Concepts in Theoretical Physics

Lecture 7: General Relativity

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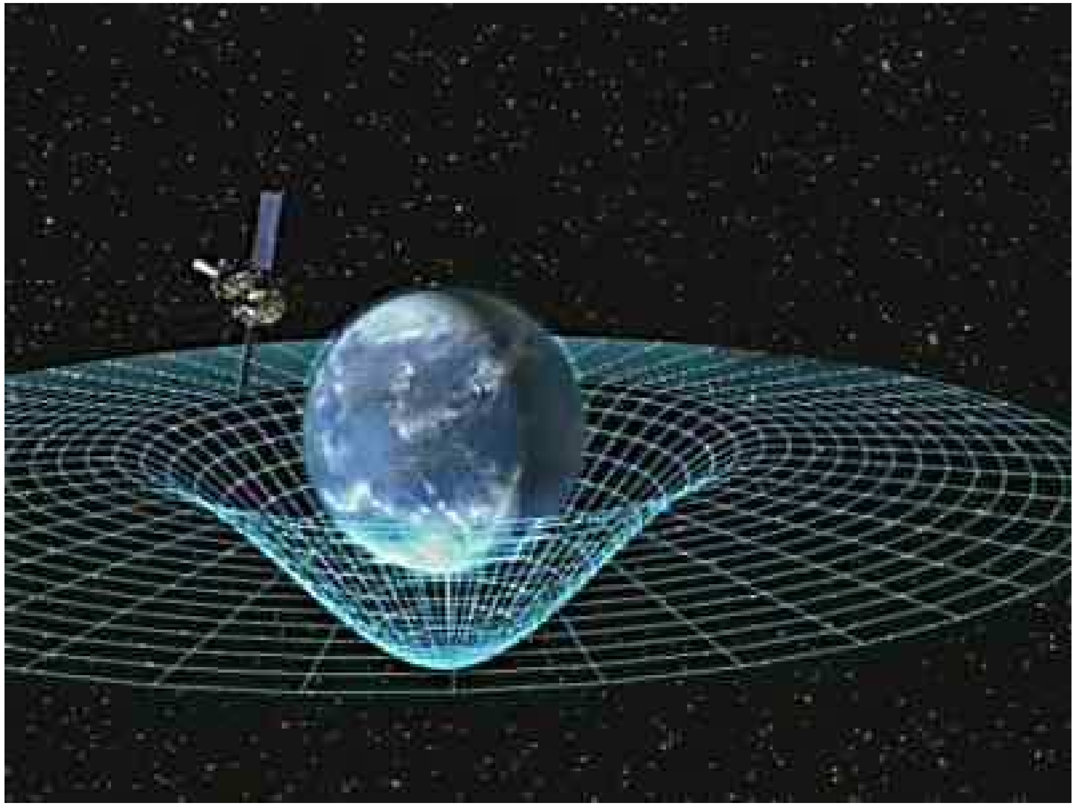
Gravity is Geometry



Inertial and Gravitational Mass

- Mass arises in two different formulae, both due to Newton

  - Gravitational Mass: $F = -GmM/r^2$

  - Inertial Mass: $F = ma$

- Yet the meaning of mass in these two formulae is very different.

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- Yet the meaning of mass in these two formulae is very different.
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- Experimentally, we find $\boxed{m_I = m_G}$

To an accuracy of 1 part in 10^{13}

Inertial and Gravitational Mass

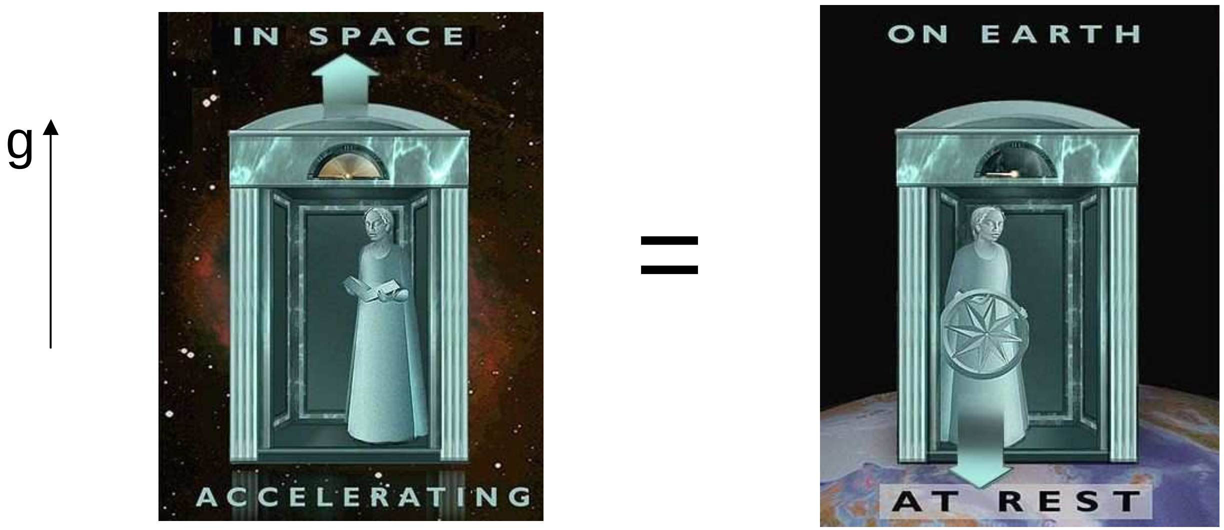
- The equality of inertial and gravitational mass is responsible for the well known fact that objects with different mass fall at the same speed under gravity.
- According to legend, this was demonstrated by Galileo dropping farm animals from the leaning tower of Pisa.
- But is there a deeper reason why the gravitational force is proportional to the inertial mass?

Fictional Forces

- There are two other forces which are also proportional to the inertial mass. These are
  - Centrifugal Force: $F = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$
  - Coriolis Force: $F = -2m\vec{\omega} \times \dot{\vec{r}}$
- But in both of these cases, we understand very well why the force is proportional to the inertial mass, m. It follows because these are "fictitious forces", arising in a non-inertial frame. (In this case, one that is rotating with frequency $\vec{\omega}$)
- Could gravity also be a fictitious force, arising only because we are in a non-inertial frame? The answer, of course, is yes!

Principle of Equivalence

- Locally, it is impossible to distinguish between a gravitational field, and acceleration



- Said another way:

> *"The gravitational field has only a relative existence... Because for an observer freely falling from the roof of a house - at least in his immediate surroundings - there exists no gravitational field".*
>
> A. Einstein

Principle of Equivalence

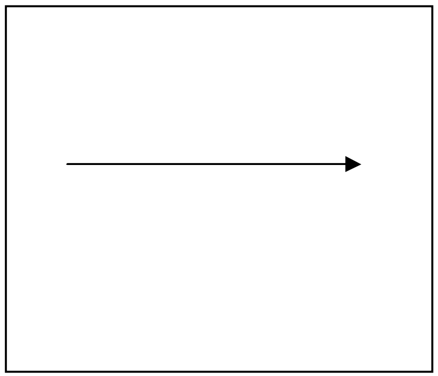
- There are some obvious difficulties with trying to identify gravitation with acceleration. It feels as if we're accelerating upwards. The people in New Zealand feel as if they're accelerating in the opposite direction. Why aren't we getting further apart?!
- A related issue is the following: there is a way to experimentally determine the difference between acceleration and gravitation. Drop two balls, one some distance above the other. The lower ball will accelerate faster if you're in gravity due to the $1/r^2$ law.
- This is not in conflict with the principle of equivalence because of the word *local*. Resolving issues like this leads us to the idea that there are different inertial frames at each point of spacetime. Patching these together gives rise to the curvature of spacetime.

Predictions

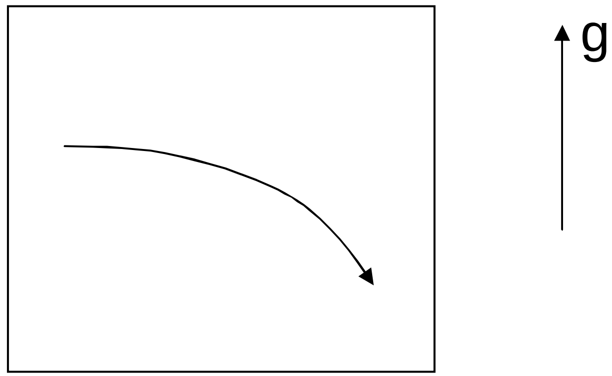
- Let's follow the principle of equivalence to derive some interesting experimental predictions.

Bending of Light

- The first prediction is simple: light bends in a gravitational field

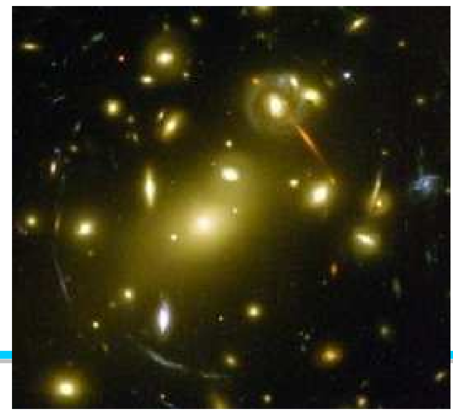


Inertial Frame



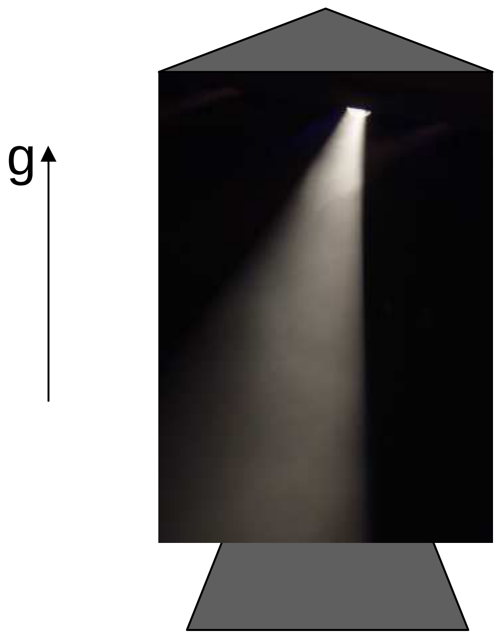
Accelerating Frame

- This is spectacularly confirmed in many experiments and observations



Gravitational Redshift

- The second prediction again involves light. It undergoes a redshift in a gravitational field.



- The rocket has height h. It starts from rest, and moves with constant acceleration g.

- Light emitted from the top of the rocket is received below. By this time, the rocket is travelling at speed

$$v = gt = gh/c$$

- This gives rise to the Doppler effect. (Neglecting relativistic effects).

$$\nu' = \nu\left(1 + \frac{v}{c}\right) = \nu\left(1 + \frac{gh}{c^2}\right)$$

Received frequency

Emitted frequency

Gravitational Redshift

- By the equivalence principle, the same effect must be observed in a gravitational field*

- The gravitational field Φ is defined to give rise to acceleration

$$\vec{a} = -\nabla\Phi$$

- For constant gravitational acceleration, $\Phi = -gh$

- This gives the formula for gravitational redshift:

$$\nu' = \nu\left(1 - \frac{\Phi}{c^2}\right)$$

- From this we can derive our third, and most dramatic, prediction…

*It is! The first indisputable experiment was in 1959

Gravitational Time Dilation

- Inverting, we get the relationship between the periods of light*

$$T' = T\left(1 - \frac{\Phi}{c^2}\right)^{-1} \approx T\left(1 + \frac{\Phi}{c^2}\right)$$

- But this holds for any time interval. Your girlfriend could be sitting at the top of the rocket, throwing popcorn at you at regular intervals *T*. In constant acceleration, you would be hit at intervals $T' < T$

- This means time goes slower at the bottom of the rocket or, by the equivalence principle, closer to the earth. Suppose she throws a piece of popcorn every time her heart beats. She will appear as if her heart beats every T', while your heart beats only at intervals *T.* You will live longer than she will.

- This "gravitational twin paradox" has been tested with atomic clocks on planes.

* (under the assumption that $\Phi \ll c^2$ which means weak gravitational field)

Spacetime Intervals

- Since time slows down in a gravitational field, we can think of writing this in a way familiar from special relativity

$$(dt')^2 = dt^2 \left(1 + \frac{\Phi}{c^2}\right)^2 \approx dt^2 \left(1 + \frac{2\Phi}{c^2}\right)$$

- Although this formula was derived under the assumption of constant acceleration, it actually holds for arbitrary gravitational potential $\Phi(\vec{x})$

- E.g. for a point particle of mass M, we have $\Phi = -GM/r$

- This means, that time runs slower on Earth than in space. If you choose to spend your life on the ground floor, and never climb the stairs, you live longer by a couple of microseconds.

$$(dt')^2 = dt^2 \left(1 - \frac{2GM}{rc^2}\right)$$

The Schwarzschild Radius

- Something special happens at

$$r_S = \frac{2GM}{c^2}$$

for, here, time stands still! It is called the Schwarzschild radius.

- For the Earth, this distance is 1cm. For the sun, it is about 1 km. Clearly neither the mass of the Earth nor the sun are contained within their Schwarzschild radius, so the concept doesn't make sense.

- However, there are objects which are so dense that they lie within their Schwarzschild radius. These are black holes. The distance, r_S where time stands still, is the *event horizon*. Stray beyond this point and you are doomed!

The Metric

- We can write down a metric, akin to the Minkowski metric that you've met in special relativity. We've already seen that the time component should look like:

$$ds^2 = \left(1 + \frac{\Phi(x)}{c^2}\right) c^2 dt^2 + \ldots$$

- But, under Lorentz transformations, time and space mix together. So the space part should also vary with space. This means that measuring sticks contract and expand at different points in space. In general, the metric of spacetime is a 4x4 matrix, whose components can be any function of t and x.

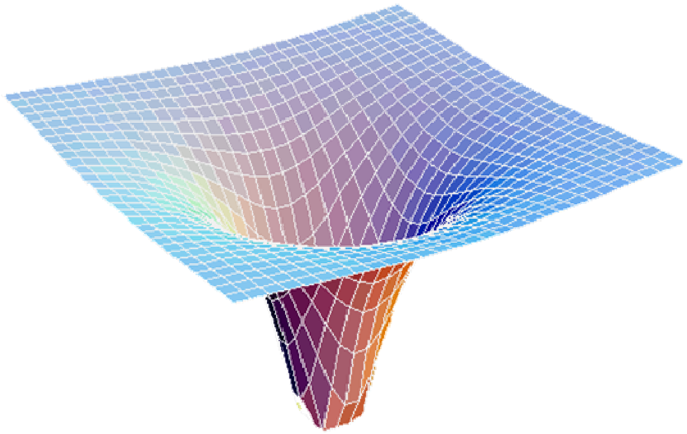
$$ds^2 = g_{\mu\nu}(t, x)\, dx^\mu dx^\nu$$

- Einstein's field equations tell you how to calculate the metric, describing how spacetime bends, for a given distribution of matter.

The Schwarzschild Solution

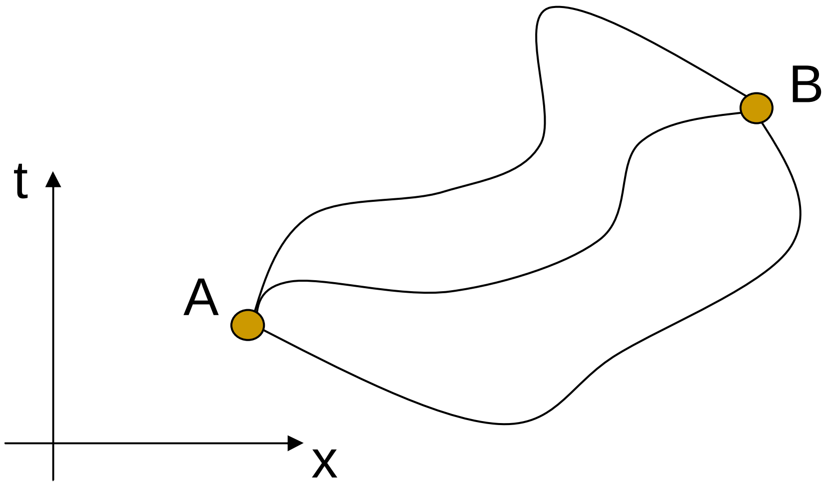
- The metric for spacetime around a black hole, or outside a spherical star, is the Schwarzchild solution

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$



Particle Orbits and Geodesics

- Suppose we have a curved spacetime, specified by a metric. How do particles move?
- It's simplest to describe this in terms of the principle of least action which we covered in the first lecture. We minimize over all paths



- The action is simply given by

$$S = -mc \int_A^B \, ds$$

- Paths which minimze this action are called *geodesics*

An Example: Flat Space

- The Minkowski metric is $ds^2 = c^2 dt^2 - d\vec{x}^2$

- Plugging this into the action, we get

$$S = -mc \int_A^B \sqrt{c^2\, dt^2 - d\vec{x}^{\,2}} = -mc^2 \int_{t_A}^{t_B} dt \sqrt{1 - \dot{\vec{x}}^{\,2}/c^2}$$

- You can minimize this to find that, in flat space, the particle travels with constant velocity

$$\dot{\vec{x}} = \mathrm{const}$$

Effects of Gravitational Time Dilation

- Now let's look at a particle moving in a metric with gravitational time dilation

$$ds^2 = \left(1 + 2\Phi(\vec{x})/c^2\right) c^2 dt^2 - d\vec{x}^2$$

- Now we can look at the action. Expanding for small velocity,

$$\begin{aligned} S &= -mc^2 \int_{t_A}^{t_B} \sqrt{(1 + 2\Phi/c^2) - \dot{\vec{x}}^2/c^2} \\ &\approx \int_{t_A}^{t_B} \frac{1}{2} m\dot{\vec{x}}^2 - m\Phi(\vec{x}) - mc^2 + \ldots \end{aligned}$$

- But the action is supposed to be "kinetic energy minus potential energy". We see that the time dilation induces a exactly the gravitational potential energy

$$V(\vec{x}) = m\Phi(\vec{x})$$