Concepts in Theoretical Physics

Lecture 1: The Principle of Least Action

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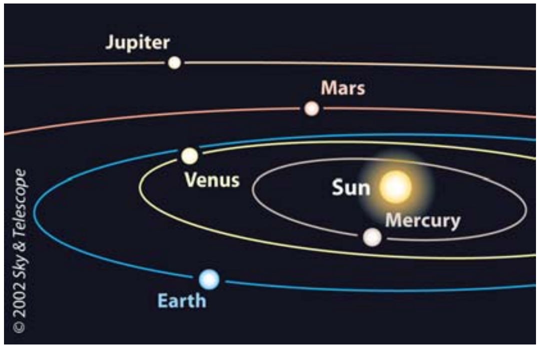
Newtonian Mechanics



You've all done a course on Newtonian mechanics so you know how to calculate the way things move.

You draw a pretty picture; you draw arrows representing forces; you add them all up; and then you use F=ma to figure out where things are heading. Then, a moment later, when the particles have moved infinitesimally, you do it all again.

Probably you express this in terms of differential equations, but basically this is what you do.

And all of this is rather impressive -- it really is the way the world works and we can use it to compute things such as the orbits of the planets. This is a big deal.

But…..there's a better way

Reformulating Newtonian Mechanics

The new way of doing things is equivalent to Newtonian mechanics, but puts the emphasis on different ideas. It was formulated 100-150 years after Newton by some of the giants of mathematical physics: people like Lagrange, Euler and Hamilton. The new way is better for a number of reasons:

Firstly, it's elegant. In fact, it's completely gorgeous. In a way that theoretical physics should be, and usually is, but in a way that the old Newtonian mechanics really isn't.

Secondly, it's more powerful. The new methods allow us to solve hard problems in a fairly straightforward manner

Finally, this new way of viewing classical dynamics provides a framework that can be extended to all other laws of physics. It reveals new facets of classical dynamics, such as chaos theory and illustrates the connections to quantum mechanics.

In this lecture I'll show you the key idea that leads to this new way of thinking. It's one of the most profound results in physics. But it has a rubbish name. It's called the "principle of least action".

Summary of Newtonian Mechanics

- Newton's equation for a single particle with position $\vec{r}$, acted upon by a force $\vec{F}$ is

$$\vec{F} = m\vec{a} \equiv m\ddot{\vec{r}}$$

- The goal of classical mechanics is to solve this differential equation for different forces: gravity, electromagnetism, friction, etc…

- *Conservative* forces are special. They can be expressed as in terms of a potential $V(\vec{r})$

$$\vec{F} = -\nabla V$$

- The potential depends on $\vec{r}$, but not $\dot{\vec{r}}$. This includes the forces of gravity and electrostatics. But not friction forces.

Summary of Newtonian Mechanics

- For conservative forces, Newton's equations read

$$m\ddot{\vec{r}} = -\nabla V$$

- The total energy is conserved

$$E = \tfrac{1}{2}m\dot{\vec{r}}^{\,2} + V(\vec{r})$$

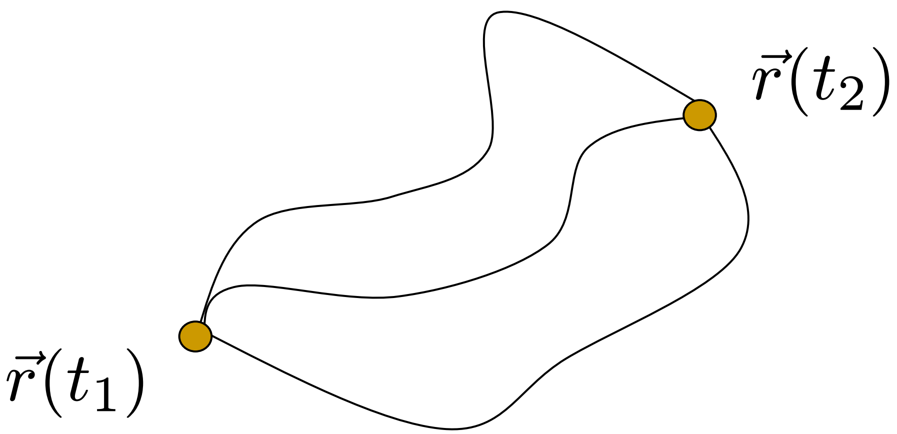
Kinetic energy

Potential energy

- Newton's equations are second order differential equations. The general solution has two integration constants. Physically this means we need to specify the initial position and momentum of the particle before we can figure out where it's going to end up.

A New Way of Looking at Things

- Instead of specifying the initial position and momentum, let's instead choose to specify the initial and final positions:



- Question: What path does the particle take?

The Action

- To each path, we assign a number which we call the *action*

$$S[\vec{r}(t)] = \int_{t_1}^{t_2} dt \, \left(\frac{1}{2} m \dot{\vec{r}}^{\,2} - V(\vec{r}) \right)$$

- This is the difference between the kinetic energy and the potential energy, integrated over the path. We can now state the main result:

- **Claim:** The true path taken by the particle is an extremum of S.

The Proof

- **Proof:** You know how to find the extremum of a function --- you differentiate and set it equal to zero. But this is a *functional*: it is a function of a function. And that makes it a slightly different problem. You'll learn how to solve problems of this type in next year's "methods" course. These problems go under the name of *calculus of variations.*

- To solve our problem, consider a given path $\vec{r}(t)$. We ask how the action changes when we change the path slightly

$$\vec{r}(t) \to \vec{r}(t) + \delta\vec{r}(t)$$

- such that we keep the end points of the path fixed

$$\delta\vec{r}(t_1) = \delta\vec{r}(t_2) = 0$$

The Proof Continued

$$S[\vec{r} + \delta\vec{r}] \quad = \quad \int_{t_1}^{t_2} dt \, \left[\frac{1}{2}m(\dot{\vec{r}}^{\,2} + 2\dot{\vec{r}}\cdot\delta\dot{\vec{r}} + \delta\dot{\vec{r}}^{\,2}) - V(\vec{r} + \delta\vec{r})\right]$$

$$V(\vec{r} + \delta\vec{r}) = V(\vec{r}) + \nabla V \cdot \delta\vec{r} + \mathcal{O}(\delta\vec{r}^{\,2})$$

$$\begin{aligned}
\delta S \equiv S[\vec{r} + \delta\vec{r}] - S[\vec{r}] \quad &= \quad \int_{t_1}^{t_2} dt \, \left[m\dot{\vec{r}}\cdot\delta\dot{\vec{r}} - \nabla V \cdot \delta\vec{r}\right] + \ldots \\
&= \quad \int_{t_1}^{t_2} dt \, \left[-m\ddot{\vec{r}} - \nabla V\right] \cdot \delta\vec{r} + \left[m\dot{\vec{r}}\cdot\delta\vec{r}\right]_{t_1}^{t_2}
\end{aligned}$$

Vanishes because we fix the end points

The end of the Proof

$$\delta S = \int_{t_1}^{t_2} dt \, \left[-m\ddot{\vec{r}} - \nabla V \right] \cdot \delta \vec{r}$$

- The condition that the path we started with is an extremum of the action is

$$\delta S = 0$$

- Which should hold for *all* changes $\delta \vec{r}(t)$ that we make to the path. The only way this can happen is if the expression in [...] is zero. This means

$$m\ddot{\vec{r}} = -\nabla V$$

- We recognize this as Newton's equations. Requiring that the action is extremized is equivalent to requiring that the path obeys Newton's equations. $\square$

The Lagrangian

- The integrand of the action is called the *Lagrangian*

$$L = \frac{1}{2} m \dot{\vec{r}}^2 - V(\vec{r})$$

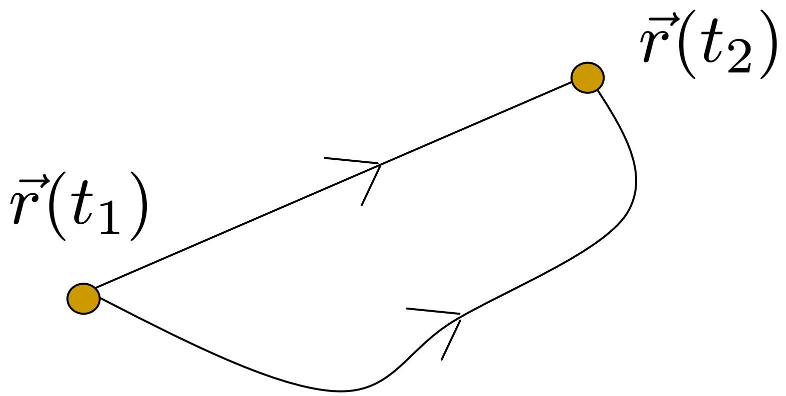
- The "*principle of least action"* is something of a misnomer. The action doesn't have to be minimal. It is often a saddle point.

- This idea is also called "*Hamilton's Principle*", after Hamilton who gave the general statement some 50 years after Lagrange.

Example 1: A Free Particle

$$L = \frac{1}{2}m\dot{\vec{r}}^2$$

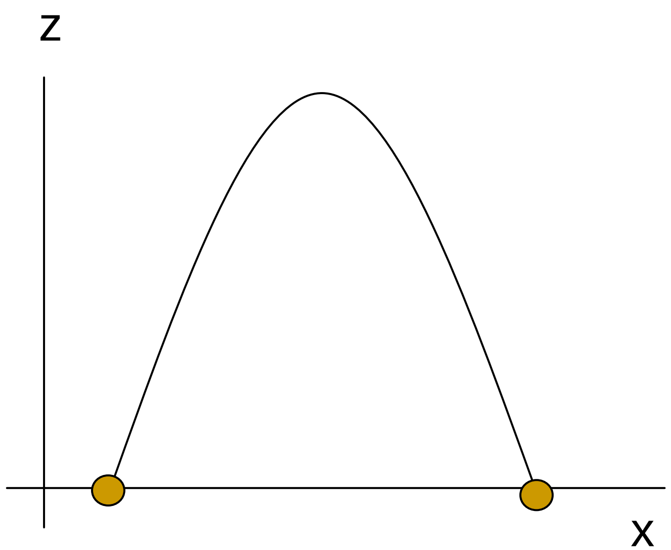
- We want to minimize the kinetic energy over a fixed time…..so the particle must take the most direct route. This is a straight line.
- But do we slow down to begin with, then speed up? Or do we go at a uniform speed?
- To minimize the kinetic energy, we should go at a uniform speed.



Example 2: Particle in Uniform Gravity

$$L = \frac{1}{2}m\dot{x}^{\,2} + \frac{1}{2}m\dot{z}^{\,2} - mgz$$

- Now we don't want to go in a straight line. We can minimize the difference between K.E. and P.E. if we go up, where the P.E. is bigger.
- But we don't want to go too high either.
- To strike the right balance, the particle takes a parabola.



The Advantages of This Approach

- There are several reasons to use this approach.
  - It is independent of the coordinates we choose to work in. The idea of minimizing the action holds in Cartesian coordinates, polar coordinates, rotating frames, or any other system of coordinates you choose to work in. This can often be very useful.
  - It is easy to implement constraints in this set-up
- This means that we can solve rather tricky problems, such as the strange motion of spinning tops, with ease.
  - All of this will be covered in the third year “Classical Dynamics” course.

Unification of Physics

- All fundamental laws of physics can be expressed in terms of a least action principle. This is true for electromagnetism, special and general relativity, particle physics, and even more speculative pursuits that go beyond known laws of physics such as string theory.

- For example, (nearly) every experiment ever performed can be explained by the Lagrangian of the standard model

$$\mathcal{L} = \sqrt{g}(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\gamma_\mu D^\mu \psi + |\mathcal{D}h|^2 - V(|h|) + \psi\psi h)$$

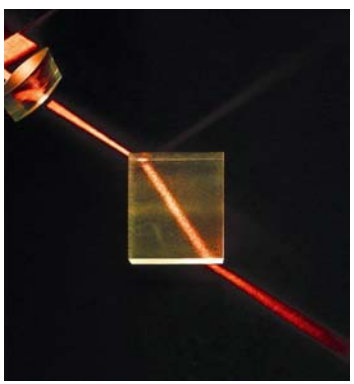
Einstein   Maxwell Yang-Mills   Dirac   Higgs   Yukawa

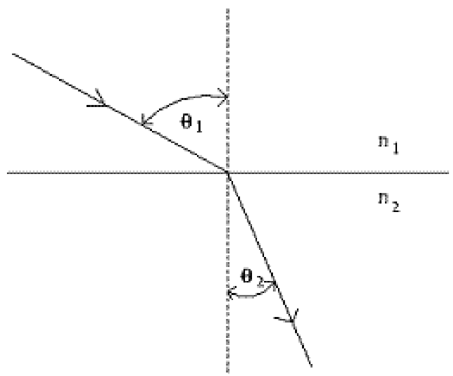
A Simple Example:

- In geometrical optics (which applies to light of negligible wavelength), the light ray travels according to *Fermat's principle*.*

Light travels so as to minimize the time it takes.

- i.e. for light, action = time. We can use this to easily derive Snell's law.





$$\frac{\sin\theta_1}{v_{\mathrm{air}}} = \frac{\sin\theta_2}{v_{\mathrm{glass}}}$$

*Same Fermat, but this time he had a big enough margin

Sniffing out Paths…

- The principle of least action gives a very different way of looking at things:

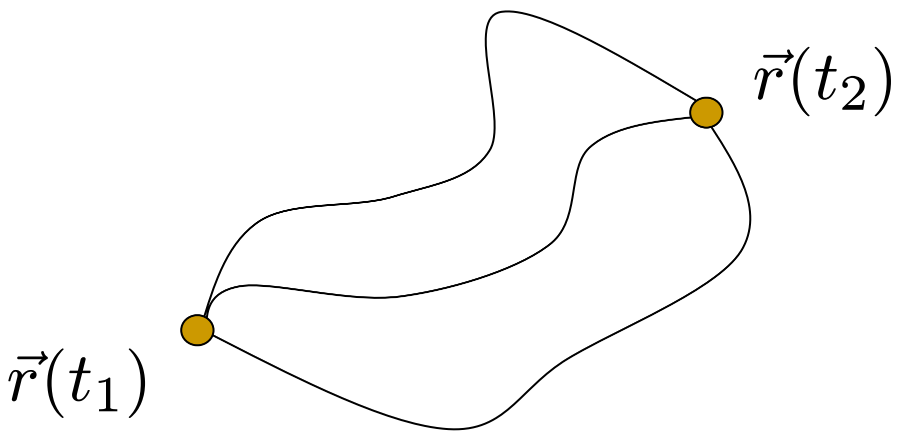
- In the Newtonian framework, you start to develop an intuition for how particles move, which goes something like this: at each moment in time, the particle thinks “where do I go now?”. It looks around, sees the potential, differentiates it and says “ah-ha….I go this way”. Then, an infinitesimal moment later, it does it all again.

- But the Lagrangian framework suggests a rather different viewpoint. Now the particle is taking the path which is minimizing the action. How does it know this is the minimal path? Is it sniffing around, checking out all paths, before it decides: “I think I’ll go this way”.

- On some level, this philosophical pondering is meaningless. After all, we just proved that the two ways of doing things are completely equivalent. However, the astonishing answer is: yes, the particle does sniff out every possible path! This is the way *quantum mechanics* works.

Feynman's Path Integral

- Nature is probabilistic. At the deepest level, things happen by random chance. This is the key insight of quantum mechanics.



- The probability that a particle starting at $\vec{r}(t_1)$ will end up at $\vec{r}(t_2)$ is expressed in terms of an *Amplitude* A, which is a complex number that can be thought of as the square root of the probability

$$Prob = |A|^2$$

Feynman's Path Integral

- To compute the amplitude, you must sum over all paths that the particle takes, weighted with by phase

$$A = \sum_{\text{paths}} \exp(iS/\hbar)$$

- Here S is the action, while $\hbar$ is Planck's constant (divided by 2π). It's a fundamental constant of Nature.

- The way to think about this is that when a particle moves, it really does take all possible paths. Away from the classical path, the action varies wildly, and the sum of different phases averages to zero. Only near the classical path do the phases reinforce each other.

- You will learn more about this in various courses on quantum mechanics over the next few years.