1. A non-relativistic, quantum mechanical particle sits in a box whose sides have length \( a(t) \times L \). Write down the wavefunctions when \( a(t) \) is constant.

It can be shown that if \( a(t) \) changes suitably slowly then the system remains in a given energy eigenstate. Show that the momentum “redshifts” as \( p(t) = p_0/a(t) \) where \( p_0 \) is the momentum when \( a(t_0) = 1 \).

A gas of non-relativistic particles at temperature \( T \) is described by the Maxwell-Boltzmann distribution at time \( t = t_0 \). Assuming the momentum-redshift above, show that the gas retains the Maxwell-Boltzmann form as the universe expands, but with a temperature that scales as \( T(t) = T_0/a(t)^2 \).

2. The thermal cosmic microwave background is assumed to be isotropic with a temperature \( T \) in an inertial frame \( S \). The same radiation is detected in another inertial frame \( S' \), moving with velocity \( v \) with respect to \( S \).

The Lorentz transformation relating the energy \( E \) and 3-momentum \( \mathbf{p} \) of a particle in the two frames is

\[
E = \gamma (E' - \mathbf{v} \cdot \mathbf{p}')
\]

where \( \gamma = 1 / \sqrt{1 - v^2/c^2} \). A photon has \( E = pc \). Show that the microwave background will also appear thermal in \( S' \), but with an anisotropic temperature

\[
T'(\theta') = \frac{T}{\gamma (1 - (v/c)\cos \theta')} = T \left(1 + \frac{v}{c} \cos \theta' + O(v^2/c^2)\right)
\]

where \( \theta' \) is the angle between the velocity \( \mathbf{v} \) and the momentum \( \mathbf{p}' \) of the photon arriving at the detector.

Let \( T'_+ \) and \( T'_- \) be the maximum and minimum temperatures seen in the inertial frame \( S' \). Show that

\[
T = \sqrt{T'_+ T'_-}
\]

The observed CMB, shown in the figure, has \( T'_+ - T'_- \approx 6.5 \times 10^{-3} \) K, with \( T = \sqrt{T'_+ T'_-} \approx 2.7 \) K. How fast are we travelling with respect to the universe’s preferred inertial frame?
Figure 1: The observed CMB dipole.

It is believed that there exists a yet-to-be-observed thermal cosmic neutrino background that is isotropic in the same frame $S$ as the CMB. The neutrino has a small mass and so $E^2 = p^2c^2 + m^2c^4$. Today, the neutrinos are travelling at non-relativistic speeds. Show that when (if!) we finally observe the cosmic neutrino background, we do not expect the energy density to be thermal, even at a fixed angle.

3. The Planck blackbody formula states that the number of photons with frequency between $\omega$ and $\omega + d\omega$ is

$$n(\omega) \, d\omega = \frac{1}{\pi^2 c^3} \frac{\omega^2}{e^{\beta\hbar\omega} - 1}$$

Show that the total number of photons is

$$n_{\gamma} = \frac{2\zeta(3)}{\pi^2 \hbar^3 c^3} (k_B T)^3$$

[You will need the integral $\int_0^{\infty} dy \, y^2/(e^y - 1) = 2\zeta(3)$ with $\zeta(3) \approx 1.2$.]

Define $n_{\text{ energetic}}$ to be the number of energetic photons with energy greater than the hydrogen binding $E_{\text{bind}} \approx 13.6$ eV. Show that, when $k_B T \ll E_{\text{bind}},$

$$\frac{n_{\text{ energetic}}}{n_{\gamma}} \approx \frac{(\beta E_{\text{bind}})^2}{2\zeta(3)} e^{-\beta E_{\text{bind}}}$$

As a naive diagnostic, suppose that recombination occurs when there is less than a single energetic photon per baryon. Use the baryon-to-photon ratio $\eta = n_B/n_{\gamma} \approx 10^{-9}$ to determine the temperature and redshift of recombination according to this criterion.
4. Assume that electrons, protons and hydrogen are in chemical equilibrium during recombination, with the chemical potentials related by $\mu_e + \mu_p = \mu_H$. Show that the number of electrons is related to the number of hydrogen atoms by

$$n_e^2 \approx n_H \left( \frac{m_e k_B T}{2\pi \hbar^2} \right)^{3/2} e^{-\beta E_{\text{bind}}}$$

with $E_{\text{bind}}$ the binding energy of the hydrogen ground state. What assumptions did you make along the way?

The ionization fraction is defined as $X_e = n_e/n_B$ with $n_B \approx n_p + n_H$, the number of baryons. Use the expression (3) to show that

$$\frac{1 - X_e}{X_e^2} = \eta \frac{2 \zeta(3)}{\pi^2} \left( \frac{2\pi k_B T}{m_e c^2} \right)^{3/2} e^{\beta E_{\text{bind}}}$$

with $\eta$ the baryon-to-photon ratio. Consider the limiting regimes $k_B T \ll E_{\text{bind}}$ and $E_{\text{bind}} \ll k_B T \ll m_e c^2$ to roughly sketch $X_e$ as a function of temperature.

5. Recombination is not instantaneous, but happens over a period of time. Some photons in the CMB come from earlier times, when the universe was hotter, and some from later times.

Why does the observed CMB exhibit a perfect blackbody spectrum at a single temperature?

6. At temperature $T$, and vanishing chemical potential, the expected number of particles with momentum $p$ is given by

$$n(p) = \frac{1}{e^{\beta E(p)} + 1}$$

where the minus sign is for bosons and the plus sign for fermions.

For ultra-relativistic particles, with $E(p) = pc$, show that the total number of fermions, $n_F$, is related to the total number of bosons, $n_B$, by $n_F = 3n_B/4$. Show that the total energy density of fermions, $\rho_F$, is related to the total energy density of bosons, $\rho_B$, by $\rho_F = 7\rho_B/8$.

[Note: you need not evaluate any integral to do this question.]
7. Consider a gas of electrons and positrons in the ultra-relativistic limit $k_B T \gg m_e c^2$. In the early universe, there must have been a slight imbalance of electrons over positrons. This is modelled by introducing a small chemical potential $\mu_e \ll k_B T$ for electrons, with an equal and opposite chemical potential for positrons. Show that this results in a small excess $\Delta n$ of electrons over positrons, given by

$$\Delta n = \frac{g(k_B T)^2 \mu_e}{6 \hbar^3 c^3} \left( 1 + \mathcal{O} \left( \frac{\mu_e^2}{(k_B T)^2} \right) \right)$$

[You will need to use the integral $\int_0^\infty \frac{dy}{y(e^y + 1)} = \frac{\pi^2}{12}$.]