Applications of Quantum Mechanics: Example Sheet 4

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1. The semi-classical equations of motion for an electron of charge $-e$ and energy $E(k)$ moving in a magnetic field $B$ are

$$\hbar \frac{dk}{dt} = -e \mathbf{v} \times \mathbf{B} \quad \text{and} \quad \mathbf{v} = \frac{1}{\hbar} \frac{\partial E}{\partial k}$$

Show that, in momentum space, the electrons orbit the Fermi surface $E(k_F)$ in a plane perpendicular to $B$. Show that the orbit of the electron in position space, projected onto the plane perpendicular to $B$, traces out the perimeter of a cross-section of the Fermi surface. [Hint: Consider the evolution of the position $\mathbf{r}_\perp = \mathbf{r} - (\hat{\mathbf{B}} \cdot \mathbf{r}) \hat{\mathbf{B}}$, perpendicular to the magnetic field.]

A free electron has $E(k) = \hbar^2 k^2 / 2m$. Use the results above to show that, for any value of $k \cdot B$, the electron orbits the Fermi surface with cyclotron frequency $\omega_B = eB/m$. Show that the time taken to orbit the Fermi surface can be written as

$$T = \frac{2\pi}{\omega_B} = \frac{\hbar^2}{eB} \frac{\partial A(E)}{\partial E} \bigg|_{k_B}$$

where $A(E)$ is the cross-sectional area of the Fermi surface with Fermi energy $E$.

[An Aside: This formula is important because it holds for Fermi surfaces of any shape.]

2. A one-dimensional crystal comprises a chain of atoms of mass $m$ equally spaced by a distance $a$ when in equilibrium. The forces between the atoms are such that the effective spring constants are alternately $\lambda$ and $\alpha \lambda$. Show that the dispersion relation for phonons has the form

$$\omega_{\pm}(k)^2 = \frac{\lambda}{m} \left[ (1 + \alpha) \pm \sqrt{1 + 2\alpha \cos 2ka + \alpha^2} \right]$$

where the wavenumber $k$ satisfies $-\pi/2a \leq k \leq \pi/2a$. What is the speed of sound in this crystal?
3. The Schrödinger equation for a particle of mass \(m\) and charge \(q\) in an electromagnetic field is

\[
i \hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \left( \nabla - \frac{i q}{\hbar} \mathbf{A} \right)^2 \right) \psi + q \phi \psi
\]

Under a gauge transformation,

\[
\phi \to \phi - \frac{\partial \alpha}{\partial t}, \quad \mathbf{A} \to \mathbf{A} + \nabla \alpha.
\]

Show that, with a suitable transformation of \(\psi\), the Schrödinger equation transforms into itself. Show that the probability density \(|\psi|^2\) is gauge invariant. Show that the mechanical momentum \(\mathbf{\pi} = -i\hbar \nabla - q\mathbf{A}\) is gauge invariant. What is the physical interpretation of the mechanical momentum?

4. A particle of charge \(q\) moving in a magnetic field \(\mathbf{B} = \nabla \times \mathbf{A} = (0, 0, B)\) is described by the Hamiltonian

\[
H = \frac{1}{2m} (\mathbf{p} - q\mathbf{A})^2
\]

where \(\mathbf{p}\) is the canonical momentum. Show that the mechanical momentum \(\mathbf{\pi} = \mathbf{p} - q\mathbf{A}\) obeys

\[
[\pi_x, \pi_y] = i q \hbar B
\]

Define

\[
a = \frac{1}{\sqrt{2q\hbar B}} (\pi_x + i \pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2q\hbar B}} (\pi_x - i \pi_y)
\]

What commutation relations do \(a\) and \(a^\dagger\) obey? Write the Hamiltonian in terms of \(a\) and \(a^\dagger\) and hence solve for the spectrum.

5a. Symmetric gauge is defined by \(\mathbf{A} = \frac{B}{2} (-y, x, 0)\). Confirm that this gives the magnetic field \(\mathbf{B} = (0, 0, B)\). Show that the Hamiltonian can be written as

\[
H = -\frac{\hbar^2}{2m} \nabla^2 - \frac{qB}{2m} L_z + \frac{q^2B^2}{8m} (x^2 + y^2)
\]

where \(L_z\) is the component of the angular momentum parallel to \(B\).
b. Show that the operator \( a \), defined in Question 4, takes the form

\[
a = -i\sqrt{2} \left( l_B \frac{\partial}{\partial \bar{w}} + \frac{w}{4l_B} \right)
\]

where \( l_B = \sqrt{\frac{\hbar}{qB}} \) is the magnetic length and \( w = x + iy \) is a complex coordinate on the plane, with \( \partial_\omega = \frac{1}{2} (\partial_x + i\partial_y) \) so that \( \partial_\omega \omega = 0 \) and \( \partial_\omega \bar{\omega} = 1 \). Hence show that the state

\[
\psi(w) = f(w)e^{-|w|^2/4l_B^2}
\]
sits in the lowest Landau level for any holomorphic function \( f(w) \).

6. In the presence of a magnetic field \( B = (0, 0, B) \), a particle of charge \( q \) moves in the \((x, y)\)-plane on the trajectory,

\[
x(t) = X + R \sin(\omega_B t) \quad \text{and} \quad y(t) = Y + R \cos(\omega_B t)
\]

with \( \omega_B = qB/m \). Working in symmetric gauge \( A = \frac{B}{2} (-y, x, 0) \), show that the centre of mass coordinates can be re-expressed as

\[
X = \frac{x}{2} + \frac{p_y}{m\omega_B} \quad \text{and} \quad Y = \frac{y}{2} - \frac{p_x}{m\omega_B}
\]

Viewed as quantum operators in the Heisenberg representation, show that both \( X \) and \( Y \) do not change in time. Show that

\[
[X, Y] = -i\ell_B^2
\]

where \( \ell_B^2 = \hbar/qB \) is the magnetic length. Use the Heisenberg uncertainty relation for \( X \) and \( Y \) to estimate the number of states \( N \) that can sit in a region of area \( A \).

7. A particle of charge \( e \) and spin \( \frac{1}{2} \) with g-factor \( g = 2 \) moves in the \((x, y)\)-plane in the presence of a magnetic field of the form \( B = (0, 0, B) \). Show that the Hamiltonian can be written as

\[
H = \frac{1}{2m} Q^2 \quad \text{with} \quad Q = (\pi_x \sigma_x + \pi_y \sigma_y)
\]

where \( \sigma \) are the Pauli matrices and \( \pi \) is the mechanical momentum defined in earlier questions.

Confirm that \( Q \) is Hermitian. Show that zero energy states are annihilated by \( Q \). Show that \( |\psi\rangle \) and \( Q|\psi\rangle \) are degenerate and hence deduce that the lowest Landau level
contains half the states of the higher Landau levels. What is the physical interpretation of this? (Hint: consider the effect of Zeeman splitting on Landau levels.)

Working in Landau gauge, \( A = (0, Bx, 0) \) with \( B > 0 \), show that zero energy states have spin up and take the form
\[
\psi = \begin{pmatrix} f(w) e^{-x^2/2l_B^2} \\ 0 \end{pmatrix}
\]
with \( w = x + iy \) and \( l_B^2 = \frac{\hbar}{qB} \). Show by explicit calculation that there are no zero energy spin down states.

8*. Near the Dirac point, an electron in graphene is described by the Hamiltonian
\[
H = v_F Q
\]
with \( v_F \) the Fermi velocity and \( Q \) the operator defined in Question 5. Working in Landau gauge \( A = (0, Bx, 0) \), show that the Landau level spectrum is given by
\[
E = \pm v_F \sqrt{2\hbar qB} \sqrt{n} \quad n = 0, 1, 2, \ldots
\]