Applications of Quantum Mechanics: Example Sheet 3

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1. The Schrödinger equation for a particle of mass m and charge q in an electromagnetic field is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar}\mathbf{A}\right)^2\psi + q\phi\psi$$

Under a gauge transformation,

$$\phi \to \phi - \frac{\partial\alpha}{\partial t} \quad , \quad \mathbf{A} \to \mathbf{A} + \nabla\alpha \ .$$

Show that, with a suitable transformation of ψ, the Schrödinger equation transforms into itself. Show that the probability density $|\psi|^2$ is gauge invariant. Show that the *mechanical momentum* $\boldsymbol{\pi} = -i\hbar\nabla - q\mathbf{A}$ is gauge invariant. What is the physical interpretation of the mechanical momentum?

2. A particle of charge q moving in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = (0, 0, B)$ is described by the Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2$$

where $\mathbf{p}$ is the canonical momentum. Show that the mechanical momentum $\boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}$ obeys

$$[\pi_x, \pi_y] = iq\hbar B$$

Define

$$a = \frac{1}{\sqrt{2q\hbar B}}(\pi_x + i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2q\hbar B}}(\pi_x - i\pi_y)$$

What commutation relations do a and $a^\dagger$ obey? Write the Hamiltonian in terms of a and $a^\dagger$ and hence solve for the spectrum.

3i. Symmetric gauge is defined by $\mathbf{A} = \frac{B}{2}(-y, x, 0)$. Confirm that this gives the magnetic field $\mathbf{B} = (0, 0, B)$. Show that the Hamiltonian can be written as

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{qB}{2m}L_z + \frac{q^2B^2}{8m}\,(x^2 + y^2)$$

where L_z is the component of the angular momentum parallel to B.

ii. Show that the operator a, defined in Question 2, takes the form

$$a = -i\sqrt{2}\left(l_B\frac{\partial}{\partial \bar{w}} + \frac{w}{4l_B}\right)$$

where $l_B = \sqrt{\hbar/qB}$ is the magnetic length and $w = x + iy$ is a complex coordinate on the plane, with $\partial_{\bar{\omega}} = \frac{1}{2}(\partial_x + i\partial_y)$ so that $\partial_{\bar{\omega}}\omega = 0$ and $\partial_{\bar{\omega}}\bar{\omega} = 1$. Hence show that the state

$$\psi(w) = f(w)e^{-|w|^2/4l_B^2}$$

sits in the lowest Landau level for any holomorphic function $f(w)$.

4. In the presence of a magnetic field $\mathbf{B} = (0, 0, B)$, a particle of charge q moves in the (x, y)-plane on the trajectory,

$$x(t) = X + R\sin(\omega_B t) \quad \text{and} \quad y(t) = Y + R\cos(\omega_B t)$$

with $\omega_B = qB/m$. Working in symmetric gauge $\mathbf{A} = \frac{B}{2}(-y, x, 0)$, show that the centre of mass coordinates can be re-expressed as

$$X = \frac{x}{2} + \frac{p_y}{m\omega_B} \quad \text{and} \quad Y = \frac{y}{2} - \frac{p_x}{m\omega_B}$$

Viewed as quantum operators in the Heisenberg representation, show that both X and Y do not change in time. Show that

$$[X, Y] = -il_B^2$$

where $l_B^2 = \hbar/qB$ is the magnetic length. Use the Heisenberg uncertainty relation for X and Y to estimate the number of states $\mathcal{N}$ that can sit in a region of area A.

5. A particle of charge e and spin $\frac{1}{2}$ with g-factor $g = 2$ moves in the (x, y)-plane in the presence of a magnetic field of the form $\mathbf{B} = (0, 0, B)$. Show that the Hamiltonian can be written as

$$H = \frac{1}{2m} Q^2 \quad \text{with} \quad Q = (\pi_x \sigma_x + \pi_y \sigma_y)$$

where $\boldsymbol{\sigma}$ are the Pauli matrices and $\boldsymbol{\pi}$ is the mechanical momentum defined in earlier questions.

Confirm that Q is Hermitian. Show that zero energy states are annihilated by Q. Show that $|\psi\rangle$ and $Q|\psi\rangle$ are degenerate and hence deduce that the lowest Landau level contains half the states of the higher Landau levels. What is the physical interpretation of this? (Hint: consider the effect of Zeeman splitting on Landau levels.)

Working in Landau gauge, $\mathbf{A} = (0, Bx, 0)$ with $B > 0$, show that zero energy states have spin up and take the form

$$\psi = \begin{pmatrix} f(w)\, e^{-x^2/2l_B^2} \\ 0 \end{pmatrix}$$

with $w = x + iy$ and $l_B^2 = \hbar/qB$. Show by explicit calculation that there are no zero energy spin down states.

6*. Near the Dirac point, an electron in graphene is described by the Hamiltonian

$$H = v_F Q$$

with v_F the Fermi velocity and Q the operator defined in Question 5. Working in Landau gauge $\mathbf{A} = (0, Bx, 0)$, show that the Landau level spectrum is given by

$$E = \pm v_F \sqrt{2\hbar q B}\, \sqrt{n} \qquad n = 0, 1, 2, \ldots$$

7. A particle of mass m is scattered by the symmetric square well potential given by

$$V(x) \;=\; \begin{cases} -V_0 & |x| < a/2 \\ 0 & |x| > a/2 \end{cases} \qquad \text{with } V_0 > 0$$

The incoming and outgoing plane waves of even (+) and odd (−) parity are

$$\begin{array}{lcl} \mathcal{I}_+(k;x) = e^{-ik|x|} & , & \mathcal{I}_-(k;x) = \mathrm{sign}(x)\, e^{-ik|x|} \\ \mathcal{O}_+(k;x) = e^{+ik|x|} & , & \mathcal{O}_-(k;x) = -\mathrm{sign}(x)\, e^{+ik|x|} \end{array}$$

The corresponding scattering states for $|x| > a/2$ are

$$\begin{aligned} \psi_+(k;x) &= \mathcal{I}_+(k;x) + S_{++}(k)\mathcal{O}_+(k;x) \\ \psi_-(k;x) &= \mathcal{I}_-(k;x) + S_{--}(k)\mathcal{O}_-(k;x) \end{aligned}$$

where S_{++} and S_{--} are the diagonal elements of the S-matrix in the parity basis. By imposing the boundary condition that $(d\psi/dx)/\psi$ is continuous at $x = a/2$ show that

$$S_{++} = -e^{-ika}\,\frac{q\tan(qa/2) - ik}{q\tan(qa/2) + ik} \quad , \quad S_{--} = e^{-ika}\frac{q + ik\tan(qa/2)}{q - ik\tan(qa/2)}$$

where $q^2 = k^2 + U_0$ and $U_0 = 2mV_0/\hbar^2$.. Interpret the poles and zeros of S_{++} and S_{--} in terms of bound states.

8. Carry out a similar analysis to that in Question 7, this time for the potential

$$V(x) = V_0\delta(x-1) + V_0\delta(x+1)$$

with $V_0 > 0$. Interpret the poles and zeros of S_{++} and S_{--} in the complex k-plane as resonances, in the case where $V_0 \gg 1$. Show that, approximately, the pole position in S_{++} with the smallest real part lies at

$$k = \frac{\pi}{2} - \frac{\pi}{2U_0} + \frac{\pi}{2U_0^2} - i\frac{\pi^2}{4U_0^2}$$

where $U_0 = 2mV_0/\hbar^2$.