Fluid Dynamics

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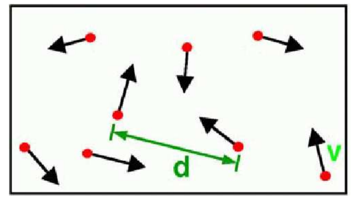
Averaging in Fluid Dynamics



Classical fluid mechanics

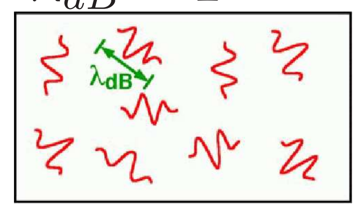
macroscopic

continuum approximation $d \ll L$

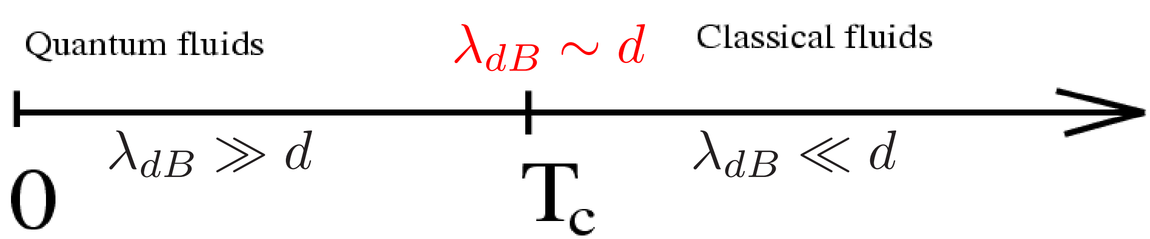
Quantum mechanics

microscopic

$\lambda_{dB} \sim T^{-1/2}$

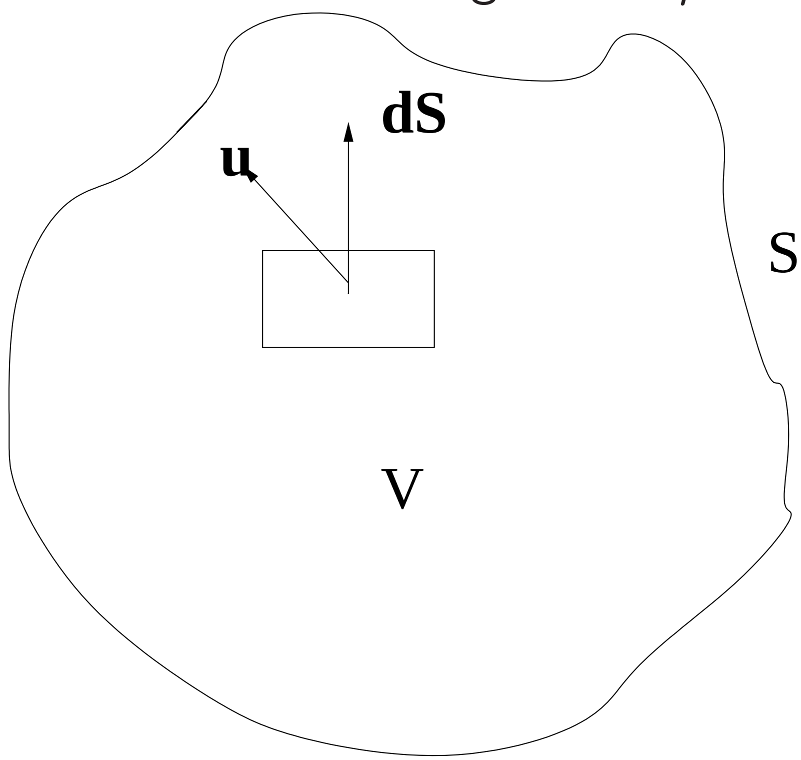


Quantum fluid mechanics – macroscopic dynamics & quantum effects



Equation of continuity (mass conservation)

This equation simply tells us that the fluid does not "get lost".
Consider the net mass flux from a volume V bounded by a surface S. If $d\mathbf{S}$ is the element of surface area (the direction of the vector being taken normally outwards), the mass flow through $d\mathbf{S}$ is $\rho\mathbf{u} \cdot d\mathbf{S}$.



Integrating over the surface of the volume, the rate of outflow of mass must equal the rate of loss of mass from within V:

$$\int_S \rho \mathbf{u} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho \, dV.$$

Using the divergence theorem,

$$\int_V \boldsymbol{\nabla} \cdot \rho \mathbf{u} \, dV + \frac{d}{dt}\int_V \rho \, dV = 0,$$

This result must be true for any elementary volume withing the fluid and hence

$$\boldsymbol{\nabla} \cdot \rho \mathbf{u} + \frac{\partial \rho}{\partial t} = 0.$$

Notice that we have changed from a total to a partial derivative, since we denote variations in the density of the fluid *at a fixed point in space.*

Example: Line/point vortices in incompressible irrotational fluid

Let's assume that the fluid flow satisfies the following conditions:
1. Fluid is incompressible, so for any fluid element $\rho = \mathrm{constant}$;
2. Flow is not time dependent, so $\partial\rho/\partial t = 0$;
3. Flow is irrotational, which is defined as $\boldsymbol{\nabla} \times \mathbf{u} = 0$.

From the equation of continuity and 1. and 2. we see that $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$.
From 3. we can express the velocity in terms of a velocity potential ϕ defined by $\mathbf{u} = \boldsymbol{\nabla}\phi$.
Therefore, the velocity potential ϕ can be found as the solution of Laplace's equation

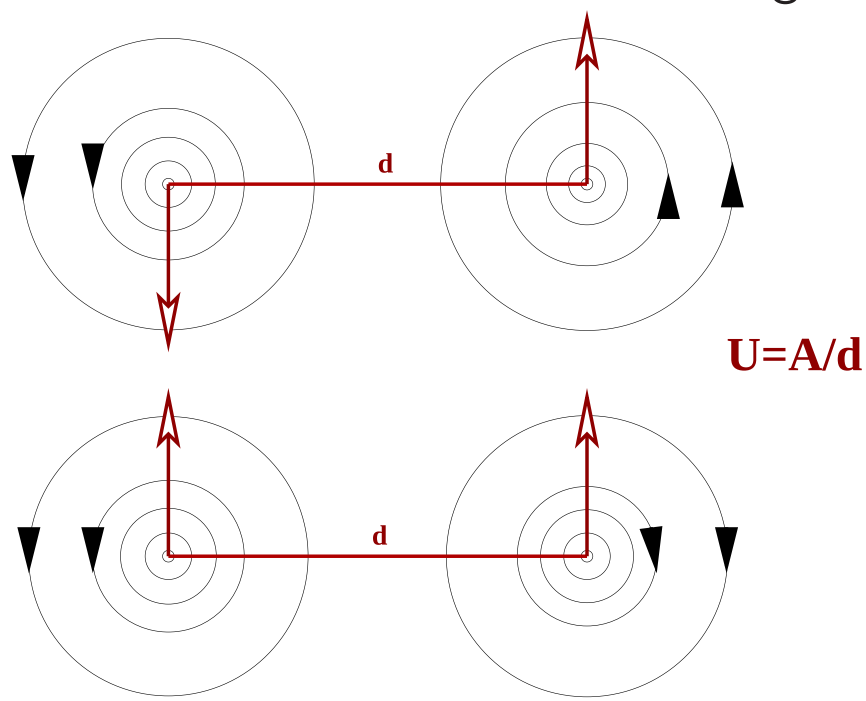
$$\nabla^2\phi = 0.$$

Cont'd Line/point vortices

A particular solution in 2D circular geometry (r, θ) is $\phi = A\theta$.
The corresponding velocity field $\mathbf{u} = A/r\,\mathbf{e}_\theta$ is that of a line/point vortex – a singularity of the velocity field with the flow moving around the singularity.

Point vortices move with the flow created by other vortices.

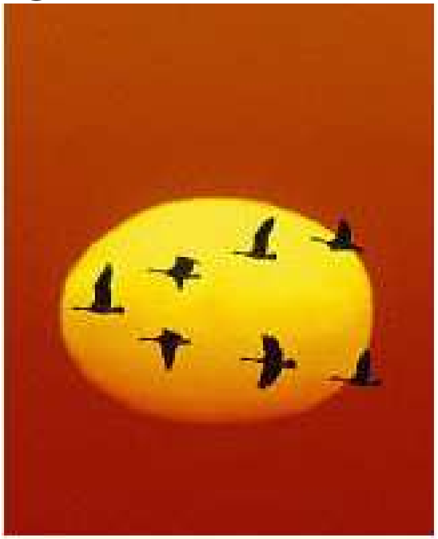
Consider two vortices of either the same sign or different sign



Wingtip vortices

The cores of wingtip vortices are sometimes visible due to condensation of water vapour in the very low pressure. The hazardous aspects of wingtip vortices are most often discussed in the context of wake turbulence. If a light aircraft is immediately preceded by a heavy aircraft, wake turbulence from the heavy aircraft can roll the light aircraft.

Migratory birds take advantage of each others' wingtip vortices by flying in a V formation so that all but the leader are flying in the upwash from the wing of the bird ahead. This upwash makes it a bit easier for the bird to support its own weight, reducing fatigue on migration flights.



Equations of motion

Consider the forces acting upon a particular unit element of volume of the fluid. Use Newton's laws of motion.

(1) Euler equation.
Neglect viscous forces:

$$\rho\frac{d\mathbf{u}}{dt} = -\boldsymbol{\nabla} p + \mathbf{F},$$

where p is pressure and $\mathbf{F}$ the volume forces (e.g. gravity).

(2) Navier-Stokes equation includes viscous forces.
For incompressible flow it takes form

$$\rho\frac{d\mathbf{u}}{dt} = -\boldsymbol{\nabla} p + \mathbf{F} + \mu\nabla^2\mathbf{u},$$

where μ is the viscosity of the fluid.
Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction.

Turbulence

Solutions of the Navier-Stokes equations often include turbulence, which remains one of the greatest unsolved problems in physics.

Even much more basic properties of the solutions to Navier-Stokes have never been proved. For the three dimensional system of equations, and given some initial conditions, mathematicians have not yet proved that smooth solutions always exist, or that if they do exist they have bounded kinetic energy. This is called the Navier-Stokes existence and smoothness problem.

The Clay Mathematics Institute offered in May 2000 a $1,000,000 prize, not to whomever constructs a theory of turbulence but (more modestly) to the first person providing a hint on the phenomenon of turbulence.

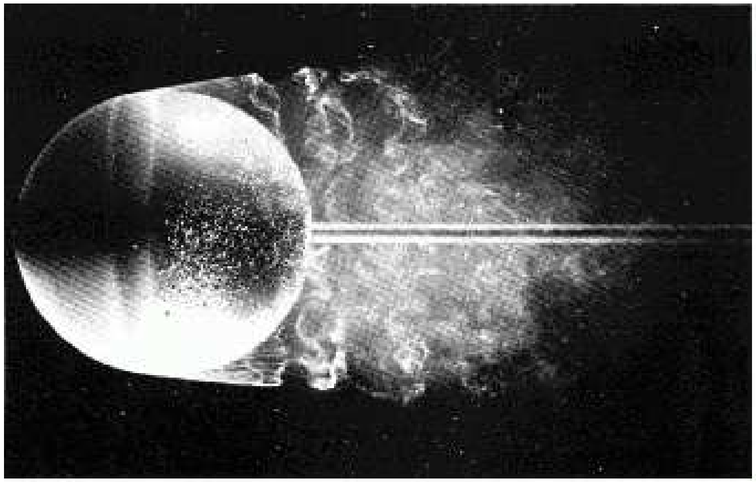
Clay Institute problem.

Prove or give a counter-example of the following statement:

In three dimensions, given an initial velocity there exist a velocity vector and a scalar pressure, which are smooth and globally defined, that solve the Navier-Stokes equations.

Kolmogorov Spectrum of Turbulence

Turbulent eddies form behind a sphere in a flow at high Reynolds number (inertia $\gg$ viscousity).



The energy of the eddies is injected on large scales (eg. scale of sphere). The large scale eddies fragment into smaller-scale eddies.
The energy injected on the largest scale cascades down through smaller and smaller eddies and dissipates at the molecular level. This process can be characterised by a spectrum of turbulence, that describes the amount of energy present in each scale.

There is no analytic theory of the spectrum of turbulence.

Progress can be made by dimensional analysis.

The amount of kinetic energy in eddies of length scales between k to $k+dk$ is $E(k)dk$, where $k = 2\pi/\lambda$.
The rate of supply of the kinetic energy per unit mass is ϵ_0.

The relevant variables and their dimensions are:
$E(k)$ with $[L]^3[T]^{-2}$ ($energy\ per\ unit\ wavenumber\ per\ unit\ mass\ of\ fluid$),
ϵ_0 with $[L]^2[T]^{-3}$ ($rate\ of\ energy\ input$) and
k with $[L]^{-1}$ ($wavenumber$).
The dimensionless group is

$$E^3(k)k^5/\epsilon_0^2.$$

The simplest assumption is that this group is a constant of order unity, so

$$E(k) \sim \epsilon_0^{2/3} k^{-5/3}.$$

This is the celebrated Kolmogorov spectrum of turbulence.