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Physics Letters A ••• (••••) •••–•••
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Soliton multidimensional equations and integrable evolutions preserving Laplace's equation

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Received 9 May 2007; accepted 17 September 2007

Communicated by A.P. Fordy

Abstract

The KP equation, which is an integrable nonlinear evolution equation in $2+1$, i.e., two spatial and one temporal dimensions, is a physically significant generalization of the KdV equation. The question of constructing an integrable generalization of the KP equation in $3+1$, has been one of the central open problems in the field of integrability. By complexifying the independent variables of the KP equation, I obtain an integrable nonlinear evolution equation in $4+2$. The requirement that real initial conditions remain real under this evolution, implies that the dependent variable satisfies a nonlinear evolution equation in $3+1$ coupled with Laplace's equation. A reduction of this system of equations to a single equation in $2+1$ contains as particular cases certain singular integro-differential equations which appear in the theory of water waves.

1. Introduction

The modern history of *integrable systems* begins with the numerical discovery of *solitons* [1] for the Korteweg–de Vries equation; this was followed by the derivation of an analytical formula describing the interaction of N-solitons and by the derivation of a linear integral equation characterizing the solution of an arbitrary initial-value problem [2]. The importance of solitons stems from the fact that they exhibit particle-type interactions, and also they characterize the long time asymptotic behavior of the solution. An explosive activity begun in this new area of mathematical physics after it was understood that the method used for the analysis of the KdV equation could also be applied to the analysis of the nonlinear Schrödinger equation [3]. This led to the investigation of several physically significant nonlinear equations, including the self-dual Yang–Mills equations [4] and the Ernst equation (the axisymmetric reduction of Einstein's vacuum equations).

The Kadomtsev–Petviashvili (KP) and Davey–Stewartson (DS) equations are physically important two-dimensional integrable generalizations of the Korteweg–de Vries and the nonlinear Schrödinger equations, respectively. For these and other integrable equations in $2+1$ (i.e., two-spatial and one-temporal dimensions), there exist several types of solitons, which include *line-solitons* (solutions decaying exponentially in every direction except on certain lines), *lumps* (solutions decaying algebraically) [5], and *dromions* (solutions decaying exponentially and driven by boundary terms) [6].

The derivation and solution of integrable analogues of the KP and DS equations has been one of the central open problems in the field of integrability. Progress in this direction was reported in [7] where it was shown that there exist integrable multidimensional analogues of the KP and DS equations, but these equations exist in $4+2$ (i.e., four-spatial and two-temporal dimensions), as opposed to $3+1$. A method for solving these $4+2$ nonlinear equations was also introduced in [7].

In this Letter I will first show that it is straightforward, starting with an integrable PDE in either $1+1$ or $2+1$, to construct an integrable PDE in either $2+2$ or $4+2$, respectively. These equations can of course be reduced to the original equations in $1+1$ and $2+1$, but they also admit more interesting reductions, namely: (a) An equation in $2+2$ can be reduced to an integrable system consisting of an equation in $2+1$ coupled with a two-dimensional *constraint*. (b) An equation in $4+2$ can be reduced to an integrable system consisting of an equa-

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0375-9601/$ – see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.physleta.2007.09.037

tion in $3+1$ coupled with a *constraint*. It is interesting that for the multidimensional equations obtained from the KdV and KP equations the associated constraint is Laplace's equation.

It must be emphasized that the above non-trivial reductions are physically motivated. Indeed, equations in $2+2$ and $4+2$ are constructed by the *complexification* of the independent variables of the associated equations in $1+1$ and $2+1$. As a result of this construction it follows that *real* initial conditions yield *complex* fields. The requirement that the fields remain real yields the reductions mentioned earlier.

For brevity of presentation I will present details only for certain generalizations of the KdV and KP equations, or actually for generalizations of their *potential* versions. Similar results for other integrable nonlinear PDEs will be mentioned at the end of this Letter.

By complexifying the independent variables of the potential KdV equation I obtain the following integrable nonlinear PDE in $2+2$,

$$q_{\bar t}=\frac{1}{4}q_{\bar x\bar x\bar x}-\frac{3}{4}q_{\bar x}^2,\qquad t=t_1+it_2,\quad x=x_1+ix_2,\tag{1}$$

where bar denotes complex conjugation and $\{x_1,x_2,t_1,t_2\}$ are real independent variables. Real initial conditions do not remain real under the evolution (1). In order that reality is preserved, I require that q, in addition to Eq. (1), it also satisfies the equation

$$q_t=\frac{1}{4}q_{xxx}-\frac{3}{4}q_x^2.\tag{2}$$

The *single* function q satisfies *two* equations, thus it is necessary that $(q_{\bar t})_t=(q_t)_{\bar t}$, which implies

$$\left(q_{\bar x\bar x\bar x}-3q_{\bar x}^2\right)_t=\left(q_{xxx}-3q_x^2\right)_{\bar t}.$$

This equation, using Eqs. (1) and (2) to eliminate q_t and $q_{\bar t}$, simplifying the resulting expression, and recalling that $\partial_{\bar x}=(\partial_{x_1}+i\partial_{x_2})/2$, yields

$$(\Delta q)(\Delta q_{x_2})=0,\quad \Delta=\partial_{x_1}^2+\partial_{x_2}^2.\tag{3}$$

Adding and subtracting Eqs. (1), (2), I find the following two evolution equations

$$q_{t_1}=\frac{1}{16}\left(\partial_{x_1}^3-3\partial_{x_1}\partial_{x_2}^2\right)q-\frac{3}{8}\left(q_{x_1}^2-q_{x_2}^2\right),\tag{4}$$

$$q_{t_2}=\frac{1}{16}\left(-\partial_{x_2}^3+3\partial_{x_2}\partial_{x_1}^2\right)q-\frac{3}{4}q_{x_1}q_{x_2}.\tag{5}$$

Eq. (3) implies that either $q_{x_2}=0$ or that q is harmonic. In the first case, Eq. (4) becomes the potential KdV equation and Eq. (5) trivializes. In the second case, Eqs. (4) and (5) yield the following two integrable systems,

$$q_{t_1}=\frac{1}{4}q_{x_1x_1x_1}-\frac{3}{8}\left(q_{x_1}^2-q_{x_2}^2\right),\quad \Delta q=0,\tag{6}$$

$$q_{t_2}=-\frac{1}{4}q_{x_2x_2x_2}-\frac{3}{4}q_{x_1}q_{x_2},\quad \Delta q=0.\tag{7}$$

It is important to note that q remains harmonic under both t_1 and t_2 evolutions. Indeed,

$$\Delta q_t=\frac{1}{4}\Delta\left[\partial_{x_1}^3q-\frac{3}{2}\left(q_{x_1}^2-q_{x_2}^2\right)\right]$$
$$=\frac{1}{4}\left[\partial_{x_1}^3-3(q_{x_1}\partial_{x_1}-q_{x_2}\partial_{x_2}+q_{x_1}-q_{x_2})\right]\Delta q,$$

and similarly for the t_2-evolution. In summary, let $q(x_1,x_2,0)=q_0(x_1,x_2)$ be a harmonic function. Then $q(x_1,x_2,t_1)$ and $q(x_1,x_2,t_2)$ satisfy the integrable systems (6) and (7), respectively. If q_0 is real, then q remains real.

By complexifying the independent variables of the potential KP equation I obtain the following integrable nonlinear equation in $4+2$,

$$q_{\bar t}=\frac{1}{4}q_{\bar x\bar x\bar x}-\frac{3}{4}q_{\bar x}^2+\frac{3}{4}\bar L q,\qquad L=\partial_x^{-1}\partial_y^2,\tag{8}$$

where

$$y=y_1+iy_2,\qquad \partial_{\bar x}^{-1}f=\frac{1}{\pi}\int_{\mathbb{R}^2}f(x_1',x_2')\frac{dx_1'\,dx_2'}{x-x'}.$$

Proceeding as with Eq. (1), I find in analogy with Eqs. (3)–(5) the following equations,

$$(\Delta q)(\Delta q_{x_2})+q_{\bar x}Lq_{\bar x}-q_x\bar L q_x+\frac{1}{2}\left(\bar L q_x^2-Lq_{\bar x}^2\right)=0,\tag{9}$$

$$q_{t_1}=N_1(q)+\frac{3}{4}(\bar L+L)q,\tag{10}$$

$$q_{t_2}=N_2(q)+\frac{3i}{4}(L-\bar L)q,\tag{11}$$

where $N_1(q)$ and $N_2(q)$ denote the right-hand side of Eqs. (4) and (5).

The investigation of whether the nonlinear Eq. (9) is preserved under the t_1 and t_2 evolutions remains open. In what follows I will concentrate on a particular solution of Eq. (9), namely on the solution characterized by $\Delta q=q_{\bar x y}=0$. It is straightforward to show that the first of these equations is preserved by the t_1 and t_2 evolutions, whether the second equation implies $q_{y_2}=\partial_{x_1}^{-1}q_{x_2y_1}$. Indeed,

$$q_{\bar x y}=\frac{1}{4}(\partial_{x_1}\partial_{y_1}+\partial_{x_2}\partial_{y_2})q+\frac{i}{4}(\partial_{x_2}\partial_{y_1}-\partial_{y_2}\partial_{x_1})q=0.$$

These equations imply the expression for q_{y_2} given earlier as well as the expression $q_{y_2}=-\partial_{x_2}^{-1}q_{x_1y_1}$; these two expressions for q_{y_2} are identical, since $\Delta q=0$. Using

$$\partial_y^2=\frac{1}{4}\left(\partial_{y_1}^2-\partial_{y_2}^2-2i\partial_{y_1}\partial_{y_2}\right)$$
$$=\frac{1}{4}\partial_{y_1}^2\partial_{x_1}^{-2}(\partial_{x_1}-i\partial_{x_2})^2=\partial_{y_1}^2\partial_{x_1}^{-2}\partial_x^2,$$

it follows that

$$L=\partial_{y_1}^2\partial_{x_1}^{-2}\partial_x=\frac{1}{2}\left(\partial_{y_1}^2\partial_{x_1}^{-1}-i\partial_{y_2}^2\partial_{x_2}^{-1}\right).$$

Hence, Eqs. (9)–(11) yield the following two integrable systems

$$q_{t_1}=\frac{1}{4}q_{x_1x_1x_1}-\frac{3}{8}\left(q_{x_1}^2-q_{x_2}^2\right)+\frac{3}{4}\partial_{x_1}^{-1}q_{y_1y_1},$$
$$\Delta q=0,\tag{12}$$

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$$q_{t_2} = -\frac{1}{4}q_{x_2x_2x_2} - \frac{3}{4}q_{x_1}q_{x_2} - \frac{3}{4}\partial_{x_2}^{-1}q_{y_1y_1},$$
$$\Delta q = 0. \tag{13}$$

Consider the particular case that q satisfies Laplace's equation in the domain $\{-\infty < x_1 < \infty, x_2 > 0\}$ with $q(x_1, 0, y_1, 0) = Q_0(x_1, y_1)$ given. The Dirichlet to Neumann map for the above domain implies [8] $q_{x_2}(x_1, 0, y_1, t) = Hq_{x_1}(x_1, 0, y_1, t)$, where H denotes the Hilbert transform in x_1. Hence, the function $q(x_1, 0, y_1, t) = Q(x_1, y_1, t)$ satisfies the equation

$$Q_{t_1} = \frac{1}{4}Q_{x_1x_1x_1} - \frac{3}{8}\left[Q_{x_1}^2 - (HQ_{x_1})^2\right] + \frac{3}{4}\partial_{x_1}^{-1}Q_{y_1y_1},$$
$$Hf = \frac{1}{\pi}\ \fint_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x_1}d\xi, \tag{14}$$

where $\fint$ denotes the principal value integral. The solution of this equation with $Q(x_1, y_1, 0) = Q_0(x_1, y_1)$ yields $Q(x_1, y_1, t)$ and then Laplace's equation with $q(x_1, 0, y_1, t) = Q(x_1, y_1, t)$, yields $q(x_1, x_2, y, t)$ which solves Eq. (12).

Eq. (14) was first derived in [9]. The dispersionless limit of this equation in the case that Q is independent of y_1 is the equation describing the formation of singularities on the free surface of an ideal fluid (see Eq. (17) of [10] and Eq. (12) of [11], where $v = Q_{x_1}$). The analogous equation associated with the t_2 evolution is the equation that appears in connection with the motion of vorticity for an ideal flow, see [12].

I recall that the nonlinear Schrödinger and the DS equations, are obtained as the particular case $q_2 = \pm\bar{q}_1$ of more general systems involving the two dependent variables q_1 and q_2. Multidimensional generalizations of the nonlinear Schrödinger and of the DS equations can be obtained by analyzing these latter systems; for these systems the requirement of $q_2 = \pm\bar{q}_1$ replaces the reality requirement used for the KdV and KP equations.

The general investigation of the analogue of Eq. (9) for the DS system in $4+2$ remains open. However, using the so-called direct linearizing method [13], it is possible to construct explicit solutions of DS in $4+2$, and to show that there exists a large class of solutions which *does* satisfy the requirement $q_2 = \sigma\bar{q}_1$. In particular, it can be shown that the following expressions are solutions of the DS system in $4+2$ in the complex variables $\{\xi, \eta, t\}$,

$$q_1 = -\frac{2c_1E_1}{1 - cE_1E_2}, \qquad q_2 = \frac{2c_2E_2}{1 - cE_1E_2}, \qquad c = \frac{c_1c_2}{(\alpha - \beta)^2},$$

where

$$E_1 = e^{2\beta\bar{\xi} + 2\alpha\bar{\eta} + (\alpha^2+\beta^2)\bar{t} - 2\bar{\alpha}\xi - 2\bar{\beta}\eta - (\bar{\alpha}^2+\bar{\beta}^2)t},$$
$$E_2 = e^{-2\alpha\bar{\xi} - 2\beta\bar{\eta} - (\alpha^2+\beta^2)\bar{t} + 2\bar{\beta}\xi + 2\bar{\alpha}\eta + (\bar{\alpha}^2+\bar{\beta}^2)t}$$

and α, β, c_1, c_2 are complex constants. The particular choices $\{c_2 = -\bar{c}_1,\ \beta_I = \alpha_I = \gamma\}$ and $\{c_2 = \bar{c}_1,\ \beta_R = \alpha_R\}$ yield $\bar{q}_2 = q_1$ and $\bar{q}_2 = -q_1$, respectively. For example, the first of these choices yields

$$q_1 = -2c_1 \times \frac{e^{2i\{2\gamma(\xi_1+\eta_1) - (\alpha_R+\beta_R)(\xi_2+\eta_2) + 2\gamma(\alpha_R+\beta_R)t_1 + (2\gamma^2-\alpha_R^2-\beta_R^2)t_2\}}}{e^{-2(\beta_R-\alpha_R)(\xi_1-\eta_1)} + \frac{|c_1|^2}{(\beta_R-\alpha_R)^2}e^{2(\beta_R-\alpha_R)(\xi_1-\eta_1)}}. \tag{15}$$

Eq. (15), which involves the three real constants α_R, β_R, γ, describes a one-soliton solution. This solution involves *only three* (as opposed to four) spatial variables, namely it involves the independent space variables x_1, x_2, x_3, where

$$x_1 = \xi_1 - \eta_1, \qquad x_2 = \xi_2 + \eta_2, \qquad x_3 = \xi_1 + \eta_1. \tag{16}$$

It is shown in [14] that there exist N-soliton solutions which also involve the three independent variables x_1, x_2, x_3, as well as the real constants $\{\alpha_{j_R}, \beta_{j_R}\}_{j=1}^N$ and γ, where $\gamma = \alpha_{j_I} = \beta_{j_I}$, $j = 1, \ldots, N$. The existence of *exponentially decaying soliton solutions* is another indication of the physical significance of the new integrable systems presented here.

Acknowledgements

This work was partially supported by the EPSRC. I thank M. Dunajski for useful suggestions.

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