3.28) at $k_1 = i\alpha_1$ and subtracting the resulting equation from (3.28) we find

$$\hat{g}^{(1)} = \frac{\hat{G}^{(1)} - \hat{G}^{(1)}(i\alpha_1, k_2, t)}{k_1 - i\alpha_1} - \frac{\left[(k_1 + i\alpha_1)e^{wt}\hat{q}(-k_1, k_2, t) - 2i\alpha_1 e^{(-i\alpha_1^2 + ik_2^2)t}\hat{q}(-i\alpha_1, k_2, t)\right]}{k_1 - i\alpha_1}$$

$$+ \frac{\left[(k_1 + i\alpha_1)\hat{g}^{(2)}(-k_1, k_2, t) - 2i\alpha_1\hat{g}^{(2)}(-i\alpha_1, k_2, t)\right]}{k_1 - i\alpha_1}.$$

The second theorem of the rhs of this equation is analytic and bounded. Thus proceeding as above we find for $\hat{g}(k,t)$ an expression similar to (3.25c) but with the additional term $-2i\alpha_1\hat{g}^{(2)}(-i\alpha_1, k_2, t)/(k_1 - i\alpha_1)$. Following the same steps used above and also evaluating equation (3.33) at $k_1 = i\alpha_1$, we find that the additional term in $\hat{g}(k,t)$ is given by (3.26b). We also note that the terms involving $\hat{q}$ appear in the form

$$-\frac{1}{k_1 - i\alpha_1}\frac{k_2 + i\alpha_2}{k_2 - i\alpha_2}\left[(k_1 + i\alpha_1)e^{wt}\hat{q}(-k_1, -k_2, t) - 2i\alpha_1 e^{i(-\alpha_1^2 + k_2^2)t}\hat{q}(-i\alpha_1, -k_2, t)\right],$$

thus they give zero contribution.

Remark 3.1

The general equation (1.6) can be treated similarly:

Suppose that $\nu_{j_1}^{(1)}(k_1)$ and $\nu_{j_1}^{(2)}(k_2)$ are the transformations which leave $w_1(k_1)$ and $w_2(k_2)$ invariant. $\hat{g}^{(1)}$ can be computed in terms of $\hat{g}^{(2)}(\nu_{j_1}^{(1)}(k_1), k_2, t)$ by solving a system of $n_1 - N_1$ algebraic equations. Similarly $\hat{g}^{(2)}$ can be computed in terms of $\hat{g}^{(1)}(k_1, \nu_{j_2}^{(2)}(k_2), t)$ by solving a system of $n_2 - N_2$ algebraic equations. Using contour deformations the k_2-integral involving $\hat{g}^{(2)}(\nu_{j_1}^{(1)}(k_1), k_2, t)$ can be deformed to a k_2-integral over the contour $\partial D_+^{(2)}$. Similarly the k_1-integral involving $\hat{g}^{(1)}(k_1, \nu_{j_2}^{(2)}(k_2), t)$ can be deformed to a k_1-integral over the contour $\partial D_+^{(2)}$. Similarly the k_1-integral involving $\hat{g}^{(1)}(k_1, \nu_{j_2}^{(2)}(k_2), t)$ can be deformed to a k_1-integral over the contour $\partial D_+^{(1)}$. This yields a combination of $\hat{g}^{(2)}(\nu_j^{(1)}(k_1), k_2, t)$ and $\hat{g}^{(1)}(k_1, \nu_{j_2}^{(2)}(k_2), t)$ over the $9K - 1, k_2)$ integral along $\partial D_+^{(1)}$ and $\partial D_+^{(2)}$. This combination can be computed by evaluating the global condition at $(k_1, k_2) = (\nu_{j_1}^{(1)}(k_1), \nu_{j_2}^{(2)}(k_2))$, $j_1 = 1, \cdots, n_1 - N_1$, $j_2 = 1, \cdots, n_2 - N_2$.

4 Systems of Evolution Equations, Forced Problems and Long Time Asymptotics

The elastic wave equation can be considered as a particular reduction of the Schrödinger equation (1.2). Indeed, if

$$q = u + iv,$$

where u and v are real, equation (1.2) yields the *system* of two equations

$$u_t = -v_{xx}, \quad v_t = u_{xx}. \tag{4.1}$$

Eliminating v from these equations we find that u satisfies the elastic wave equation (1.5). This implies that an IBV problem for the elastic wave equation can be mapped to an IBV problem for equation (1.2). Indeed, the initial conditions $u(x,0)$ and $u_t(x,0)$ imply the initial condition

$$q(x,0) = u(x,0) + iv(x,0), \tag{4.2}$$

where $v(x,0)$ is computed from $u_t(x,0)$ as follows: Given $u_t(x,0)$ equation (4.1a) implies $v_{xx}(x,0)$, which in turn implies $v(x,0)$ via integration. We now discuss two typical boundary conditions.

1. *Given $u(0,t)$ and $u_{xx}(0,t)$.* This implies

$$q(0,t) = u(0,t) + iv(0,t). \tag{4.3}$$

Indeed, given $u_{xx}(0,t)$ equation (4.1b) implies $v_t(0,t)$, which in turn implies $v(0,t)$ via integration.

2. *Given $u(0,t)$ and $u_x(0,t)$.* This implies that the rear part of q and of q_x are given at $x = 0$,

$$(\mathrm{Re}q)(0,t) = u_0(t), \quad (\mathrm{Re}q_x)(0,t) = u_1(t), \quad t > 0. \tag{4.4}$$

The Schrödinger equation (1.2) with given initial and boundary conditions $q(x,0)$ and $q(0,t)$ was solved in Schrödinger 2. In what follows we solve equation (1.2) with the boundary conditions (4.4).

The solution of the elastic wave equation.

Let $q(x,t)$ satisfy equation (1.2) with the initial condition $q(0,0) = q_0(x)$ and the boundary conditions, (4.4) where $q_0(x)$, $u_0(t)$, $u_1(t)$ are given functions and $u_0(0) =$Re$q_0(0)$, $u_1(0) =$ Re$\partial_x q_0(0)$. The solution of the IBV problem is given by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx-ik^2t}\hat{q}_0(k) + \frac{1}{2\pi}\int_{\partial D_+} dk e^{ikx-ik^2t}\hat{g}(k,t), \tag{4.5}$$

where ∂D_+ is the boundary of the first quadrant of the complex k-plane (see Figure 1.1a) and $\hat{q}_0(k)$, $\hat{g}(k,t)$ can be obtained as follows: Given q_0, u_0, u_1, compute $\hat{q}_0, \hat{u}_0, \hat{u}_1$ by

$$\hat{q}_0(k) = \int_0^{\infty} dx e^{-ikx} q_0(x), \hat{u}_0(k,t) = \int_0^t d\tau e^{ik^2\tau}u_0(\tau), \hat{u}_1(k,t) = \int_0^t d\tau e^{ik^2\tau}u_1(\tau). \tag{4.6}$$

Then compute $\hat{g}(k,t)$ by

$$\hat{g}(k,t) = 2(1-i)k\hat{u}_0 + 2(1-i)\hat{u}_1 - (i+1)\hat{\bar{q}}_0(-ik) + i\hat{q}_0(-k), \tag{4.7}$$

where bar denotes complex conjugation, i.e. $\bar{q}_0 = u_0 - iv_0$.

Indeed it was shown in §2 that q is given by equation (4.5) with $\hat{q}_0$ and $\hat{g}$ given by equations (4.6a) and

$$\hat{g} = -\hat{q}_0(-k) \neq 2k\hat{g}_0. \tag{4.8}$$

Using the notations

$$q(0,t) = u_0(t) + iv_0(t), \quad q_x(0,t) = u_1(t) + iv_1(t),$$

denoting by $\hat{v}_0(k,t)$, $\hat{v}_1(k,t)$ the associated t-integrals of v_0, v_1, and replacing k by $-k$ in the global relation (2.9) we find

$$\hat{v}_1 - ik\hat{v}_0 = i\hat{u}_1 + k\hat{u}_0 - \hat{q}_0(-k) + e^{ik^2t}\hat{q}(0k,t), \quad \mathrm{Im}k \geq 0. \tag{4.9}$$

Taking the complex conjugate of this equation and replacing *bark* by $-ik$ in the resulting equation we find

$$\hat{v}_1 + k\hat{v}_0 = -i\hat{u}_1 - ik\hat{u}_0 - \bar{\hat{q}}_0(-ik) + e^{ik^2t}\bar{\hat{q}}(-ik,t), \quad \mathrm{Re}k \geq 0. \tag{4.10}$$

Equations (4.9) and (4.10) are two algebraic equations for $\hat{v}_0$ and $\hat{v}_1$. Solving these equations for $\hat{v}_0$ and substituting the resulting expression into equation (4.8), where $\hat{g}_0 = \hat{u}_0 + i\hat{v}_0$, we find equation (4.7). We note the terms $\hat{q}(-k,t)$ and $\hat{q}(-ik,t)$ do not contribute to $q(x,t)$ due to analytic considerations (they involve the terms $\exp(ikx)$ and $\exp(-kx)$ which are bounded in the first quadrant of the complex k-plane.

Fourier Problems

Suppose that $q(x,t)$ satisfies the forced version of equation (1.1), i.e.

$$(\partial_t + w(-i\partial_x))q(x,t) = h(x,t), \quad 0 < x < \infty, \quad t > 0, \tag{4.11}$$

where $4h(x,t)$ is a given function with sufficient smoothness and decay.

The solution of this equation can be reduced to the solution of equation (1.1). Indeed using the x-Fourier transform it follows that a particular solution of equation (4.11) is given by

$$H(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx-w(k)t}\left(\int_0^t d\tau \int_0^{\infty} d\xi e^{-ik\xi+w(k)\tau}h(\xi,\tau)\right). \tag{4.12}$$

The general solution of equation (4.11) is given by

$$q(x,t) = \tilde{q}(x,t) + H(x,t), \tag{4.13}$$

where $\tilde{q}(x,t)$ satisfies the homogeneous equation (1.1). If $q(x,t)$ satisfies the initial condition $q(x,0) = q_0(x)$ and the boundary conditions $\partial_x^j q(0,t) = g_j(t)$, $j = 0, 1, \cdots, N-1$ (see equations (1.13)), then $\tilde{q}(x,t)$ satisfies equation (1.1) with the following initial and boundary conditions:

$$\tilde{q}(x,0) = q_0(x), \quad \partial_x^j\tilde{q}(0,t) = g_j(t) - \partial_x^j H(0,t). \tag{4.14}$$

Similar considerations are valid for equation (1.6), where the analogue of equation (4.12) is now

$$H(x,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 e^{ikx-w(k)t}\left(\int_0^t d\tau \int_0^{\infty} d\xi_1 \int_0^{\infty} d\xi_2 e^{-ik\xi j+w(k)\tau}h(\xi,\tau)\right). \tag{4.15}$$

An Alternative Representation

Suppose that equation (1.1) is valid for $0 < t < T$, where T is a given positive constant. It is straightforward to show that an alternative representation for the solution $q(x,t)$ is given by equation (1.15) where $\hat{g}(k,t)$ is replaced by $\hat{g}(k,T)$. This function is computed in terms of $\hat{q}_0$ and $\hat{g}_j(k,t)$,

$$\hat{g}_j(k,T) = \int_0^T d\tau e^{w(k)\tau} g_j(\tau). \tag{4.16}$$

Indeed this alternative representation and the rhs of equation (1.15) differ by terms of the type

$$\int_{\partial D_+} dk e^{ikx-w(k)t} \int_t^T d\tau e^{w(k)\tau} g_j(\tau).$$

However, since $T \geq t$, the coefficient $(\tau - t)$ of $w(k)$ is non-negative, thus $\exp[ikx + w(k)(\tau - t)]$ is analytic and bounded in D_+, and Cauchy's theorem implies that the above integral vanishes.

In particular if $T = \infty$, it follows that $q(x,t)$ can be written in the form

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx-w(k)t} \hat{q}_0(k) + \frac{1}{2\pi} \int_{\partial D_+} dk e^{ikx-w(k)t} \hat{q}(k), \tag{4.17}$$

where $\hat{g}(k)$ is computed in terms of $\hat{q}_0(k)$ and of $\hat{g}_j(k)$,

$$g_j(k) = \int_0^\infty dt e^{w(k)t} g_j(t).$$

The representation (4.17) is very convenient for computing the long time behavior of $q(x,t)$. Indeed, since x and t appear only in the exponential form $\exp[ikx - w(k)t]$, the large time asymptotics of the rhs of equation (4.17) can be computed using standard asymptotic techniques such as the steepest descent method. In the case when equation (1.1) is a dispersive equation (i.e. $w(k)$ is imaginary), this is discussed in [5].

Similar considerations are valid for equation (1.6): If $T = \infty$, $q(x,t)$ can be written in the form

$$q(x,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 e^{ikx-w(k)t} \hat{q}_0(k) + \frac{1}{(2\pi)^2} \int_{\partial D_+^{(1)}} dk_1 \int_{\partial D_+^{(2)}} dk_2 e^{ikx-w(k)t} \hat{g}(k)$$

$$+ \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_2 \int_{\partial D_+^{(1)}} dk_1 e^{ikx-w(k)t} \hat{g}^{(1)}(k) + \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dk_1 \int_{\partial D_+^{(2)}} dk_2 e^{ikx-w(k)t} \hat{g}^{(2)}(k), \tag{4.18}$$

where $\hat{g}^{(1)}(k)$, $\hat{g}^{(2)}(k)$, $\hat{g}(k)$ can be computed in terms of $\hat{q}_0(k)$ and of $\hat{g}_j^{(1)}(k)$, $\hat{g}_j^{(2)}(k)$,

$$\hat{g}_j^{(1)}(k) = \int_0^\infty dt \int_0^\infty dx_2 e^{-ik_2x_2+w(k)t} g_j^{(1)}(x_2,t),$$

$$\hat{g}_j^{(2)}(k) = \int_0^\infty dt \int_0^\infty dx_1 e^{-ik_1x_1+w(k)t} g_j^{(2)}(x_1,t). \tag{4.19}$$

5 Conclusions

We have introduced a new transform method for solving linear evolution PDEs and have illustrated the applicability of this method by solving several concrete IBV problems. It appears that this method is both general and simple to implement. Indeed, the only mathematical tools used in this paper are the Fourier transform and Cauchy's theorem. For evolution equations in one space dimension this method is based on the following: (i) It is possible to express $q(x,t)$ as *an integral in the complex k-plane*. This integral involves the x-transform of the initial condition and the t-transform of the boundary values,

$$\hat{q}_0(k) = \int_0^\infty dx e^{-ikx} q(x,0), \quad \hat{g}_j(k,t) = \int_0^t d\tau e^{w(k)\tau} \partial_x^j q(0,\tau), \quad j = 0, \cdots, n-1, \tag{5.1}$$

where n is the order of the highest spatial derivative. (ii) For a given IBV problem only a subset of $n/2$ or $(n+1)/2$ or $(n-1)/2$ of the $\hat{g}_j$'s are known. (iii) It is possible to compute the unknown $\hat{g}_j$'s by using the *global relation* satisfied by $\hat{q}_0$, by the $\hat{g}_j s$'s, and by the unknown function $\hat{q}(k,t)$. This is based on the observation that $\hat{g}_j$ is invariant under the transformations $k \to \nu(k)$, where $w(k) = w(\nu(k))$. Using these transformations, the global relation yields a system of $n/2$ or $(n-1)/2$ or $(n+1)/2$ linear algebraic equations for the unknown $\hat{g}_j$'s. This system also involves $\hat{q}(k,t)$ evaluated at $\nu(k)$, but using the representation of $q(x,t)$ in the complex k-plane mentioned in (i), it can be shown that the terms involving $\hat{q}(\nu(k),t)$ give either a zero contribution or a discrete contribution whose residue can be calculated in terms of $\hat{q}_0$ and of the known $\hat{g}_j$'s.

The situation for equations in the space dimensions is similar.

Some simple IBV problems can be solved by appropriate x-transform. In these cases it is straightforward to obtain these classical formulae by deforming the representation (1.15) from the complex k-plane to the real axis. Consider for example the solution of equation (1.2) with Dirichlet boundary conditions,

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty dk e^{ikx - ik^2 t} \hat{q}_0(k) + \frac{1}{2\pi} \int_{\partial D_+} dk e^{ikx - ik^2 t} \left(-\hat{q}_0(-k) + 2k \int_0^t d\tau e^{ik^2\tau} g_0(\tau) \right), \tag{5.2}$$

where ∂D_+ is the boundary of the first quadrant of the complex k-plane. Since $\exp[ikx]$, $\hat{q}_0(-k)$, $\exp[ik^2(\tau - t)]$ are bounded in the second quadrant of the complex k-plane, the integral along the positive imaginary axis can be deformed to an integral along the negative real axis

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty dk e^{ikx - ik^2 t} \left[\hat{q}_0(k) - \hat{q}_0(-k) + 2k \int_0^t d\tau e^{ik^2\tau} g_0(\tau) \right].$$

Splitting the integral to positive and negative k's and using $k \to j - k$, we find the usual sine transform representation,

$$q(x,t) = \frac{2}{\pi} \int_0^\infty dk \sin kx e^{-ik^2 t} \left[\tilde{q}_0(k) + ik \int_0^t d\tau e^{ik^2\tau} g_0(\tau) \right], \tag{5.3}$$

where $\tilde{q}_0(k)$ is the sine-transform of $q_0(x)$.

Appropriate x-transforms do *not* exist for complicated IBV problems. In these cases the representation for $q(x,t)$ *cannot* be deformed to the real axis. For example, the solution of equation (1.3) with Dirichlet boundary conditions contains $\hat{q}_0(i-k)$, see equation (1.17); this term involves the exponential $\exp[-i(i-k)x]$ which is bounded for $k_I \geq 1$ (i.e. in D_+), but since it is not bounded for $0 \leq k_I < 1$ this term cannot be deformed to the real axis.

It appears that even in the cases that it is possible to deform the contour to the real axis the representation in the complex k-plane has certain advantages. For example the integral of the rhs of equation (5.3), in contrast to the integral in the rhs of equation (5.2), is *not* uniformly convergent at $x = 0$ (otherwise $q(0,t) = 0$). Also a convenient way to study the long time behavior of the rhs of equation (5.3) is to *transform* (5.3) to the form given by (5.2).

We note that the alternative representation (4.17) has the disadvantage that it depends on the constant T, but it has the advantage that it depends on (x,t) only in the exponential form $\exp[ikx - w(k)t]$. This is very convenient for studying the long time asymptotics of $q(x,t)$.

An effort has been made to minimize technical considerations with the hope that this will help the interested reader become familiar with the new method. Unfortunately this has been done at the expense of rigor. However, it is possible to make the results presented here rigorous. For evolution equations in one space variable this has been done in [1], where general theorems are proven in appropriate Sobolev spaces.

There exists several possible generalizations: (i) The integral representation of non-separable PDEs in two space variables is discussed in Appendix B. (ii) It is possible to obtain similar integral representations for (x_1, x_2) in a convex polygon. However, in this case the analysis of the global relation is more complicated. (iii) The application of this method to elliptic PDEs in two variables with (x_1, x_2) in a convex polygon is discussed in [6]. In this case for simple polygons and simple boundary conditions the global relation can be solved by algebra, but in general it leads to a Riemann-Hilbert problem [7]. (iv) Evolution PDEs in one space variable with a moving boundary are discussed in [8].

Historical Remarks

1. The discovery of this method was the byproduct of the investigation of boundary value problems for nonlinear integrable PDEs [9]. Actually the original derivation of equation (1.15) did not use the Fourier transform and contour deformation but it used the *spectral analysis* of an auxiliary function $\mu(x,t,k)$ see 2. below. Although the spectral analysis is indispensable for nonlinear integrable PDEs, it can be avoided for linear PDEs. An alternative derivation of equation (1.15) (avoiding the spectral analysis of μ) based on a reformulation of Green's theorem is presented in [10]. The derivation presented here, which is different than the one of [10], appears to be the simplest.

2. Equation (2.1) implies that the following differential 1-form W is closed

$$W(x,t) = e^{-ikx+w(k)t}(q\,dx - X\,dt). \tag{5.4}$$

Indeed,

$$dW = \left[\left(e^{ikx+w(k)t}q\right)_t + \left(e^{-ikx+w(k)t}X\right)_x\right] dt \wedge dx = 0.$$

For a simply connected domain this implies: (a)

$$\int_{\partial D} W = 0, \tag{5.5a}$$

where ∂D denotes the boundary of the domain. This equation is precisely the global condition. (b) There exists a 0-form in such that $dM = W$. Letting $M = \mu \exp[-ikx + w(k)t]$ this yields

$$d\left[e^{-ikx+w(k)t}\mu\right] = e^{-ikx+w(k)t}(qdx - Xdt). \tag{5.5b}$$

The original derivation of equation (1.15) is based on the spectral analysis of this equation. In this respect we note that this equation can be considered as a Lax pair [11] for the linear PDE (1.1) [12],

$$\mu_x - ik\mu = q, \quad \mu_t + w(k)\mu = -X.$$

For equation (1.6) the analogue of equation (5.5b) is

$$d\left[e^{-ikx+w(k)t}\mu\right] = e^{-ikx-w(k)t}\left[qdx_1 \wedge dx_2 + X^{(1)}dx_2 \wedge dt + X^{(2)}dt \wedge dx_1\right].$$

3. The representation (4.17) is consistent with the celebrated Euler-Ehrenpreis-Palamolev principle [13].

Appendix A

Let q satisfy

$$\partial_t q + \partial_x q - \partial_x^5 q = 0, \quad 0 < x < \infty, \quad t > 0. \tag{A.1}$$

$$q(x,0) = q_0(x), \quad 0 < x < \infty$$

$$\partial_x^j q(0,t) = g_j(t), \quad j = 0,1, \quad t > 0.$$

1. Determination of the domain Re $w < 0$.

Substituting $q = \exp(ikx - w(k)t)$ into equation (A.1) we find

$$w(k) = ik - ik^5. \tag{A.2}$$

Thus for large k, $w(k) \sim -ik^5$, and the domain Re $w < 0$ asymptotically is determined by

$$\frac{\pi}{2} + 2m\pi < 5\arg k - \frac{\pi}{2} < \frac{3\pi}{2} + m\pi.$$

Thus it consists of the five shaded sectors depicted in Figure A.1; each of these sectors is of angle $\pi/5$. In the upper half plane

Figure A.1

there exist two sectors denoted by D_1^+, D_2^+ while in the lower half sector there exist three sectors denoted by D_1^-, D_2^-, D_3^-.

The precise form of the domain Re $w < 0$ is depicted in Figure A.2.

Figure A.2

2. Determination of $\hat{g}(k,t)$.

The equation

$$X = -i\frac{w(k) - w(l)}{k - l}q,$$

implies

$$X = \frac{(k - k^5) - (l - l^5)}{k - l}q = [1 - (k^4 + k^3 l + k^2 l^2 + kl^3 + kl^4)]q,$$

thus

$$X = [(1 - k^4) + ik^3\partial_x + ik^3\partial_x + k^2\partial_x^2 - ik\partial_x^3 - k\partial_x^4]q.$$

Hence

$$\hat{g} = (1 - k^4)\hat{g}_0 + ik^3\hat{g}_1 + k^2\hat{g}_2 - ik\hat{g}_3 - k\hat{g}_4, \tag{A.3}$$

where

$$\hat{g}_j = \int_0^t d\tau e^{ikx + i(k - k^5)t}\partial_x^j q(0,t), \quad j = 0, 1, \cdots, 4. \tag{A.4}$$

3. The integral representation

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx - i(k-k^5)t}\hat{q}_0(k) + \frac{1}{2\pi}\int_{\partial D_+} dk e^{ikx - i(k-k^5)t}\hat{g}(k,t), \tag{A.5}$$

where ∂D_+ is the contour shown in Figure A.2, $\hat{g}(k,t)$ is given by (A.3) and $\hat{q}_0(k)$ is the Fourier transform of the initial condition (equation (1.21a)).

4. The analysis of the global relation

The global relation (2.9) becomes

$$(1-k^4)\hat{g}_0 + ik^3\hat{g}_1 + k^2\hat{g}_2 - ik\hat{g}_3 - k\hat{g}_4 = -\hat{q}_0 + e^{i(k-k^5)t}\hat{q}(k,t). \qquad (A.6)$$

The equation

$$k - k^3 = \nu - \nu^3$$

has four non-trivial roots. For large k these roots are given by

$$\nu_1 \sim ke^{\frac{2i\pi}{5}}, \quad \nu_2 \sim ke^{\frac{4i\pi}{5}}, \quad \nu_3 \sim ke^{\frac{6i\pi}{5}}, \quad \nu_4 \sim ke^{\frac{8i\pi}{5}}. \qquad (A.7)$$

If ν_1 is in D_1^-, ν_2 is in D_2^-, ν_3 is in D_3^- then k is in D_1^+.

If ν_2 is in D_1^-, ν_3 is in D_2^-, ν_4 is in D_3^- then k is in D_2^+.
Replacing in equation (A.6) k by ν_1, ν_2, ν_3 we find three algebraic equations for the 3 unknown functions $\hat{g}_2, \hat{g}_3, \hat{g}_4$. This yields $\hat{g}$ for k in D_1^+. Similarly replacing in equation (A.6) k by ν_2, ν_3, ν_4, we find three algebraic equations which yield $\hat{g}$ for k in D_2^+.

Appendix B

Let $q(x,t)$ satisfy

$$\left(\partial_t + w(-i\partial_{x_1}, -i\partial_{x_2})\right) q(x_1, x_2, t) = 0, \quad 0 < x_j < \infty, \quad t > 0. \qquad (B.1)$$

where $w(k_1, k_2)$ is a polynomial. The solution of this equation can be represented in the form

$$q(x,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 e^{ikx - w(k)t}\hat{q}_0(k) +$$

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_2 \int_{\partial D_+^{(1)}((k_2)_R)} dk_1 e^{ikx-w(k)t}\hat{g}^{(1)}(k,t) + \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1 \int_{\partial D_+^{(2)}((k_1)_R)} dk_2 e^{ikx-w(k)t}\hat{g}^{(2)}(k,t), \qquad (B.2)$$

where $\hat{q}_0(k)$ is defined by (1.21a), $\hat{g}^{(1)}$ and $\hat{g}^{(2)}$ are defined in terms of $X^{(1)}$ and $X^{(2)}$ by equations (3.7), and $X^{(m)}$, $\partial D_+^{(m)}$, $m = 1, 2$, are defined as follows:

$$X^{(1)} = -i\frac{w(k_1,k_2) - w(l_1,k_2)}{k_1 - l_1}q, \quad l_1 = -i\partial_{x_1}, \qquad (B.3a)$$

$$X^{(2)} = -i\frac{w(l_1,k_2) - w(l_1,l_2)}{k_2 - l_2}q, \quad l_1 = -i\partial_{x_1}, \quad l_2 = -i\partial_{x_2}; \qquad (B.3b)$$

$\partial D_+^{(m)}$ are the oriented boundaries of the domains,

$$D_+^{(1)}((k_2)_R): \quad k_2 = (k_2)_R, \quad (k_1)_I > 0, \quad \mathrm{Re}w(k) < 0,$$

$$D_+^{(2)}((k_1)_R): \quad k_1 = (k_1)_R, \quad (k_2)_I > 0, \quad \mathrm{Re}w(k) < 0, \qquad (B.4)$$

We emphasize that in contrast to the separable case, $D_+^{(1)}$ depends on $(k_2)_R$ and $D_+^{(2)}$ depends on $(k_1)_R$.

Indeed we first note that equation (B.1) can be rewritten in the form (3.1) where $X^{(m)}$ are given by equations (B.3): Using (B.1), equation (3.1) becomes

$$(w(k_1,k_2) - w(-i\partial_{x_1}, -i\partial_{x_2}))\, q + (\partial_{x_1} - ik_1)X^{(1)} + (\partial_{x_2} - ik_2)X^{(2)} = 0.$$

Adding and subtracting the term $w(-i\partial_{x_1}, k_2)q$ we find

$$\left\{(w(k_1,k_2) - w(-i\partial_{x_1}, k_2))\, q + (\partial_{x_1} - ik_1)X^{(1)}\right\} + \left\{(w(-i\partial_{x_1}, k_2) - w(-i\partial_{x_1}, -i\partial_{x_2})))\, q + (\partial_{x_2} - ik_2)X^{(2)}\right\} =$$

Thus if $X^{(m)}$, $m = 1, 2$, are defined by equations (B.3), this equation is satisfied identically.

It is again possible to deform the relevant integrals to the complex (k_1, k_2)-planes. For example, the term involving $\hat{g}^{(1)}$ can be deformed through the domain defined by $(k_1)_I > 0$ and $\mathrm{Re} w(k) > 0$; this yields an integral along $\partial D_+^{(1)}$. Similarly for the term involving $\hat{g}^{(2)}$.

Example B.1 For the linearized Kadomtsev-Petviashvilli equation

$$q_t + q_{x_1x_1x_1} - 3\partial_{x_1}^{-1} q_{x_2x_2x_2} = 0, \tag{B.5}$$

$X^{(1)}$ and $X^{(2)}$ are given by

$$X^{(1)} = \left(\partial_{x_1}^2 + ik_1\partial_{x_1} - k_1^2 + 3i\frac{k_2^2}{k_1}\partial_{x_1}^{-1}\right) q, \tag{B.6a}$$

$$X^{(2)} = -3\left(\partial_{x_1}^{-1}\partial_{x_2} + ik_2\partial_{x_1}^{-1}\right) q. \tag{B.6b}$$

Also

$$D_+^{(1)}((k_2)_R): \quad (k_1)_I > 0, \quad -(k_1)_I^2 + e(k_1)_R^2 - 3\frac{(k_1)_I}{|k_1|^2}(k_2)_R^2 < 0,$$

$$D_+^{(2)}((k_1)_R): \quad (k_2)_I > 0, \quad \frac{(k_2)_R}{(k_1)_I} < 0. \tag{B.7}$$

Indeed, using

$$w(k_1,k_2) = -i\left(k_1^3 + \frac{3k_2^2}{k_1}\right),$$

we find

$$X^{(1)} = -\frac{k_1^3 + \frac{3k_2^2}{k_1} - l_1^3 - \frac{3k_2^2}{l_1}}{k_1 - l_1} q = \left[-(k_1^2 + l_1^2 + k_1 l_1) + \frac{3k_2^2}{k_1 l_1}\right] q,$$

$$X^{(2)} = -\frac{l_1^3 + \frac{3k_2^2}{l_1} - l_1^3 - \frac{3l_2^2}{l_1}}{k_2 - l_2} q = -\frac{3}{l_1}(k_2 + l_2).$$

Remark B.1

It is sometimes convenient to use an alternative parameterization for (k_1, k_2). This is particularly the case for degenerate dispersion relations [14]. For example in the case of equation (B.5) using

$$k_1 = \lambda_1 - \lambda_2, \quad k_2 = \lambda^2 - \lambda_2^2$$

we find

$$w = -4i(\lambda_1^3 - \lambda_2^3).$$

Thus, the solution of equation (B.5) can be expressed in the form

$$q(x,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda_2 E \hat{q}_0(\lambda) + \frac{1}{(2\pi)^2} \int_{D_+^{(1)}} d\lambda_1 \int d\lambda_2 E g^{(1)}(\lambda, t)$$

$$+ \frac{1}{(2\pi)^2} J \int d\lambda_1 \int_{\Delta_+^{(2)}} d\lambda_2 E g^{(2)}(\lambda, t),$$

where

$$E = 2e^{i(\lambda_1 - \lambda_2)x_1 + i(\lambda_1^2 - \lambda_2^2)x_2 + 4i(\lambda_1^3 - \lambda_2^3)t}(\lambda_1 - \lambda_2),$$

$$\hat{q}_0(\lambda) = \int_0^{\infty} dx_1 \int_0^{\infty} dx_2 e^{-i(\lambda_1 - \lambda_2)x_1 - i(\lambda_1^2 - \lambda_2^2)x_2} q_0(x_1, x_2),$$

$$g^{(1)}(\lambda, t) = \int_0^t d\tau \int_0^{\infty} dx_2 e^{-i(\lambda_1^2 - \lambda_2^2)x_2 - 4i(\lambda_1^3 - \lambda_2^3)\tau} X^{(1)}(0, x_2, \tau, \lambda),$$

$$g^{(2)}(\lambda, t) = \int_0^t d\tau \int_0^{\infty} dx_1 e^{-i(\lambda_1 - \lambda_2)x_1 - 4i(\lambda_1^3 - \lambda_2^3)\tau} X^{(2)}(x_1, 0, \tau, \lambda),$$

$X^{(1)}$, $X^{(2)}$ are given by equations (B.5) with $k_1 = \lambda_1 - \lambda_2$ and $k_2 = \lambda_1^2 - \lambda_2^2$ and the domains $\Delta_+^{(1)}$, $\Delta_+^{(2)}$ are determined by

$$\mathrm{Im}(\lambda_1 - \lambda_2) > 0, \quad (\mathrm{Im}(\lambda_1^3 - \lambda_2^3) < 0,$$

$$\mathrm{Im}(\lambda_1^2 - \lambda_2^2) > 0, \quad (\mathrm{Im}(\lambda_1^3 - \lambda_2^3) < 0.$$

The solution of IBV problems for nonseparable evolution equations again involves the analysis of the global relation. This will be illustrated elsewhere.

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