Fluid Mechanics

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Recommended Books and Resources

There are many books on fluid mechanics, ranging from the eminently accessible to the dauntingly comprehensive. Here are a collection that I found useful.

- Van Dyke, *An Album of Fluid Motion*

If you're going to look at one book on fluid mechanics then it should be this one. It's a book of pictures, many of them very pretty, While this likely sounds lightweight, in this case a picture really does paint 20 equations and helps build intuition for fluid flow. It's difficult to buy at a reasonable price (at the time of writing, Amazon offer a paperback version for £833.82) but you can find versions on the internet.

- Acheson, *Elementary Fluid Dynamics*
- Childress, *An Introduction to Theoretical Fluid Mechanics*

If you're going to look at a second book on fluid dynamics, it should probably be one of these, or something similar. Both are aimed at the beginner. They are clear and easygoing. I have a slight preference for Acheson which focuses more on the physics.

- George Batchelor, *An Introduction to Fluid Dynamics*

This is considered the bible of fluid mechanics by many practitioners. It's not particularly cuddly, but the explanations are clear enough and it is certainly comprehensive (unless you care about turbulence).

- Landau and Lifshitz, *Fluid Mechanics*

An astonishing amount of physics is packed into this book, but it's not the easiest read. Like Batchelor, it puts thermodynamics front and centre which is useful in making contact with other areas of physics which can otherwise feel hidden. (In these lectures, we only bring thermodynamics into the game when we describe sound waves.)

- Drazin and Reid, *Hydrodynamic Stability*

For all your instability needs.

- Frisch, *Turbulence*

A look at symmetries and scaling in turbulent flow.

Contents

Acknowledgements

I'm no expert on fluid mechanics. I wrote these notes primarily to teach myself the basics of the subject and I hope that others may find them useful. If, however, you would prefer to learn from someone who actually knows what they're talking about then I put together a collection of resources that I found helpful on this webpage.

My thanks to Matt Davison, Mihalis Dafermos, Sean Hartnoll and Jorge Santos for helpful discussions on some of the topics included in these notes. I am supported by a Simons Investigator award.

1 Introduction

Take anything in the universe, throw a bunch of it in a box, and turn up the heat. Then it doesn't matter what you started with, the motion of this substance will be governed by the equations of fluid dynamics.

This is a remarkable statement. There are lots of different things in the universe and we go to great lengths to understand their properties. Yet if you heat them, most of the differences disappear. When things get hot, everything looks the same.

Here are some examples. Take any element in the periodic table and heat it until it melts, so that it is either a liquid or a gas. The motion of every element is governed by the same set of equations. The only reminder of what you started with is to be found in a handful of parameters of these equations which describe, among other things, the density and viscosity of the fluid. These will differ from element to element. But the basic set of equations are the same, regardless of whether you started with an alkaline earth metal or an inert gas.

This same story holds if we turn our attention to more exotic substances. For example, inside every proton and neutron sit three quarks. They have been trapped there since the Big Bang, held in place by the grip of the strong nuclear force. However, earlier this century, experimenters succeeded in colliding nuclei together with energies that were so high that the protons and neutrons themselves melted, freeing their imprisoned quarks and forming a novel state of matter known as a the quark-gluon plasma. This plasma only lasts for a fraction of a second before it cools and once again forms protons and neutrons. But during that fraction of a second it moves. And the movement is described by the laws of fluid mechanics.

Here is an even more extreme example. Take spacetime itself. It is possible for spacetime to collapse in on itself to form a black hole and, due to the work of Hawking, we know that these black holes are hot objects. So a black hole can be viewed as a way to heat spacetime. Surprisingly, if you look at the equations that govern the event horizon of a black hole, you will once again find the laws of fluid mechanics.

All of which is to say that there is a wonderful universality to the laws that govern fluids. In certain circumstances, these laws describe literally everything. And this makes them interesting.

The reasons underlying this universality are well understood. At the microscopic level, fluids are ridiculously complicated objects, consisting of, say, 10^{23} atoms, each

following its own path, while acting through various forces on the atoms around it. But much of this motion is fleeting and we lose little if we ignore it. Instead, we care only about patterns in the collective motion of the atoms that survive over long time scales. It turns out that these long-lived modes are all related to familiar conservation laws – conservation of mass, momentum and energy – and these conservation laws are universal and obeyed by all substances. This, ultimately, is why all fluids look the same: the equations of fluid dynamics are essentially the equations that govern how conserved quantities evolve in time. (This is a theme that will rear its head at various places in this course, but is not something that we dwell upon. In contrast, the idea that conservation laws underlie fluid mechanics will be the focal point of the lectures on Kinetic Theory which derive the Navier-Stokes equation starting from 10^{23} atoms, each obeying Newton's laws.)

In addition to the universal aspect of fluid mechanics, the subject also has enormous practical applications. It explains, for example, why planes fly. (As we will recount later in these lectures, one of the more embarrassing episodes in the history of theoretical physics occurred in 1903 when the Wright brothers took to the air before physicists were able to adequately explain either lift or drag!) Fluid mechanics explains how oil flows through pipes and how the motion of the atmosphere manifests itself in the climate, and how many decades of focussing on the former has resulted in an urgent and desperate need to better understand the latter.

In this course we explore the basics of fluid mechanics. Our focus will not be on quarks and black holes, but nor will it be any particular application of fluid mechanics. Instead our goal is simply to understand the different things that fluids can do. Fluids are everywhere and they have a tendency to move. The purpose of these lectures is simply to construct and explore the equation governing this motion.

As we've stressed above, the motion of all fluids is described by the same basic set of equations. Prominent among these is the Navier-Stokes equation, accompanied by one or two of further equations describing the conservation of mass and, in some cases, the flow of heat. One of the themes of fluid mechanics is that a wonderful diversity of different behaviour emerges from these equations. As these lectures progress, we will find ourselves falling into a routine. Like Monet and his haystacks, we will return to these same theme over and over again, not because we did anything wrong the first time but because there is always something new to see. Attacking the same set of equations, but with slight change to the boundary condition, or a novel approximation scheme, will often yield something new and surprising. One of the delights of the subject lies in finding such riches sitting inside such simple equations.

1.1 The Basics

When we were kids, we are told that there are three phases of matter: solid, liquid and gas. As we grow older, we learn that this is a hopelessly naive view of the world. Nonetheless, it is the one that we will adopt in this course which is concerned only with the latter two. Liquids and gases are both examples of fluids. Roughly speaking, a fluid is a substance that flows when pushed. More rigorously, fluids are objects that are well described by the equations of these lectures.

The subject of fluid mechanics starts with a lie. (Applied mathematicians prefer the term "approximation".) The lie, sometimes dubbed the *continuum hypothesis*, is that fluids are indivisible continuous objects. The fluid can be then described by two smooth, continuous fields,

- The density $\rho(\mathbf{x}, t)$
- The velocity $\mathbf{u}(\mathbf{x}, t)$.

Of course, we know that in reality fluids are made of molecules and this approximation must break down on atomic scales. But we also know from experience that if we look on suitably large scales, where we are coarse graining over a many many molecules, then the continuum description is remarkably good.

It is appropriate to start these lectures by stressing that we are dealing with an approximation. It will not be our last. The study of fluids is all about the art of approximation. The equations of fluid mechanics, simple as they are, cannot be solved in full generality and we will make progress only by simplifying. The skill is in learning what to keep and what to ignore. And we start by ignoring the existence of atoms.

It's not just the discreteness of matter that is swept under the rug in the continuous description. We also ignore the vast majority of the motion of the constituent atoms and molecules that make up the fluid. At room temperature, these constituents are flying around at speeds of 100 ms^{-1} or so. (This is certainly true of gases. For liquids, the molecules are more closely bound to their neighbours and we have to think more carefully about what the velocity of a single molecule really means.) But most of this underlying atomic motion is neglected in our coarse-grained description. Instead, the velocity field $\mathbf{u}(\mathbf{x}, t)$ describes the average, macroscopic motion of the fluid. In particular, there is a state of the fluid in which $\mathbf{u}(\mathbf{x}, t) = 0$ and we pretend that the fluid is completely still, even though the underlying particles are still flying around, just with no direction preferred over any other.

(As an aside: the internal motion of the constituents doesn't show up in the velocity field $\mathbf{u}(\mathbf{x}, t)$, but it does manifest itself in the temperature of the fluid which is another field $T(\mathbf{x}, t)$. We'll elaborate on the role that temperature plays as these lectures progress but for now, and indeed for much of the lectures, we will be able to ignore it.)

It is also worth elaborating on how to think about the position $\mathbf{x}$ that appears in the argument of the fields $\rho(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$. This is some fixed position in space. This means, in particular, that $\mathbf{u}(\mathbf{x}, t)$ is the velocity that would be measured by some fixed array of sensors embedded in the fluid, as opposed to sensors that drift along with the fluid. The use of fields $\rho(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$ is called the *Eulerian* description.

We will also have use for a slightly different viewpoint, in which we think of individual "parcels of fluid", each initially sitting at some position $\mathbf{x}$ and then following the flow by travelling at speed $\mathbf{u}(\mathbf{x}, t)$. It's not so easy to define what we mean by these"parcels of fluid" given that the underlying atoms are, as we described above, wandering off in all sorts of directions, often at high speed, with only the most scant regard for the velocity field $\mathbf{u}(\mathbf{x}, t)$. But the concept of a fluid parcel that keeps its identity as the fluid moves is an extremely useful pretence. We will sometimes talk about a "particle" of fluid and we have in mind these parcels rather than the underlying atoms. The perspective in which we follow the trajectories of these parcels, and study the forces that act on them as if they were particles in classical mechanics, is called the *Lagrangian* description.

Throughout these lectures, all our equations will be written in the Eulerian description, using the velocity field $\mathbf{u}(\mathbf{x}, t)$, but some intuition will come from a more Lagrangian way of thinking. Moreover, we will certainly have a need to understand the trajectories of particles that are embedded within the fluid. Indeed, we kick off with some simple observations.

1.1.1 Path Lines and Streamlines

There are a number of ways to visualise the flow $\mathbf{u}(\mathbf{x}, t)$ of a fluid. Here are the two most useful:

- A *pathline* is the trajectory followed by a particle embedded within the fluid.
- A *streamline* is a tangent to $\mathbf{u}(\mathbf{x}, t)$ at every point $\mathbf{x}$ for fixed time t. In general, the tangents to a vector field $\mathbf{F}(\mathbf{x})$ are said to be integral curves for $\mathbf{F}$. So the streamlines are integral curves for the velocity field at a fixed time.

If the flow is *steady*, meaning that $\partial\mathbf{u}/\partial t = 0$, then the pathlines and streamlines coincide. But, for time dependent flows, they differ. To see this, let's drape some equations around the definitions above.

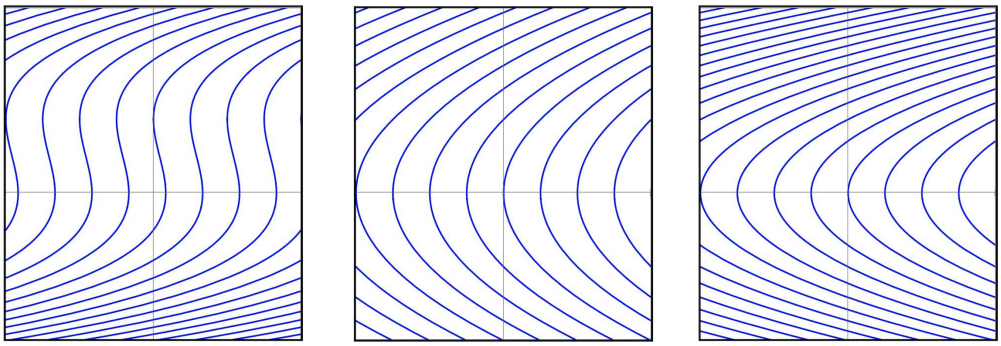


Figure 1. The pathlines for particles in the flow $\mathbf{u} = (yt, 1)$ are shown on the the left. These are a history of the flow. The middle and right hand figures show streamlines, with the right-hand figure at a later time.

First consider the pathline. A particle within the fluid will follow some trajectory $\mathbf{x}(t)$. At any time t, the velocity of this particle is given by the velocity field $\mathbf{u}$ evaluated at the position of the particle, meaning

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{u}(\mathbf{x}(t), t) \tag{1.1}$$

Given some initial starting point $\mathbf{x}(t = 0) = \mathbf{x}_0$, we can solve this equation to find the pathline.

In contrast, a streamline is a trajectory $\mathbf{x}(s)$ such that the tangents of $\mathbf{x}(s)$ coincide with the velocity field at a *fixed* time t,

$$\frac{d\mathbf{x}}{ds}(s) = \mathbf{u}(\mathbf{x}(s), t)$$

In words, the streamline is a snapshot of the flow at some fixed time, while the pathline tells us about the actual history of the particle.

An Example

Consider the two-dimensional flow given by

$$\mathbf{u}(\mathbf{x}, t) = \begin{pmatrix} \alpha y t \\ \beta \end{pmatrix}$$

for some fixed coefficients α and β. The pathline obeys

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha y t \\ \beta \end{pmatrix}$$

The y component is solved by $y = y_0 + \beta t$, while the equation for the x component becomes $\dot{x} = \alpha y t = \alpha(y_0 t + \beta t^2)$, which gives $x = x_0 + \frac{1}{2}\alpha y_0 t^2 + \frac{1}{3}\alpha\beta t^3$. To get the pathline, we eliminate t to get the family of curves in the (x, y) plane

$$x = x_0 + \frac{\alpha}{2\beta^2} y_0 (y - y_0)^2 + \frac{\alpha}{3\beta^2}(y - y_0)^3$$

These are plotted on the left-hand plot of Figure 1 for various values of the starting point (x_0, y_0).

In contrast, to find the streamlines we instead solve

$$\frac{d\mathbf{x}}{ds} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha y t \\ \beta \end{pmatrix}$$

where the prime means d/ds. These now have the solutions $y = y_0 + \beta s$ and $x = x_0 + \alpha y_0 t s + \frac{1}{2}\alpha\beta t s^2$ where t is now some fixed parameter. These are shown in the middle and right-hand plots of Figures 1 for $t > 0$. Note that the pathlines and streamlines are not similar in this example: the former is a cubic curve, the latter a parabola. (Or, in the special case of $t = 0$, straight lines.) Moreover, the streamlines are time-dependent: the right-hand figure is a snapshot of the flow at a later time than the middle figure.

1.1.2 The Material Time Derivative

As we stressed above, the density $\rho(\mathbf{x}, t)$ and velocity field $\mathbf{u}(\mathbf{x}, t)$ are measured in the Eulerian sense at some fixed point $\mathbf{x}$. But this leaves us with the question: how do we see things change in time if we're drifting along with the fluid?

Specifically, suppose that there is some field $\phi(\mathbf{x}, t)$ that we would like to measure. This might be the density of the fluid itself, or something else. The explicit time dependence in $\phi(\mathbf{x}, t)$ tells us how this quantity changes with time if we're sitting at some fixed position $\mathbf{x}$. But if we're drifting with the fluid, then we follow a pathline $\mathbf{x}(t)$ defined by (1.1). The value of field along this trajectory is given by $\phi(\mathbf{x}(t), t)$ and the total time derivative is

$$\frac{d}{dt}\phi(\mathbf{x}(t), t) = \frac{\partial \phi}{\partial t} + \dot{\mathbf{x}} \cdot \nabla\phi = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla\phi$$

The additional $\mathbf{u} \cdot \nabla\phi$ term captures the change in ϕ because of the way we're swept along by the fluid. The transport of some object as it's carried along by a fluid is known as *advection* and, correspondingly, $\mathbf{u} \cdot \nabla\phi$ is called the *advective* rate of change. This idea of a total time derivative will be important, so much so that we introduce some

new notation for it (even though we already have perfectly good notation in $d\phi/dt$!). We write

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi$$

and call this the *material derivative*. It can be thought of as a bridge between the Eulerian description in terms of a fixed point $\mathbf{x}$ and the Lagrangian description which moves with the fluid.

1.1.3 Conservation of Mass

Our first equation of fluid mechanics is the simplest: it captures the fact that mass is conserved. Moreover, like all conservation laws in physics, mass is conserved *locally*. This means that if the mass of the fluid decreases at some point in space then it must have moved to a neighbouring point.

This fact is captured by the conservation equation, relating the density ρ and the velocity $\mathbf{u}$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1.2}$$

Equations of this kind are commonplace in physics because they appear whenever we have a conservation law. In particular, an identical equation appears in Electromagnetism where, in that context, ρ is the electric charge density and $\mathbf{J} = \rho\mathbf{u}$ is the electric current density. For us, ρ is the mass density and $\rho\mathbf{u}$ is the *mass flux density*.

To see why (1.2) captures the conservation of mass, consider the mass M of fluid in some fixed region V,

$$M = \int_V \rho \, dV$$

The change of this mass is given by

$$\frac{dM}{dt} = \int_V \frac{\partial \rho}{\partial t} \, dV = -\int_V \nabla \cdot (\rho \mathbf{u}) \, dV = -\int_S \rho \mathbf{u} \cdot d\mathbf{S}$$

where we have used the divergence theorem and $S = \partial V$ is the boundary of the region V. This tells us that if there is no net flow of mass flux through the boundary S then the total mass M inside the region V remains constant. In other words, mass is conserved.

We can also write the mass conservation equation (1.2) using our new material derivative notation. It becomes

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0 \tag{1.3}$$

Incompressible Fluids

Throughout much of these lectures notes we will make one further approximation: we will assume that fluids are *incompressible*, meaning that $\rho(\mathbf{x}, t)$ is a constant. In this case, $\dot{\rho} = \nabla\rho = 0$ and the continuity equation (1.2) becomes simply

$$\nabla\cdot\mathbf{u} = 0 \tag{1.4}$$

In the language of our Vector Calculus lectures, we say that the fluid flow is solenoidal or divergence free. The vast majority of these lectures will be devoted to finding the wonderfully diverse solutions to the equation (1.4).

In the fact, the requirement that $\dot{\rho} = \nabla\rho = 0$ can be loosened slightly. We see from (1.3) that we only really require $D\rho/Dt = 0$ for the incompressible condition (1.4) to be enforced. This means that any individual parcel of fluid should not change its density as it's swept along, but different parts of the larger fluid may have different densities. Such a situation is said to be *stratified* and arises, for example, in the ocean where the water is more dense at the bottom than the top. We'll meet situations like these when we discuss some aspects of waves in Section 4.

The assumption that fluid flow is incompressible is not totally innocent. In fact, the phenomenon of fluids compressing and expanding as their density changes is so common that we give it a special name. This name is "sound"! It turns out that that assumption of incompressibility is good when the speed of the fluid $|\mathbf{u}|$ is much less than the speed of sound. For air at atmospheric pressure, the speed of sound is 340 ms^{-1}; for water at room (or ocean) temperature it is around 1500 ms^{-1}. For much of these lectures, we will restrict ourselves to flows much below these speeds and assume that $\nabla\cdot\mathbf{u} = 0$. But, in Section 4.4, we will discuss the propagation of sound waves and then we will be forced to look more closely at the equations that govern compressible fluids.

1.1.4 The Stream Function

For incompressible fluids, satisfying $\nabla\cdot\mathbf{u} = 0$, we can write the velocity field as

$$\mathbf{u} = \nabla\times\mathbf{A}$$

For many fluid flows, this isn't particularly helpful since we have just swapped one vector field $\mathbf{u}$ for another $\mathbf{A}$. However, when the flow is two-dimensional (in some sense) this provides a very useful simplification because it means that we get to exchange the vector field $\mathbf{u}$ for a scalar field Ψ called the *stream function*.

For example, suppose that the flow is independent of the z-direction, so that the velocity field takes the form

$$\mathbf{u} = (u_1(x, y, t), u_2(x, y, t), 0)$$

Then the vector potential $\mathbf{A}$ can be written as

$$\mathbf{A} = (0, 0, \Psi(x, y, t)) \quad \Rightarrow \quad u_1 = \frac{\partial \Psi}{\partial y} \text{ and } u_2 = -\frac{\partial \Psi}{\partial x}$$

and the degrees of freedom are captured by the stream function $\Psi(x, y, t)$. It has the nice property that lines of constant Ψ are streamlines of the flow. To see this, note that lines of constant Ψ have a normal $\mathbf{n}$ given by

$$\mathbf{n} = \nabla \Psi = \left(\frac{\partial \Psi}{\partial x}, \frac{\partial \Psi}{\partial y}, 0 \right)$$

and so $\mathbf{u} \cdot \mathbf{n} = 0$. This is telling us that vectors that are normal to lines of constant Ψ are also normal to streamlines. But in 2d, the enemy of an enemy is necessarily a friend. So lines of constant Ψ are streamlines.

We can also use a stream function in cylindrical polar coordinates, with $\mathbf{A} = (0, 0, \psi(r, \theta, t))$. In this case, the resulting flow is

$$\mathbf{u} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{\mathbf{r}} - \frac{\partial \psi}{\partial r} \hat{\boldsymbol{\theta}} \ . \tag{1.5}$$

We'll make good use of the stream function in a number of places throughout this book, starting when we discuss potential flows in section 2.3.

2 Inviscid Flows

Fluids have a property known as *viscosity*. This is an internal, friction force acting within the fluid as different layers rub together. It is crucially important in many applications.

Nonetheless, we will start our journey into the world of fluids by ignoring viscosity altogether. Such flows are called *inviscid*. This will allow us to build intuition for the equations of fluid mechanics without the complications that viscosity brings. Moreover, the flows that we find in this section will not be wasted work. As we will see later in Section 3, they give a good approximation to viscous flows in certain regimes where the more general equations reduce to those studied here.

2.1 The Euler Equation

We have already met the mass conservation equation (1.2)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{2.1}$$

We will assume that the fluid is incompressible, so that this becomes

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

But we need one more equation to describe the motion of fluids. This second equation comes from what fluid dynamicists sometimes call "momentum balance". It is what everyone else calls "$F = ma$".

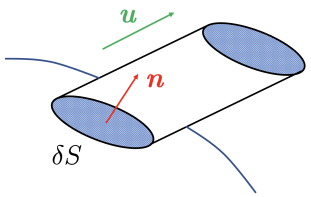
Consider some fixed region in space that we call V. The momentum in this volume is $\int_V \rho \mathbf{u} \, dV$ and Newton's second law tells us that the rate of change of momentum is equal to the force. The novelty here is that the momentum inside V might change simply because the fluid leaves (or enters) the region V. To write down the equation of motion, we need to take this into account.

Claim: The momentum flux across the boundary in some time δt is

$$\text{Momentum Flux} = \delta t \times \int_S (\rho \mathbf{u}) \, \mathbf{u} \cdot d\mathbf{S} \tag{2.3}$$

with $S = \partial V$ is the boundary of the volume V.

Proof: Consider some small area of fluid δS lying on the surface S and watch it evolve. In some small time interval δt it sweeps out a volume $\delta(\text{Vol}) = \mathbf{u}\cdot\mathbf{n}\,\delta t\,\delta S$ where $\mathbf{n}$ is the normal to the surface as shown in following figure. As usual we write the vector area element $\delta\mathbf{S} = \mathbf{n}\,\delta S$, so we have $\delta(\text{Vol}) = \mathbf{u}\cdot\delta\mathbf{S}\,\delta t$. This means that the momentum departing through the surface is $\rho\mathbf{u}\,\delta(\text{Vol})$. This gives the claimed result. □



Including this extra term for the momentum that leaks through the sides, the "F=ma" equation of motion for the fluid is

$$\frac{d}{dt}\int_V \rho\mathbf{u}\; dV = -\int_S (\rho\mathbf{u})\,\mathbf{u}\cdot d\mathbf{S} + \text{Force}$$

The additional term from the leaking momentum flux (2.3) sits on the right-hand side with the minus sign there, as always, to signify loss.

We'll get to the force shortly. Before we do, we can use the divergence theorem to convert the surface integral over momentum flux into a volume integral. Taking this term over to the other side, and resorting to index notation for the vectors $\mathbf{u}$, we have

$$\int_V \rho\frac{\partial u_i}{\partial t}\; dV + \int_V \rho\frac{\partial}{\partial x^j}(u_i u_j)\; dV = \text{Force}$$

Here we've used the fact that we're working with an incompressible fluid, both in the fact that ρ is constant and also in the derivative in the second term which reads $\partial_j(u_i u_j) = u_i\partial_j u_j + u_j\partial_j u_i$. But the first of these vanishes because incompressible fluids obey $\nabla\cdot\mathbf{u} = 0$. We're left with

$$\int_V \rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) dV = \int_V \rho\frac{D\mathbf{u}}{Dt}\; dV = \text{Force} \tag{2.4}$$

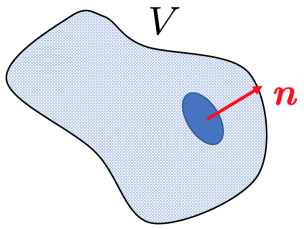
In other words, the "ma" part of our equation involves the material derivative of the velocity. In hindsight, this is not surprising. The material derivative is the rate of change when you follow a parcel of fluid through the flow. This is the appropriate meaning of "rate of change" in Newton's second law.

2.1.1 Under Pressure

Next we come to the question of forces that the fluid experiences. As we've already mentioned, we'll postpone any discussion of friction forces to Section 3. The fluid may be exposed to some external force, with gravity the most obvious, and we'll come to these shortly. But the most important force comes from within: this is *pressure*.

From a microscopic perspective, the pressure in a fluid comes from the motion of the underlying atoms or molecules. But, as we've already stressed, in these notes we shy away from the fundamentals and focus on the macroscopic. Here pressure manifests itself as a force acting on the surface of any fluid element.

The *pressure* is defined as the force per unit area. Consider a small parcel of fluid, contained within a fixed volume V. The pressure $P(\mathbf{x}, t)$ acts on the surface $S = \partial V$ of this volume[1]. It's an isotropic force, meaning that it is the same in all directions. The force exerted by the fluid outside V on some small region δS on the surface is



$$\mathbf{F}_{\text{pressure}} = -P\mathbf{n}\,\delta S$$

where $\mathbf{n}$ is the outward-pointing normal as shown in the figure.

The pressure $P(\mathbf{x}, t)$ should be viewed as a dynamical field that must be solved, subject to certain boundary conditions, at the same time as the velocity field $\mathbf{u}(\mathbf{x}, t)$. Indeed, for certain simple flows we'll see that there's a direct relationship between the pressure and velocity.

Including the pressure, the equation of motion for the fluid (2.4) becomes

$$\int_V \rho \frac{D\mathbf{u}}{Dt}\, dV = -\int_S P\, d\mathbf{S} + \text{Other Forces} \tag{2.5}$$

We will assume that these other forces act on the volume of the fluid rather than the surface (this is true for external forces like gravity) and so take the integral form

$$\text{Other Forces} = \int_V \mathbf{f}\, dV$$

The pressure acts on the surface of the volume V but we can massage it into a volume-type force through use of the divergence theorem. This gives

$$\int_V \rho \frac{D\mathbf{u}}{Dt}\, dV = \int_V (-\nabla P + \mathbf{f})\, dV$$

[1] I am apparently alone in the world in thinking that the lower case p for pressure looks way too much like the density ρ for them to happily cohabit in the same equation.

The final step is to recall that this whole derivation holds for an arbitrary volume V within the fluid. Since it holds for all such V, the integrand itself must vanish. So we're left with the differential equation of motion for the fluid

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mathbf{f} \tag{2.6}$$

This is the *Euler equation*. Finding solutions to this simple equation will occupy us for the rest of this section although we will, ultimately, replace it in Section 3 by the Navier-Stokes equation which includes the effects of viscosity. Fluids that obey the Euler equation are said to be *ideal*.

Importantly, the Euler equation is non-linear in the velocity field, although this is somewhat hidden in the notation above since the non-linearity sits in the material derivative: $D\mathbf{u}/Dt = \partial\mathbf{u}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{u}$.

Note that a constant pressure P throughout the fluid does nothing. This is because the pressure is isotropic: if one piece of fluid pushes on a neighbour, the neighbour pushes back with equal force. Interesting dynamics only arises when we have pressure differences across the fluid, as captured by ∇P.

The Euler equation is a vector equation. Combined with the requirement of incompressibility, $\nabla\cdot\mathbf{u} = 0$, we have four equations in total. We will use these to solve for the four dynamical variables: P and $\mathbf{u}$.

Looking Forwards: the Equation of State

If you know one thing about gases, then it will be the ideal gas law. This relates the pressure P, volume V and temperature T of a gas by

$$PV = Nk_BT$$

where N is the number of molecules in the gas and k_B a universal constant of nature called Boltzmann's constant that relates energy to temperature. (For what it's worth, $k_B \approx 1.4\times 10^{-23}$ JK^{-1}.) For our purposes, it's more useful to think of the ideal gas law in terms of the density $\rho = Nm/V$ rather than volume, where m is the mass of the constituent molecule,

$$P = \frac{k_B\rho T}{m}$$

The ideal gas law is an example of an *equation of state*. It holds for strictly non-interacting gases. If we take into account interactions, either in gases or in liquids, it will be replaced by some other equation of state that again relates pressure P, density ρ and temperature T. (You can learn more about how to calculate the equation of state from first principles in the lectures on Statistical Physics.)

When we first meet the ideal gas law, we think of P, ρ and T as constants that characterise the whole system. But it also holds if they are promoted to the kind of local fields $P(\mathbf{x},t)$, $\rho(\mathbf{x},t)$ and $T(\mathbf{x},t)$ that we work with in these lectures. For incompressible fluids, with ρ constant, the equation of state tells us that the temperature $T(\mathbf{x},t)$ simply tracks the pressure $P(\mathbf{x},t)$. For this reason we won't need to consider it separately.

Things are more interesting if we have compressible fluids, in which $\rho(\mathbf{x},t)$ is another dynamical variable. In this case the mass conservation equation (1.2) and Euler equation aren't enough information to tell us what happens and we need another equation. It turns out that in this situation the right way forward is to use the equation of state to replace $\rho(\mathbf{x},t)$ with the temperature field $T(\mathbf{x},t)$ and then write down a separate equation for how heat flows in the system. (Roughly speaking, it is a version of the heat equation, with the material derivative replacing the usual time derivative.) We'll explain this further in Section 4.4 when we discuss sound waves and we will be forced to think more carefully about the thermodynamics of fluids. (A fuller derivation can be found in the lectures on Kinetic Theory.)

2.1.2 The Euler Equation is Just Momentum Conservation

Suppose that there is no external force on our fluid, so $\mathbf{f}=0$. Then the Euler equation can be written in the characteristic form of a conservation law

$$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla P=0 \quad \Rightarrow \quad \frac{\partial \rho u_i}{\partial t}+\frac{\partial}{\partial x^j}\left(\rho u_i u_j+P\delta_{ij}\right)=0 \tag{2.7}$$

where we've used the assumption that the fluid is incompressible, both in taking ρ inside the derivatives and in using $\partial_j u_j=0$.

It's clear what is conserved here: it is simply the momentum in each of the three directions: $\int_V \rho u_i \, dV$. Associated to each conserved quantity is a current. The novelty here is that because the conserved quantity is itself a vector, the associated current is a tensor Π_{ij}. This tells us how the momentum in the i^{th} direction is transported in the j^{th} direction. The form of the momentum current can be read off from the equation above,

$$\Pi_{ij}=\rho u_i u_j+P\delta_{ij}$$

The first, advective contribution describes the momentum due to the motion of the fluid. The pressure contribution to momentum is perhaps more surprising. It is a hint, even at this macroscopic level, that pressure is associated to something moving around. This something is, of course, the constituent atoms of molecules of the fluid that we have declared irrelevant for fluid mechanics.

There is a simple way of seeing why pressure is related to momentum. Take a box with some fluid inside and make a little hole in it. The pressure inside the box will force the fluid out of the hole. The rate at which momentum escapes from the box is equal to the pressure. (Or, more strictly, the pressure difference between the inside and outside of the box.)

2.1.3 Archimedes' Principle

Before exploring the full content of the Euler equation, we can extract some familiar and long-known results. To kick off, suppose that the fluid sits in a gravitational field. (Which, let's face it, most do.) This means that we have an external force density

$$\mathbf{f} = \rho\mathbf{g}$$

where $\mathbf{g} = -g\hat{\mathbf{z}}$ is the gravitational acceleration and points downwards.

We can now look for the trivial solution to the Euler equation (2.6) in which the fluid is at rest, so $\mathbf{u} = 0$. We see that the fluid must have a pressure gradient to counteract the gravitational field

$$\nabla P = \rho\mathbf{g} \quad \Rightarrow \quad P = P_0 - \rho g z \tag{2.8}$$

This is known as *hydrostatic pressure*. It is the pressure that pushes against the weight of the fluid above. (If you're worried about the minus sign and the possibility of the pressure becoming negative, think of the surface of the fluid as sitting at $z = 0$, so that pressure only increases as we move down to $z < 0$.)

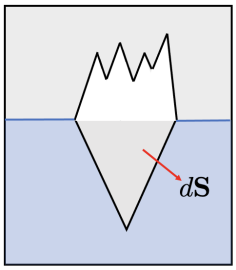
Suppose that we have some object partially immersed in a fluid as shown in the figure. We'll set $P = P_0$ at $z = 0$ to be atmospheric pressure. Then we can ask: what is the force that the fluid exerts on the body? This is simply

$$\mathbf{F} = -\int_S P(z)\, d\mathbf{S}$$

where the minus sign is because $d\mathbf{S}$ is taken to have outward-pointing normal as shown in the figure, and the integral should be taken over the surface of the object that is immersed in the fluid. We can use the divergence theorem, together with our expression for the hydrostatic pressure (2.8) to write this as

$$\mathbf{F} = -\int_V \nabla P\, dV = -\int_V \rho\mathbf{g}\, dV$$

where the integral is now over the volume of displaced fluid. This is telling us that the force exerted by the fluid on the object is equal to the weight of the displaced fluid. Eureka! This, of course, is Archimedes principle. In equilibrium, the force $\mathbf{F}$ must balance the weight of the object itself. This can be achieved if the object is less dense than water, in which case it floats. Otherwise it sinks. This discussion hasn't brought anything new to Archimedes idea. It's really just the old argument wrapped in the language of vector calculus.



The results above also give us a reason to ignore gravity for much of this course. In the presence of a gravitational field, the pressure simply adapts as in (2.8) to cancel it. Therefore, in the presence of gravity, we can think of the pressure as

$$P = P_0 - \rho g z + P'$$

and Euler's equation becomes

$$\rho \frac{D\rho}{Dt} = -\nabla P'$$

and we proceed from there.

2.1.4 Energy Conservation and Bernoulli's Principle

In classical mechanics, it's often useful to identify conserved quantities. The same is true in fluid mechanics and there is a way to rewrite the Euler equation that highlights one such conserved quantity. We start with the vector identity

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{u}$$

We use this to substitute for the non-linear $(\mathbf{u}\cdot\nabla)\mathbf{u}$ term in the Euler equation to get

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u}\times(\nabla\times\mathbf{u})\right) = -\nabla P + \mathbf{f} \tag{2.9}$$

So far this doesn't look any more useful. But now we dot with $\mathbf{u}$ to make the curly term disappear. We have

$$\rho\mathbf{u}\cdot\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\left(\frac{1}{2}\rho|\mathbf{u}|^2 + P\right) = \mathbf{u}\cdot\mathbf{f}$$

At this stage, we make one further assumption: we take the force to be conservative, meaning that we can write it in terms of a potential energy $\Phi(\mathbf{x}, t)$,

$$\mathbf{f} = -\nabla\Phi \tag{2.10}$$

For example, the gravitational force can be written in this way. We then have

$$\frac{1}{2}\rho\frac{\partial|\mathbf{u}|^2}{\partial t} + \mathbf{u}\cdot\nabla\left(\frac{1}{2}\rho|\mathbf{u}|^2 + P + \Phi\right) = 0$$

This is again of the form of a conservation equation. To see this, we again pull the $\mathbf{u}$ inside the ∇ using the fact that the fluid is incompressible so $\nabla\cdot\mathbf{u} = 0$. (This is the same step that we did for the momentum conservation equation in (2.7).) We get the final form

$$\frac{1}{2}\rho\frac{\partial|\mathbf{u}|^2}{\partial t} + \nabla\cdot(\mathbf{u}H) = 0 \tag{2.11}$$

where

$$H = \frac{1}{2}\rho|\mathbf{u}|^2 + P + \Phi \tag{2.12}$$

There's no mystery in what is being conserved here: the time derivative is acting on $\frac{1}{2}\rho|\mathbf{u}|^2$ which we recognise as the kinetic energy density of the fluid. The equation (2.11) is simply capturing energy conservation of the continuous fluid, with $\mathbf{u}H$ the energy flux.

For a steady fluid, satisfying $\partial\mathbf{u}/\partial t = 0$, we have

$$\mathbf{u}\cdot\nabla H = 0 \tag{2.13}$$

This is *Bernoulli's Theorem*. It states that the quantity H is constant along streamlines. Roughly speaking, the fluid flows quickly in places where the pressure is low, and more slowly when the pressure builds.

An Example: Drinking from a Firehose

Consider water flowing down a pipe which, at some point, narrows as shown in Figure 2. This might, for example, be the nozzle on a firehose. We'll take the narrowing to be gradual so that the streamlines are smooth and follow the pipe.

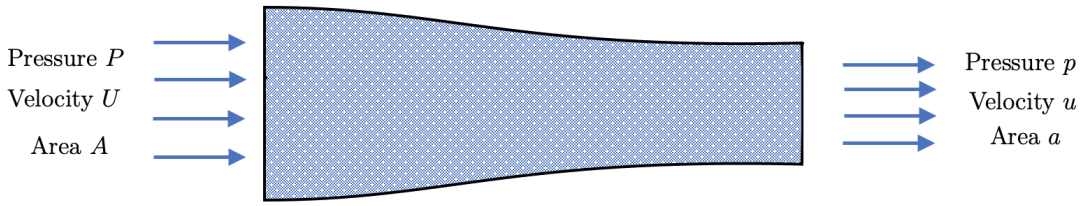


Figure 2. As a pipe narrows, the velocity must increase and Bernoulli's theorem tells us that, for steady flows, the pressure also increases.

Initially, the pipe has area A and the fluid has speed U. By the end the area has reduced to $a < A$ and the speed to u. For incompressible fluids, the speed is dictated by the conservation of mass which tells us that the volume of fluid passing through any given slice of the pipe must remain the same, so

$$UA = ua$$

This immediately tells us that the speed of the flow in the narrow section is faster than in the initial section: $u = UA/a$. Meanwhile, Bernoulli's theorem tells us that

$$\frac{\rho}{2}U^2 + P = \frac{\rho}{2}u^2 + p$$

where P and p are the initial and final pressure respectively and we are ignoring any external forces. Rearranging, we have

$$p = P + \frac{\rho}{2}U^2\left(1 - \frac{A^2}{a^2}\right)$$

We see that because $A > a$, the pressure actually decreases as the pipe narrows. This makes sense: the decrease in pressure in the narrow section means that there is a pressure difference and this is precisely what causes the fluid to accelerate from speed U to speed u.

More Qualitative Applications

There are other situations where Bernoulli's principle gives us some useful intuition. For example, it's possible to levitate a ping pong ball on a fast jet of air. You can achieve this by blowing through a straw or by using a hairdryer. The question is: why is the ball stable? Why doesn't it fall off to one side? In this situation, the airflow is turbulent and it's not entirely clear that Bernoulli's principle, which requires a steady flow, can be invoked. Nonetheless, it does provide an answer. Suppose that the ball

did move slightly off to one side and out of the main flow. Then the air will be moving faster in the middle of the flow, resulting in a lower pressure and the ball gets pushed back into the middle.

The most famous application of Bernoulli's principle is to explain the lift experienced by an aerofoil. The air travels faster over the top of the wing than the bottom and the pressure difference results in a net upwards force. But this begs the question: why does the air travel faster over the top of the wing? One popular explanation (and one that I was told in school) is that the flow must reach the trailing edge of the wing at the same time, regardless of whether it goes up or down. But that doesn't sound right! There's no principle in physics that says you must reach your goal at the same time regardless of the path you take. (If there were, we wouldn't need maps.) We will revisit this later in the course when we study flows around objects in some detail.

2.2 Vorticity

To characterise the shape of a velocity field $\mathbf{u}$, we look at its derivatives. In general there are nine such derivatives, $\partial_i u_j$, with $i, j = 1, 2, 3$. But, for incompressible flows, we know that one linear combination vanishes: $\nabla \cdot \mathbf{u} = 0$. The remaining derivatives can be decomposed as a symmetric and anti-symmetric tensor. The symmetric one is known as the *rate of strain tensor*,

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) \tag{2.14}$$

The anti-symmetric tensor is

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i}\right)$$

It contains the same information as vector field, $\omega_i = -\epsilon_{ijk}\Omega_{jk}$, which is more familiarly written as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

This is the *vorticity*. It tells us how the fluid swirls at each point in space. The integral curves associated to $\boldsymbol{\omega}$ (i.e. the lines that are tangent to $\boldsymbol{\omega}$ at each point $\mathbf{x}$) are called *vortex lines*. Because $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, the vortex lines are perpendicular to streamlines.

Examples of Flows

To get a feel for what the vorticity $\boldsymbol{\omega}$ and rate of strain E are telling us, we can look at a couple of examples.

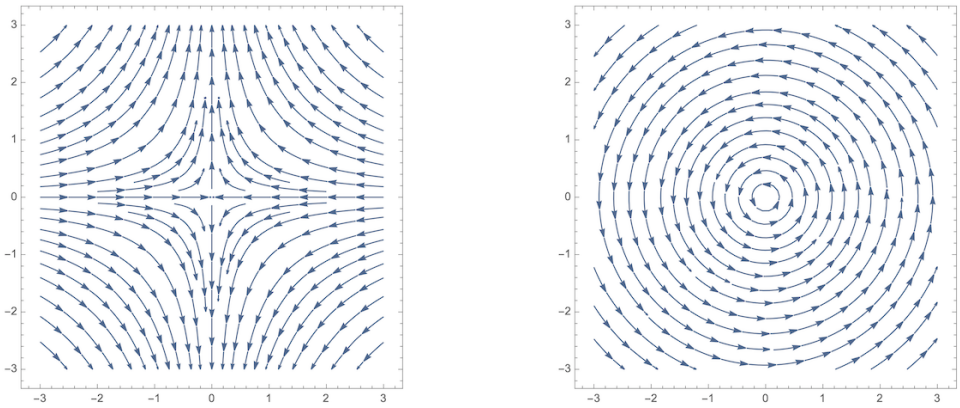


Figure 3. On the left, a flow with strain and no vorticity. On the right, a flow with vorticity and no strain.

First consider the 2d flow

$$\mathbf{u} = \alpha(-x, y, 0)$$

with α a constant. This is plotted on the left of Figure 3. The velocity field has $\nabla\cdot\mathbf{u} = 0$ and also $\boldsymbol{\omega} = 0$, while the rate of strain tensor is

$$E = \alpha \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the figure, you can see that the fluid is squeezed in one direction (the x-direction in this case) and stretched in the other (the y-direction). This is the characteristic feature of flows with a rate of strain. To see this, note that the rate of strain tensor is symmetric and so can always be diagonalised so that it takes the form

$$E = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}$$

But, for incompressible fluids with $\nabla \cdot \mathbf{u} = 0$, we must have $E_1 + E_2 + E_3 = 0$. So one eigenvalue is necessarily positive and another necessarily negative. These are the directions in which the flow is, respectively, stretched and squeezed.

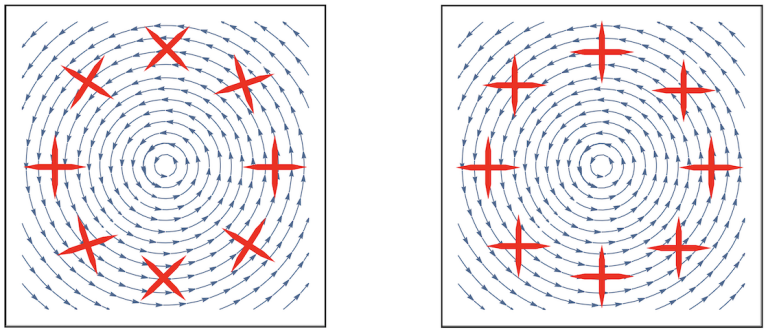


Figure 4. On the left, a flow with vorticity. On the right, a flow that rotates around the origin but with vanishing vorticity.

Next consider the flow

$$\mathbf{u} = \alpha(-y, x, 0) \tag{2.15}$$

This has $\nabla \cdot \mathbf{u} = E = 0$ and a constant vorticity everywhere in the fluid, $\boldsymbol{\omega} = (0, 0, 2\alpha)$. It is depicted on the right of Figure 3. Unsurprisingly, it exhibits a rotation.

However, one should be wary of simply eyeballing a flow to decide on vorticity. To illustrate this, consider the example

$$\mathbf{u} = f(r)(-y, x, 0)$$

where $f(r)$ is any function of $r^2 = x^2 + y^2$. (Note that we're keeping the flow essentially two dimensional.) This is a generalisation of our previous flow (2.15) and the streamlines look identical for any choice $f(r)$. The vorticity is $\boldsymbol{\omega} = (0, 0, \omega(r))$, with

$$\omega = \frac{1}{r}\frac{d}{dr}(r^2 f) \tag{2.16}$$

Now the vorticity $\omega(r)$ varies in the radial direction. This means that if we take the specific choice of $f = 1/r^2$, then the vorticity vanishes, $\boldsymbol{\omega} = 0$, even though the flow is clearly rotating around the origin. This is because a non-zero vorticity $\boldsymbol{\omega}(\mathbf{x}) \neq 0$ at some point $\mathbf{x}$ means that the fluid is rotating locally around $\mathbf{x}$, not just around the origin.

To build a more physical understanding for what vorticity means, suppose that we drop some propellers in the fluid, like those plastic windspinners that you can buy at the seaside. If you drop them in the fluid, they will move around the origin with the flow. But if the fluid has a vorticity then their orientation will also rotate as the move,

as shown on the left-hand side of Figure 4. If the fluid has no vorticity, as is the case for $f = 1/r^2$, then they will remain in the same orientation as they move around, as shown in the right-hand figure.

In fact, things are a little more subtle than this. The specific choice $\mathbf{u} = (-y/r^2, x/r^2, 0)$ has the property that the integral of the velocity field around any circle C that surrounds the origin always gives

$$\oint_C \mathbf{u} \cdot d\mathbf{x} = 2\pi$$

This is because the velocity field drops off as $1/r$, while the perimeter of the circle grows as r. But, by Stokes' theorem, we have

$$\oint_C \mathbf{u} \cdot d\mathbf{x} = \int_S \boldsymbol{\omega} \cdot d\mathbf{S} = 2\pi$$

where S is a surface with boundary $\partial S = C$. So it can't quite be true that the vorticity $\boldsymbol{\omega}$ vanishes everywhere! Indeed, the flow is singular at the origin $x = y = 0$ (which, in three dimensions, means that it is singular along the entire z-axis.) For the above calculation to be consistent, the vorticity must be non-zero along this axis, with

$$\boldsymbol{\omega} = 2\pi\delta^2(r)\hat{\mathbf{z}}$$

This is sufficient for the flow to have rotation around the origin, even though it doesn't have vorticity at any other point. This slightly subtle example will arise in some later applications. In fact, it's not a bad approximation for what happens when you empty the bath, with the (admittedly finite size) plughole taking the place of $r = 0$.

The Biot-Savart Law

We can invert the equation $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ to get an expression for the velocity in terms of the vorticity. In fact, this is a calculation that we've done elsewhere and it's worth taking the opportunity to remind ourselves of this.

In Electromagnetism, the magnetic field obeys $\nabla \cdot \mathbf{B} = 0$ which means that it can be written in terms of a vector potential $\mathbf{B} = \nabla \times \mathbf{A}$. In the case of magnetostatics, the magnetic field is given by Ampére's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \Rightarrow \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

with $\mathbf{J}$ the current density. This is just the Poisson equation for each component of $\mathbf{A}$ and can be solved using the Green's function,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

If we subsequently take the curl of this equation, then we get an expression for the magnetic field $\mathbf{B}$ in terms of the current density

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_V d^3x' \, \frac{\mathbf{J}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \tag{2.17}$$

This is the *Biot-Savart law*.

But we can now repeat each of these steps for the fluid velocity. If the fluid is incompressible, so $\nabla\cdot\mathbf{u}=0$, then we can introduce a vector potential $\mathbf{A}$ such that $\mathbf{u}=\nabla\times\mathbf{A}$. This way of writing the velocity is at the heart of the idea of a stream function, as we saw in Section 1.1.4. The curl of the velocity is the vorticity, so we have

$$\nabla\times\mathbf{u}=\boldsymbol{\omega} \quad\Rightarrow\quad \nabla^2\mathbf{A}=-\boldsymbol{\omega}$$

Following the same steps that we took above, the vector potential can then be expressed as

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{4\pi}\int_V d^3x' \, \frac{\boldsymbol{\omega}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}$$

Again taking the curl gives the fluid analog of the Biot-Savart law

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{4\pi}\int_V d^3x' \, \frac{\boldsymbol{\omega}(\mathbf{x}',t)\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

In fact, there's an additional subtlety that's important for fluids. While the expression above is true if the vorticity field $\boldsymbol{\omega}(\mathbf{x},t)$ is defined everywhere in $\mathbb{R}^3$, often that's not the case for fluids. We may have boundaries, or obstacles in the fluid, that require us to impose certain boundary conditions. The most general form of the velocity is then

$$\mathbf{u}(\mathbf{x},t) = \nabla\phi(\mathbf{x},t) + \frac{1}{4\pi}\int_V d^3x' \, \frac{\boldsymbol{\omega}(\mathbf{x}',t)\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \tag{2.18}$$

where the $\mathbf{u}\sim\nabla\phi$ piece doesn't contribute to the vorticity because $\nabla\times\nabla\phi=0$. We can only reconstruct the velocity field from the vorticity up to this subtlety. In particular, there are situations – such as those we will meet in Sections 2.3 and 2.4 – where all the physics is sitting in the $\mathbf{u}\sim\nabla\phi$ term.

While the mathematics leading to the electromagnetic and fluidic versions of the Biot-Savart law is identical, there are some differences. The first is conceptual. In electromagnetism, one thinks of the current $\mathbf{J}$ as something fixed and external, which determines the magnetic field $\mathbf{B}$. In contrast, in fluid mechanics the vorticity $\boldsymbol{\omega}$ is thought of as an object derived from the velocity field $\mathbf{u}$. Nonetheless, there will be times in these lectures when it's useful to think of vorticity as an object in its own right.

The second difference is more technical. The electromagnetic Biot-Savart law (2.17) holds only for static currents. There is a generalisation to time-dependent currents, but it requires us to take into account the time that it takes light to travel from the current to the place where the magnetic field is measured. (See Section 6 of the lectures on Electromagnetism.) In contrast, as shown, the fluid version (2.18) holds for time dependent flows, with the velocity and vorticity fields evaluated at the same time.

2.2.1 The Vorticity Equation

It is interesting to ask how the vorticity $\boldsymbol{\omega}$ evolves. We return to the equation (2.9) that we previously used on the way to deriving Bernoulli's formula, again restricted to a conservative force $\mathbf{f} = -\nabla\Phi$,

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\rho\nabla|\mathbf{u}|^2 = \rho\mathbf{u}\times\boldsymbol{\omega} - \nabla P - \nabla\Phi \tag{2.19}$$

If we take the curl of this, and use the fact that $\nabla\times(\nabla\,\text{anything}) = 0$, we have

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega})$$

We now use the vector identity

$$\nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = (\nabla\cdot\boldsymbol{\omega})\mathbf{u} + (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\boldsymbol{\omega} - (\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$$

We have $\nabla\cdot\boldsymbol{\omega} = 0$ because the vorticity $\boldsymbol{\omega}$ is itself a curl. And $\nabla\cdot\mathbf{u} = 0$ because we're dealing with an incompressible fluid. Rearranging the remaining terms, we have

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} \tag{2.20}$$

This is the *vorticity equation*. It tells us how the vortex lines stretch and twist as the fluid evolves.

Using $\nabla\cdot\mathbf{u} = \nabla\cdot\boldsymbol{\omega} = 0$, the vorticity equation can be rewritten as

$$\frac{\partial\omega^i}{\partial t} + \frac{\partial}{\partial x^j}\left(u^j\omega^i - u^i\omega^j\right) = 0$$

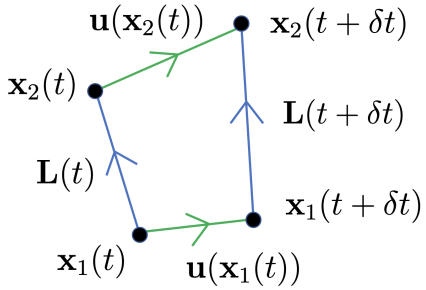
This is the standard form of a continuity equation, telling us that vorticity is conserved.

To try to get a feel for what the vorticity equation (2.20) is telling us, first suppose that the right-hand side vanished. Then the vorticity would simply drift with the fluid. We can get a sense for what the right-hand side means by considering two nearby points $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ at some time t, separated by a small distance

$$\mathbf{L}(t) = \mathbf{x}_2(t) - \mathbf{x}_1(t)$$

We'll think about how this material line segment evolves with the flow. At a later time $t+\delta t$, each of these end points has been swept along and now sit at

$$\mathbf{x}_i(t+\delta t) \approx \mathbf{x}_i(t) + \delta\mathbf{x}_i \approx \mathbf{x}_i(t) + \mathbf{u}(\mathbf{x}_i(t))\delta t$$



So the line segment $\mathbf{L}$ has evolved as

$$\begin{aligned}\mathbf{L}(t+\delta t) &\approx \mathbf{x}_2(t+\delta t) - \mathbf{x}_1(t+\delta t)\\ &\approx \mathbf{L}(t) + \big(\mathbf{u}(\mathbf{x}_2(t)) - \mathbf{u}(\mathbf{x}_1(t))\big)\delta t\end{aligned}$$

We now Taylor expand $\mathbf{u}(\mathbf{x}_2) = \mathbf{u}(\mathbf{x}_1 + \mathbf{L}) \approx \mathbf{u}(\mathbf{x}_1) + \mathbf{L}\cdot\nabla\mathbf{u}(\mathbf{x}_1)$ to write this as

$$\mathbf{L}(t+\delta t) \approx \mathbf{L}(t) + (\mathbf{L}\cdot\nabla)\mathbf{u}(\mathbf{x}(t))\,\delta t$$

where we have evaluated the gradient of the velocity field at $\mathbf{x}$, which could be either $\mathbf{x}_1$ or $\mathbf{x}_2$ or anywhere in between: it doesn't matter as they are close. In the limit $\delta t \to 0$, all the $\approx$ signs become $=$ signs. We see that a small line segment of the fluid evolves as

$$\frac{d\mathbf{L}}{dt} = (\mathbf{L}\cdot\nabla)\mathbf{u}$$

But the right-hand-side is the same form as we find in the vorticity equation (2.20). This is telling us that the lines of vorticity are stretched and twisted like the material lines of the fluid itself. We usually say that the vortex lines "move with the fluid".

We can get a more direct expression for the change in the magnitude of the vorticity. First take the dot product of (2.20) with $\boldsymbol{\omega}$. This tells us how the magnitude (squared) of the vorticity $|\boldsymbol{\omega}|^2$ changes,

$$\frac{1}{2}\frac{D|\boldsymbol{\omega}|^2}{Dt} = \boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla)\mathbf{u} = \omega_i\omega_j\frac{\partial u_i}{\partial x^j}$$

where, in the second term, we've resorted to index notation to clarify what is inner-producted with what. Note, however, that $\omega_i\omega_j$ is symmetric in i and j so this picks out the strain of the flow defined in (2.14). We have

$$\frac{1}{2}\frac{D|\boldsymbol{\omega}|^2}{Dt} = \boldsymbol{\omega}\cdot E\boldsymbol{\omega} \tag{2.21}$$

We learn that vorticity is increased or decreased by the rate of strain in the flow.

Note that if, at some time, the vorticity vanishes everywhere, say $\boldsymbol{\omega}(\mathbf{x}, t = 0) = 0$, then it will vanish everywhere at all subsequent times. This holds regardless of any conservative forces that might be at play. This prompts the question: where does vorticity come from in the first place? The answer is that it comes from non-conservative forces. These includes friction forces, as captured through the viscosity of the fluid, and the Coriolis force. We will devote Section 3 to understanding the effects of viscosity and see in a number of explicit examples how it gives rise to vorticity.

An Example

To illustrate how vortex lines stretch and twist, consider the flow

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_{\text{strain}}(\mathbf{x}) + \mathbf{u}_{\text{rot}}(\mathbf{x}, t) \quad \text{with} \quad \begin{cases} \mathbf{u}_{\text{strain}} = \alpha(-x, -y, 2z) \\ \mathbf{u}_{\text{rot}} = f(r, t)(-y, x, 0) \end{cases}$$

Both of these flows are similar to the examples given above. The strain flow stretches the fluid in the z direction, while squeezing in the (x, y)-plane; the rotational flow clearly rotates in the (x, y)-plane, with an angular velocity determined by the function $f(r, t)$ where $r^2 = x^2 + y^2$.

The vorticity lies on z-direction, with $\boldsymbol{\omega} = (0, 0, \omega)$ and ω given by (2.16),

$$\omega = \frac{1}{r}\frac{d}{dr}(r^2 f)$$

The vorticity equation (2.20) is then a partial differential equation for $\omega(r, t)$,

$$\frac{\partial \omega}{\partial t} - \alpha r \frac{\partial \omega}{\partial r} = 2\alpha\omega$$

This is solved by

$$\omega(r, t) = e^{2\alpha t}\, W(re^{\alpha t}) \tag{2.22}$$

for an arbitrary function $W(r)$, which is the initial vorticity at time $t = 0$. We see that the strain indeed increases the vorticity, with an exponential growth in time. But the time dependence in the function $W(re^{\alpha t})$ gives a corresponding squeezing of the vorticity in the (x, y) plane. This effect is known as *vortex stretching*.

In this example, the vorticity is aligned with one of the principal axes of the rate of strain tensor. When this isn't the case, the vortex lines get twisted by the strain.

Bernoulli's Theorem Revisited

There is a version of Bernoulli's theorem for the vortex lines, tangent to $\boldsymbol{\omega}$. To see this, we take the inner product of (2.19) with $\boldsymbol{\omega}$ to find that, in a steady flow with $\partial \mathbf{u}/\partial t = 0$, we have

$$\boldsymbol{\omega} \cdot \nabla H = 0$$

We learn that the Bernoulli function H, defined in (2.12), is constant both along streamlines (as in (2.13)) and along vortex lines.

If the vorticity vanishes everywhere, then the fluid is said to be *irrotational*. In this case, we can say more. For a steady, irrotational flow, the equation (2.9) tells us that Bernoulli's function

$$H = \frac{1}{2}\rho \mathbf{u}^2 + P + \Phi$$

is actually constant everywhere in the fluid, not just along streamlines and vortex lines. We will explore these flows further in Section 2.3.

2.2.2 Kelvin's Circulation Theorem

The *circulation* of a flow around a closed curve C is defined by

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{x}$$

Now consider a *material* curve $C(t)$, meaning that it follows the flow of the underlying fluid elements. We want to understand how the associated circulation $\Gamma(t)$ changes. We have

$$\frac{D\Gamma}{Dt} = \oint_{C(t)} \left(\frac{D\mathbf{u}}{Dt} \cdot d\mathbf{x} + \mathbf{u} \cdot \frac{D(d\mathbf{x})}{Dt} \right) \tag{2.23}$$

We can replace $D\mathbf{u}/Dt$ in the first term using the Euler equation (2.6). Assuming a conservative force $\mathbf{f} = -\nabla\Phi$, this gives

$$\oint_{C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{x} = \frac{1}{\rho} \oint_{C(t)} (-\nabla P - \nabla\Phi) \cdot d\mathbf{x} = 0$$

which vanishes because it is the integral of a gradient around a closed path. That leaves us with the second term in (2.23). The notation $D(d\mathbf{x})/Dt$ is a little formal because the material derivative D/Dt was defined to act on fields, while here it's acting on a line element. But the meaning is straightforward: it captures the way that the line element $d\mathbf{x}$ changes under the flow.

To see what this means in practice, we can return to the fundamentals. Consider a small, moving line element $\delta\mathbf{x}(t)$, with end points $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, so $\delta\mathbf{x} \approx \mathbf{x}_2 - \mathbf{x}_1$. We want to know how this line segment evolves. But this is the calculation that we just saw when building intuition for the meaning of the vorticity equation: there we called the material line segment $\mathbf{L}(t)$, but it is the same thing as $\delta\mathbf{x}$ in the present context. This tells us how the line element changes and gives meaning to the expression $D(d\mathbf{x})/Dt$: it is

$$\frac{D(d\mathbf{x})}{Dt} = (d\mathbf{x}\cdot\nabla)\mathbf{u}$$

Using this in (2.23), we have

$$\frac{D\Gamma}{Dt} = \oint_{C(t)} \mathbf{u}\cdot(d\mathbf{x}\cdot\nabla)\mathbf{u} = \oint_{C(t)} u_i \frac{\partial u_i}{\partial x^j}\, dx^j$$

where we've again resorted to index notation to clarify which objects are dotted together. This can be written as

$$\frac{D\Gamma}{Dt} = \frac{1}{2}\oint_{C(t)} \nabla(\mathbf{u}\cdot\mathbf{u})\cdot d\mathbf{x} = 0$$

which again vanishes because it is the integral of a gradient around a closed path. The upshot is that the circulation around any closed loop $C(t)$ does not change when we follow this loop with the flow,

$$\frac{D\Gamma}{Dt} = 0$$

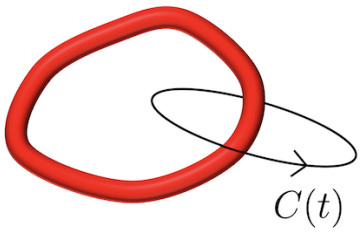
This is *Kelvin's Circulation Theorem*.

To see the consequences of this result, first note that the circulation is related to the vorticity by Stokes theorem

$$\Gamma = \int_S \boldsymbol{\omega}\cdot d\mathbf{S} \tag{2.24}$$

where S is any surface with boundary $\partial S = C$. (It's worth remembering at this point that Stokes learned about Stokes' theorem from his friend William Thomson, later known as Lord Kelvin!) So the circulation theorem again tells us that a fluid that starts off as irrotational, with $\boldsymbol{\omega} = 0$, will remain irrotational.

More intuition comes if we focus on flows in which vorticity is localised. To this end, suppose that $\boldsymbol{\omega}$ is non-vanishing only in some region of the fluid. Find a surface S such that the circulation defined in (2.24) is non-vanishing. As we vary the surface S, Γ can't change. This means that the vorticity can't be localised in a co-dimension three region of space: it must be extended along a tube-like region. This tube might extend to infinity, which is the case in the example of vorticity that we saw earlier in this section. Or it might form a vortex loop, as shown in the figure to the right. In either case, it can't just end.



We learned previously that the magnitude of the vorticity can change due to the strain in the fluid (2.21). Now we see that, in a certain sense, vorticity must be conserved. There's no contradiction here. As the magnitude of the vorticity increases, the area of the flux tube must decrease so that the vortex flux (2.24) remains unchanged. Indeed, we saw precisely this effect at play in the vortex (2.22). At heart, this is just the conservation of angular momentum: it is the fluid version of an ice skater who spins faster when they pull in their arms.

An Historical Aside

I think it's fair to say that Kelvin got a little carried away with his results on vortices. He was so taken with the stability of vortices, and smoke rings in particular, that he proposed that they may form the basis of all matter, with different atoms arising as different knots of vortices. Some pictures from one of Kelvin's original papers are shown in Figure 5.

With hindsight, Kelvin's idea looks overly optimistic. Nonetheless, modern ideas in physics suggest that they may contain a grain of truth. In quantum field theories, certain particles arise as so-called "solitons" in which the fields wrap themselves in some stable configuration, not unlike vortices in fluids. From a certain perspective, the proton and neutron can be viewed as solitons of an underlying pion field, known as a *Skyrmion*. (Admittedly, the more familiar story of the proton and neutron as made from three quarks is a more fundamental perspective.) Magnetic monopoles, if they exist, would be examples of solitons.

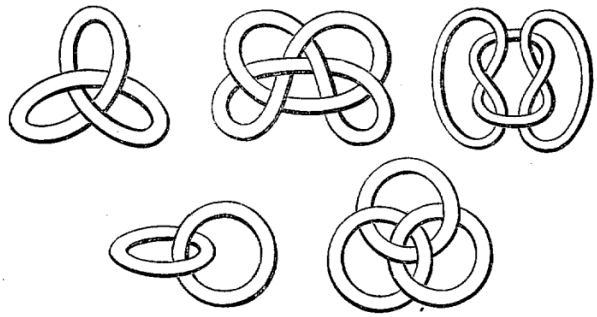


Figure 5. Taken from the 1867 paper "On Vortex Motion" by Sir William Thomson, better known by his later name Lord Kelvin.

2.3 Potential Flows in 3d

In this section we restrict ourselves to flows that are steady, so $\partial\mathbf{u}/\partial t = 0$, incompressible and irrotational. These latter two properties mean that

$$\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \nabla \times \mathbf{u} = 0$$

This suggest two different vector calculus routes to attack the problem. We could use the first condition to write $\mathbf{u} = \nabla \times \mathbf{A}$. This was our previous stream function approach. However, it turns out to be more useful to use the irrotational property. If the domain of the flow is simply connected, then a vector field that obeys $\nabla \times \mathbf{u} = 0$ can be written in terms of a potential ϕ such that

$$\mathbf{u} = \nabla\phi$$

The requirement that the flow is incompressible, $\nabla \cdot \mathbf{u} = 0$, then tells us that

$$\nabla^2\phi = 0$$

This is a very familiar: it is just the Laplace equation. A flow that is steady, incompressible and irrotational is called, for obvious reasons, a *potential flow*. Importantly, the Laplace equation is linear. That means that if we have two solutions then we can simply superpose them to get a third. The non-linearity of the Euler equation disappeared by virtue of the irrotational assumption.

To understand potential flows, all we have to do is solve the Laplace equation. The devil in the details is, as we shall see, is largely in the boundary conditions imposed on the flow.

2.3.1 Boundary Conditions

In many courses in theoretical physics, boundary conditions are relatively unimportant beyond the usual requirement that things fall off asymptotically. (There are, of course, counterexamples such as the study of electromagnetic waves in materials.) For fluids, however, many of the most important results come from imposing the right boundary conditions.

We'll meet various kinds of boundary conditions in this course. For example, later when we come to discuss waves we'll think about dynamical interfaces between two fluids. But, for now, we will restrict to the simplest kind: a solid boundary.

Suppose that the fluid comes into contact with a solid object. Maybe there's a wall at the edge of the container. Or maybe there's some object, like the wing of an aircraft, sitting in the fluid flow. What boundary condition should we impose?

Our first condition is completely obvious. The fluid can't flow into the solid. To describe this mathematically, we introduce a normal vector $\mathbf{n}(\mathbf{x})$ at each point $\mathbf{x}$ on the boundary. If the boundary is flat, then $\mathbf{n}$ is constant. If the boundary curves in some way, then $\mathbf{n}$ changes accordingly. Provided that the boundary itself does not move, we must have

$$\mathbf{n} \cdot \mathbf{u} = 0$$

at each point of the boundary. This is the statement that nothing seeps into the solid. It is also the statement that the boundary of a fluid is a streamline.

We will also be interesting in situations in which the boundary does move, with some velocity $\mathbf{U}$. In this case, we place ourselves in the frame of the moving boundary, where the fluid velocity is $\mathbf{u}' = \mathbf{u} - \mathbf{U}$ and the boundary condition is $\mathbf{n} \cdot \mathbf{u}' = 0$. Back in the original frame, we have

$$\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \mathbf{U} \tag{2.25}$$

This simple statement that the solid is impermeable is sometimes called the *kinematic boundary condition*. It fixes the component of the fluid velocity perpendicular to the boundary.

We haven't yet said anything about the component of the velocity that is tangential to the boundary. For example, we might think that a "no-slip" boundary condition should be imposed, which says that the layer of fluid right next to the boundary is stationary. Indeed, this will be important in certain fluid flows (actually, very important!) but

these kinds of boundary conditions arise only when take the viscosity of the fluid into account. For that reason we postpone their discussion to Section 3.

2.3.2 Flow Around a Sphere

Perhaps the most familiar solution to the Laplace equation (and certainly the one most useful for Electromagnetism), is the spherically symmetric potential

$$\phi(r) = -\frac{q}{r} \tag{2.26}$$

for some constant q. This corresponds to a radial, three-dimensional flow

$$\mathbf{u} = \frac{q}{r^2}\hat{\mathbf{r}}$$

Strictly speaking, this doesn't satisfy the Laplace equation everywhere. Instead, it is the Green's function, obeying

$$\nabla^2\phi = 4\pi q\delta^3(\mathbf{x})$$

The delta-function should be thought of as a source (for $q > 0$) or a sink (for $q < 0$) for the fluid.

This radially symmetric solution is simple, but of little immediate utility in the context of fluid dynamics because it's hard to think of a situation in which a fluid spews out radially in 3d from some source. Instead we turn to (slightly) more complicated solutions. Our strategy is going to be a little bit cheap: rather than trying to solve a particular problem, we'll instead write down some simple potentials and then try to interpret the results in terms of some fluid flow that might be of interest. We then declare success at having solved something important!

To make progress, we work with spherical polar coordinates

$$x = r\sin\theta\,\cos\varphi \quad , \quad y = r\sin\theta\,\sin\varphi \quad , \quad z = r\cos\theta$$

In these coordinates, the Laplacian takes the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

We'll look for solutions that are independent of the coordinate φ. The most general such solution can be written in terms of Legendre polynomials $P_n(\cos\theta)$,

$$\phi(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

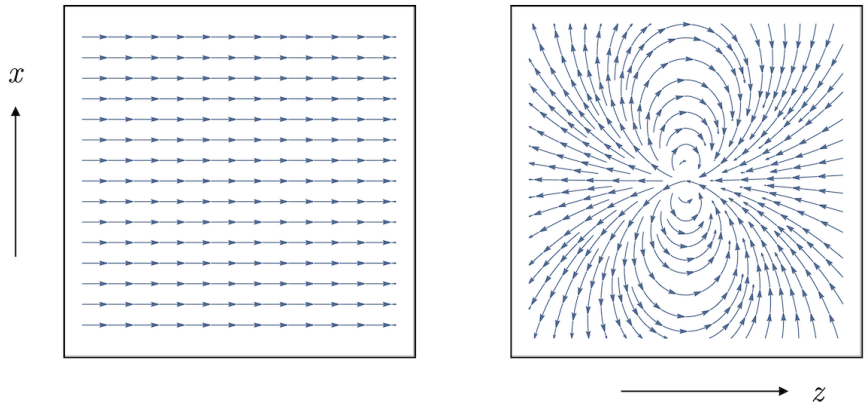


Figure 6. On the left, the constant flow with $A \neq 0$ and $B = 0$. On the right, the dipole flow with $A = 0$ and $B \neq 0$.

The radial solution that we saw above corresponds to the $n = 0$ term (with $P_0(\cos\theta) = 1$). The next simplest is the $n = 1$ term. Recalling that $P_1(\cos\theta) = \cos\theta$, this solution depends on two constants A and B,

$$\phi(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta \tag{2.27}$$

Both of these terms have a natural interpretation in terms of fluid flow. The first term can be rewritten as $\phi = Az$, which tells us that it's simply a straight, constant flow in the z-direction. This is shown in the left-hand side of Figure 6. The flow runs left to right in the figure, which means that I've made the slightly disorienting choice of the taking the z-axis to lie horizontally. At large distances, this term dominates so we identify

$$A = U$$

as the asymptotic velocity.

The second term can be viewed, in the language of electromagnetism, as a dipole. To see this, consider a source and sink of the form (2.26) displaced slightly in some direction $\mathbf{d}$. The potential is

$$\phi = \frac{q}{r} - \frac{q}{|\mathbf{r}+\mathbf{d}|} \tag{2.28}$$

We then look at this at distances $r \gg |\mathbf{d}|$. We Taylor expand the second term as

$$\frac{1}{|\mathbf{r}+\mathbf{d}|} \approx \frac{1}{r} + \mathbf{d}\cdot\nabla\frac{1}{r} + \ldots = \frac{1}{r} - \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} + \ldots$$

The potential (2.28) then becomes

$$\phi \approx q\frac{\mathbf{d}\cdot\mathbf{r}}{r^3} + \dots$$

If we take the displacement to be aligned with the z-direction, so $\mathbf{d} = d\hat{\mathbf{z}}$ and $\mathbf{d}\cdot\mathbf{r} = dr\cos\theta$, and subsequently take the limit $|\mathbf{d}| \to 0$ keeping the product qd fixed, then we get the second term in (2.27) with $B = qd$. The velocity field can be computed in spherical polar coordinates,

$$\mathbf{u} = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}}$$

The resulting fluid flow is shown on the right in Figure 6.

Because the Laplace equation is linear, we can simply add these two flows together for any choice of $A = U$ and B. The result is shown on the left-hand side of Figure 7. So far it's not immediately obvious that we've constructed something useful. However, if we look at the velocity, we find something interesting. The radial and angular velocity are given by

$$u_r = \frac{\partial\phi}{\partial r} = \left(U - \frac{2B}{r^3}\right)\cos\theta \quad \text{and} \quad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\left(U + \frac{B}{r^3}\right)\sin\theta \tag{2.29}$$

Crucially, the radial velocity vanishes at a radius R where

$$R^3 = \frac{2B}{U}$$

This means that the flow has the appropriate boundary conditions to hold if there is a solid sphere of radius R at the at the origin. Nothing is flowing into the sphere! We then just ignore the previous flow inside the sphere at $r < R$ completely. It is only what sits outside that matters. This is shown in the right-hand side of Figure 7. The point $\theta = 0$ sits on the right of the sphere, and the point $\theta = \pi$ sits on the left, where the fluid comes from.

The upshot is that the potential

$$\phi = U\left(r + \frac{R^3}{2r^2}\right)\cos\theta \tag{2.30}$$

describes a flow of asymptotic velocity U past a solid sphere of radius R. Standard uniqueness theorems then tell us that it is *the* flow with these properties.

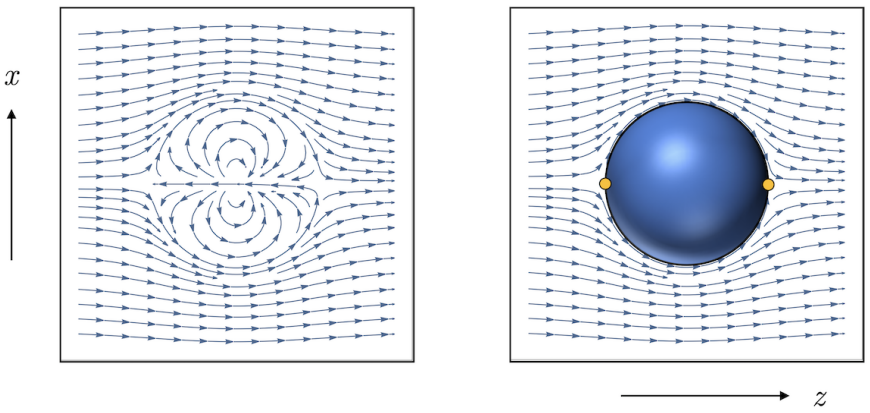


Figure 7. One the left, the constant flow superposed with the dipole flow. On the right, a well-placed solid sphere, hiding the messy bit.

We've chosen to describe a flow with asymptotic velocity U and a stationary sphere. Alternatively, we could boost by U. This means that we remove the constant U term in (2.30) to describe a fluid that is asymptotically stationary, but with a sphere moving through it at speed U.

The velocity perpendicular to the sphere vanishes, but the velocity u_θ tangent to the surface of the sphere does not vanish when $r = R$. We may wonder how realistic this is for actual fluids and the answer, in many situations, is not very! We'll revisit this when we come to discuss viscosity.

There are a number of interesting features of the flow (2.30). First, there are two points where the flow stops completely and $\mathbf{u} = 0$. This happens on the surface of the sphere, $r = R$, at $\theta = 0$ (on the right) and $\theta = \pi$ (on the left) as depicted by orange dots in Figure 7. This occurs when the fluid comes in with vanishing impact parameter and, on symmetry grounds, can't tell whether to go up or down. So instead it stops. Points where the local fluid velocity vanishes are called *stagnation points*.

Next, we can look at the top and bottom of the sphere with $\theta = \pm\pi/2$. From (2.29), we see that the velocity on the boundary of the sphere is

$$|\mathbf{u}_{\rm top}| = \frac{3}{2}U$$

In other words, the fluid speeds up as it moves past the sphere. In fact, this follows from Bernoulli's principle as we explain below. Relatedly, you can see that the streamlines get squeezed together at the top and bottom of the sphere. This is familiar in other situations: stand at the top of a hill and it's windier than it was at the bottom.

2.3.3 D'Alembert's Paradox

Next we calculate the pressure that the fluid exerts on the sphere. For this we use Bernoulli's principle which says that the function H defined in (2.12) remains constant along streamlines (and, because the flow is irrotational, throughout the fluid). Asymptotically,

$$H = \frac{1}{2}\rho U^2 + P_\infty$$

where P_∞ is the asymptotic pressure of the flow. Meanwhile, on the surface of the sphere

$$H = \frac{9}{8}\,\rho U^2 \sin^2\theta + P(\theta)$$

So the pressure on the surface of the sphere is

$$P(\theta) = P_\infty + \frac{1}{2}\rho U^2 \left(1 - \frac{9}{4}\sin^2\theta\right) \tag{2.31}$$

Here's the weird thing: the pressure depends only on $\sin^2\theta$. This means that the pressure exerted on the sphere at the front, where $\pi/2 < \theta \leq \pi$ (this is on the left in the figure) is identical to the pressure exerted behind, where $0 \leq \theta < \pi/2$ (on the right in the figure). And that doesn't sound right at all! We know from experience that an object placed in a stream will suffer a *drag force* which, in this case, should serve to carry the sphere along with the flow. But that's not what we find! Instead the flow finds a way to move seamlessly around the object, exerting no force.

Said differently, we can always boost our solution by speed U so that the fluid is stationary and the sphere is moving through it with speed U. The result above says that the sphere experiences no friction. It just glides through the fluid unimpeded. Tantalising as this sounds, it's simply not right. The fact that the maths differs so wildly from observation is known as the D'Alembert paradox, after the mathematician Jean le Rond d'Alembert who first uncovered this puzzle in 1752.

Another Historical Aside

D'Alembert concluded his paper with:

> "It seems to me that the theory, developed in all possible rigour, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future Geometers to elucidate"

Future geometers (and physicists) took their time. Even though the Euler equation was replaced by the Navier-Stokes equation, which includes the effects of viscosity, there are arguments that suggest that, at least for fast fluid flows, the effects of viscosity are negligible. Which, if true, would mean that the solution we've described here provides a good approximation for fast moving fluids and the drag force should be close to zero. Needless to say, that's not what's observed.

As the years wore on, this curious mathematical puzzle turned into something of an embarrassment. And this embarrassment, in turn, grew into a sense of genuine shame as controlled, powered flight became a reality. The Wright brothers made their famous first flight in December 1903, a century and a half after D'Alembert's work, yet the basic resolution of his paradox was still not fully understood, leaving theoretical physicists at something of a disadvantage in explaining the most important technology of the day. The breakthrough came only in 1905 (a good year for physics) and the work of Prandtl on boundary layers. We will describe this in Section 3.5.

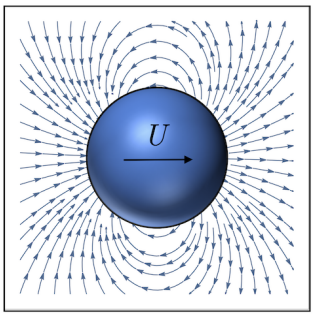
Yet Pradtl's discovery was far from a proof and its full importance took some time to seep in. Even as late as 1915, the great Rayleigh finished a review of a book on hydrodynamics with

> "'We may hope that before long [artificial flight] may be co-ordinated and brought into closer relation with theoretical hydrodynamics. In the meantime one can scarcely deny that much of the latter science is out of touch with reality"

Part of the goal of these lectures is to explain that, happily, theoretical hydrodynamics is very much in touch with reality. It's just a little more subtle than the simple approach we've taken here.

2.3.4 A Bubble Rising

A small variation on the calculation allows us to understand how bubbles rise to the surface in a fluid, at least in the inviscid approximation that underlies this section.



Consider a sphere of radius R and mass M moving through the a stationary fluid with speed U. At the time when the bubble is centred at the origin, the flow is described by the

potential

$$\phi = -\frac{UR^3}{2r^2}\cos\theta$$

To see that this is the appropriate flow function, note that $\mathbf{u} \to 0$ as $r \to \infty$, so the fluid is indeed asymptotically stationary. Moreover, on the surface of the sphere $\mathbf{u} = U\cos\theta\,\hat{\mathbf{r}} + \frac{1}{2}U\sin\theta\,\hat{\boldsymbol{\theta}}$ so $\mathbf{u}\cdot\hat{\mathbf{r}} = U\cos\theta$ which is indeed the relevant boundary condition (2.25) for a sphere moving with speed U. The flow is shown in the figure to the right.

We can calculate the kinetic energy of the fluid,

$$T_{\rm fluid} = \frac{1}{2}\rho\int_{r>R}(\nabla\phi)^2\,dV = \frac{1}{2}\rho\int_{r>R}\big(\nabla\cdot(\phi\nabla\phi) - \phi\nabla^2\phi\big)\,dV$$

The second term vanishes because the potential obeys $\nabla^2\phi = 0$. The first term can be evaluated by the divergence theorem and gives

$$T_{\rm fluid} = \frac{1}{2}\rho\int\phi\,\mathbf{u}\cdot d\mathbf{S} = \frac{1}{4}\rho R^3U^2\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\ \cos^2\theta\sin\theta = \frac{\pi}{3}\rho R^3U^2$$

To this we should add the kinetic energy $\frac{1}{2}MU^2$ of the sphere itself, so that the total kinetic energy of the sphere and the fluid is

$$T_{\rm total} = \frac{1}{2}MU^2 + T_{\rm fluid} = \frac{1}{2}M_{\rm eff}U^2$$

Here we've introduced the effective mass $M_{\rm eff}$ is the combined mass of the sphere, together with the surrounding fluid,

$$M_{\rm eff} = M + \frac{2\pi}{3}\rho R^3$$

Note that the additional contribution $\frac{2\pi}{3}\rho R^3$ is precisely half the mass of the fluid displaced by the sphere.

Now consider the case of a bubble, with $M = 0$. The effective mass is just $M_{\rm eff} = \frac{2\pi}{3}\rho R^3$. But this bubble is an absence of water. This means that if we raise it by some height z then we *lose* potential energy! This is most easily seen because moving a bubble upwards is the same as moving the displaced water downwards. The potential energy is then

$$V = -\left(\frac{4\pi}{3}\rho R^3\right)gz$$

where the factor of $\frac{4\pi}{3}\rho R^3 = 2M_{\rm eff}$ is the mass of the displaced water. The total energy of the bubble is then

$$E = \frac{1}{2}M_{\rm eff}U^2 - 2M_{\rm eff}gz$$

The minus sign means that the bubble rises, rather than falls, due to gravity. Of course, we knew that anyway. The factor of 2 is perhaps more surprising: it says that the bubble accelerates upwards at twice the usual gravitational acceleration,

$$\dot{U} = 2g$$

The idea that the absence of something can itself be viewed as a new object – the bubble – is rather intuitive in this context. A more quantum version of the same idea also arises in the theory of solids where the absence of an electron acts very much like a particle with positive electric charge, known as a "hole".

Looking (Far) Forwards: Renormalisation

Before we go on, it's worth pausing to point out that, hidden inside this simple calculation, is an idea that will later blossom into something rather deep. This is the observation that, when immersed in a fluid, an object acts as if it has an effective mass $M_{\rm eff}$, which is a combination of its original mass M together with the mass of the fluid that it drags along with it.

This same phenomena occurs at a much more fundamental level for elementary particles such as electrons, quarks, and the Higgs boson. This arises because our universe is filled with different fields, each of which acts in many ways like a fluid. These fields include the familiar electric and magnetic fields as well as many others. When a particle moves through space, these fields become excited and get dragged along with the particle, much like a ball moving through water. The upshot is that mass of an elementary particle has two contributions: an inherent mass, analogous to M that in this context is called the "bare mass", and an additional contribution from the other fields. These combine to give the mass $M_{\rm eff}$ which is what we measure in experiment.

The calculations that give rise to this shift in the mass go by the name of *renormalisation*. They can be rather challenging and you will have the pleasure of meeting them in later courses on Quantum Field Theory. While the underlying mathematics can seem daunting, it's worth keeping in mind that what's really going on is little different from the effective mass of a ball moving through water.

2.4 Potential Flows in 2d

In this section, we again look at potential flows with $\mathbf{u} = \nabla\phi$ and $\nabla^2\phi = 0$, but this time in two dimensions. You might think that things are simpler for 2d flows. However, as we will explain, there is a novelty that doesn't arise in the 3d case.

We work in 2d polar coordinates (r, θ) so Laplace's equation takes the form

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

We will explore the different solutions to this equation.

Radial and Angular Flows

Once again, the simplest rotationally invariant solution is not particularly useful: it is the radial, planar flow

$$\phi = q\log\left(\frac{r}{r_0}\right) \quad \Rightarrow \quad \mathbf{u} = \frac{q}{r}\hat{\mathbf{r}}$$

This is again the Green's function, now obeying the 2d Poisson equation

$$\nabla^2\phi = 2\pi q\,\delta^2(\mathbf{x})$$

However, this time there is a second, rotationally invariant flow. It arises from the potential

$$\phi = \frac{\Gamma}{2\pi}\theta \tag{2.32}$$

with Γ a constant. First note that this is *not* a single-valued potential because $\phi(\theta + 2\pi) \neq \phi(\theta)$ and you may wonder about the validity of such potentials. To see the consequence, we can simply compute the velocity field

$$\mathbf{u} = \nabla\phi = \frac{\Gamma}{2\pi r}\hat{\boldsymbol{\theta}}$$

We've met this velocity field before! It was given in (2.15) (where you should substitute $f(r) = \Gamma/2\pi r^2$ in (2.15)). This is the flow that has the property that it swirls around the origin, even though it is irrotational, with $\nabla \times \mathbf{u} = 0$, at least away from $r = 0$. The parameter Γ measures the circulation of the flow

$$\Gamma = \oint_C \mathbf{u}\cdot d\mathbf{x}$$

where the integral is taken around any curve that surrounds the origin. Usually the integral of any conservative vector field like $\mathbf{u} = \nabla\phi$ around a closed curve is necessarily vanishing. The reason that it's not the case here is because ϕ is not single-valued. It is the ability to have circulation in 2d flows that adds some extra spice to the proceedings. We'll now see how this manifests itself in a simple example.

2.4.1 Circulation Around a Cylinder

We consider the flow around an infinite cylinder, aligned along the y direction. This ensures that the flow is effectively two-dimensional: we care only about the velocity in the (x, z)-plane.

The start of our story is the same as the flow around a sphere that we saw in the previous section. The most general solution to the 2d Laplace equation is

$$\phi(r,\theta) = (A_0 + B_0 \log r)(C_0 + D_0\theta) + \sum_{n=1}^{\infty} \left(A_n r^n \cos(n\theta + \alpha_n) + \frac{B_n}{r^n} \cos(n\theta + \beta_n) \right)$$

Here we focus on the flows with $n = 1$. The integration constants α_1 and β_1 will play no role so we set them to zero and look at:

$$\phi = U\left(r + \frac{R^2}{r}\right)\cos\theta \tag{2.33}$$

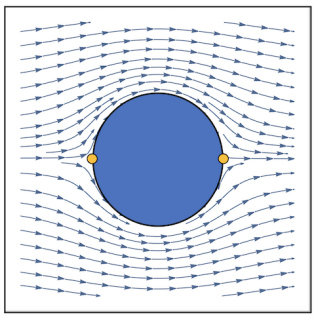
This is very similar to the 3d potential (2.27). Again, the first term describes a constant flow with asymptotic velocity U, a fact that we've anticipated in labelling the overall coefficient. The second term is now a two-dimensional dipole. Combined, they give rise to the velocity field

$$\mathbf{u} = \frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}} = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta\hat{\mathbf{r}} - U\left(1 + \frac{R^2}{r^2}\right)\sin\theta\,\hat{\boldsymbol{\theta}} \tag{2.34}$$

We see that the radial component has the property that

$$u_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta = 0 \text{ when } r = R$$

This means that this potential describes the flow past a solid cylinder of radius R. The velocity field $\mathbf{u}$ is shown on the right, with the two stagnation points shown in orange. The details are slightly different, but the qualitative features are the same as for the sphere.



Adding Circulation

Things get more interesting if we add some circulation. Because the Laplace equation is linear, we can superpose the flow around the cylinder (2.33) with the rotation (2.32),

$$\phi = U\left(r + \frac{R^2}{r}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta \tag{2.35}$$

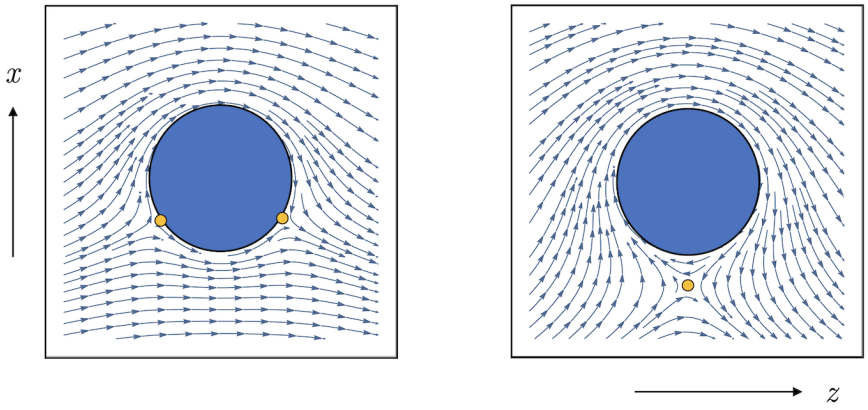


Figure 8. On the left: the flow around a cylinder when the circulation is small; on the right, when the circulation is big.

The extra term affects only the angular part of the velocity, which now takes the form

$$\mathbf{u} = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta\,\hat{\mathbf{r}} + \left[\frac{\Gamma}{2\pi r} - U\left(1 + \frac{R^2}{r^2}\right)\sin\theta\right]\hat{\boldsymbol{\theta}}$$

You can check that the associated stream function is

$$\Psi = Ur\left(1 - \frac{R^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi}\log\left(\frac{r}{r_0}\right) \tag{2.36}$$

To understand the effect on the flow, we can search for the stagnation points at which $|\mathbf{u}| = 0$. Clearly $u_r = 0$ provided that we sit at radius $r = R$. The angular velocity then vanishes at the angle θ such that

$$\Gamma = 4\pi UR\sin\theta$$

But this has a solution only when $|\Gamma| < 4\pi UR$ (where we're taking $U > 0$). This suggests that the flow will be different for small and large circulation Γ.

We start by looking at small $|\Gamma| < 4\pi UR$ so that there are two stagnation points on the surface of the cylinder at $\sin\theta = \Gamma/4\pi UR$. The corresponding flow is shown on the left hand side of Figure 8. (I've taken $\Gamma < 0$ in this figure for reasons that will become apparent below.) Note that the stagnation point plays an important role: this is where the fluid separates, with stream lines on either side taking different paths, one above the cylinder and the other below.

Meanwhile, when $|\Gamma| > 4\pi UR$, there is no stagnation point on the surface of the cylinder. Instead it now occurs at $\theta = \pi/2$ (which ensures that $u_r = 0$) and a distance r from the centre, given by the solution to the quadratic

$$r^2 - \frac{\Gamma}{2\pi U} + R^2 = 0$$

This ensures that $u_\theta = 0$. The quadratic is guaranteed to have one positive root sitting outside the sphere provided that $|\Gamma| > 4\pi UR$. The flow is shown on the right-hand side of Figure 8, again with the stagnation point shown in orange.

2.4.2 Lift and the Magnus Force

Now we can repeat the calculation that we performed for the sphere to answer the quesion: what's the pressure that the fluid exerts on the cylinder? We use Bernoulli's principle and the conservation of H throughout the flow. At infinity we have

$$H = \frac{1}{2}\rho U^2 + P_\infty$$

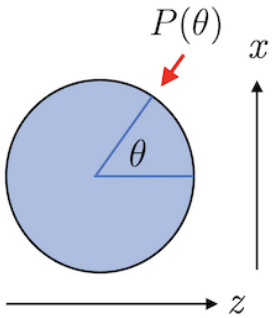
while, on the surface of the sphere, it is

$$H = \frac{1}{2}\rho\left(\frac{\Gamma}{2\pi R} - 2U\sin\theta\right)^2 + P(\theta)$$

So the pressure on the surface of the sphere is

$$P(\theta) = P_\infty + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta\right) + \frac{U\Gamma\rho}{\pi R}\sin\theta - \frac{\Gamma^2\rho}{8\pi^2R^2} \tag{2.37}$$

The pressure acts radially on the sphere. We want to decompose this force to compute the component forces F_z in the z-direction (horizontal in the flow diagrams in Figure 8) and the x-direction (vertically in Figure 8). From the diagram on the right, we see that



$$F_z = \int_0^{2\pi} P(\theta)R\cos\theta\, d\theta = 0$$

So there is no force in the direction of the flow.
Or, said differently, there is no drag force. This is the same result that we saw for the sphere and leads to D'Alembert's paradox. The novelty is that the force perpendicular

to the asymptotic flow is non-vanishing: it receives a contribution from the $\sin\theta$ term in (2.37),

$$F_x = -\int_0^{2\pi} P(\theta) R \sin\theta \, d\theta = -\frac{U\Gamma\rho}{\pi}\int_0^{2\pi} \sin^2\theta = -U\Gamma\rho$$

The minus sign means that, for $\Gamma < 0$ as shown in Figure 8, the force is upwards. This makes sense: if you look at the figure, you see that the streamlines are closer together at the top of the cylinder. This means that the fluid is travelling faster at the top and, correspondingly, there is a lower pressure. Hence the upwards force. This force is called *lift*. (We took $\Gamma < 0$ in Figure 8 to save ourselves the embarrassment of having a force called "lift" that acts downwards.)

In the calculation above, we took the fluid to be circulating and the cylinder to be stationary. However, the same effect occurs if the cylinder rotates while the fluid has no circulation. In this situation, the lift force is referred to as the *Magnus force*. It is the same force that makes a ball swerve when you put spin on it.

2.5 A Variational Principle

All laws of physics can be expressed using the principle of least action. What about the laws of fluid mechanics?

The action principle is best suited to fundamental laws of physics where there is no friction at play. The full Navier-Stokes equation for fluids (that we will meet in Section 3) includes a friction term and so isn't immediately amenable to a formulation using an action. But the Euler equation that we've studied in this section has no such friction term which suggests that it should be possible to write down an action that gives rise to the Euler equation. The question is: how?

This, it turns out, is not quite as straightforward as one might think. But it is possible and, moreover, gives some insight into the mathematical structure of the Euler equation. The purpose of this section is to describe this.

This section is something of a tangent to the rest of these notes and we won't be returning to the action principle later in these lectures, not least because we'll be embracing the full Navier-Stokes equation. Also, the terminology in this section can be a little confusing simply because Euler and Lagrange were rather impressive mathematicians. To give you a sense of this, our goal is to work in the Eulerian framework of fluid mechanics, rather than the Lagrangian framework, and then write down a Lagragian and derive the Euler-Lagrange equations to reproduce the Euler equation. All clear? Good.

2.5.1 The Principle of Least Action

We start by giving a review of the principle of least action, both in the framework of classical mechanics and also in classical field theory. You can read more about this in the lectures on Classical Dynamics and in the first section of the lectures on Quantum Field Theory.

First, Newtonian mechanics. We'll consider a single particle with a position given by $\mathbf{x} \in \mathbb{R}^3$. The position changes with time, so the trajectory of a particle traces out a curve $\mathbf{x}(t)$. Of all these possible trajectories, there is a typically one that obeys the laws of physics. We want to know which one.

If the particle has mass m then its kinetic energy is $T = \frac{1}{2}m\dot{\mathbf{x}}^2$. We'll assume that the particle experiences a potential energy $V(\mathbf{x})$. We then define the *Lagrangian*

$$L(\mathbf{x}, \dot{\mathbf{x}}) = T - V \tag{2.38}$$

and, from this, the *action*

$$S[\mathbf{x}(t)] = \int dt\ L(\mathbf{x}, \dot{\mathbf{x}}) = \int dt\ \left[\frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x})\right] \tag{2.39}$$

The action assigns a number S to each trajectory $\mathbf{x}(t)$. (Strictly speaking, we should consider the action for all trajectories with certain boundary conditions specified, such as $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$. This is important, but we'll sweep it under the rug in what follows.)

The principle of least action states that the true trajectory $\mathbf{x}(t)$ followed by the particle is the one that extremises the action S. Mathematically, this means the following. Suppose that you have a putative trajectory $\mathbf{x}(t)$ with some action S. We look at all neighbouring trajectories $\mathbf{x}(t) + \delta\mathbf{x}(t)$ and compute their action $S + \delta S$. The original trajectory is the one taken by the particle if $\delta S = 0$ for *all* variations $\delta\mathbf{x}(t)$.

For our action (2.39), we have

$$\begin{aligned} S[\mathbf{x}(t) + \delta\mathbf{x}(t)] &= \int dt\ \left[\frac{1}{2}m(\dot{\mathbf{x}} + \delta\dot{\mathbf{x}})^2 - V(\mathbf{x} + \delta\mathbf{x})\right] \\ &\approx \int dt\ \left[\frac{1}{2}m(\dot{\mathbf{x}}^2 + 2\dot{\mathbf{x}}\cdot\delta\dot{\mathbf{x}}) - V(\mathbf{x}) - \nabla V\cdot\delta\mathbf{x}\right] = S + \delta S \end{aligned}$$

where, in going to the second line, we've ignored all terms of order $\delta\mathbf{x}^2$ and higher. This gives us an expression for the variation of the action δS which we can now play with

$$\delta S = \int dt\ \left[m\dot{\mathbf{x}}\cdot\delta\dot{\mathbf{x}} - \nabla V\cdot\delta\mathbf{x}\right] = \int dt\ \left[-m\ddot{\mathbf{x}} - \nabla V\right]\cdot\delta\mathbf{x}$$

In the second equality we've integrated by parts and thrown away the boundary terms. (We've been careless about why one can throw away boundary terms after integration by parts; that's the bit we're sweeping under the rug.) The principle of least action states that the true trajectory has $\delta S = 0$ for all possible variations $\delta\mathbf{x}$. This can only be true if the expression in square brackets vanishes, meaning

$$m\ddot{\mathbf{x}} = -\nabla V \tag{2.40}$$

This, of course, is the Newtonian equation of motion. The principle of least action is just a recasting of this familiar result.

The action for a given equation of motion is not necessarily unique. Here, for example, is a different action that yields the same equation of motion (2.40). We will initially think of the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ of the particle as independent quantities. We'll then enforce the requirement $\mathbf{v} = \dot{\mathbf{x}}$ through the use of a Lagrange multiplier. The upshot is that we can write down the action

$$S[\mathbf{x}(t), \mathbf{v}(t), \boldsymbol{\beta}(t)] = \int dt \ \left[\frac{1}{2}m\mathbf{v}^2 - V(\mathbf{x}) - \boldsymbol{\beta}\cdot(\mathbf{v} - \dot{\mathbf{x}})\right] \tag{2.41}$$

The equation of motion for $\boldsymbol{\beta}$ reproduces the constraint $\mathbf{v} = \dot{\mathbf{x}}$, while the equation of motion for $\mathbf{v}$ tells us that we should identify the Lagrange multiplier with the velocity: $m\mathbf{v} = \boldsymbol{\beta}$. Finally, the equation of motion for $\mathbf{x}$ is $\dot{\boldsymbol{\beta}} = -\nabla V$. Combining these reproduces (2.40).

For the Newtonian particle, there's clearly no advantage to writing the action (2.41) over (2.39). Indeed, it seems a little perverse to do so. But these kind of tricks can prove useful in other contexts and one of these turns out to be fluid dynamics.

An Action for Fields

The next conceptual step is to move from particles to fields. We will consider a scalar field $\varphi(\mathbf{x}, t)$ which associates a number to each point in space and time. Note, in particular, that the role of the spatial coordinate $\mathbf{x}$ has changed. In the context of Newtonian mechanics, $\mathbf{x}$ was the dynamical degree of freedom, something that evolves over time. But in field theory that's no longer the case. Now $\mathbf{x}$ is just a label, like time t, and the field φ is the dynamical degree of freedom whose values depends on both space and time.

We would like to write down an action for this field. This means that we want to associate a number S to each possible field configuration $\varphi(\mathbf{x}, t)$. We start by defining

the *Lagrangian density* $\mathcal{L}$ (although everyone simply refers to it as the "Lagrangian"). A natural choice, which is the analog of (2.38), is

$$\mathcal{L}(\varphi, \dot{\varphi}, \nabla\varphi) = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}c^2(\nabla\varphi)^2 - V(\varphi)$$

We have a kinetic energy type term, $\dot{\varphi}^2$, but now we have two different kinds of potential energy. The first, proportional to $\nabla\varphi^2$, is an energy arising from spatial gradients of the field. It comes with a constant coefficient c which has dimension $[c] = LT^{-1}$. In many situations, this is the speed of ripples of the field. In addition, we have a second potential energy $V(\varphi)$ which depends only on φ and not on its derivatives. We pick different potentials $V(\varphi)$ to model the situation that we're interested in, just like $V(\mathbf{x})$ in Newtonian mechanics. Typically one picks $V(\varphi)$ so that it penalises large values of φ, e,g. $V(\varphi) \sim \varphi^2$. Here we'll keep $V(\varphi)$ general.

We associate an action S to a given field configuration $\varphi(\mathbf{x}, t)$ by integrating the Lagrangian over both space and time,

$$S = \int dt\, d^3x\ \mathcal{L} = \int dt\, d^3x\ \left[\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}c^2(\nabla\varphi)^2 - V(\varphi)\right] \tag{2.42}$$

It's worth stressing, for the second time, the different roles that the spatial coordinate plays in (2.39) and (2.42). It has been demoted from its role as a dynamical degree of freedom in the former to a mere integration variable in the latter.

At this point, we proceed in much the same way as for the Newtonian particle. We take a reference field configuration $\varphi(\mathbf{x}, t)$ and compute its action S. Then we look at all nearby field configurations $\varphi(\mathbf{x}, t) + \delta\varphi(\mathbf{x}, t)$ and compute their action $S + \delta S$. The original field configuration obeys the classical equations of motion if and only if $\delta S = 0$ for all $\delta\varphi$. In equations, we have

$$\begin{aligned} S[\varphi + \delta\varphi] &= \int dt\, d^3x\ \left[\frac{1}{2}(\dot{\varphi} + \delta\dot{\varphi})^2 - \frac{1}{2}c^2(\nabla\varphi + \nabla\delta\varphi)^2 - V(\varphi + \delta\varphi)\right] \\ &\approx \int dt\, d^3x\ \left[\frac{1}{2}(\dot{\varphi}^2 + 2\dot{\varphi}\,\delta\dot{\varphi}) - \frac{1}{2}c^2(\nabla\varphi^2 + \nabla\varphi\cdot\nabla\delta\varphi) - V(\varphi) - \frac{\partial V}{\partial\varphi}\delta\varphi\right] \end{aligned}$$

where, as before, we've truncated our expansion at leading order in $\delta\varphi$ in the second line. From this we can extract the variation of the action

$$\begin{aligned} \delta S &= \int dt\, d^3x \left[\dot{\varphi}\,\delta\dot{\varphi} - c^2\nabla\varphi\cdot\nabla\delta\varphi - \frac{\partial V}{\partial\varphi}\delta\varphi\right] \\ &= \int dt\, d^3x\ \left[-\ddot{\varphi} + c^2\nabla^2\varphi - \frac{\partial V}{\partial\varphi}\right]\delta\varphi \end{aligned}$$

Here we've again integrated by parts, now with respect to both temporal and spatial derivatives, so that all terms are proportional to $\delta\varphi$. Requiring that $\delta S = 0$ for all possible $\delta\varphi$ tells us that the expression in square brackets must vanish, so

$$\frac{\partial^2\varphi}{\partial t^2} - c^2\nabla^2\varphi = -\frac{\partial V}{\partial\varphi} \tag{2.43}$$

This is the simplest equation of motion for a classical scalar field.

The equation of motion (2.43) doesn't play a particularly prominent role in classical physics, where our heads are more likely to be turned by more sophisticated theories such as Electromagnetism or General Relativity. It does however, arise in various cameos and we'll meet it briefly in Section 4.3.2 when discussing a certain kind of wave that is driven by the Coriolis force. The equation only really comes to the fore when we turn to Quantum Field Theory, where it plays more of a starring role.

2.5.2 An Action Principle for Fluids

Now we are in a position to construct an action principle for fluids. Our goal is to write down an action which reproduces the Euler equation for an incompressible fluid

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0 \tag{2.44}$$

We could also include further forces, such as gravity, but since this doesn't add extra conceptual issues we will just ignore it and focus on the simplest equations above.

The first question that we should ask is: what are the dynamical degrees of freedom for a fluid? Until now, we have viewed (2.44) as four equations for four variables, $\mathbf{u}$ and P. But we might suspect that these aren't quite the right variables to construct an action. After all, when writing down an action for the Newtonian particle, it's important that we vary with respect to the position $\mathbf{x}$ rather than the velocity $\dot{\mathbf{x}}$. And the same is true for a fluid. To build an action, we need to start thinking about the "position" of the fluid.

To this end, we will think of the configuration of the fluid as a map from $\mathbb{R}^3 \mapsto \mathbb{R}^3$

$$\mathbf{x} \mapsto \alpha^i(\mathbf{x}, t) \quad i = 1, 2, 3$$

Here $\mathbf{x}$ label the fixed positions in space, while $\alpha^i(\mathbf{x}, t)$ label parcels of the fluid. This is the Eulerian (as opposed to Lagrangian) description of a fluid. We will refer to α^i as the embedding coordinate of the fluid.

We will think of $\alpha^i(\mathbf{x}, t)$ as the fields of our Lagrangian although, as we will see, they will need to be augmented by several more. But even before we get going, it's worth pointing out that $\alpha^i(\mathbf{x}, t)$ aren't quite like other fields. This is because the map from $\mathbb{R}^3 \mapsto \mathbb{R}^3$ that describes our fluid must be invertible. For example, there's no configuration of the fluid with, say, $\alpha^i(\mathbf{x}, t) = 0$. That would describe the entire fluid as sitting at a single point and that's not allowed. In fact, because our fluid is incompressible, we should require that the map from $\mathbb{R}^3 \to \mathbb{R}^3$ is volume preserving. This is assured if

$$\det\left(\frac{\partial \alpha^i}{\partial x^j}\right) = 1 \tag{2.45}$$

We will have to find a way to impose a constraint like this on our map.

(As an aside: these kind of constraints are not entirely unfamiliar. In general relativity, the dynamical degree of freedom is a metric $g_{\mu\nu}(\mathbf{x}, t)$ but, as with a fluid, we're not allowed to set $g_{\mu\nu} = 0$. Instead, we must have $\det(g_{\mu\nu}) \neq 0$.)

We describe the fluid by the maps α^i. How do we define the velocity? You might naively think that it's just $\dot{\alpha}^i$, but that's not right. Instead, we need to think more physically. Suppose that you focus on one particular parcel of fluid, say the one labelled by $\alpha^i = (3, 7, 4)$. Then we can follow this parcel through the fluid. If $\alpha^i(\mathbf{x}, t)$ changes then the parcel of fluid must have moved to a some neighbouring point, which means that the velocity $\mathbf{u}$ is non-zero. This velocity is defined implicitly as

$$\frac{\partial \alpha^i}{\partial t} + \mathbf{u} \cdot \nabla \alpha^i = 0 \tag{2.46}$$

Because the map from $\mathbb{R}^3 \mapsto \mathbb{R}^3$ is invertible, we can get an explicit expression for $\mathbf{u}$ in terms of α^i. Using the fact that the map preserves volumes (2.45), this is given by

$$u^i(\mathbf{x}, t) = -\frac{1}{2}\epsilon^{ijk}\epsilon_{lmn}\frac{\partial \alpha^l}{\partial t}\frac{\partial \alpha^m}{\partial x^j}\frac{\partial \alpha^n}{\partial x^k}$$

To see this, you just need to use the definition of the determinant in terms of ϵ^{ijk}. It's also straightforward to show that the condition (2.45) ensures that $\nabla \cdot \mathbf{u} = 0$ as expected. (You should use the expression for the determinant of a 3×3 matrix in terms of ϵ_{ijk}.)

Note that for these incompressible flows, with $\nabla \cdot \mathbf{u} = 0$, the equation (2.46) takes the form of a conservation law $\partial \alpha^i / \partial t + \nabla \cdot (\mathbf{u}\alpha^i) = 0$. There is a simple physical intuition for this: it is just the statement that you can trace the evolution of a given parcel of fluid, a kind of "conservation of particle identity" if you like.

Now we've set-up the basic kinematical structure for fluids, our next job is to write down the action. Here a number of choices await us. It is possible to write down an action just for the embedding coordinates $\alpha^i(\mathbf{x}, t)$, with the constraint (2.45) imposed by a Lagrange multiplier. While it's possible, it's also a little messy. It turns out to be more straightforward to write down an action for α^i and u^i, together with a collection of Lagrange multipliers. This is analogous to the slightly daft action (2.41) that we introduced for the Newtonian particle.

We take as our action

$$S[\boldsymbol{\alpha}, \mathbf{u}, \phi, \boldsymbol{\beta}] = \int dt\, d^3x \ \left[\frac{1}{2}\rho\mathbf{u}^2 + \phi\nabla\cdot\mathbf{u} - \beta_i\left(\frac{\partial\alpha^i}{\partial t} + \mathbf{u}\cdot\nabla\alpha^i\right)\right] \tag{2.47}$$

The equations of motion arise from varying the action with respect to $\alpha^i(\mathbf{x}, t)$, $u^i(\mathbf{x}, t)$ and the Lagrange multipliers $\phi(\mathbf{x}, t)$ and $\beta^i(\mathbf{x}, t)$.

The Lagrange multipliers are easiest to deal with. Varying with respect to ϕ gives the incompressibility condition $\nabla\cdot\mathbf{u} = 0$, now directly in terms of velocity rather than the more abstract (2.45). Meanwhile, varying with respect to β^i gives us the relation (2.46) between the embedding coordinate and velocity. That leaves us with the equations of motion that come from varying the action with respect to α^i and u^i. If we vary with respect to α^i, we have

$$\frac{\partial\beta_i}{\partial t} + \mathbf{u}\cdot\nabla\beta_i = 0 \tag{2.48}$$

So we see that the Lagrange multiplier β^i obey the same equation (2.46) as the embedding coordinates. Meanwhile, varying with respect to the components of the velocity $\mathbf{u}$ gives the expression

$$\rho\mathbf{u} = \nabla\phi + \beta_i\nabla\alpha^i \tag{2.49}$$

This is a curious equation, relating the velocity to ϕ, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Note that the first term is familiar: it is just the kind of potential flow that we met in Sections 2.3 and 2.4, with the Lagrange multiplier playing the role of the potential. But the the second term is less familiar and it's not immediately obvious how this is related to the Euler equation. In particular, we haven't yet seen how the pressure emerges in this framework.

To make progress, we compute $D\mathbf{u}/Dt$ using the expression (2.49). There's a little bit of algebra involved, but it's not too hard to show that

$$\rho\frac{Du^i}{Dt} \equiv \rho\left(\frac{\partial u^i}{\partial t} + u^j\frac{\partial u^i}{\partial x^j}\right) = \frac{\partial}{\partial x^i}\left(\frac{\partial\phi}{\partial t} + \beta_j\frac{\partial\alpha^j}{\partial t} + \frac{1}{2}\rho\mathbf{u}^2\right) + \frac{D\beta_j}{Dt}\frac{\partial\alpha^j}{\partial x^i} - \frac{D\alpha^j}{Dt}\frac{\partial\beta_j}{\partial x^i}$$

But the last two terms vanish by virtue of (2.46) and (2.48). We're left,

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P \quad \text{where} \quad P = -\frac{\partial\phi}{\partial t} - \beta_j\frac{\partial\alpha^j}{\partial t} - \frac{1}{2}\rho\mathbf{u}^2 + \text{constant}$$

with the pressure given, as shown, by a combination of the velocity and Lagrange multipliers. This is the promised Euler equation, now derived from an action principle.

A Slightly Simpler Action

As we mentioned above, there are slightly simpler versions of the fluid action. Here we describe one that succeeds in eliminating the need for embedding coordinates completely. Instead, it uses the fact that a general velocity field $\mathbf{u}(\mathbf{x}, t)$ in $\mathbb{R}^3$ can be written as

$$\mathbf{u} = \nabla\phi + \beta\nabla\alpha \tag{2.50}$$

for some functions ϕ, β and α. (These functions are not unique.) This is sometimes known as the *Clebsch representation*. Note that it's very similar to the form of the velocity (2.49) that arose from our previous variational principle, except now there is just a single α and β function rather than a triplet.

There is a nice way of visualising the form of the velocity field (2.50). The first term is clearly the irrotational, potential flow that we met previously. The second term gives vorticity

$$\boldsymbol{\omega} = \nabla\times\mathbf{u} = \nabla\beta\times\nabla\alpha$$

This is telling us that vortex lines (i.e. integral curves of $\boldsymbol{\omega}$) lie on the intersection of surfaces of constant α and constant β.

Now consider the action

$$S = \int dt\, d^3x\ \left[-\beta\frac{\partial\alpha}{\partial t} - \frac{1}{2}\left(\nabla\phi + \beta\nabla\alpha\right)^2\right] \tag{2.51}$$

This is closely related to our previous action (2.47): it's what you get if you substitute the expression (2.49) for $\mathbf{u}$ into the action and drop the $i = 1, 2, 3$ indices on α^i and β_i.

Now when varying the action, we must remember that the velocity $\mathbf{u}$ is defined by (2.50). The equation of motion for ϕ then tells us that $\nabla\cdot\mathbf{u} = 0$. Meanwhile, the equations of motions for α and β are, respectively,

$$\frac{D\beta}{Dt} = 0 \quad \text{and} \quad \frac{D\alpha}{Dt} = 0$$

We can now repeat our previous calculation to once again find the Euler equation

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P \quad \text{with} \quad P = \rho\left(-\frac{\partial\phi}{\partial t} - \beta\frac{\partial\alpha}{\partial t} - \frac{1}{2}\left(\nabla\phi + \beta\nabla\alpha\right)^2\right)$$

There is one rather pretty consequence of this: the Lagrangian that appears in (2.51) is recognised as the pressure,

$$S[\phi,\alpha,\beta] = \frac{1}{\rho}\int dt\, d^3x\ P[\phi,\alpha,\beta]$$

If you're thermodynamically inclined, this makes sense. In an appropriate ensemble, the pressure is equal to the free energy and there are situations where the action and free energy sit on the same footing.

3 The Navier-Stokes Equation

So far we've built up some intuition for how fluids flow. But we've missed an ingredient and it turns out that this ingredient is rather important. This is the internal friction experienced by a fluid, known as *viscosity*. We'll give a more careful discussion of viscosity shortly in Section 3.1. But first we give a quick, slightly handwavy derivation of the relevant equation.

The Euler equation describing fluid motion is, as we've seen, just the continuum version of "$F = ma$". It is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mathbf{f} \tag{3.1}$$

where $\mathbf{f}$ describes any other forces experienced by the fluid. We want to write down the force that arises from friction.

We already met friction briefly in our introduction to Newtonian Mechanics. There we explained that friction is not a fundamental force, but one that arises from the underlying interactions of many (say, 10^{23}) particles. We stated, without proof, that particles moving slowly with speed v through very viscous fluids experience linear drag, with $F \sim -v$, while particles moving more quickly through less viscous fluids experience quadratic drag $F \sim -v^2$. (For what it's worth, we'll recover the equations for linear drag in Section 3.4, while quadratic drag involves turbulent flows and is not so easy to derive from first principles.) The first question that we want to ask in these lectures is: what kind of friction force does the fluid experience when it rubs against itself?

The answer is that the viscosity is *linear* in the fluid velocity $\mathbf{u}(\mathbf{x}, t)$. At heart, this statement follows simply from Taylor's theorem. The underlying atomic interactions are complicated and they surely result in a friction force that is some arbitrarily complicated function of $\mathbf{u}$. But, for suitably small velocities, the linear term is always larger than the quadratic term. This simple but powerful argument is known as *linear response*. It's the same argument that leads to Ohm's law with the current given by $I = V/R$ rather than some more complicated function of voltage. (The lecture notes on Kinetic Theory contain a section devoted to linear response in more general settings.) Those fluids for which the linear approximation is a good one are called *Newtonian*.

So the viscosity should be a force $\mathbf{f}$ that is linear in the fluid velocity $\mathbf{u}$. What can we write down? First, the force can't be proportional to $\mathbf{u}$ itself: that's in contradiction with Gallilean relativity which says that the equations of fluid mechanics must be invariant under a boost of the whole fluid (and any container) by a constant $\mathbf{u}$. This

tallies with the idea that viscosity is associated to the friction force experienced by the fluid when one part rubs up against another. That means that the different parts of the fluid should be moving at different speeds for viscosity to kick in. Or, in other words, the friction force must depend on spatial changes of $\mathbf{u}$.

The simplest possibility is that the friction force depends on the first derivative of $\mathbf{u}$, but there is only one vector that we can form in this way and that's the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. But this isn't a good candidate for a force. This is because of the symmetry of *parity* or reflections, $\mathbf{x} \to -\mathbf{x}$. Under parity $\mathbf{u} \to -\mathbf{u}$ and each term in the Euler equation is odd. This should continue to be true of any force that acts on the fluid. But vorticity is even: $\boldsymbol{\omega} \to \boldsymbol{\omega}$. It cannot arise as a force.

This means that we must look to two-derivative terms if we are to build a force linear in velocity. There are two possibilities: $\nabla^2\mathbf{u}$ and $\nabla(\nabla \cdot \mathbf{u})$. But, given that we are dealing with an incompressible fluid, the second of these necessarily vanishes. We're left with just one choice for our friction force,

$$\mathbf{f}_{\text{viscous}} = \mu \nabla^2 \mathbf{u}$$

The coefficient μ is a constant known as the *dynamic (shear) viscosity*[2]. The resulting equation describing the motion of fluids is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{u} + \mathbf{f} \tag{3.2}$$

where, in $\mathbf{f}$, we retain the option to include any other external force such as gravity. This is the famous *Navier-Stokes equation*. Its solutions will occupy us for the rest of this course.

In what follows, we will also frequently come across the ratio

$$\nu = \frac{\mu}{\rho}$$

This arises so frequently that it is given its own name: it is called the *kinematic viscosity*.

Simple as the Navier-Stokes equation is, there is a great deal about it that remains to be understood. This includes the most basic questions about the existence and uniqueness of solutions. We will touch upon some of these issues in these lectures although much our focus will be on the things that are known rather than those that are not.

[2]Annoyingly, the shear viscosity was denoted as η in the lectures on Kinetic Theory. Sorry.

Viscosity Causes Diffusion of Momentum

As these lectures progress, we'll develop some intuition for what viscosity does. But even before we get going, we can gain some insight through analogy. If we focus on two terms in the Navier-Stokes equation, ignoring the others (and also ignoring why we might be allowed to ignore them!) we have

$$\rho\frac{\partial \mathbf{u}}{\partial t} \sim \mu\nabla^2\mathbf{u}$$

It's useful to compare this to the heat equation which describes how temperature changes

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c_V}\nabla^2 T$$

where (for what it's worth) κ is the thermal conductivity and c_V the specific heat. Solutions to the heat equation are well studied: if we start with a hot spot somewhere, a place where there is a localised increase in temperature, then it will spread out increasing in size as $L \sim \sqrt{t}$. This kind of behaviour is called *diffusion*. It is the characteristic behaviour of any conserved quantity when undergoing random bombardment by some microscopic process.

This also tells us how to think of the viscosity term in the Navier-Stokes equation. It is causing momentum $\rho\mathbf{u}$ to diffuse. It doesn't cause the momentum to change direction. But if there was some place in the fluid where the momentum density was higher then the viscosity term will cause this to spread out, much like temperature in the heat equation. This also makes physical sense: if the momentum density is higher in one region of the fluid then that region will be rubbing against its neighbouring regions. Viscosity is the friction force induced by this rubbing and results in a transfer of momentum from one region into neighbouring regions.

Viscosity Breaks Time Reversal

More intuition comes from the observation that the additional viscosity term breaks the symmetry of *time reversal*. This acts as

$$T : t \to -t \quad , \quad T : \mathbf{x} \to \mathbf{x} \quad , \quad T : \mathbf{u} \to -\mathbf{u}$$

You can check that all the terms in the Euler equation are invariant under this transformation (at least if the external force is time-reversal invariant, so $T : \mathbf{f} \to \mathbf{f}$). But the extra viscosity term is not invariant under time-reversal: it transforms as

$$T : \nabla^2\mathbf{u} \to -\nabla^2\mathbf{u}$$

This is important. If we're given any solution to the Euler equation, then we can always run it backwards in time and we will get another solution. This is not true of solutions to the Navier-Stokes equation which exhibits an arrow of time. This is because, as advertised above, the viscosity is a friction force which, like other friction forces in classical mechanics, causes the system to lose energy. We'll see this explicitly in Section 3.1.2 where we compute the energy lost due to viscosity.

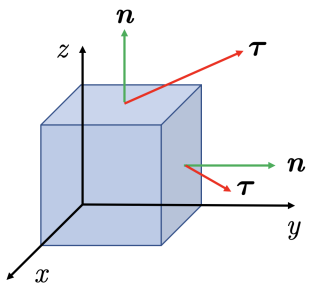
3.1 Stress, Strain and Viscosity

We'll now give a slightly more involved derivation of the Navier-Stokes equation which will allow us to unpick the meaning of viscosity. To do this, we go back to basics and start thinking afresh about the kinds of forces that act on a fluid. Recall from (2.5) that, for a volume V of fluid, the equation $F = ma$ becomes

$$\rho \int_V \frac{D\mathbf{u}}{Dt}\, dV = -\int_S P\, d\mathbf{S} + \text{Other Forces} \tag{3.3}$$

with $S = \partial V$ the surface of the volume. Importantly, the pressure P acts on the surface of the volume and this ensures that it appears in the Navier-Stokes equation as the gradient ∇P as the surface integral is converted to a volume integral by the divergence theorem. Similarly, the friction forces also naturally act on the surface of the volume as a neighbouring piece of fluid brushes past. So our first task is to better understand what the general kind of force acting on a surface looks like.

Consider a small cubic volume V as shown in the figure. Obviously there are six sides, and there may be a force acting on each. The pressure force is special because it acts parallel to the normal on each side. But that not necessarily the case for all forces. In general, the force might act in any direction. Moreover, the direction of the force will generally depend on the orientation of the surface. For example, this is obviously true of pressure which is parallel to the normal. The figure to the right shows the normals to two faces in green, while the force acting on those faces is shown in red.



These considerations mean that to specify the force that acts on a surface, we first have to specify the orientation to the surface. This is achieved through the introduction of the *stress tensor*, σ_{ij}. It is defined so that the force $\mathbf{F}$ acting on a small surface of

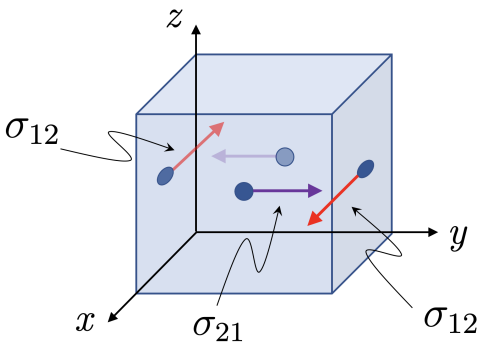


Figure 9. The torque experienced by a small parcel of fluid. The red arrows depict σ_{12}, the purple arrows σ_{21}. (The purple arrow on the furthest face is unlabelled to avoid clutter.)

area δA and normal $\mathbf{n}$ is given by

$$\mathbf{F} := \mathbf{f}\,\delta A = \sigma \mathbf{n}\,\delta A$$

Here $\mathbf{f}$ is the force per unit area. (Like pressure). In index notation, we have

$$f_i = \sigma_{ij} n_j \tag{3.4}$$

For pressure, the stress tensor takes the simple diagonal form

$$\sigma_{ij} = -P\,\delta_{ij}$$

But, in general, it may take a more complicated form. Furthermore, for a fluid the stress tensor is itself a field $\sigma_{ij}(\mathbf{x}, t)$ that may vary in both space and time. This means that the forces acting on various parts of the fluid depend both on their position in the fluid and on the orientation of the surface that is considered.

The stress tensor has an important property: it is symmetric

$$\sigma_{ij} = \sigma_{ji}$$

We will now show this. Consider the (slightly messy) Figure 9 depicting a small cube with side lengths L. The two red lines depict the force (per unit area) in the x direction on the faces that are normal to the y direction. From (3.4) this force is σ_{12}. Meanwhile, the two purple lines depict the force in the y-direction on the faces that are normal to the x direction. This force is σ_{21}.

These four forces give rise to a torque. Each σ_{ij} is a force-per-unit-area, so the actual force is $L^2\sigma_{ij}$. Furthermore, the moment of each force about the centre of the cube is $L/2$. This means that the total torque around a line parallel to the z-axis, through the centre of the cube, is

$$\tau = L^3(\sigma_{12} - \sigma_{21}) + \mathcal{O}\left(\frac{\partial\sigma_{12}}{\partial y}L^4, \frac{\partial\sigma_{21}}{\partial x}L^4\right)$$

The leading term comes from the difference between σ_{12} and σ_{21}. The sub-leading terms come from the difference of, say, σ_{12} on the left and right-hand faces. (The statement that the cube is small is the assumption that σ_{ij} does not vary much over the length scale L.)

Further torque may come from bulk forces whose strength varies over the inside of the cube. But this torque will always be of order L^4 (times some suitable dimensionful parameter) so, for small cubes, the leading contribution to the torque is proportional to the difference $(\sigma_{12} - \sigma_{21})$ and scales as L^3.

But torques that scale as L^3 are bad. To see this, recall that the angular acceleration is given by $\dot{\omega} = \tau/I$ where I is the moment of inertia. But the moment of inertia of any object always scales as L^5 (which is mass $\times$ $L^2 = \rho L^5$) and so $\dot{\omega} \sim 1/L^2$. The actual speed of the object is $v \sim \omega L$ so if the torque scales as L^3, the acceleration will diverge as $\dot{v} \sim 1/L$ for small L. That makes no sense. To avoid this we must have

$$\sigma_{12} = \sigma_{21}$$

Obviously the same argument works for all other components: $\sigma_{ij} = \sigma_{ji}$. The stress tensor is necessarily symmetric.

3.1.1 Newtonian Fluids

With the technology of the stress tensor, it is straightforward to describe the effect of friction. A *Newtonian fluid* is one where the friction forces are linear in velocity. If we assume that the fluid is isotropic then the form of the force is pretty much fixed by rotational invariance: it must be a symmetric tensor constructed from ∇ and $\mathbf{u}$ and the only option is $\partial_i u_j + \partial_j u_i$. In fact, a symmetric tensor can be decomposed into its trace and a traceless piece (see the lectures on Vector Calculus) so in general we have, including the pressure term,

$$\sigma_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - \frac{2}{3}\nabla\cdot\mathbf{u}\,\delta_{ij}\right) + \zeta\nabla\cdot\mathbf{u}\,\delta_{ij}$$

where, as we saw previously, μ is the *dynamical shear viscosity*. This time we've included the extra term proportional to $\nabla \cdot \mathbf{u}$ with a coefficient ζ known as the *bulk viscosity* or sometimes the *volume viscosity*. Importantly, it can be shown that each of these coefficients is necessarily positive. For μ, this follows from energy dissipation and we will give the argument shortly. For ζ it turns out that this follows from considerations of entropy. However, in this course we are dealing only with incompressible fluids with $\nabla \cdot \mathbf{u} = 0$ which means that we can forget all about the bulk viscosity. We have

$$\sigma_{ij} = -P\delta_{ij} + 2\mu E_{ij} \tag{3.5}$$

where E_{ij} is the rate of strain tensor that we met previously (2.14)

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) \tag{3.6}$$

We can now use this form of the stress tensor in the equation of motion for the fluid. With a general surface force, captured by σ_{ij}, the equation of motion for a fluid (3.3) become

$$\rho \int_V \frac{Du_i}{Dt}\, dV = \int_S \sigma_{ij}\, dS^j$$

where we have neglected other forces such as gravity. We use the divergence theorem to change the surface integral into a volume integral

$$\rho \int_V \frac{Du_i}{Dt}\, dV = \int_V \frac{\partial \sigma_{ij}}{\partial x^j}\, dV$$

This formula holds for arbitrary volume V, so the equation of motion is

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x^j} \tag{3.7}$$

From our equation (3.5), the right-hand side becomes

$$\frac{\partial \sigma_{ij}}{\partial x^j} = -\frac{\partial P}{\partial x^i} + \mu\left(\frac{\partial^2 u_i}{\partial x^j \partial x^j} + \frac{\partial^2 u_j}{\partial x^j \partial x^i}\right)$$

The second of these vanishes, again using our incompressibility condition $\nabla \cdot \mathbf{u} = 0$, and we're left with promised Navier-Stokes equation,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{u}$$

In what follows, we'll often divide by the density to write the Navier-Stokes equation as

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{u}$$

where, as defined earlier, $\nu = \mu/\rho$ is the kinematic viscosity. We can also add further forces on the right-hand side to taste.

The derivation of the Navier-Stokes equation that we described above sits entirely within the continuum language that underlies this course. There is another remarkable, and ultimately better, derivation that really goes back to basics. This is due to Boltzmann. The derivation starts with the underlying $\sim 10^{23}$ atoms and tracks their interactions, albeit in a statistical way. It explains why the variables of the Navier-Stokes equation are the right thing to focus on if you care only about long-time physics and gives a microscopic explanation of the various terms. You can find this derivation in the lectures on Kinetic Theory.

3.1.2 Momentum and Energy Conservation Revisited

For inviscid fluids, the Euler equation is simply the statement that momentum is conserved, while energy conservation (2.11) led to Bernoulli's principle. What becomes of these in the presence of viscosity?

First momentum. Here there is no problem: in the absence of external forces, we can write the Navier-Stokes equation in the form of a continuity equation, telling us that momentum is conserved. The only difference from the Euler equation is that we get an extra term in the momentum current, proportional to the viscosity

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x^j} = 0 \quad \text{with} \quad \Pi_{ij} = \rho u_i u_j + P\delta_{ij} - 2\mu E_{ij}$$

As before, we've used the fact that $\nabla \cdot \mathbf{u} = 0$ for incompressible fluids. In particular, we've used this to keep Π_{ij} symmetric by taking the extra term proportional to the rate of strain tensor (3.6) rather than just $\partial_j u_i$.

This gives us another perspective on the Navier-Stokes equation: it is, like the Euler equation, simply conservation of momentum, but with an additional term in the momentum current coming from gradients of the velocity. The idea that gradients drive currents is something that also occurs in other, perhaps more familiar, contexts where it goes by the name of *Fick's law*. For example, differences in temperature result in a heat current $\mathbf{J} \sim \nabla T$.

What about energy? We will ignore other bulk forces for now. (We've already seen in Section 2.1.4 that conservative forces don't spoil conservation of energy.) However, it's useful to briefly return to the form of the Navier-Stokes equation (3.7) in which we allow for general stress forces σ_{ij}. Taking the inner product with the velocity $\mathbf{u}$, the matter derivative becomes

$$\mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} = u_i \cdot \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x^j} \right) = \frac{1}{2}\frac{\partial |\mathbf{u}|^2}{\partial t} + \frac{1}{2}\mathbf{u} \cdot \nabla |\mathbf{u}|^2$$

and our proto-Navier-Stokes equation (3.7) becomes

$$\frac{\rho}{2}\left(\frac{\partial|\mathbf{u}|^2}{\partial t}+\mathbf{u}\cdot\nabla|\mathbf{u}|^2\right)=u_i\frac{\partial\sigma_{ij}}{\partial x^j}$$

Remember the game that we're playing: we'd like to massage this into the continuity equation to see the conservation of energy. Using the fact the fluid is incompressible, so $\nabla\cdot\mathbf{u}=0$, we have

$$\frac{\rho}{2}\frac{\partial|\mathbf{u}|^2}{\partial t}+\frac{\partial}{\partial x^j}\left(\frac{\rho}{2}|\mathbf{u}|^2u_j-u_i\sigma_{ij}\right)=-\sigma_{ij}\frac{\partial u_i}{\partial x^j}=-\sigma_{ij}E_{ij}$$

where, in the second equality, we've used the fact that $\sigma_{ij}=\sigma_{ji}$ so the contraction picks out the symmetric part of $\partial u_i/\partial x^j$ which is E_{ij}, the rate of strain tensor defined in (3.6). The two terms on the left-hand side take the form of a continuity equation. But now the right-hand side is not zero. This tells us that, in contrast to the Euler equation, energy is *not* conserved in the Navier-Stokes equation.

We can get an expression for how energy is low. If we integrate over some fixed volume V then we have

$$\frac{\rho}{2}\frac{\partial}{\partial t}\int_V|\mathbf{u}|^2dV+\int_S\left(\frac{\rho}{2}|\mathbf{u}|^2u_j-u_i\sigma_{ij}\right)dS^j=-\int_V\sigma_{ij}E_{ij}\;dV \tag{3.8}$$

with $S=\partial V$. The volume term on the left-hand side is clearly the change in the kinetic energy in V. The surface term accounts for (some of) this change: the $|\mathbf{u}|^2u_j$ term captures the energy that flows out through the surface, while the $\sigma_{ij}u^j$ is the work done by the surface forces on the fluid contained in V. This includes the work done by the both the pressure and by the viscous forces. All of this is consistent with the conservation of energy. However, because the right-hand side of (3.8) doesn't vanish is telling us that energy is, in fact, no longer conserved. Instead, the right-hand side tells us the rate at which energy is dissipated.

$$\text{Dissipation}=\int_V\sigma_{ij}E_{ij}\;dV=2\mu\int_VE_{ij}E_{ij}\;dV \tag{3.9}$$

where, in the second equality, we've used the explicit form of the stress tensor (3.5). We see that the pressure doesn't contribute to energy dissipation (because $\delta_{ij}E_{ij}=\nabla\cdot\mathbf{u}=0$). This, of course, is something that we found when studying the Euler equation. But we now see that one important consequence of viscosity is we no longer have energy conservation. Correspondingly, the Bernoulli's principle no longer holds when the effects of viscosity are important

The dissipation is the integral of a total square, so it clearly positive provided that $\mu > 0$. And the minus sign on the right-hand side of (3.8) is telling us that energy is lost to friction, rather than gained. This is reason why we should take $\mu > 0$.

It is natural to ask: where did the energy go?! After all, energy is certainly conserved at a fundamental level. The answer is that it went into heat. The dissipation (3.9) is a transfer of energy from the macroscopic, coherent kinetic energy of the fluid, captured by the coarse-grained velocity field $\mathbf{u}$, to some microscopic, incoherent internal motion of the underlying atoms. This internal motion is still kinetic energy, but not with any overall preferred direction. To properly account for this, we should understand how the temperature and entropy of the fluid changes due to these dissipative effects. As with friction forces in classical mechanics, we won't attempt to do this here: we will simply count this as lost energy. (We will, however, return to the interplay of heat and energy in Section 4.4 when we discuss sound waves.)

3.2 Some Simple Viscous Flows

Our first task is to explore some very simple solutions to the Navier-Stokes equation (3.2). This will allow us to build some intuition for the role that viscosity plays.

3.2.1 The No-Slip Boundary Condition

We've already seen the importance of boundary conditions in constructing fluid flows. For an inviscid flow, we introduced the obvious "you shall not pass" condition in Section 2.3

$$\mathbf{n} \cdot \mathbf{u} = 0 \tag{3.10}$$

where $\mathbf{n}$ is the normal to a solid surface. This solid surface might be the walls of the container, or an obstacle sitting in the fluid like the sphere and cylinder we studied previously. If the solid object is itself moving with some velocity $\mathbf{U}$ then this condition becomes

$$\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \mathbf{U}$$

For viscous fluids, we introduce a further boundary condition that restricts the flow *tangent* to a solid. This is the *no-slip* condition that states

$$\mathbf{t} \cdot \mathbf{u} = \mathbf{t} \cdot \mathbf{U} \tag{3.11}$$

where $\mathbf{t}$ is now the vector tangent to the boundary. This states that the velocity of the fluid along the boundary must match the velocity of the boundary itself. It is sometimes written as the requirement that $\mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$ is continuous at the boundary.

The no-slip condition (3.11) doesn't follow from the Navier-Stokes equation. Instead, it is something additional that we assert. It is, however, physically sensible and arises from the friction forces between the fluid and the boundary. Importantly, it is also the boundary condition that is observed to be correct for most experiments.

Note that the flows that we met in Section 2 describing fluids moving around spheres and cylinders do *not* obey the no-slip condition. Of course, they also failed miserably in explaining drag forces. This is our first hint that we should do a better job of describing the flows close to the boundary of an object. You might wonder why we just don't search for other solutions to the Euler equations that include the no-slip condition. The reason is that there simply aren't any such solutions. This is because the Euler equation is first order in spatial derivatives and we only get to impose one boundary condition, namely the inpenetrability condition (3.10). In contrast, the Navier-Stokes equation is second order. This means that we must impose an additional boundary condition when solving the equation. The no-slip condition is the boundary condition of choice.

3.2.2 Couette Flow

Take two, infinite parallel plates lying in the (x, y) plane and separated by some distance h in the z-direction. The bottom plate is stationary while the top plate moves with a constant speed U in the x-direction. What happens to fluid trapped between them?

We will look for a steady flow with $\partial\mathbf{u}/\partial t$ with the velocity lying solely in the x-direction. The speed of the fluid depends only on the z direction, meaning

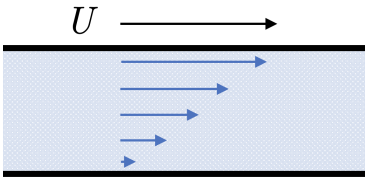
$$\mathbf{u}(\mathbf{x}, t) = (u(z), 0, 0)$$

With this ansatz $(\mathbf{u}\cdot\nabla)\mathbf{u} = 0$ so the material time derivative vanishes: $D\mathbf{u}/Dt = 0$. There are no pressure gradients in the fluid, so the only surviving term in the Navier-Stokes equation comes from the viscosity,

$$\mu\frac{d^2u}{dz^2} = 0$$

The boundary conditions are $u(0) = 0$ and $u(h) = U$. This is an easy equation to solve and the velocity profile must increase linearly to match the speeds of the two plates,

$$u(z) = \frac{Uz}{h}$$



The result is known as *Couette flow* and is shown in the figure. Flows of this kind, in which adjacent layers of fluids move at different speeds, are collectively referred to as *shear flows*.

Couette flow is not a potential flow. The simplest way to see this is to note that, even though the flow doesn't look like its rotating, it has vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = (0, U/h, 0)$$

This vorticity arises because we've implemented the no-slip boundary condition, ensuring that the upper plate drags the fluid along with it. This suggests that the no-slip boundary condition may be a way to generate vorticity. We will see later that this is an important observation.

It is a simple matter to compute the stress exterted on the fluid using (3.5),

$$\sigma = \begin{pmatrix} -P & 0 & \mu U/h \\ 0 & -P & 0 \\ \mu U/h & 0 & -P \end{pmatrix}$$

This tells us that the force per unit area exerted by the top plate with $\mathbf{n} = \hat{\mathbf{z}}$ is

$$\mathbf{f} = (\mu U/h, 0, -P)$$

while the bottom plate, with $\mathbf{n} = -\hat{\mathbf{z}}$ exerts an equal and opposite force. We usually think of the bottom plate, and the distance h between the plates, as fixed externally. We then ask what force we have to exert on the upper plate to keep it moving if it has (large) area A. The answer is

$$\frac{F}{A} = \frac{\mu U}{h}$$

This is operational definition of viscosity μ that we met in our first course on Newtonian Mechanics. The work done by this pushing (again, per unit area) is just $\mu U^2/h$. You can check that this agrees with the more formal definition of dissipation given in (3.9).

Circular Couette Flow

The same basic idea arises in different geometries. Consider, for example, two concentric, infinite cylinders, aligned along the z-direction. The inner cylinder has radius R_1 and rotates with angular velocity Ω_1. The outer cylinder has radius R_2 and rotates with angular velocity Ω_2.

From the geometry, we see that the flow should be rotationally invariant, meaning that it takes the form

$$\mathbf{u} = \Omega(r)\,(y, -x, 0)$$

where $r^2 = x^2 + y^2$ and $\Omega(r)$ is the angular velocity of the fluid. The no-slip condition implements the boundary conditions $\Omega(R_1) = \Omega_1$ and $\Omega(R_2) = \Omega_2$.

This time the story is a little different because we can no longer ignore the non-linear term in the Navier-Stokes equation,

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -r\Omega^2\hat{\mathbf{r}}$$

But this is something familiar, it is just the outward pointing centrifugal force that comes from the rotation of the fluid. It gives rise to a pressure gradient in the fluid, with the radial pressure $P(r)$ obeying

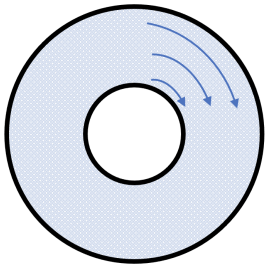
$$\frac{\partial P}{\partial r} = r\Omega^2 \quad \Rightarrow \quad \frac{D\mathbf{u}}{Dt} = -\nabla P$$

Such a flow obeys the Euler equation for any choice of $\Omega(r)$. But to satisfy the Navier-Stokes equation we must have, in addition,

$$\mu\nabla^2\mathbf{u} = 0$$

A quick calculation shows that $\nabla^2\mathbf{u} = (3\Omega'/r + \Omega'')(y, -x, 0)$ so the angular velocity of the flow must take the form

$$\Omega'' = -\frac{3\Omega'}{r} \quad \Rightarrow \quad \Omega = A + \frac{B}{r^2}$$



The first term is just a constant rotation, while the second term corresponds to the irrotational line vortex that we met in Section 2.2. The no-slip boundary conditions fix these coefficients to be

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \quad \text{and} \quad B = (\Omega_1 - \Omega_2)\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}$$

This circular Couette flow is also known as *Taylor-Couette flow*. (Taylor gets his name attached because he discovered certain instabilities in the flow.)

3.2.3 Poiseuille Flow

Here's another simple example. Again, take a fluid sitting between two, infinite parallel plates lying in the (x, y) plane. This time is will be slight more convenient if we separate them by distance $2h$ in the z-direction. We take them to sit at $z = \pm h$.

In contrast to Couette flow, both plates are now stationary. However, this time we induce a constant pressure gradient through the fluid

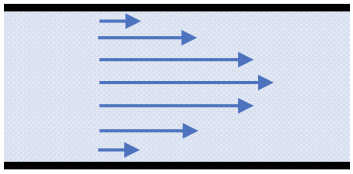
$$\frac{dP}{dx} = \text{constant}$$

We again look for a steady, shear flow of the form $\mathbf{u} = (u(z), 0, 0)$. The Navier-Stokes equation is now

$$\mu \frac{d^2u}{dz^2} = \frac{dP}{dx} = \text{constant}$$

With the no-slip boundary conditions $u(z = \pm h) = 0$, the solution is

$$u(z) = -\frac{1}{2\mu}\frac{dP}{dx}(h^2 - z^2) \qquad (3.12)$$



This is known as *Poiseuille flow*. The minus sign is sensible: it tells us that if the pressure is greatest to the left, so $dP/dx < 0$, then the fluid moves to the right. Clearly the speed increases as we move away from the edges and is maximum in the middle where $z = 0$. Again, the flow has vorticity induced by the no-slip boundary condition.

The stress (3.5) is

$$\sigma = \begin{pmatrix} -P(x) & 0 & z\,dP/dx \\ 0 & -P(x) & 0 \\ z\,dP/dx & 0 & -P(x) \end{pmatrix}$$

and, perhaps surprisingly, is independent of the viscosity. The top and bottom plates have normal $\mathbf{n} = \pm\hat{\mathbf{z}}$ and sits at $z = \pm h$, giving a force per unit area

$$\mathbf{f} = (h\frac{dP}{dx}, 0, \mp P(x))$$

The force exerted by each plate is now in the negative x direction, as it should be.

Circular Poiseuille Flow

A simple generalisation of this story describes flow down a circular pipe of radius R with a constant pressure gradient. We work in cylindrical polar coordinates, (r, θ, x) with

$$\frac{dP}{dx} \neq 0$$

The velocity takes the form

$$\mathbf{u} = u(r)\hat{\mathbf{x}}$$

The Navier-Stokes equation is

$$\mu\nabla^2 u = \frac{dP}{dx} \quad \Rightarrow \quad \frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{dP}{dx}$$

The solution with the appropriate boundary conditions is

$$u(r) = -\frac{1}{4\mu}\frac{dP}{dx}(R^2 - r^2)$$

This is known as *Hagen-Poiseuille flow*.

3.2.4 Vorticity Revisited and the Burgers Vortex

As our final example of a flow, we will look at something that swirls. This gives us the opportunity to revisit vorticity in the presence of viscosity.

Previously we derived the vorticity equation (2.20) from the Euler equation. We can follow the same steps, now taking the curl of the Navier-Stokes equation to find

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} \tag{3.13}$$

This is the *vorticity equation* for a viscous fluid. The term due to viscosity, naturally written in terms of $\nu = \mu/\rho$, should be viewed as analogous to the diffusion term in the heat equation. Just as viscosity gives rise to diffusion of momentum, so it gives rise to diffusion of vorticity too. It is telling us that if there is some vorticity localised in some region of space, the viscosity will tend to make it diffuse into neighbouring regions. For example, if you blow a smoke ring then the size of the ring will grow over time as the vorticity diffuses into neighbouring regions.

For inviscid fluids, the Kelvin circulation theorem told us that $\Gamma = \oint_{C(t)} \mathbf{u}\cdot d\mathbf{x}$ doesn't change for curves $C(t)$ that move with the fluid. You can check that the addition of the viscosity term means that the circulation is no longer conserved in the full Navier-Stokes equations.

Burgers Vortex

To highlight how viscosity changes the physics, we can return to the vortex solution that we saw back in Section 2.2. There we looked at a combination of a strain and rotation,

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_{\text{strain}}(\mathbf{x}) + \mathbf{u}_{\text{rot}}(\mathbf{x},t) \quad \text{with} \quad \begin{cases} \mathbf{u}_{\text{strain}} = \alpha(-x,-y,2z) \\ \mathbf{u}_{\text{rot}} = f(r,t)(-y,x,0) \end{cases}$$

The strain part of the flow stretches the fluid in the z-direction, while squeezing in the (x,y)-plane; the rotational flow clearly rotates in the (x,y)-plane, giving rise to a vorticity $\boldsymbol{\omega} = (0,0,\omega)$ with ω given by (2.16),

$$\omega = \frac{1}{r}\frac{d}{dr}(r^2 f) \tag{3.14}$$

The vorticity equation (3.13) is a partial differential equation for ω,

$$\frac{\partial\omega}{\partial t} - \alpha r\frac{\partial\omega}{\partial r} - 2\alpha\omega = \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\omega}{\partial r}\right) \tag{3.15}$$

Previously we solved this equation when $\nu = 0$ to find an example of vortex stretching (2.22). The solution we found was time dependent, with $\omega(r,t) = e^{2\alpha t}\,W(re^{\alpha t})$ and shows the magnitude of vorticity increasing exponentially, while being squeezed in the (x,y) plane so that the overall flux is conserved, in a way that is consistent with the circulation theorem.

Now we want to solve the vorticity equation with $\nu \neq 0$ to include the effect of viscosity. We already noted that the contribution $\nu\nabla^2\boldsymbol{\omega}$ to the vorticity equation looks like a diffusion term. This suggests that we might be able to find a time independent solution in which the squeezing of vorticity is balanced by an outward diffusion caused by the viscosity. For steady solutions, the equation (3.15) becomes

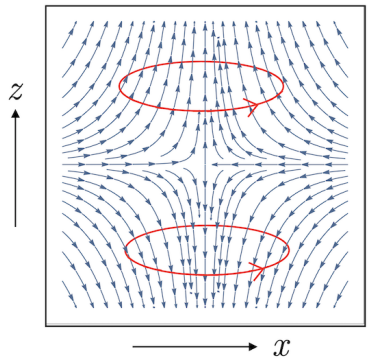
$$\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha r^2\omega + \nu r\frac{\partial\omega}{\partial r}\right) = 0$$

We can integrate once to get

$$\frac{\partial\omega}{\partial r} = -\frac{\alpha r}{\nu}\omega$$

where we've set the integration constant to zero by requiring that ω and ω' decay suitably quickly asymptotically. This equation gives an exponentially localised vorticity

$$\omega(r) = \frac{\Gamma\alpha}{2\pi\nu}\, e^{-\alpha r^2/2\nu}$$



Here Γ is a constant that the determines the overall magnitude of vorticity. The slightly strange combination of constants that accompany it ensure that Γ can also be identified with the asymptotic circulation of the flow,

$$\Gamma = \int_S \boldsymbol{\omega}\cdot d\mathbf{S} = 2\pi\int_0^\infty dr\; r\omega(r)$$

We can then solve (3.14) to get the associated profile function for the angular velocity,

$$f(r) = \frac{\Gamma}{2\pi r^2}\left(1 - e^{-\alpha r^2/2\nu}\right)$$

This is *Burgers vortex solution*. It is the simplest model for a hurricane.

We can compute the dissipation due to the vortex. We first rewrite our previous formula (3.9) as

$$\text{Dissipation} = 2\mu\int d^3x\; E_{ij}E_{ij} = \mu\int d^3x\; |\boldsymbol{\omega}|^2$$

where some simple algebra shows that the difference is a boundary term which vanishes at infinity. It is now a simple computation to get the dissipation per unit length

$$\text{Dissipation per unit length} = 2\pi\mu\int_0^\infty dr\; r\omega^2 = \frac{\Gamma^2\alpha\rho}{4\pi}$$

where the density ρ has made an appearance through $\mu = \nu\rho$. Curiously, for fixed circulation Γ, the dissipation is independent of the viscosity ν.

3.3 Dimensional Analysis

The Navier-Stokes equation is

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{u} \tag{3.16}$$

Each term has dimension LT^{-2}. This means that the dimension of the kinematic viscosity ν is

$$[\nu] = L^2T^{-1}$$

Fluid	Kinematic Viscosity (m^2s^{-1})	Dynamic Viscosity (Nsm^{-2})
Air	1.5×10^{-5}	1.8×10^{-5}
Water	10^{-6}	10^{-3}
Honey	$\sim 10^{-3}$	~ 10
Pitch	$\sim 10^{5}$	$\sim 10^{8}$

Table 1. The viscosities of some substances at room temperature.

Meanwhile, the dimension of dynamic viscosity $\mu = \rho\nu$ is

$$[\mu] = ML^{-1}T^{-1}$$

Values of these viscosities for various fluids are shown in Table 1. To get a sense of the scales involved, we can do some further dimensional analysis. The kinematic viscosity has dimension of velocity times distance. For a fluid, the relevant internal velocity (as opposed to the velocity of some flow) is the speed of sound, c_s. On dimensional grounds, this is given by

$$c_s \sim \sqrt{\frac{k_B T}{m}}$$

where T is the temperature, k_B is Boltzmann's constant and m is the mass of the constituent atom or molecule. (We'll derive this formula, together with the overall coefficient, in Section 4.4. You can also find a derivation in the lectures on Kinetic Theory.) Meanwhile, the relevant distance scale is the average separation a of atoms in the fluid. This suggests that the viscosity should be of order

$$\nu \sim c_s a$$

For water, $c_s \sim 1000$ ms^{-1}, with a characteristic separation between molecules of $a \sim 10^{-9}$ m. This gives $\nu \sim 10^{-6}$ m^{2}s^{-1} which is, indeed, in the right ballpark.

For some fluids, the internal molecular forces are strong, resulting in a much higher viscosity. Honey is a particularly familiar example. One of the most viscous fluids is pitch, also known as tar, which has a viscosity many orders of magnitude higher than water[3].

[3]A pitch drop experiment was set up in the University of Queensland, Australia in 1927. The flow of the pitch is ten times slower than continental drift. To date, nine drops have fallen. None have been witnessed. A webcam was set up in the 1990s, but was offline when the eighth drop fell in 2000. The ninth drop was accidently broken off before it fell in 2014. You can watch and wait for the tenth drop here although there's likely to be another eight years or so before anything actually happens.

At the other end of the spectrum, superfluids, such as Helium-4 at low temperatures, have strictly zero viscosity. This is very much a quantum mechanical effect and a proper description requires us to leave the comfortable classical realm of these lectures.

3.3.1 The Reynolds Number

Solving the Navier-Stokes equation (3.16) in full generality is, to put it mildly, a challenging problem. We make progress only by making some approximation. This involves deciding which terms, if any, can be ignored in any given situation. The obvious thing to do is to ask whether the viscosity is small or large. But this question in itself doesn't make any sense. Viscosity is dimensionful. There's no meaning to it being absolutely small or absolutely large. It can only be small or large relative to something else.

That something else depends on the flow. Suppose that the flow has a characteristic speed U and length L. Here U could be the speed of the fluid relative to some boundary, or the rotational speed of the fluid. Similarly L could be some geometrical distance over which the flow changes. From this we can construct a dimensionless ratio called the *Reynolds number*

$$Re = \frac{UL}{\nu} \tag{3.17}$$

Roughly speaking, this captures the relative importance of the inertial term $\mathbf{u}\cdot\nabla\mathbf{u}$ and the viscosity term $\nu\nabla^2\nu$,

$$\frac{\text{intertial term}}{\text{viscosity term}} = \frac{|\mathbf{u}\cdot\nabla\mathbf{u}|}{|\nu\nabla^2\mathbf{u}|} \sim \frac{U^2/L}{\nu U/L^2} \sim Re$$

With very broad brush, fluid flows can be characterised in one of two different types:

- High Reynolds Number, $Re \gg 1$: In this case, the flow is inertia dominated. In many cases, we can drop the viscosity term and return to the Euler equation that we studied in Section 2. Flows at high Reynolds number have an associated time scale that comes from equating the kinetic term $\partial\mathbf{u}/\partial t$ with the inertial term. This time scale is simply the time it takes the fluid to move some distance: $T \sim L/U$.

  For example, for the flow past an aircraft wing, $U \sim 100$ ms^{-1} while $L \sim 1$m is the width of the wing. Using the value $\nu \sim 10^{-5}$, we have $Re \sim 10^7 \gg 1$ which suggests that the viscosity term is unimportant for such flows.

  However, this example also suggests that we should be nervous about such simple arguments. If we can really neglect viscosity at high Reynolds number then we

run smack into the d'Alembert paradox that we met previously because, as we saw in Section 2.3, the Euler equation doesn't correctly capture the drag force that a fluid exerts on an object. Indeed, the argument that we can ignore the viscosity term is precisely what led to physicists being unable to understand how planes fly! We'll resolve these issues in Section 3.5 where we will see that, even at high Reynolds number, there can be situations where the viscosity term is important after all because it qualitatively changes certain aspects of the flow, in particular through the introduction of a so-called "boundary layer".

- Low Reynolds Number, $Re \ll 1$: In this case, the flow is dominated by viscosity. If we ignore both the inertial term and the pressure term, then the Navier-Stokes equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \nabla^2 \mathbf{u} \tag{3.18}$$

  As we've seen previously, this is heat equation that describes diffusion. It's telling us that flows at low Reynolds number exhibit diffusive transport of momentum, with the kinematic viscosity understood as a measure of momentum diffusivity. In this regime, the time scale associated to the flow is $T \sim L^2/\nu$.

  For example, consider a bug of size $L \sim 10^{-5}$ m moving in water. It could be bombing along at a whopping $U \sim 10^{-5}$ ms^{-1} – that's one body length every second – but the associated Reynolds number is still $Re \sim 10^{-4}$. The bug's world, viscosity rules. We'll explore the low Reynolds world further in Section 3.4

Other Dimensionless Ratios

In different circumstances, there are further dimensionless ratios that we can form to characterise a flow and help us formulate good approximations to the equations. For example, if there is some characteristic time scale T to the flow – perhaps because the flow is being forced in some way – then we can form the *Strouhal number*

$$Sr = \frac{L}{UT}$$

This is also written as $Sr = L\omega/U$, with ω the frequency of oscillation. The Strouhal number tells us the relative importance of the acceleration term $\partial \mathbf{u}/\partial t \sim U/T$ and the inertial term $\mathbf{u} \cdot \nabla \mathbf{u} \sim U^2/L$, with the acceleration term dominant when $Sr \gg 1$.

We get further dimensionless numbers when we add further forces. For example

- The *Euler number* captures the relative importance of pressure gradients to the inertial term
$$Eu = \frac{\Delta P}{\rho U^2}$$
- The *Froude number* captures the relative importance of the inertial term $\sim U^2/L$ to the gravitational force $\sim g$
$$Fr = \frac{U}{\sqrt{gL}}$$

We'll meet other dimensionless quantities as these lectures progress. In Section 4.6, we'll come across the *Mach number* which measures how fast the flow is compared to the speed of sound, and in Section 5.3 the *Rayleigh number* and *Prandtl number*, both of which play a role when temperature differences are important.

3.3.2 Scaling

We upgraded ourselves from the Euler equation to the Navier-Stokes equation by adding a higher derivative term $\nabla^2\mathbf{u}$. But if we're happy to add a term with two derivative, why not further terms with four derivatives? Or sixteen derivatives? Why should we stop here?

In fact there is a reason why higher derivative terms are irrelevant, at least if we look on suitably large distance scales. (The term "irrelevant" has a technical meaning in the language of physics, but happily it coincides with the usual meaning in this context!) To see this note that, in the absence of any external force, the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{u}$$

has a novel scaling symmetry

$$t \to \lambda^2 t \quad , \quad \mathbf{x} \to \lambda\mathbf{x} \quad , \quad \mathbf{u} \to \lambda^{-1}\mathbf{u} \quad , \quad P \to \lambda^{-2}P \tag{3.19}$$

The whole Navier-Stokes equation scales with an overall factor of λ^{-3} under this scaling. But, crucially, all terms scale in the same way. This means that if we find one solution to the Navier-Stokes equations, then we can always rescale by some factor λ and get another solution. Because the spatial coordinate scales as $\mathbf{x} \to \lambda\mathbf{x}$, as we increase λ any features in the flow – for example, vortices – will clearly get bigger. Note that the Reynolds number (3.17) is invariant under this scaling: $Re \to Re$. This, in large part, is why it's important: the Reynolds number is a scale-invariant way of characterising a flow.

Now suppose that, in a fit of excitement, you decide that you'd like to add further terms to the Navier-Stokes equation. You should retain rotational symmetries and Galilean boosts (i.e. constant shifts of $\mathbf{u}$) but otherwise you can write down anything you like. The terms that you add will contain some number of time derivatives, spatial derivatives and factors of the fields $\mathbf{u}$ and P. Schematically, we might have

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{u} + \mathcal{O}\left(\partial_t^{n_1}, \nabla^{n_2}, \mathbf{u}^{n_3}, P^{n_4}\right)$$

where the integers n_i, with $i = 1, 2, 3, 4$, tell us the number of the various objects that appear. We can ask how this new term fares under the scaling symmetry (3.19). We have

$$\mathcal{O}\left(\partial_t^{n_1}, \nabla^{n_2}, \mathbf{u}^{n_3}, P^{n_4}\right) \to \lambda^{-(2n_1+n_2+n_3+2n_4)}\,\mathcal{O}\left(\partial_t^{n_1}, \nabla^{n_2}, \mathbf{u}^{n_3}, P^{n_4}\right)$$

The key point is that the Navier-Stokes equation already contains the leading terms, each of which scales as λ^{-3}. Any new term that you try to construct scales away more quickly than the λ^{-3}. This means that if you try to scale a flow to larger length scales, then these additional terms play an increasingly diminished role in determining the form of the solution. In particular, on suitably large length scales they will always be less important than those that appear in the Navier-Stokes equation. This is what we mean when we say that they are irrelevant.

This isn't to say that higher derivative terms are never important under any circumstances. If the gradients of fields are large enough, then higher derivative terms will surely compete with the others. But how large do they need to be? The answer to that is governed by the coefficients of these higher derivative terms which characterise the fluid. On dimensional grounds, these coefficients must have certain length or time dimensions, with the relevant scale set by some microscopic interactions. But these new scales are then likely to be set by microscopic physics – say the mean free path of the underlying molecules – and we certainly don't expect fluid mechanics to be relevant if we have large changes on such scales.

The upshot of this discussion is that the Navier-Stokes equation is special because each of the terms scales as λ^{-3} and any other term is always more irrelevant. For this reason, we never add any higher derivative terms. Instead we go the other way! Much of our work in the remainder of these lectures will be in figuring out what terms in the Navier-Stokes equation we get to drop in certain circumstances, in the hope that the equations may actually become easy enough to solve.

3.4 Stokes Flow

At low Reynolds number, $Re \ll 1$, the flow is dominated by viscosity. In many situations, we can ignore the matter derivative $D\mathbf{u}/Dt$ completely: it is unimportant for the physics of interest. What remains are the *Stokes equations*

$$\nabla P = \mu \nabla^2 \mathbf{u} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \tag{3.20}$$

We view these as four equations in four unknowns: $\mathbf{u}$ and P. They should certainly be augmented with the no-slip boundary condition, as appropriate for our uber-viscous Stokesian world. In some circumstances, we may wish to add further external forces $\mathbf{f}$ to the first of the equations.

Solutions to the Stokes equations are known as *Stokes flows*, or sometimes *creeping flows*. They describe, among many other things, micro-organisms swimming in water.

The lack of time derivatives is unusual when solving dynamical equations. It means that the fluid reacts instantaneously to any imposed force. In some sense, the fluid has no life of its own as there are no propagating waves. Instead, it just does what it's told.

More surprising, the lack of any time derivatives means that any flow is reversible. Act with an external force $\mathbf{F}$ for some time and the fluid will evolve. Then act with the opposite force $-\mathbf{F}$ for an equal amount of time and it will evolve back again, returning to its original state. There are dramatic demonstrations of this in which some ink is dropped in a fluid at low Reynolds number. The ink doesn't disperse, but just sits there. The fluid is then stirred, and the ink swirls and mixes with the fluid as expected. But when the stirring is reversed, so too is the mixing until the ink returns to its original starting point[4]. It's the kind of behaviour that the second law of thermodynamics usually prohibits. But life is different at low Reynolds number.

There's something a little disconcerting about this reversible behaviour. Not least because, as we've seen above, dissipation in fluids only arises because of viscosity. This means that the increase of entropy in a fluid is also due to viscosity. Yet, when viscosity completely dominates, the dynamics becomes reversible and there is no increase in entropy! Or, said better, there is no dynamics since we have neglected the time derivative term.

[4]Here's a rather wonderful video demonstrating this effect. It can also be seen in this old school documentary, with the great fluid dynamicist G.I.Taylor doing the mixing.

Solving the Stokes Equations

In the remainder of this section, we'll explore Stokes flow in a number of different settings. There are some simple manipulations that we can make to highlight the mathematical structure of the Stokes equations. Taking the divergence of both sides of $\nabla P = \mu\nabla^2\mathbf{u}$, and using the fact that $\nabla\cdot\mathbf{u} = 0$, tells us that the pressure is necessarily a harmonic function

$$\nabla^2 P = 0 \tag{3.21}$$

Meanwhile, taking the curl of both sides tells us that the vorticity $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is also harmonic

$$\nabla^2\boldsymbol{\omega} = 0 \tag{3.22}$$

Finally, acting with ∇^2 on both sides, and using the fact that $\nabla^2 P = 0$, tells us that the velocity itself is "biharmonic", meaning that

$$\nabla^4\mathbf{u} := \nabla^2\nabla^2\mathbf{u} = 0$$

In some situations, this is a useful starting point for solving the equations. But, for our first application, we'll take a different route.

3.4.1 Flow Around a Sphere

We want to repeat the calculation that we did for an inviscid fluid in Section 2.3 for the flow around a sphere. In that case, we found the solution by superimposing a constant flow with a dipole flow, and then hiding the singularity behind the sphere. Because the Stokes equations are linear, it's perfectly possible that a similar strategy will again work, now for very viscous fluids. We'll see that this is indeed the case, albeit with some of the details changed.

To kick things off, we'll look for the Green's function to the equations (3.20). This is a velocity field $\mathbf{u}$ and a pressure P obeying

$$\mu\nabla^2\mathbf{u} - \nabla P = -\mathbf{a}\,\delta^3(\mathbf{x}) \tag{3.23}$$

together with the requirement that $\nabla\cdot\mathbf{u} = 0$. The right-hand-side of (3.23) includes an arbitrary constant vector $\mathbf{a}$.

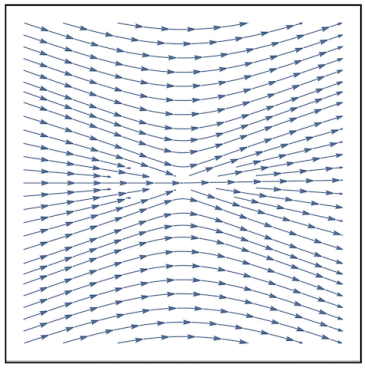
Claim: The Green's function for the Stokes equations is

$$\mathbf{u} = G\mathbf{a} \quad \text{and} \quad P = \frac{\mathbf{x}\cdot\mathbf{a}}{4\pi r^3} \tag{3.24}$$

where G is the matrix

$$G_{ij} = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right)$$

The tensor G is known as the *Stokeslet*. The two terms in G conspire to ensure that $\nabla \cdot \mathbf{u} = 0$. The Stokeslet flow is shown in the figure, with $\mathbf{a} = \hat{\mathbf{z}}$ pointing to the right.



Proof: First, we'll check that the solution (3.24) obeys $\mu\nabla^2\mathbf{u} = \nabla P$ everywhere except for $r = 0$. Then we'll check that the coefficient of the delta function works out. First look at the velocity term. We have

$$\mu\nabla^2 u_i = \frac{1}{8\pi}\nabla^2\left(\frac{a_i}{r} + \frac{x_i x_j a_j}{r^3}\right)$$

We recognise the first $1/r$ term as the Green's function for ∇^2 (we already met this interpretation when we discussed potential flows in Section 2.3), with $\nabla^2(1/r) = -4\pi\delta(\mathbf{x})$. Clearly this contributes to the delta function on the right-hand side of (3.23), but only with a coefficient of $\frac{1}{2}$. We'll see that another $\frac{1}{2}$ comes from the other terms. Staying away from $r = 0$ for now, a little bit of algebra is needed to differentiate the second term twice. We have

$$\mu\nabla^2 u_i = \frac{1}{8\pi}\partial_k\partial_k\left(\frac{x_i x_j a_j}{r^3}\right) = \frac{1}{4\pi}\left(\frac{a_i}{r^3} - 3\frac{x_i x_j a_j}{r^5}\right) \qquad \text{for } r \neq 0 \tag{3.25}$$

But now it's simple to check that this is cancelled by the pressure

$$(\nabla P)_i = \frac{1}{4\pi}\partial_i\left(\frac{x_j a_j}{r^3}\right) = \frac{1}{4\pi}\left(\frac{a_i}{r^3} - 3\frac{x_i x_j a_j}{r^5}\right) \qquad \text{for } r \neq 0$$

So we do indeed have a solution to (3.23) away from the origin. Now we just need to check that the $1/8\pi$ normalisation of G gives the correct strength for the delta function. For this we integrate over a ball of radius R centred at the origin, and use the divergence theorem to convert this into an integral over the sphere S^2_R of radius R,

$$\begin{aligned}
\int d^3x\ (\mu\nabla^2 u_k - \partial_k P) &= \int_{S^2_R} d^2S_i\ \left(\mu\partial_i G_{kj}a_j - \frac{1}{4\pi}\delta_{ik}\frac{x_j a_j}{r^3}\right) \\
&= \frac{a_j}{8\pi}\int_{S^2_R} d^2S_i\left(\partial_i\left(\frac{\delta_{kj}}{r} + \frac{x_k x_j}{r^3}\right) - 2\frac{\delta_{ik}x_j}{r^3}\right) \\
&= \frac{a_j}{8\pi}\int_{S^2_R} d^2S_i\ \left(-\frac{\delta_{jk}x_i}{r^3} - \frac{\delta_{ik}x_j}{r^3} + \frac{\delta_{ij}x_k}{r^3} - 3\frac{x_i x_j x_k}{r^5}\right)
\end{aligned}$$

At this stage, it's all about the placement of indices. The first term is straightforward: it is the usual integral of a radial field over a sphere and gives

$$\frac{a_j}{8\pi}\int_{S^2_R} d^2S_i\ \left(-\frac{\delta_{jk}x_i}{r^3}\right) = -\frac{1}{2}a_k$$

This is the same factor of $\frac{1}{2}$ contribution that we noted above. The remaining three terms in the integral must, ultimately, be proportional to δ_{kj} because that's the only invariant tensor available. A standard trick (see, for example, the lectures on Vector Calculus) is to take the trace over k and j indices and evaluate the integral: this then gives $3\times$ the coefficient in front of δ_{kj}. If we do this, we find that the second and third terms cancel, while the final term is

$$\frac{a_j}{8\pi}\int_{S^2_R} d^2S_i\ \left(-3\frac{x_ix_jx_k}{r^5}\right) = -\frac{1}{2}a_k$$

That's the extra factor of $\frac{1}{2}$ that we were looking for. We learn that our flow and pressure do indeed satisfy (3.23). □

Given a basic solution like (3.23), we can always generate further solutions by differentiating. These solutions will be more singular at the origin, but drop off quicker asymptotically. This is how the dipole solution is generated for potential flow (and, in fact, for electromagnetism). And it turns out to be what we need to solve our problem of the sphere. The relevant flow is again referred to as a dipole and is given by

$$\mathbf{u}_{\rm dipole} = (\nabla^2 G)\mathbf{a} \quad \text{with} \quad (\nabla^2 G)_{ij} = \frac{1}{4\pi\mu}\left(\frac{\delta_{ij}}{r^3} - 3\frac{x_ix_j}{r^5}\right)$$

where we computed $\nabla^2 G$ previously in (3.25). The associated pressure field is simply $P_{\rm dipole} = 0$ because, as we saw in (3.21), the original pressure (3.23) is necessarily a harmonic function.

We now have all the ingredients to solve our problem of interest: a Stokes flow around a sphere of radius R. Importantly, this flow must satisfy the no-slip condition which means that $\mathbf{u} = 0$ for all $|\mathbf{x}| = R$.

We start with a superposition of the different flows that we've found. We take a constant flow $\mathbf{u} = \mathbf{U}$, together with some combination of the Stokeslet and dipole flows. Both the latter flows involve some constant vector $\mathbf{a}$ and, on symmetry grounds, this must be proportional to the asymptotic velocity $\mathbf{U}$. We're left with

$$\begin{aligned}\mathbf{u} &= \mathbf{U} + 4\pi\mu\alpha\left(G + \beta\nabla^2 G\right)\mathbf{U} \\ &= \mathbf{U}\left(1 + \frac{\alpha}{2r} + \frac{\alpha\beta}{r^3}\right) + (\mathbf{U}\cdot\mathbf{x})\mathbf{x}\left(\frac{\alpha}{2r^3} - \frac{3\alpha\beta}{r^5}\right)\end{aligned}$$

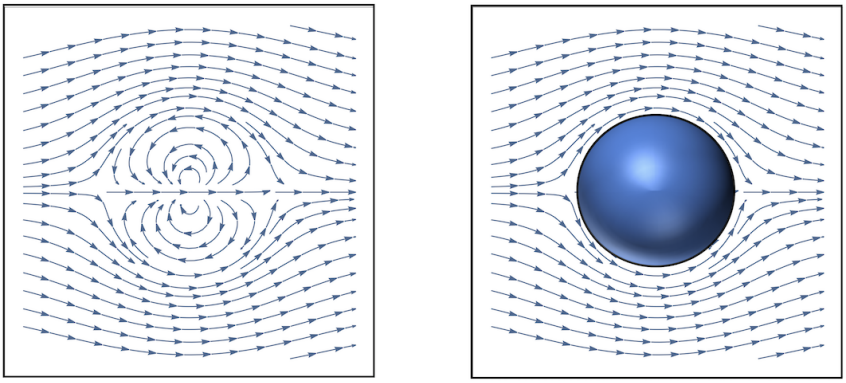


Figure 10. Stokes flow around a sphere. The left-hand figure shows the flow in the middle is in the opposite direction to the flow outside. This means that it must vanish on some surface. This surface is that of the solid sphere, as shown on the right.

where α and β are constants that are fixed by the boundary condition on the sphere. As we've seen, this requires that $\mathbf{u} = 0$ when $|\mathbf{x}| = R$, which is achieved only if both terms are individually vanishing. So we must have

$$\beta = \frac{R^2}{6} \quad \text{and} \quad \alpha = -\frac{3R}{2}$$

so our final flow for a very viscous fluid around a sphere is

$$\mathbf{u} = \mathbf{U}\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right) + (\mathbf{U}\cdot\mathbf{x})\mathbf{x}\left(-\frac{3R}{4r^3} + \frac{3R^3}{4r^5}\right) \tag{3.26}$$

This is shown in Figure 10. By eye, the flow outside the sphere doesn't look wildly different from the potential flow that we saw in Section 2.3. But there is a key difference that is clear if you look closely at the left-hand figure, before we placed the sphere over it. The fluid inside is moving in the opposite direction to the flow outside. (In contrast, for the potential flow shown in Figure 7, the fluid inside moves in the same direction as the fluid outside.) This is what ensures the existence of a surface $r = R$ for which the flow is strictly vanishing, as befits the no-slip boundary condition. This, it turns out, makes a big difference.

The difference first manifests itself in the pressure field, which is

$$P = P_\infty - \frac{3}{2}R\mu\frac{\mathbf{U}\cdot\mathbf{x}}{r^3}$$

If we take $\mathbf{U} = U\hat{\mathbf{z}}$ and work in spherical polar coordinates, the pressure on the surface of the sphere is

$$P = P_\infty - \frac{3U\mu\cos\theta}{2R}$$

This means that the pressure is bigger than P_∞ on the front of the sphere (the left in the figure) where $\pi/2 < \theta \leq \pi$ and $\cos\theta < 0$. The pressure is less than P_∞ at the back of the sphere where $0 \leq \theta < \pi/2$. This, of course, sounds very reasonable: it's simply because the flow is exerting pressure on the sphere. But it was this simple physics that was noticeably absent in the potential flow of Section 2.3 (see equation (2.31) for the analogous equation in that case). This is the first hint that we may be on the way to finally understand the drag force.

Such a Drag

The pressure is not the only force that the sphere experiences. The technology to compute the drag force comes from the stress tensor (3.5)

$$\sigma_{ij} = -P\delta_{ij} + 2\mu E_{ij}$$

where E_{ij} is the rate of strain tensor. We can compute this for the flow (3.26). It simplifies somewhat when evaluated on the surface of the sphere:

$$E_{ij}(|\mathbf{x}| = R) = \frac{3}{4R^2}(U_i x_j + U_j x_i) - \frac{3}{2R^4}(\mathbf{U}\cdot\mathbf{x})x_i x_j$$

To compute the force experienced by any point on the sphere, we consider $\sigma_{ij}n_j = \sigma_{ij}(x_j/R)$ where $\mathbf{n} = \mathbf{x}/R$ is the unit normal to the surface of the sphere. Using our expressions above, we have (ignoring the asymptotic pressure P_∞ which has no net effect on the sphere),

$$\sigma_{ij}n^j = \frac{3\mu(\mathbf{U}\cdot\mathbf{x})x_i}{2R^3} + 2\mu\left(\frac{3U_i}{4R} - \frac{3(\mathbf{U}\cdot\mathbf{x})x_i}{4R^3}\right)$$

We see that, rather nicely, the first term from the pressure cancels the final term from the strain. This means that the force acting on any point of the sphere is constant, and in the direction $\mathbf{U}$ of the asymptotic flow

$$\sigma\mathbf{n} = \frac{3\mu}{2R}\mathbf{U}$$

It's now very easy to compute the drag force: we just integrate this over the whole sphere, getting an additional factor of the surface area $4\pi R^2$. The total drag force acting on the sphere is

$$\text{Drag Force} = 6\pi\mu R\mathbf{U} \tag{3.27}$$

This is known as *Stokes' law*. It is the drag experienced by a sphere moving at very low Reynolds number.

3.4.2 Uniqueness and the Minimum Dissipation Theorem

We found a solution for the flow around the sphere. But it turns out that it is *the* solution: there is no other with the same boundary conditions. This follows from a uniqueness theorem that is proven in the same way as the uniqueness of solutions to the Laplace equation (see the lectures on Vector Calculus).

Suppose that we have two solutions, $\mathbf{u}_1$ and $\mathbf{u}_2$, both obeying non-slip boundary conditions on the surface. Then the difference $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ necessarily vanishes on the boundary. With $\tilde{P} = P_1 - P_2$ the difference in the pressure fields, we have

$$0 = \int_V \mathbf{v} \cdot (\mu \nabla^2 \mathbf{v} - \nabla \tilde{P})\, dV = \int_V \partial_i \left(\mu v_j \partial_i v_j - v_i \tilde{P} \right)\, dV - \int_V (\partial_i v_j)^2\, dV$$

The first term on the right-hand side vanishes because it's a total derivative and $\mathbf{v} = 0$ on the boundary ∂V. Moreover, the second term is the integral of a total square so this can be zero only if the integrand vanishes: $\partial_i v_j = 0$. Hence $v_j = 0$ everywhere and our original solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ were the same.

Stokes Flow Dissipates Less Than Any Other Flow

Here's a cute mathematical result. Among all the incompressible flows with the same boundary condition, the Stokes flow dissipates the least energy.

To prove this, suppose that we have a solution $\mathbf{u}$ and P to the Stokes equations with no external force (3.20), and a second flow $\tilde{\mathbf{u}}$ that satisfies the same boundary conditions but is otherwise arbitrary. Recall from (3.9) that the energy dissipated by an arbitrary flow $\tilde{\mathbf{u}}$ is (3.9)

$$\begin{aligned}
\text{Dissipation} &= 2\mu \int_V \tilde{E}_{ij}\tilde{E}_{ij}\, dV \\
&= 2\mu \int_V \left[E_{ij}E_{ij} + (E_{ij} - \tilde{E}_{ij})^2 + 2E_{ij}(\tilde{E}_{ij} - E_{ij}) \right] dV \\
&\geq 2\mu \int_V \left[E_{ij}E_{ij} + 2E_{ij}(\tilde{E}_{ij} - E_{ij}) \right] dV \\
&= \text{Stokes Dissipation} + 4\mu \int_V E_{ij}(\tilde{E}_{ij} - E_{ij})\, dV
\end{aligned}$$

We'll now show that this second integral actually vanishes. To see this, recall that the stress tensor for the Stokes flow is

$$\sigma_{ij} = -P\delta_{ij} + 2\mu E_{ij}$$

Importantly, the stress tensor is divergence free for the Stokes flow,

$$\partial_i \sigma_{ij} = -\partial_j P + \mu \nabla^2 u_j = 0 \tag{3.28}$$

where the other term in E_{ij} vanishes because it involves $\partial_i u_i = 0$ and the whole thing is equal to zero by virtue of the Stokes equations. This is the special property of the Stokes flow that we need. If we now contract the Stokes stress with the strain tensor $\tilde{E}_{ij}$ for any other flow, we have

$$\sigma_{ij}\tilde{E}_{ij} = 2\mu E_{ij}\tilde{E}_{ij}$$

where the other term $-P\tilde{E}_{ii}$ vanishes because the flow is incompressible and $\tilde{E}_{ii} = \nabla\cdot\tilde{\mathbf{u}}$. We now have

$$\begin{aligned} 4\mu \int_V E_{ij}(\tilde{E}_{ij} - E_{ij})\, dV &= 2\int_V \sigma_{ij}(\tilde{E}_{ij} - E_{ij})\, dV \\ &= 2\int_V \sigma_{ij}\left(\partial_i \tilde{u}_j - \partial_i u_j\right)\, dV \\ &= 2\int_V \partial_i\left[\sigma_{ij}(\tilde{u}_j - u_j)\right]\, dV = 0 \end{aligned}$$

where, in the second line, we've used the fact that σ_{ij} is symmetric and, in the final line, we've used the special property of the Stokes flow (3.28), together with the divergence theorem which means that the integral only cares about the boundary where, by assumption, $\tilde{\mathbf{u}} = \mathbf{u}$. The upshot is that for any flow $\tilde{\mathbf{u}}$ that is *not* a Stokes flow, we necessarily have

$$\int_V \tilde{E}_{ij}\tilde{E}_{ij}\, dV > \int_V E_{ij}E_{ij}\, dV$$

The dissipation from other flows is always greater than the corresponding Stokes flow. This is the Helmholtz minimum dissipation theorem.

There is, it turns out, a deep relationship between drag and dissipation, known as the *fluctuation dissipation theorem*. (We describe this in the lectures on Kinetic Theory.) The fact that the Stokes flow has the smallest dissipation translates into the statement that it also results in the smallest drag. This means that, as we increase the Reynolds number, the drag on the sphere will only increase beyond that given by Stokes law (3.27). Indeed, one can set up a perturbation expansion to understand the effects of the terms in the Navier-Stokes equation that we neglected. This is an expansion in the Reynolds number $Re \ll 1$ and the leading order term turns out to be

$$\text{Drag Force} = 6\pi\mu R\mathbf{U}\left(1 + \frac{3}{8}Re + \ldots\right)$$

3.4.3 Eddies in the Corner

As you might imagine, there are many different flows that exhibit interesting properties. Here is another one. We simply look at fluid passing around a corner. This corner has an opening angle that we denote as 2α. We want to know what happens.

This problem is effectively two-dimensional and can be solved quite straightforwardly by working in cylindrical polar coordinates and introducing a stream function $\Psi(r,\theta)$. Recall from Section 1.1.4 that the stream function allows us to construct a vector field $\mathbf{A} = \Psi\hat{\mathbf{z}}$ and, from that, an incompressible flow $\mathbf{u} = \nabla \times \mathbf{A}$. In cylindrical polar coordinates, the resulting flow is

$$\mathbf{u} = \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{\mathbf{r}} - \frac{\partial\Psi}{\partial r}\hat{\boldsymbol{\theta}} \tag{3.29}$$

The associated vorticity is

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = -(\nabla^2\Psi)\hat{\mathbf{z}} \quad \text{with} \quad \nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}$$

But we've seen in (3.22) that the vorticity $\boldsymbol{\omega}$ is harmonic for Stokes flows, which means that the stream function must be biharmonic

$$\nabla^4\Psi = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\Psi = 0$$

The form of the equation suggests that it might be fruitful to look for scale-invariant, separable solutions of the form

$$\Psi(r,\theta) = r^\lambda\, f(\theta)$$

for some exponent λ and some function $f(\theta)$. The biharmonic condition then becomes a differential equation for f,

$$\nabla^4\Psi = r^{\lambda-4}\left(\frac{\partial^2}{\partial\theta^2} + \lambda^2\right)\left(\frac{\partial^2}{\partial\theta^2} + (\lambda-2)^2\right) f(\theta) = 0$$

The solution is simply

$$f(\theta) = A\sin\lambda\theta + B\cos\lambda\theta + C\sin(\lambda-2)\theta + D\cos(\lambda-2)\theta$$

with four integrations constants as well as the exponent λ still to be determined. At this point we bring out some boundary conditions. We'll arrange the geometry so that the boundaries lie at $\theta = \pm\alpha$. The fluid comes in close to one boundary, and out close to the other, meaning that the radial component of the flow should be an odd function of θ. The expression (3.29) then tells us that the stream function should be an even function of θ, so $A = C = 0$.

We now have two further boundary conditions since both components of $\mathbf{u}$ must vanish along the boundary. The requirement that no fluid moves into the boundary is

$$\left.\frac{\partial\Psi}{\partial r}\right|_{\theta=\pm\alpha} = 0 \quad \Rightarrow \quad B\cos\lambda\alpha + D\cos(\lambda-2)\alpha = 0$$

Meanwhile, the no-slip condition tells us that

$$\left.\frac{\partial\Psi}{\partial\theta}\right|_{\theta=\pm\alpha} = 0 \quad \Rightarrow \quad B\lambda\sin\lambda\alpha + D(\lambda-2)\sin(\lambda-2)\alpha = 0$$

Or, combined,

$$\lambda\,\sin\lambda\alpha\,\cos(\lambda-2)\alpha = (\lambda-2)\cos\lambda\alpha\,\sin(\lambda-2)\alpha$$

This equation always has the solution $\lambda = 1$, but the conditions above tell us that if $\lambda = 1$ then $B = -D$ and, correspondingly, $\Psi = 0$. This is not what we want. So we'll look for solutions with $\lambda \neq 1$. Expand each sin and cos above in terms of $e^{i(\text{whatever})}$ and rearrange to get

$$\frac{\sin 2(\lambda-1)\alpha}{\lambda-1} = -\sin 2\alpha$$

This equation determines the exponent λ in terms of the opening angle of the corner 2α, admittedly in a slightly opaque form. To understand what it's telling us, write $x = 2(\lambda-1)\alpha$, so the equation becomes

$$\frac{\sin x}{x} = -\frac{\sin 2\alpha}{2\alpha} \tag{3.30}$$

Suppose that the opening angle α is small. Then, as you can see from Figure 11, the value of $\sin 2\alpha/2\alpha$ is large. But there is no value of x for which $\sin x/x$ has the equal negative value. So for small opening angles, we can't solve (3.30), at least not for real x.

As the opening angle gets bigger, we do get solutions. The smallest value of $\sin x/x$ occurs at the first minimum as shown in Figure 11, which sits at

$$x \approx 1.43\pi \quad \Rightarrow \quad \frac{\sin x}{x} \approx -0.217$$

This corresponds to a value of 2α given by

$$2\alpha \approx 0.813\pi \approx 146^\circ \quad \Rightarrow \quad \frac{\sin 2\alpha}{2\alpha} \approx +0.217$$

We learn that there is a critical value of the opening angle, given by

$$2\alpha_{\text{crit}} \approx 146^\circ$$

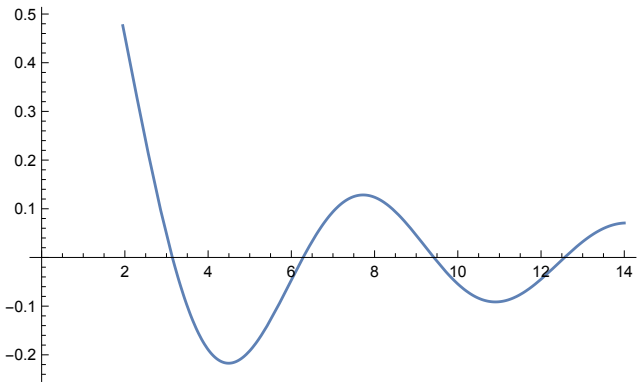
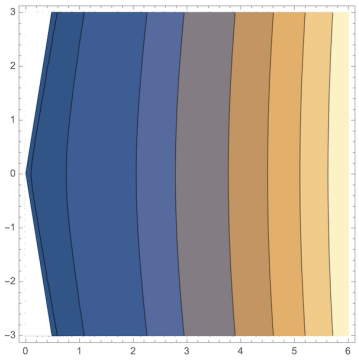


Figure 11. The graph of $\sin x/x$ with the value at the minimum around -0.217.



For opening angles larger than this, we can find solutions to (3.30). A contour plot of the stream function for $2\alpha = 160^\circ$ is shown in the figure to the right. The lines of constant value are the streamlines and they simply flow around the corner undisturbed.

What happens when the opening angle is smaller than 146°? Now, no solutions to (3.30) exist. Or, said more precisely, no *real* solutions exist! There are, however, always complex solutions. For example, suppose that we have a right angle corner, with $2\alpha = \pi/2 < 2\alpha_{\rm crit}$. Then there is an infinite sequence of complex solutions to (3.30), starting with

$$2\alpha = \frac{\pi}{2} \quad \Rightarrow \quad \begin{aligned} \lambda &\approx 3.74 + 1.12i \\ \lambda &\approx 7.84 + 1.66i \ \ldots \end{aligned}$$

What is the interpretation of these solutions? If we have a solution with

$$\lambda = \lambda_1 + i\lambda_2$$

then, because the velocity (3.29) is a linear function of Ψ, we can take the real part of the stream function to get

$$\Psi(r,\theta) = \text{Re}\left[r^\lambda f(\theta)\right] = r^{\lambda_1}\left[\cos(\lambda_2 \log r)\,\text{Re}f(\theta) - \sin(\lambda_2 \log r)\,\text{Im}f(\theta)\right]$$

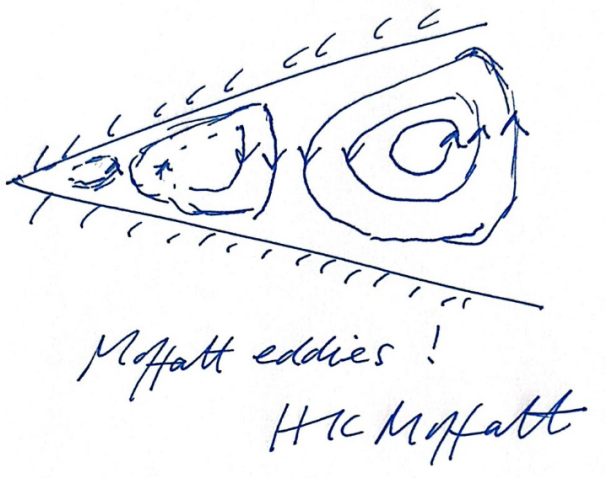
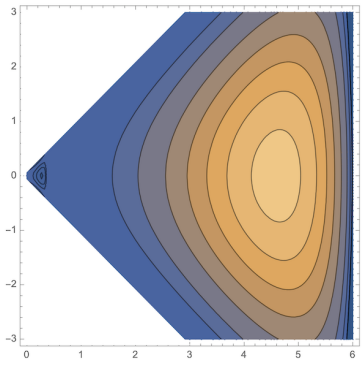


Figure 12. Not just any old Moffatt eddies, but Moffat's Moffatt eddies. My thanks to Keith Moffatt for graciously humouring my fanboy request to sketch these.



That $\cos\log r$ behaviour is striking! For a fixed angle θ, it gives rise to increasingly wild oscillations as $r \to 0$, albeit with decreasing amplitude because of the overall r^{λ_1} scaling. You can check that this means that the angular velocity $\mathbf{u} \cdot \hat{\boldsymbol{\theta}}$ is also oscillating in sign as $r \to 0$. This is telling us that the flow no longer takes the simple form, as shown in the figure for large opening angle, but instead develops eddies. In fact, there are an infinite number of these eddies, becoming increasingly small as $r \to 0$. These are known as *Moffatt eddies*.

The stream function for a right-angle corner is shown in the figure, clearly exhibiting one such eddy. The logarithm means that both the size of the eddies, and the amplitude of the stream function, vary exponentially. The centres of consecutive eddies lie at

$$\lambda_2 \log r_{n+1} = \lambda_2 \log r_n - \pi \quad \Rightarrow \quad \frac{r_{n+1}}{r_n} = e^{-\pi/\lambda_2}$$

and this also characterises the size of the eddies. (If you squint, you can just see a second eddie in the figure centred around $x \approx 0, 2$.) Meanwhile, the size of the stream function scales as

$$\frac{|\Psi(r_{n+1})|}{|\Psi(r_n)|} \sim \left(\frac{r_{n+1}}{r_n}\right)^{\lambda_1} = e^{-\lambda_1 \pi/\lambda_2}$$

The magnitude of velocities involves a derivative of stream function, $u_\theta \sim \partial\Psi/\partial r$, and so scale as $(r_{n+1}/r_n)^{\lambda_1-1} = e^{-(\lambda_1-1)\pi/\lambda_2}$. For the right-angle corner shown in this figure, this ratio is around 2000. This exponential scaling doesn't just make it difficult to plot the eddies; it also makes it difficult to experimentally observe more than two or three.

Although the eddies get smaller as you approach the vertex, the flow also becomes slower so it takes significantly longer for a particle to orbit the smaller eddies than the larger ones.

3.4.4 Hele-Shaw Flow

In this short section, we look at a particular way of restricting Stokes flow to two dimensions. However, rather than simply solving the 2d version of Stokes equations, we instead do something more physical. We trap the fluid between two parallel, stationary plates, separated by a distance h. This scale will be much smaller than any other scale, such as the size of any object that the fluid moves around

We separate the plates in the z-direction and consider situations in which the fluid flows only in the (x, y)-plane

$$\mathbf{u} = (u(x, y, z), v(z, y, z), 0)$$

and we now solve the Stokes equation

$$\nabla P = \mu\nabla^2\mathbf{u}$$

The first thing to realise is that gradients in the z direction are of order $\partial/\partial z \sim 1/h$ and so are much bigger than anything else. (These gradients can't vanish because the no-slip condition means that $\mathbf{u}$ vanishes at $z = 0$ and $z = a$ but we want it to be non-vanishing in the middle.) We work in the approximation that these z-gradients are entirely accounted for by the pressure

$$\mu\frac{\partial^2\mathbf{u}}{\partial z^2} = \nabla P \quad \Rightarrow \quad u = \frac{1}{2\mu}\frac{\partial P}{\partial x}z(z-a) \quad \text{and} \quad v = \frac{1}{2\mu}\frac{\partial P}{\partial y}z(z-a)$$

where the boundary conditions have been chosen so that the no-slip condition is satisfied. This is the same kind of velocity profile that we saw for Poiseuille flow (3.12), but now in 2d rather than 1d. In the present context, it is known as *Hele-Shaw* flow. (One person, not two! He chose, I think rather unusually, to adopt both his father's and his mother's name.)

But Hele-Shaw flow is something very familiar: we have a situation where the 2d velocity field $\mathbf{u}_{2d} = (u, v)$ is given by

$$\mathbf{u}_{2d} = \nabla_{2d}\phi \quad \text{with} \quad \phi(x, y; z) = \frac{1}{2\mu} P(x, y) z(z - a)$$

and with $\nabla_{2d} = (\partial_x, \partial_y)$. In other words, we're back in the realm of 2d potential flow that we solved in Section 2.4. This means, for example, that if you place a cylinder between the plates, with its axis pointing in the z-direction, then the velocity flow around it coincides with the velocity (2.34) that we previously calculated.

There is an irony here. We originally introduced potential flow as a description of completely inviscid fluids. Yet the same solutions also describe extremely viscous fluids when sandwiched between plates! In fact the irony runs deeper. If you attempt to go to a regime where viscosity can be neglected – which means high Reynolds number – then another effect, known as the boundary layer, kicks in and the flows don't look at all like potential flows near objects. (We describe this in Section 3.5.) So, in fact, the only way to genuinely manufacture the inviscid potential flows of Section 2.4 is to work with very viscous fluids.

There is, however, a difference between Hele-Shaw flows and the general 2d potential flow. Hele-Shaw flows can have no circulation in the (x, y)-plane,

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{x} = 0$$

This is because, as we showed in Section 2.4, circulation arises only from potentials that are not single-valued. In contrast, the potential for Hele-Shaw flows is effectively the pressure $P(x, y)$ and this is certainly single-valued. The upshot is that Hele-Shaw flows don't include those flows shown in Figure 8 which induce a lift force on the obstacle.

3.4.5 Swimming at Low Reynolds Number

Given the obvious constraints of their biology, scallops are remarkably elegant swimmers[5]. They open their shells, then quickly close them, forcing water out through the hinges to propel themselves forward.

This strategy works in the ocean. But it would be hopeless at low Reynolds number. This is because, as we mentioned at the beginning of this section, the lack of time derivatives in the Stokes equations means that motion at low Reynolds number is reversible. When friction dominates, the speed at which a scallop opens or closes its

[5] as this video shows.

shell is irrelevant. It moves in one direction when the shell opens, and comes back the same amount when it closes. A scallop dropped in honey can no longer swim. Although it surely tastes nice.

To swim at low Reynolds number, you need a different strategy. You can't just flap your arms (or your legs or your fins) back and forth, because what is done by the forward flap is undone by the backward one. Instead, you need to change your shape in some way that is not, itself, time reversible. In other words, you need something like breaststroke.

Such non-reversible strategies have been developed by micro-organisms living in water, for whom life is lived at low Reynolds number. For example, the bacterium E. coli has a helical flagellum which rotates to make it swim[6].

In this section, we describe a very simple model that captures the essence of swimming at low Reynolds number. It also captures the tension between finding a mathematical model that is easy to solve, and finding one that looks vaguely like a living creature.

An Infinite, Wavey Plate

As our proxy micro-organism, we take an infinite thin plate, lying in the (x, z)-plane. (Admittedly, this object is unlikely to make a good pet.) The plate "swims" by wriggling so that a wave passes down in the x-direction, meaning that the position of the plate in the y-direction, which is perpendicular to the flat plate, is

$$y(x) = A \sin(kx - \omega t)$$

Here A is the amplitude of the wave, which has wavenumber k and frequency ω. Said another way, the wave has wavelength $\lambda = 2\pi/k$ and travels with speed

$$c = \frac{\omega}{k}$$

We want to understand the flow that results from this wriggling. Our goal is to show that the wriggling induces a constant asymptotic velocity in the fluid. This isn't quite swimming of course: it's staying still and making all the universe move around you. But, by Galilean relativity, this is equivalent to the fluid staying still and the plate moving. And that's what we mean by swimming.

[6] A number of videos of micro-organisms swimming can be found on the webpage of Howard Berg, one of the pioneers of low Reynolds number physics. The story is described further in a famous and charming paper by E. Purcell called "Life at Low Reynolds Number". The underlying theory is closely related to the rotation of deformable (as opposed to rigid) bodies that is covered in the lectures on Classical Dynamics.

To proceed, we introduce a stream function $\Psi(x, y)$, so that $\mathbf{u} = (u, v, 0)$ with

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \Psi}{\partial x}$$

Repeating the argument that we saw for corner flows, we know that $\boldsymbol{\omega} = \nabla \times \mathbf{u} = -\nabla^2 \Psi \, \hat{\mathbf{z}}$ and $\nabla^2 \boldsymbol{\omega} = 0$, which, combined, tell us that

$$\nabla^4 \Psi = 0$$

We must solve this, subject to the no-slip requirement that the velocity of the flow matches that of the plate,

$$u = 0 \quad \text{and} \quad v = -A\omega \cos(kx - \omega t) \quad \text{on} \quad y = A \sin(kx - \omega t) \tag{3.31}$$

We'll also impose suitable boundary conditions asympotically. We'll flag these up as we go along.

Simple as the equations above are, it's not straightforward to solve them because the boundary condition (3.31) is evaluated on the waving plate. To proceed, we need an approximation. Roughly speaking, we want the amplitude of the wave A to be small in the hope that the boundary condition is easier to implement. But A is dimensionful, so it has to be small relative to something else and the only other length scale we have is the wavelength. So the relevant dimensionless expansion parameter is

$$\epsilon = Ak \ll 1$$

To understand how to write our equations in terms of a Taylor expansion, it's useful to introduce dimensionless distances and a dimensionless stream function

$$\tilde{x} = kx \quad , \quad \tilde{y} = ky \quad , \quad \tilde{\Psi} = \frac{k\Psi}{A\omega}$$

The boundary conditions (3.31) then become

$$\frac{\partial \tilde{\Psi}}{\partial \tilde{y}} = 0 \quad \text{and} \quad \frac{\partial \tilde{\Psi}}{\partial \tilde{x}} = \cos(\tilde{x} - \omega t) \quad \text{on} \quad \tilde{y} = \epsilon \sin(\tilde{x} - \omega t)$$

Now we can see that, for $\epsilon \ll 1$, we do indeed impose the boundary condition on a value of $\tilde{y}$ that is small. It means that we can Taylor expand the function $\tilde{\Psi}$ around $\tilde{y} = 0$, so these boundary conditions read

$$\left.\frac{\partial \tilde{\Psi}}{\partial \tilde{y}}\right|_{\tilde{y}=\epsilon \sin(\tilde{x}-\omega t)} = \left.\frac{\partial \tilde{\Psi}}{\partial \tilde{y}}\right|_{\tilde{y}=0} + \epsilon \sin(\tilde{x} - \omega t) \left.\frac{\partial^2 \tilde{\Psi}}{\partial \tilde{y}^2}\right|_{\tilde{y}=0} + \ldots = 0 \tag{3.32}$$

and

$$\left.\frac{\partial\tilde{\Psi}}{\partial\tilde{x}}\right|_{\tilde{y}=\epsilon\sin(\tilde{x}-\omega t)} = \left.\frac{\partial\tilde{\Psi}}{\partial\tilde{x}}\right|_{\tilde{y}=0} + \epsilon\sin(\tilde{x}-\omega t)\left.\frac{\partial^2\tilde{\Psi}}{\partial\tilde{x}\partial\tilde{y}}\right|_{\tilde{y}=0} + \ldots = \cos(\tilde{x}-\omega t) \qquad (3.33)$$

We now expand the stream function itself in powers of ϵ,

$$\tilde{\Psi} = \tilde{\Psi}_0 + \epsilon\tilde{\Psi}_1 + \ldots$$

Each $\tilde{\Psi}_n$ is biharmonic, meaning that it satisfies $\nabla^4\tilde{\Psi}_n = 0$ for $n = 0, 1, 2, \ldots$. But each $\tilde{\Psi}_n$ obeys different boundary conditions at $\tilde{y} = 0$. We start with $\tilde{\Psi}_0$ which has boundary conditions

$$\frac{\partial\tilde{\Psi}_0}{\partial\tilde{y}} = 0 \quad \text{and} \quad \frac{\partial\tilde{\Psi}_0}{\partial\tilde{x}} = \cos(\tilde{x}-\omega t) \quad \text{on} \quad \tilde{y} = 0$$

The biharmonic function obeying these boundary conditions in the region above the plate, $\tilde{y} > 0$, is

$$\tilde{\Psi}_0 = (1+\tilde{y})e^{-\tilde{y}}\sin(\tilde{x}-\omega t) \qquad (3.34)$$

which obeys $\tilde{\nabla}^2\tilde{\Psi}_0 = -2e^{-\tilde{y}}\sin(\tilde{x}-\omega t)$ and so $\tilde{\nabla}^4\tilde{\Psi}_0 = 0$. Note that we've thrown away a similar solution that scales as $e^{+\tilde{y}}$ on the grounds that it gives an unbounded velocity field as $\tilde{y} \to +\infty$. A solution of this kind is relevant below the plate for $\tilde{y} < 0$.

The first correction to this solution is $\tilde{\Psi}_1$ which, from (3.32) and (3.33), obeys

$$\frac{\partial\tilde{\Psi}_1}{\partial\tilde{y}} + \sin(\tilde{x}-\omega t)\frac{\partial^2\tilde{\Psi}_0}{\partial\tilde{y}^2} = 0 \quad \text{and} \quad \frac{\partial\tilde{\Psi}_1}{\partial\tilde{x}} + \sin(\tilde{x}-\omega t)\frac{\partial^2\tilde{\Psi}_0}{\partial\tilde{x}\partial\tilde{y}} = 0$$

Both boundary conditions should again be imposed at $\tilde{y} = 0$. Using our solution (3.34) for $\tilde{\Psi}_0$, these become

$$\frac{\partial\tilde{\Psi}_1}{\partial\tilde{y}} = \sin^2(\tilde{x}-\omega t) \quad \text{and} \quad \frac{\partial\tilde{\Psi}_1}{\partial\tilde{x}} = 0 \quad \text{on} \quad \tilde{y} = 0$$

The $\sin^2$ term is where our interest lies. We decompose this into Fourier modes by using the double angle formula $\sin^2\tilde{x} = \frac{1}{2}(1-\cos 2\tilde{x})$. The $\cos 2\tilde{x}$ term is just telling us that the second harmonic is excited. That's little surprise. The constant term is more interesting as it tells us that there must be a constant component to the fluid motion. Indeed, you can check that the biharmonic function obeying these boundary conditions is

$$\tilde{\Psi}_1 = \frac{1}{2}\tilde{y} - \frac{1}{2}\tilde{y}e^{-2\tilde{y}}\cos(2(\tilde{x}-\omega t))$$

Again, this solution holds above the plate for $\tilde{y} > 0$. There is again an analogous solution with a $e^{+\tilde{y}}$ below the plate but with the same constant $\frac{1}{2}\tilde{y}$ term. That linear term is what we're after. Putting the various constants back in, it gives a contribution to the stream function that looks like $\Psi = \frac{1}{2}A^2k^2cy + \ldots$ where the $\ldots$ are the oscillatory terms that drop off as e^{-ky} or e^{-2ky} as we move away from the plate. The constant term is telling us that, far away from the plate, there is necessarily a constant fluid velocity

$$\mathbf{u} \to \frac{1}{2}A^2k^2c\,\hat{\mathbf{x}} \quad \text{as } y \to +\infty$$

Alternatively, if we boost to another frame so that the fluid is asymptotically stationary, then the plate must be moving to the left with speed $U = \frac{1}{2}A^2k^2c$. In other words, the plate is swimming. The speed is proportional to the speed c with which waves propagate down the plate but, at least in this approximation, suppressed by $\epsilon^2 = A^2k^2 \ll 1$.

3.5 The Boundary Layer

In the previous section, we focussed on very viscous flow at low Reynolds number. Now we turn to the opposite regime of high Reynolds number. We're going to revisit the question of flows around some fixed object, like a sphere or the wing of an aircraft.

When the Reynolds number is large, the inertia term in the Navier-Stokes equation should dominate over the viscosity term,

$$Re = \frac{\text{intertial term}}{\text{viscosity term}} = \frac{|\mathbf{u}\cdot\nabla\mathbf{u}|}{|\nu\nabla^2\mathbf{u}|} \gg 1$$

For example, for a plane flying we have $Re \sim 10^7$. Given this, it's tempting to think that we can drop the viscosity term completely. But this brings us back to the Euler equation and, as we have seen in Section 2, inviscid flows do not give rise to any drag on an object. Something is amiss! In fact it turns out that, no matter how small the viscosity, it still plays an important role.

Mathematically this is because the character of the Navier-Stokes equation changes if we set $\nu = 0$. With $\nu \neq 0$, we have an equation that is second order in spatial derivatives. When $\nu = 0$, it changes to an equation that is first order. As we have commented previously, this means that we must impose two boundary conditions when solving the $\nu \neq 0$ Navier-Stokes equation, but only a single boundary condition when solving the Euler equation. The boundary condition that is expendable is the no-slip condition and, in its absence, solutions exhibit no drag force. However, as soon as we have ν, no matter how small, we're back in business and we can impose the no-slip condition to our heart's content.

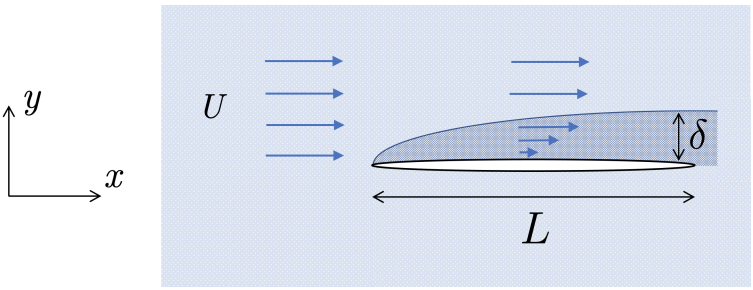


Figure 13. An exaggerated picture of the boundary layer forming over a thin plate. A second boundary layer will, of course, also form below.

Physically, a continuous flow with a no-slip boundary condition must have a layer of almost-stationary fluid sitting next to the object. This is the *boundary layer*. The purpose of this section is to understand some of its properties.

We can make progress with some simple dimensional analysis, coupled with a little intuition built on what we've learned so far. For example, one of the most basic questions that we can ask is: what is the width of the boundary layer? It seems plausible that when the fluid first hits the leading edge of the object, only those molecules immediately in contact know about its existence. But, as we look further down the flow, more and more of the fluid should be affected. How much?

Suppose that our object has length L, and travels relative to the fluid with speed U. The Reynolds number is then

$$Re = \frac{UL}{\nu}$$

By assumption, $Re \gg 1$.

Any fluid element takes a time $T = L/U$ to move past the object. Close to the object, the fluid will be affected by the no-slip condition and it is reasonable to think that the near-boundary behaviour mimics that of Couette or Poiseuille flow. One of the simple, yet important facts about these flows is that they have vorticity, as the fluid near the boundary travels at different speeds. And we know from the vorticity equation (3.13) that viscosity causes vorticity to diffuse, with diffusion constant ν. Importantly, diffusion spreads as $\sqrt{\text{time}}$ rather than linearly in time. This means that in the time scale T, the vorticity will diffuse a distance

$$\delta \sim \sqrt{\nu T} \sim \sqrt{\frac{\nu L}{U}} \sim \frac{L}{\sqrt{Re}} \tag{3.35}$$

This is the result for the width of the boundary layer that we wanted. It suggests that, at high Reynolds number, there are actually two length scales in the game. The first is the size L of the object. The second, $\delta \ll L$, is the width of a boundary layer that surrounds the object where the effects of both viscosity and vorticity are important. The existence of this thin boundary layer is the 1905 insight of Prandtl.

Outside of the boundary layer, we may neglect viscosity and the fluid is well described by the Eulerian flows of Section 2.3. But much of the physics is dictated by what happens inside the boundary layer where there are large velocity gradients. We want to better understand the properties of this boundary layer.

3.5.1 Prandtl's Boundary Layer Equation

As usual, we don't want to attack the full Navier-Stokes equations. Instead, we will extract the relevant equations that will suffice to model the boundary layer.

We'll set things up as follows. We consider a two-dimensional flow in the (x, y)-plane. As shown in Figure 13, we'll take a thin plate that extends in the x-direction sitting at $y = 0$. The flow is two-dimensional and we write

$$\mathbf{u} = (u, v)$$

Asymptotically, $\mathbf{u} \to (U, 0)$. We impose the no-slip boundary condition $u = v = 0$ on the plate at $y = 0$. Incompressibility tells us that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.36}$$

We'll also look only at steady flows, so there are no time derivatives. The full Navier-Stokes equations then read

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.37}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3.38}$$

We want to ask: which of these terms can we safely ignore? And which should we keep in the boundary layer?

We look at how the flow changes over a horizontal scale L. We start with the assumption that velocities vary in the x-direction only over the scale L, but may vary in the y-direction on the much smaller scale $\delta \ll L$. Our goal is to construct a consistent truncation of (3.37) and (3.38) such that the terms we're omitting are systematically smaller by a factor of the dimensionless parameter δ/L.

Our first piece of information comes from the incompressibility condition (3.36), with the terms scaling as

$$\frac{\partial u}{\partial x} \sim \frac{U}{L} \quad \text{and} \quad \frac{\partial v}{\partial y} \sim \frac{v}{\delta} \quad \Rightarrow \quad v \sim \frac{\delta}{L} U \tag{3.39}$$

So the vertical velocity v is much smaller than the horizontal velocity U. This equation is telling us that the fluid flow is deflected only through a small angle $\sim \delta/L$.

Now let's look to the Navier-Stokes equations (3.37) and (3.38). Both terms on the left-hand side of (3.37) scale as U^2/L, while both terms on the left-hand side of (3.38) scale as $U^2\delta/L^2$. This means that the equation (3.38) is significantly less important than (3.37). In particular, if we assume that the pressure terms have the same order of magnitude then this tells us that

$$\left|\frac{\partial P}{\partial y}\right| \sim \frac{\delta}{L}\left|\frac{\partial P}{\partial x}\right|$$

So, to leading order, pressure becomes a function only of the horizontal distance: $P = P(x)$.

Now we turn to the second order terms on the right-hand-side of (3.37). We have

$$\frac{\partial^2 u}{\partial x^2} \sim \frac{U}{L^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} \sim \frac{U}{\delta^2}$$

The second of these is clearly the most important, and we may ignore the $\partial^2 u/\partial x^2$ term. Moreover, assuming that the $\partial^2 u/\partial y^2$ term has the same order of magnitude as those on the left-hand side tells us that

$$\frac{U^2}{L} \sim \frac{\nu U}{\delta^2} \quad \Rightarrow \quad \delta \sim \sqrt{\frac{\nu L}{U}} \sim \frac{L}{\sqrt{Re}}$$

which confirms our earlier estimate (3.35) and reassures us that the whole approximation scheme is valid at large Reynolds number.

The upshot is that, when solving for the fluid in the boundary layer, we may ignore the y-component of the Navier-Stokes equation (3.38) and the x-component (3.37) simplifies to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{3.40}$$

This is the *Prandtl boundary layer equation*. It should be solved in conjunction with the incompressibility condition (3.36).

There is one final finesse. We know that the pressure is approximately a function only of x. This means that we are at liberty to evaluate the pressure $P(x)$ far from the boundary layer, $y \gg \delta$. But here the viscosity terms may be neglected completely, and the flow is governed by the Euler equation. The velocity field takes some profile

$$\mathbf{u} \to (U(x), 0) \quad \text{as} \quad y/\delta \to \infty$$

where $U(x) \to U$ as $x \to -\infty$. The Euler equation then tells us that, for a steady flow,

$$-\frac{1}{\rho}\frac{dP}{dx} = U\frac{\partial U}{\partial x} \tag{3.41}$$

which can be substituted into (3.40).

Our next task is to solve (3.40). Far from the plate, the term proportional to ν is unimportant. There is a mathematical framework to solve equations of this kind equation, whose characteristic form differs in some limit such as $\nu \to 0$. This is the theory of "matched asymptotic expansion". We won't need this in what follows. Instead, we'll look just at some simple examples.

3.5.2 An Infinite Flat Plate

Our simple example is a semi-infinite flat plate. The plate starts at $x = 0$, which we refer to as the leading edge. It then continues indefinitely.

Asymptotically, the flow is constant, $\mathbf{u} \to (U, 0)$ as $y/\delta \to \infty$ so, from (3.41), we have $dP/dx = 0$ and the Prandtl equation becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{3.42}$$

The flow is two-dimensional so we can again use a stream function $\Psi(x, y)$, such that $\mathbf{u} = (u, v)$ with

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \Psi}{\partial x}$$

If we take the stream function to scale as $\Psi \sim U\delta$ then, with the scalings described above, we expect to get $u \sim U$ and $v \sim (\delta/L)U$ which is what we want. In looking for a solution, we'll be guided by Figure 13. We know that as we move further in the x-direction, the width δ of the boundary layer grows. We will search for "self-similar" solutions in which the velocity profile within the boundary layer remains the same, but

gets stretched in the y direction as the the layer grows. Mathematically, this means that we'll search for solutions of the form

$$\Psi(x, y) = U\,\delta(x)\,f(\eta)$$

where η is the rescaled y coordinate,

$$\eta = \frac{y}{\delta(x)}$$

where $\delta(x)$ is the size of the boundary layer (3.35)

$$\delta(x) = \sqrt{\frac{\nu x}{U}} \tag{3.43}$$

(Note: for once $\delta(x)$ has nothing to do with the Dirac delta function!) For our whole approximation to be valid, we required $\delta \ll L$ which, in the present context means

$$\delta(x) \ll x \quad \Rightarrow \quad x \gg \frac{\nu}{U}$$

In other words, we can only trust what follows a distance ν/U from the leading edge of the plate. It only gives a good description beyond that point.

The velocity in the x-direction is

$$u = Uf'$$

Meanwhile, the y-direction, we have

$$v = -U\delta' f - U\delta f' \frac{\partial \eta}{\partial x} = -U(f - \eta f')\delta' \tag{3.44}$$

Now we can start building the various terms in the Prandtl equation (3.42). We have

$$\frac{\partial u}{\partial x} = -Uf'' \frac{\eta}{\delta}\delta' \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{U}{\delta} f'' \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \frac{U}{\delta^2} f'''$$

So putting it all together, the Prandtl equation (3.42) becomes

$$-U^2 \eta \frac{\delta'}{\delta} f'' f' - U^2 \frac{\delta'}{\delta}(f - \eta f') f'' = \nu \frac{U}{\delta^2} f'''$$

Two of the terms happily cancel, and we're left with

$$U\delta'\delta f f'' + \nu f''' = 0$$

But, from (3.43), we have $\delta'\delta = \nu/2U$ so our problem reduces to an ordinary, third order differential equation for $f(\eta)$,

$$f''' + \frac{1}{2} f f'' = 0 \tag{3.45}$$

We need to solve this subject to the no-slip boundary condition

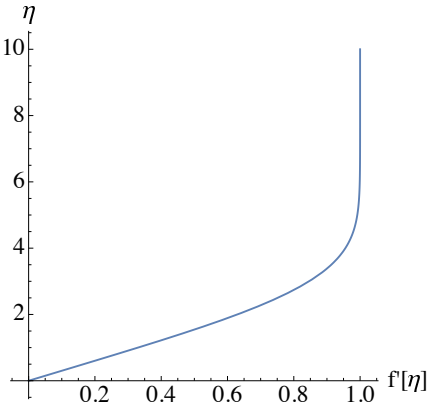
$$f = f' = 0 \quad \text{at} \quad \eta = 0$$

and the asymptotic requirement

$$f' \to 1 \quad \text{as} \quad \eta \to \infty$$

which ensures that, far from the plate, $\mathbf{u} \to (U, 0)$.

There's no analytic solution to this equation. But it's straightforward to solve the equation numerically. The resulting velocity profile is shown in the figure on the right and is known as the *Blasius boundary layer*. The distance from the plate $y \sim \eta$ is plotted vertically and the velocity $u \sim f'[\eta]$ plotted horizontally. You can see that the velocity interpolates from its zero value on the plate, to the asymptotic value. The graph also gives a more accurate estimate of the thickness of boundary layer as something like $\sim 4 - 5$ times δ, by which point the velocity is pretty much at its asymptotic value.



The numerical solution tells us something else. Asymptotically, as $\eta \to \infty$, we find that

$$f(\eta) \approx \eta - 1.72 + \mathcal{O}(1/\eta)$$

This means that, far from the plate, there is vertical component to the velocity (3.44),

$$v \approx 1.72\sqrt{\frac{\nu U}{4x}} \quad \text{as } y/\delta \to \infty$$

This is capturing what we saw previously in (3.39): the fluid is deflected by an angle $\sim \delta/L$. This angle gets smaller as we get further from the leading edge. This is because the boundary layer increases, and so the velocity gradient – which is always such that the velocity changes from zero to U – decreases as x gets larger, and this fact is reflected by the velocity component in the y-direction infinitely far from the plate. The would-be divergence at $x = 0$ is mitigated by the fact that, as we have seen, our solution only makes sense for distances $x \gg \nu/U$ from the leading edge.

The Drag Force on a Finite Plate

Strictly, the calculation above holds for an infinite plate. We've also seen that it fails within a distance ν/U of the leading edge, and one may expect that it similarly fails near the trailing edge. But we may hope that, for large L, it gives a suitable approximation of the boundary layer over much of a finite plate. With this assumption, we can compute the drag force.

The force on the plate comes from the appropriate component of the stress tensor (3.5). For a single boundary layer, we have

$$\sigma_{ij} = \rho\nu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)_{y=0} = \rho\nu \frac{U}{\delta} f''(0) \tag{3.46}$$

where only the $\partial u/\partial y$ term contributes because $\partial v/\partial x$ vanishes at $y = 0$. We use the numerical solution to evaluate $f''(0) \approx 0.33$. We also need to remember that there are two boundary layers, one on each side. So the total drag force is

$$F_{\rm drag} = 2 \times 0.33 \times \rho\nu^{1/2} U^{3/2} \int_0^L dx \, \frac{1}{\sqrt{x}} = 1.33 \, \rho\nu^{1/2} U^{3/2} L^{1/2}$$

Note that the drag force increases as $\sqrt{L}$ rather than proportional to L as one might naively expect. This is because, as the boundary layer thickens, the velocity gradients decrease and, hence, so too does the stress on the plate.

This is our first honest resolution of d'Alembert's paradox: the drag force for an object at high Reynold's number, where one might think that the Euler equation is sufficient, is non-zero. We see explicitly that the drag does vanish if we set $\nu = 0$. If we embed the viscosity in the dimensionless Reynolds number $Re = UL/\nu$, we have

$$F_{\rm drag} = 1.33 \rho \frac{U^2 L}{\sqrt{Re}}$$

Taken at face value, this says that the drag force is, in fact, vanishing in the limit $Re \to \infty$. But, sadly, there's another catch awaiting us. The calculation above breaks down at large Reynolds numbers due to the effects of turbulence. Experimentally, this is found to happen at $Re \sim 10^5$ or 10^6.

3.5.3 Boundary Layers with Pressure Gradients

There is a generalisation of the ideas above that exhibits some novel behaviour within the boundary layer. This will be important in the next section when we look at the fate of the boundary layer when it leaves an object.

The generalisation involves looking at boundary layers in flows that are accelerating or decelerating asymptotically. We will again take a semi-infinite flat plate. Far from the boundary layer, the fluid flow takes the form $\mathbf{u} \to (U(x), 0)$, now with

$$U(x) = U\left(\frac{x}{l}\right)^m \tag{3.47}$$

with l some length scale and m a parameter that determines the acceleration. Note that when $m < 0$ our velocity profile (3.47) diverges at $x = 0$. We deal with this by ignoring it: our interest is only in the behaviour of the boundary layer downstream at $x > 0$.

From (3.41), we must have a pressure gradient driving this flow

$$\frac{1}{\rho}\frac{dP}{dx} = -U\frac{dU}{dx} = -\frac{mU^2}{l}\left(\frac{x}{l}\right)^{2m-1}$$

There are two distinct cases that will interest us:

- $m > 0$: Accelerating flow with $dP/dx < 0$.
- $m < 0$: Decelerating flow with $dP/dx > 0$.

This is the asymptotic pressure gradient. But, by the arguments of Section 3.5.1, there is no change in the pressure in the y-direction, perpendicular to the plate. This means that the boundary layer also experiences the pressure gradient dP/dx. Our goal is to understand how the boundary layer reacts to this gradient.

The Prandtl equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{3.48}$$

We again seek a self-similar solution, now of the form

$$\Psi(x, y) = U(x)\,\delta(x)\,f(\eta)$$

Here $U(x)$ is given by (3.47), while $\delta(x)$ is a generalisation of our previous expression for the boundary layer thickness,

$$\delta(x) = \sqrt{\frac{\nu x}{U(x)}}$$

which takes into account the x-dependence of the asymptotic velocity. Note that, for accelerating flows, the boundary layer becomes thinner, relative to the $m = 0$ case, as the flow proceeds. It becomes thicker for decelerating flows. Finally, $\eta = y/\delta(x)$ is the rescaled y-coordinate, as before.

The velocity in the x-direction and y-directions are now

$$u = Uf' \quad \text{and} \quad v = -(U\delta)'f + U\eta f'\delta'$$

After a small amount of algebra, the Prandtl equation (3.48) becomes

$$UU'f'^2 - \frac{U}{\delta}(U\delta)'ff'' = UU' + \frac{\nu U}{\delta^2}f'''$$

Now we use the explicit expression for the asymptotic velocity (3.47), which tells us that $U \sim x^m$ and $U\delta \sim x^{(m+1)/2}$. Substituting these into the equation above, we see that all terms scale as U^2/x and we may divide by this. Happily, the partial differential equation reduces once again to an ordinary differential equation,

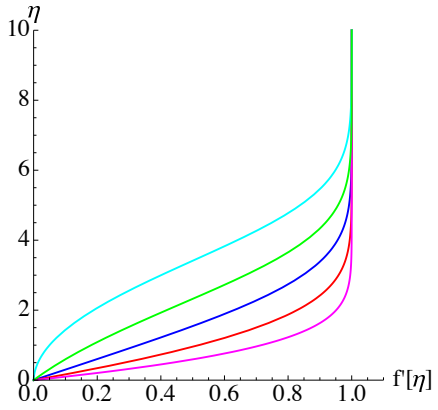
$$mf'^2 - \frac{1}{2}(m+1)ff'' = m + f'''$$

This reduces to our previous equation (3.45) when $m = 0$.

Again, we solve this subject to the boundary conditions

$$f = f' = 0 \quad \text{at } \eta = 0 \quad \text{and} \quad f' \to 1 \quad \text{as} \quad \eta \to \infty$$

The solutions are known as the *Falkner-Scan family of boundary layers*. The velocity profiles $u \sim f'(\eta)$ for a number of different flows are shown in the figure. The colours correspond, from top to bottom, to $m = -0.09$ (in cyan), $m = -0.07$ (in green), $m = 0$ (in blue), $m = 0.2$ (in red) and $m = 0.7$ (in magenta).



For accelerating flows, with $m > 0$, there isn't a great deal of difference from our previous results. One can show that the solution to the equations is unique and, as you can see from the graph, the velocity profiles all live underneath the $m = 0$ curve, coming in at ever more acute angles at the origin. This can be understood because there is a greater transfer of momentum from the accelerating fluid above. It also has consequence: the angle at which the graph intersects the origin is related to (the inverse of) $f''(0)$. As the acceleration increases, so too does $f''(0)$. But, from (3.46), means that the force imparted on the plate due to the boundary layer also increases.

At first glance, things don't look too different for decelerating flows with $m < 0$ either. Two are shown in the figure: $m = -0.07$ (in green) and $m = -0.09$ (in cyan). Now the graphs come in more steeply at the origin, corresponding to a smaller value of $f''(0)$ and, correspondingly, a smaller stress on the plate. But when we look more closely, there is a surprise waiting us: numerically, we find that for some critical value $m_{\rm crit}$, the solution actually comes into the origin vertically,

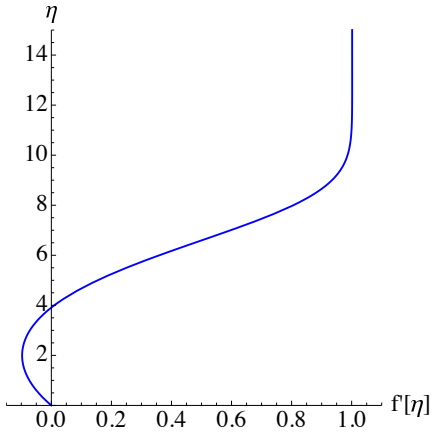
$$m = m_{\rm crit} \approx -0.0904 \quad \Rightarrow \quad f''(0) = 0$$

In other words, for a critical deceleration, there is no friction force between the plate and fluid!

What's going on here? Consider an element of fluid near the boundary. It has a force to the right due to the fluid moving above it. But there are also forces to the left, both from the pressure gradient $dP/dx > 0$ and from the viscous force of the boundary. At $m = m_{\rm crit}$, these precisely cancel. The result is that not only is $u = 0$ on the boundary, but also $du/dy = 0$.

What happens if we decrease m below the value $m_{\rm crit}$? Naively, one might have thought that one would find solutions with $du/dy < 0$, which would mean the the fluid closest to the boundary actually flows in the opposite direction. It turns out that this *doesn't* happen. There are no solutions for $m < m_{\rm crit}$.

However, there are further solutions that do exhibit reverse flows. It turns out that these solutions exist for any $m_{\rm crit} < m < 0$ where there are two branches of solutions. The first, given above, has $u > 0$ everywhere. The second has a region with $u < 0$ close to the plate. An example is shown in the figure for $m = -0.05$. It has $f''(0) \approx -0.1$. In this case, a fluid element in the region closest to the boundary has a velocity in the opposite direction to the rest of the flow. This reverse flow can be understood as the pressure gradient pushing to the left, while the force from both the fluid above it, and also from the plate, pushes to the right.



It seems that these boundary solutions with reverse flow cannot be set-up in experiment because they are thought to be unstable in this particular context. Nonetheless, the existence of such reversed boundary layers is crucial to understand the next topic that we turn to. This is the fate of the boundary layer when the boundary ends.

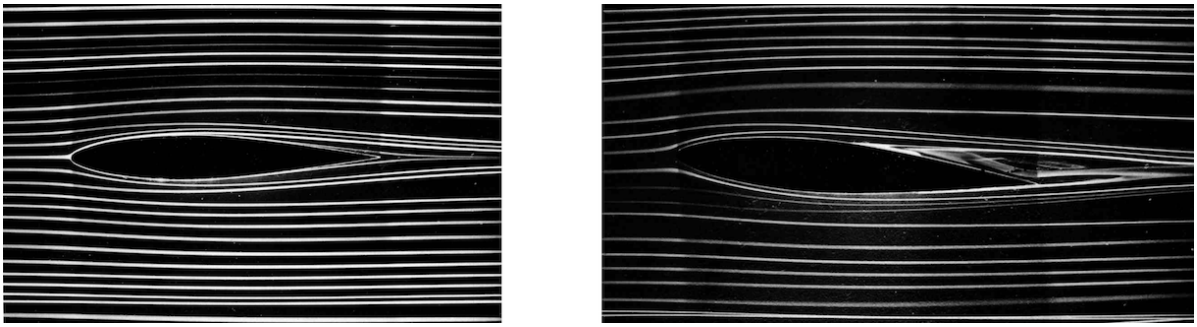


Figure 14. The flow, from left to right, around a streamlined object at $Re \approx 7000$. On the left, the object is aligned with the streamlines and the boundary layer merges smoothly into the flow at the trailing edge. On the right, the object is inclined by 5°. The boundary layer separates from the object on the upper edge.

3.5.4 Separation

So far we've understood how the boundary layer develops, but only by restricting to a flat, semi-infinite plate. Needless to say, that's not particularly realistic. Most objects are neither flat, nor semi-infinite. Clearly, we need to understand the physics of the boundary layer for objects that are curved and finite.

This, it turns out, is not so easy. Until now, we've made progress by finding clever ways to reduce the Navier-Stokes equations to an ordinary differential equation which can then easily be solved. But the problem that we're now interested in offers no such simplification. That means that to get a complete handle on the problem we must resort to solving partial differential equations numerically. Which is possible, but challenging, and beyond the scope of these lectures. Instead we will make do with some rather qualitative arguments, piecing together various bits of physics that we've learned so far.

First, we can gain some intuition for what's going on by turning to experiment[7]. Figure 14 shows the stream lines for a high Reynolds number flow ($Re \approx 7000$) around an elegantly pointy object. In the figure on the left, the object is aligned with the streamlines, which glide around much like the flows that we've discussed so far in these lectures. Such flows, where there is little mixing between adjacent layers of the fluid, are called *laminar*. A boundary layer forms around the object but, at least as far as

[7]The photos in Figures 14, 15, 16 and 18 are taken from the beautiful book "An Album of Fluid Motion" by Milton Van Dyke.

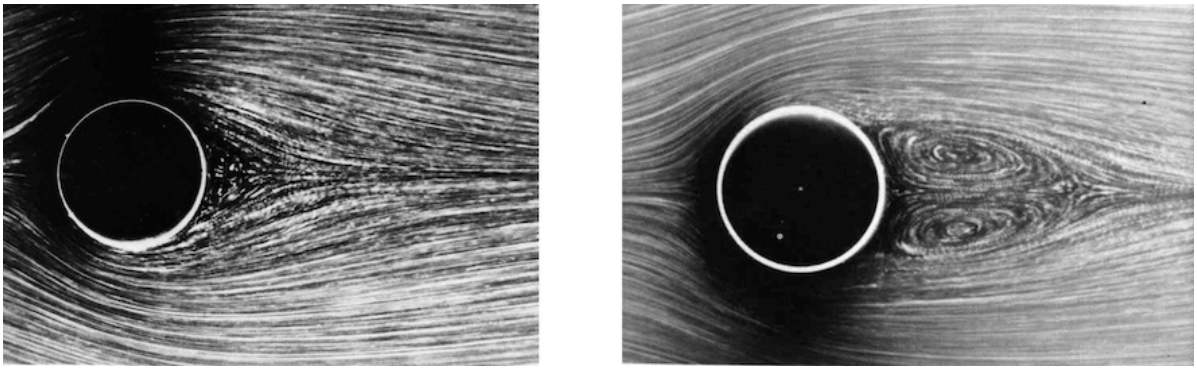


Figure 15. The flow around a circular cylinder with $Re \approx 10$ on the left, and $Re \approx 26$ on the right. In both cases, the flow separates from the cylinder at some point, leaving two trailing eddies in the wake (more visible in the second picture).

the photograph shows, appears to merge seamlessly back into the bulk fluid at the tail end.

On the right of Figure 14 is the same object, again at $Re \sim 7000$, but now tilted at an angle of 5°. The flow is again laminar at the front and below the object. But you can see that something screwy is happening on the upper trailing edge. There is clearly a streamline that moves away from the object, leaving a swirling indeterminate flow beneath it.

The same phenomenon occurs for less aerodynamic objects. Figures 15 and 16 show flows moving past a circular cylinder. The first flow, at $Re \approx 10$, clearly shows an anti-symmetry between the front and back of the cylinder as the streamlines separate from the body. This is unsurprising, but sits in stark contrast to the potential flows and Stokes flows that we've seen previously, where it's difficult to see by eye the difference between the front and back of the flow. (See, for comparison, Figure 7 or Figure 10.) In the second picture in Figure 15, the Reynolds number has increased to $Re \approx 26$ and we again see the flow separating from the body, this time clearly leaving two counter-circulating eddies in its wake.

The Reynolds numbers in Figure 15 are fairly low and it's not at all obvious that we can use the theory of boundary layers, which relies on the approximation $Re \gg 1$. But this is surely valid for the picture in Figure 16, now at $Re \approx 2000$. Now we clearly see that the laminar flow at the front of the cylinder separates somewhere near the top of the cylinder, leaving a turbulent flow in its wake.

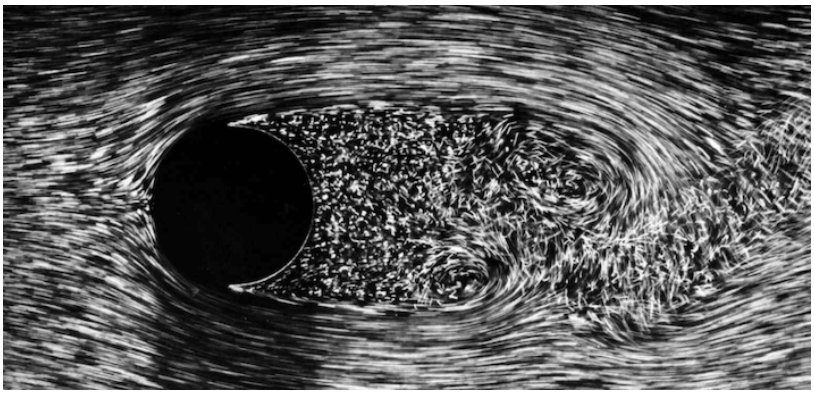


Figure 16. A flow around the circular cylinder, now with $Re \approx 2000$. We're now at values where the boundary layer theory should work. The picture clearly shows laminar flow at the front of the cylinder, where the boundary layer remains attached. It separates somewhere near the top and bottom of the cylinder, leaving a turbulent wake.

There are a bunch of things to unpack here. First, how do we extend the theory of a boundary layer to a curved object like those shown in the figures? Second, why does the flow separate from the object at some point? And, finally, how can we understand the physics of the wake left behind? We'll deal with each of these in turn.

Here is a cartoon of the physics. First, extending the theory of the boundary layer to a curved object turns out to be fairly straightforward. We use the same equations as before, but with x and y now curvilinear coordinates: x is the coordinate along the boundary and y the coordinate perpendicular. The boundary layer is so thin that, locally, it barely notices the curvature. All we must do is ensure that the pressure in the boundary layer is given by (3.41),

$$-\frac{1}{\rho}\frac{dP}{dx} = U\frac{\partial U}{\partial x}$$

Here, as a first approximation to $U(x)$, we should take the *near*-boundary limit of the flow that surrounds the boundary layer. Provided that this flow isn't turbulent, we can use the near-boundary limit of the inviscid potential flows that we described in Section 2.3. But we know how the pressure changes over the sphere or cylinder due to a potential flow. (The answer for the sphere was given in (2.31) and the result for the cylinder is similar.) There we saw that the pressure directly at the front and back is the same as the asymptotic pressure, but the pressure reduces as you move up or down over the sphere and takes its minimum value at the top and bottom. Crucially,

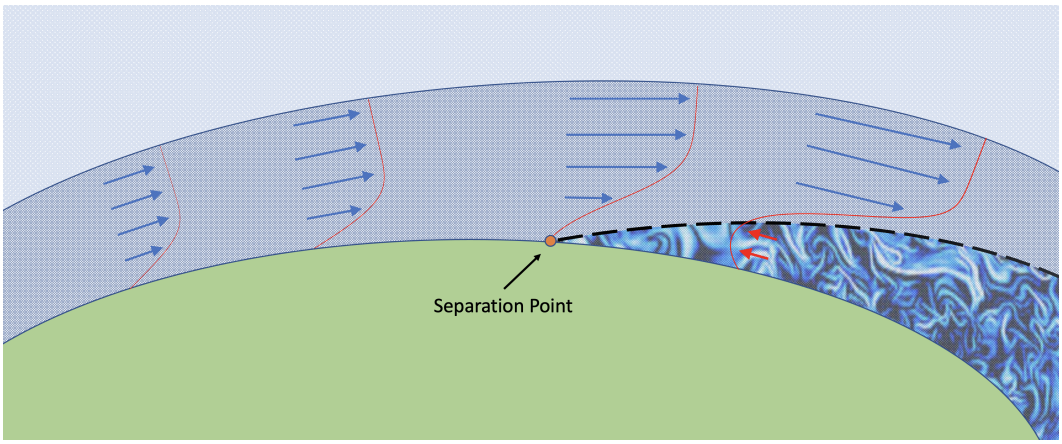


Figure 17. A cartoon of the evolution of the boundary layer and its ultimate separation from the boundary as the flow becomes reversed.

the pressure for an inviscid potential flow is symmetric on the front and back: this, of course, was what lead to d'Alembert's paradox.

Now we can see what this means for the boundary layer. On the front edge of the cylinder, the pressure is decreasing, $P' < 0$. This corresponds to an accelerating flow. But on the back edge, the pressure is increasing, $P' > 0$, and the flow is decelerating. This suggests that we might get the kind of behaviour that we observed for decelerating flows in the Falkner-Scan family of boundary layers. In particular, at some point the velocity u tangential to the boundary will obey

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$$

where y is the direction perpendicular to the boundary. This is the *separation point*, with the streamline bifurcating and leaving the boundary. Beyond this point, one expects reverse flow close to the boundary. Beyond the separation point, the boundary layer moves off into the bulk of the fluid, leaving behind the wake. A sketch of the scenario is shown in Figure 17.

The boundary layer itself cannot just dissolve once it has separated from the boundary. One might reasonably wonder what distinguishes it from the bulk of the fluid. After all, they're made from the same stuff. The answer is that the boundary layer has vorticity, generated by the no-slip condition

$$\boldsymbol{\omega} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{z}} \approx -\frac{\partial u}{\partial y}\hat{\mathbf{z}}$$

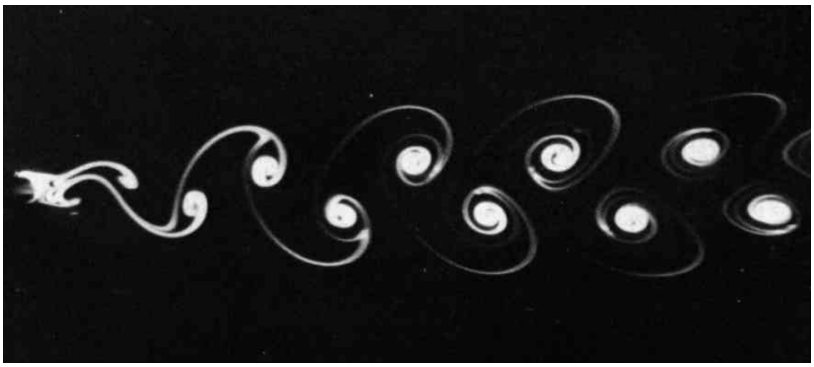


Figure 18. The von Kármán vortex street from air passing over a circular cylinder at Reynolds number $Re \approx 105$.

where the first term dominates in the boundary layer approximation. For the boundary layers described above, we have $|\boldsymbol{\omega}| = Uf''(0)/\delta$. Meanwhile, as we saw previously, the outer laminar flow is irrotational. The vorticity persists in the wake that trails the objects.

For low Reynolds number, the stream flow is low and this vorticity has time to diffuse due to the effects of viscosity. The result is the two large eddies trailing the object seen in Figure 15. The flow is steady. These are steady eddies.

But as the Reynolds number is increased to around $Re \sim 100$, something more interesting happens. One of the eddies grows until it peels off from the boundary in a process known as *vortex shedding*. The flow then curls back around the boundary and a new eddy forms. Meanwhile, the eddie on the other side then undergoes the same process. The result is a gorgeous flow pattern of alternating eddies known as the *von Kármán vortex street*. An example is shown in Figure 18. At these Reynolds numbers, there is no steady flow of the kind that we've searched for in these lectures. Instead, the flow is time dependent, but periodic.

There is much that we have swept under the carpet in the discussion above. The elephant in the room is turbulence. As the pictures clearly show, for large Reynolds number the flow is far from laminar. Indeed, the flow is no longer even two dimensional, but twists and turns in a noisy fashion in three dimensions. This occurs for $Re \gtrsim 10^4$ when the wake becomes turbulent as shown in Figure 16. A process known as *turbulent mixing* causes the pressure to be uniform across the turbulent wake, and equal to its value at the point of separation. This means that there is a much lower pressure behind the object and, correspondingly, a much larger drag force.

As the Reynolds number is increased yet further to $Re \gtrsim 10^5$ something novel happens: now the boundary layer itself becomes turbulent. The same turbulent mixing means that vorticity can be transferred vertically much more efficiently, and the result is that the boundary layer gets thicker. This has two, competing effects. The first is that the drag due to the turbulent boundary layer increases compared to the laminar boundary layer. The second is that the separation of the boundary layer is delayed, with the reversed flow happening further downstream. This results in a narrower wake which reduces the drag. It turns out that this reduced drag from the narrower wake is more than sufficient to compensate for the increased drag due to the turbulent boundary layer, and the result is that, surprisingly, the drag force actually drops suddenly at this Reynolds number. This goes by the name of the *drag crisis*.

4 Waves

Our story so far has involved the bulk motion of fluids, flowing from one place to another, sometimes trying to negotiate obstacles in their way. But fluids are more subtle and interesting than this. They contain mechanisms to transfer energy through space, but *without* the bulk of the fluid travelling very far. This is achieved this through oscillatory behaviour known as *waves*.

Waves are familiar, both from our everyday experience as well as from other areas of physics. Our purpose in this section is to explore some of large variety of waves that can occur in fluids. This includes, in Section 4.4, sound waves which gives us an opportunity to look at some of the novelties that arise with compressible fluids.

4.1 Surface Waves

> "Now, the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light; they have all the complications that waves can have."
>
> *Richard Feynman*

We start with waves travelling on the surface of a fluid. These include waves on the ocean. As Feynman points out, there are a surprisingly large number of subtleties that arise in understanding these waves.

Viscosity will not play a leading role in our story, so we return to the Euler equation of Section 2,

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P + \rho\mathbf{g} \tag{4.1}$$

We've included the effects of gravity on the right-hand side. As we will see, this provides the restoring force needed to create waves.

We will shortly solve the Euler equation using the same techniques that we met in Section 2. All of the novelties come, like so many things in fluid dynamics, from the boundary conditions. So before we get going, we need to think about the kind of boundary condition we should impose on the surface of a fluid.

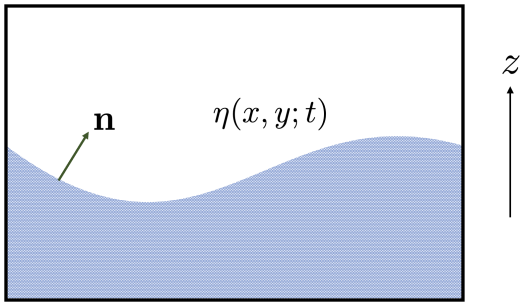


Figure 19. An interface between two fluids.

4.1.1 Free Boundary Conditions

The surface of a fluid is best viewed as the interface between two different fluids. In the case of the ocean, this is the water and the air above. But we could also have a situation where we have two immiscible liquids, like oil and water. The surface is free to move, and so is sometimes referred to as a *free boundary*.

Suppose that the boundary lies close to some $z \approx$ constant surface, as shown in Figure 19. Clearly this is appropriate for the surface of the ocean. The surface can fluctuate and, in general, is described by some function

$$F(\mathbf{x}, t) = z - \eta(x, y; t) = 0 \tag{4.2}$$

The normal to such a surface is parallel to ∇F (as shown, for example, in the lectures on Vector Calculus),

$$\mathbf{n} \sim \nabla F = (-\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1)$$

Meanwhile, the velocity of the interface is, by construction, in the z direction and given by

$$\mathbf{U} = (0, 0, \frac{\partial \eta}{\partial t})$$

The appropriate boundary condition on the fluid velocity $\mathbf{u}$ is the same as we saw in (2.25) for a solid surface moving with some velocity $\mathbf{U}$: one fluid cannot permeate the other. This means that if we write the fluid velocity as $\mathbf{u} = (u_x, u_y, u_z)$, then we have

$$\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \mathbf{U} \quad \Rightarrow \quad -u_x \frac{\partial \eta}{\partial x} - u_y \frac{\partial \eta}{\partial y} + u_z = \frac{\partial \eta}{\partial t}$$

We can alternatively write this as

$$u_z - \mathbf{u} \cdot \nabla \eta = \frac{\partial \eta}{\partial t} \quad \Rightarrow \quad \frac{D\eta}{Dt} = u_z \tag{4.3}$$

Alternatively, if we return to our original definition of the interface as the surface $F(\mathbf{x}, t) = 0$ given in (4.2), the fact that one fluid cannot invade the other can be written in the elegant form,

$$\frac{DF}{Dt} = 0$$

In addition, there is a further dynamical boundary condition that comes from the requirement that the stress tensor is continuous over the surface, ensuring that all forces are balanced. For an inviscid fluid, there is no tangential stress. The component of the stress tensor perpendicular to the interface simply tells us that the pressure must be continuous

$$P(x, y, \eta(x, y)) = P_0 \tag{4.4}$$

where, for the waves on the ocean, P_0 is atmospheric pressure. For example, if the water is stationary, so $\mathbf{u} = 0$, with a flat, free boundary at $z = 0$, then the Euler equation, together with this boundary condition, tells us that

$$P(z) = P_0 - \rho g z$$

We'd like to understand how to generalise this to the case where waves propagate on the boundary.

There are further complications that we could add to this story. In particular, there is an additional force that acts on the boundary known as *surface tension*. We'll postpone a discussion of this to Section 4.1.3.

4.1.2 The Equations for Surface Waves

We will look for irrotational flows which, as in Section 2, allows us to introduce a velocity potential

$$\nabla \times \mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{u} = \nabla \phi$$

The incompressibility of the fluid then tells us that the potential ϕ must obey the Laplace equation

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \nabla^2 \phi = 0$$

All the subtleties lie in the boundary conditions.

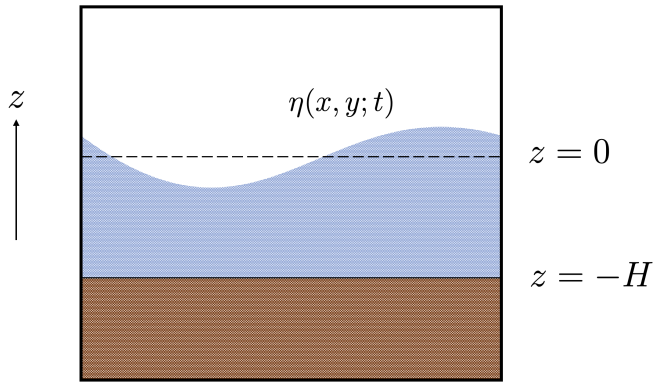


Figure 20. Surface waves.

We'll take the waves the be propagating in an ocean of height H, as shown in Figure 20. The bottom of the ocean lies at $z = -H$ while the surface of the ocean is at $z = \eta(x, y, t)$, some height above or below the equilibrium value of $z = 0$. The boundary condition at the bottom of the ocean is straightforward: the water can't flow into the ocean floor so

$$u_z(z = -H) = \left.\frac{\partial \phi}{\partial z}\right|_{z=-H} = 0 \tag{4.5}$$

Meanwhile, on the surface $z = \eta(x, y, t)$, we impose the free boundary condition (4.3)

$$u_z = \frac{D\eta}{Dt} \quad \Rightarrow \quad \left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = \frac{\partial \eta}{\partial t} + \left.\frac{\partial \phi}{\partial x}\right|_{z=\eta} \frac{\partial \eta}{\partial x} + \left.\frac{\partial \phi}{\partial y}\right|_{z=\eta} \frac{\partial \eta}{\partial y} \tag{4.6}$$

This boundary condition is all well and good if we're given the equation of the surface $z = \eta(x, y, t)$. But here, of course, the surface is something that arises dynamically and our goal is to find it. The equation that relates the velocity potential ϕ and the surface η comes from the continuity of pressure (4.4). To implement this, we write the Euler equation (4.1) as

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla |\mathbf{u}|^2 - \mathbf{u} \times \boldsymbol{\omega} \right) = -\nabla (P + \rho g z)$$

But we're dealing with irrotational flows, so $\boldsymbol{\omega} = 0$, and this becomes

$$\rho \left(\frac{\partial \nabla \phi}{\partial t} + \frac{1}{2} \nabla \left| \nabla \phi \right|^2 \right) = -\nabla (P + \rho g z)$$

But both sides are now total derivatives, so we have

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho |\nabla \phi|^2 + P + \rho g z = f(t)$$

where the function $f(t)$ on the right-hand-side can depend on time, but not on space. This is the time-dependent version of Bernoulli's principle that we derived earlier in (2.12) for stationary flows. For our final boundary condition, we simply require this condition on the surface $z = \eta(x, y, t)$, but with the pressure replaced by the atmospheric pressure (4.4),

$$\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right)_{z=\eta} + P_0 + \rho g\eta = f(t) \tag{4.7}$$

This completes our setting up of the equations. We must solve the Laplace equation $\nabla^2\phi = 0$ subject to the boundary conditions (4.5), (4.6) and (4.7). The Laplace equation is easy, but these boundary conditions look hard. As with so many other problems in this course, we need to find an appropriate approximation scheme.

The Linearised Approximation

To make progress, we will assume that the waves are small and flat. The first condition is the statement that the amplitude is small,

$$|\eta| \ll H$$

The second condition is the statement that the derivatives of the amplitude are also small

$$\frac{\partial\eta}{\partial x}, \frac{\partial\eta}{\partial y} \ll 1 \tag{4.8}$$

The boundary condition (4.6) is imposed at $z = \eta(x, y, t)$ but, since η is small, we can view this as "close" to a boundary condition at $z = 0$ by Taylor expanding

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0} + \eta\left.\frac{\partial^2\phi}{\partial z^2}\right|_{z=0} + \ldots$$

The second order term is smaller than the first and can be dropped. Relatedly, the velocities u_x and u_y will be assumed to be small and we will drop all quadratic terms in the above boundary conditions. This means that we can ignore the $(\partial\phi/\partial x)(\partial\eta/\partial x)$ terms in (4.6) and the $|\nabla\phi|^2$ term in (4.7). The upshot is that our rather complicated set of boundary conditions reduce to the linear equations

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=-H} = 0 \quad \text{and} \quad \left.\frac{\partial\phi}{\partial z}\right|_{z=0} = \frac{\partial\eta}{\partial t} \quad \text{and} \quad \left.\frac{\partial\phi}{\partial t}\right|_{z=0} + g\eta = \tilde{f}(t) \tag{4.9}$$

where, in the last of these, we have absorbed the pressure P_0 and density ρ into the redefined function $\tilde{f}(t)$. It is, it turns out, a significantly easier task to solve the Laplace equation subject to these conditions.

Finally, Some Waves

We will consider wave solutions that move in the x-direction and are independent of the y-direction. We make the ansatz

$$\phi(x,z,t) = \phi_0(z)\, e^{ikx-i\omega t} \quad \text{and} \quad \eta(x,t) = \eta_0\, e^{ikx-i\omega t} \tag{4.10}$$

You may be surprised that the right-hand side is suddenly complex, while both quantities on the left-hand side are clearly real! There's nothing deep going on here, only laziness. Because the equations are linear, if we find a complex solution then the real and imaginary parts are also solutions. But it's often simpler to work with complex numbers $e^{i(\text{something})}$ rather than cos and sin functions. Moreover, this will be particularly useful in Section 5 when we come to study instabilities since these manifest themselves as complex frequencies or wavenumbers for which the solution grows exponentially in time or space. For now, whenever you see equations like those above, you should implicitly think that we are taking the real (or imaginary) part. We'll use the same conventions in other lectures, including those on Electromagnetism.

The ansatz (4.10) depends on two numbers in the exponent, k and ω. Here k is called the *wavenumber*. From the equation, we see the successive peaks – also known as *wavecrests* – are spaced a distance apart given by

$$\lambda = \frac{2\pi}{k}$$

The distance λ is known as the *wavelength*. Meanwhile, ω is the *frequency* of the wave. Part of our goal is to determine the relationship between these. In particular, that will tell us the speed c at which waves travel,

$$c = \frac{\omega}{k} \tag{4.11}$$

The other part of our goal is to fix the function $\phi_0(z)$.

If we substitute the ansatz above into the Laplace equation, we get

$$\frac{d^2\phi_0}{dz^2} = k^2\phi_0$$

This has the solution

$$\phi_0(z) = A\cosh(kz + kH)$$

where we've chosen one integration constant to ensure that the first boundary condition in (4.9) is satisfied, and the overall amplitude A is still to be fixed. The second boundary condition in (4.9) tells us

$$Ak\sinh(kH) = -i\omega\eta_0$$

while the third, Bernoulliesque, condition in (4.9) tells us

$$(-iA\omega\cosh(kH) + g\eta_0)\, e^{ikx-i\omega t} = \tilde{f}(t) \quad \Rightarrow \quad A\omega\cosh(kH) = -ig\eta_0$$

where the second equality holds because the function $\tilde{f}(t)$ is a function only of time and so must be independent of x. Dividing these two equations to eliminate A/η_0, we get the relationship between the frequency ω and wavenumber k,

$$\omega^2 = gk\tanh(kH)$$

Equations of this kind are called *dispersion relations*. They are important in many different places in physics. In later courses that take place in the quantum world, we will see similar equations that relate energy (associated to frequency) and momentum (associated to wavenumber).

We find that, for surface waves, the frequency depends on the wavelength. This means that waves of different wavelength travel at different speeds (4.11),

$$c = \sqrt{\frac{g}{k}\tanh(kH)}$$

For many kind of waves, including sound and light, the speed is independent of the wavelength. Not so for surface waves. For fixed H, the speed is a monotonically decreasing function of k. In other words, long wavelength waves travel faster than short wavelength waves.

There are two interesting limits that we can take

- In deep water, $H \gg \lambda = 2\pi/k$ so we have the limit $kH \gg 1$. Here the speed becomes

$$c \approx \sqrt{\frac{g}{k}} \tag{4.12}$$

  We could have anticipated this. When the ocean is very deep, we don't expect the speed of surface waves to depend on H simply because the floor is a long way from the surface. Then, on dimensional grounds, the only thing that we can write down is $c \sim \sqrt{g/k}$.

- For long wavelengths, we have $kH \ll 1$ and the speed becomes

$$c \approx \sqrt{gH} \tag{4.13}$$

  Now the speed is independent of the wavelength of the wave. In this limit, the wave goes faster in deeper water than in shallow. There is a nice consequence of this. When waves come into the beach at an angle, the wave front that is further out travels faster and so the wave rotates until it is parallel to the beach.

Group Velocity and Dispersion

The waves described above extend to infinity in the x-direction, varying over wavelength $\lambda = 2\pi/k$. We can create a localised wavepacket by summing over many different wavenumbers and, because the equations are linear (after linearisation!) this too is a solution. In this way, a bump in the fluid surface can be written as

$$\eta(x,t) = \int \frac{dk}{2\pi}\, a(k) e^{ikx - i\omega t}$$

With some Fourier coefficients $a(k)$. This wavepacket may have some particular shape at time $t = 0$. But because the individual Fourier modes travel at different speeds, this shape will be distorted over time. This is known as *dispersion*.

We can ask: what is the right way to characterise the speed at which the wavepacket moves, rathe than the individual Fourier modes? Suppose that the Fourier modes $a(k)$ are peaked around some particular wavenumber $k = \bar{k}$. Then we can Taylor expand the frequency and write

$$\omega(k) = \omega(\bar{k}) + (k - \bar{k}) \left.\frac{\partial \omega}{\partial k}\right|_{k=\bar{k}} + \ldots$$

Substituting this into the expression for the wavepacket, we have

$$\eta(x,t) \approx e^{-i\omega(\bar{k})t} \int \frac{dk}{2\pi}\, a(k)\, e^{ik(x - v_g t)} \tag{4.14}$$

where

$$v_g = \left.\frac{\partial \omega}{\partial k}\right|_{k=\bar{k}} \tag{4.15}$$

This is called the *group velocity* of the wave. It's clear from the form (4.14) that v_g is the speed at which the wavepacket moves. If $\omega \sim k$ then the wave doesn't disperse and the group velocity coincides with the speed $c = \omega/k$ that we defined previously,

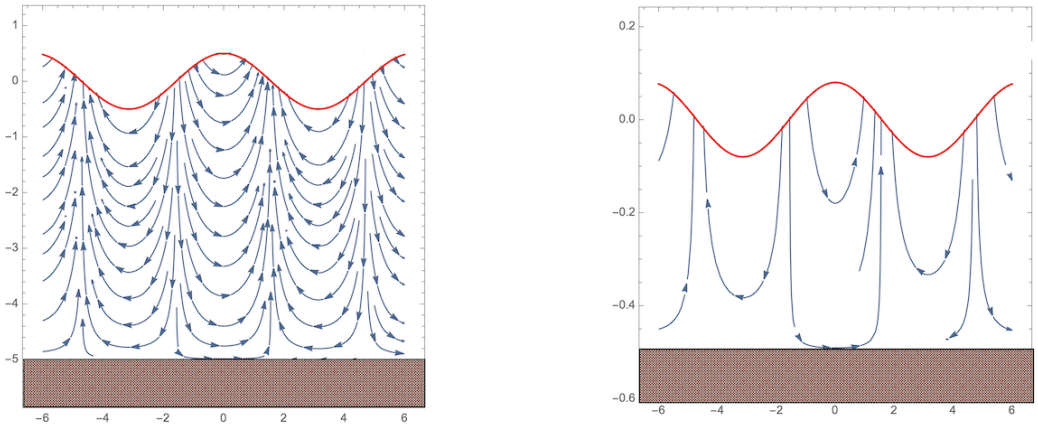


Figure 21. The velocity field for deep water waves on the left, with $kH = 5$, and shallow water on the right with $kH = 0.5$. (Note the different scales on the vertical axes!) The streamplot shows only the direction of the velocity field, not its size. In deep water, the velocity is exponentially smaller at the bottom than the top.

known as the *phase velocity*. But, in general, the two differ. The group velocity is the speed at which energy (and, in other contexts, information) is transported by the wave. For the surface waves considered here, $\omega \sim k^{1/2}$ and so the group velocity and phase velocity are related by $v_g(k) = \frac{1}{2}c(k)$. The wavepackets travel at half the speed of the individual Fourier modes.

The Velocity Field

It is a simple matter to compute the velocity field of the fluid. Substituting for the various integration constants, we have the potential

$$\phi = \text{Re}\left[-i\frac{\omega\eta_0}{k}\frac{\cosh(kz+kH)}{\sinh(kH)}e^{ikx-i\omega t}\right]$$

which now just has a single undetermined integration constant η_0 that fixes the amplitude of the wave. Our approximations above mean that the solution should be trusted only when $\eta_0 k \ll 1$. For once we've explicitly reminded ourselves that we should take the real part of the potential when computing the velocity $\mathbf{u} = \nabla\phi$. We have

$$\begin{pmatrix} u_x \\ u_z \end{pmatrix} = \frac{\omega\eta_0}{\sinh(kH)}\begin{pmatrix} \cosh(kz+kH)\cos(kx-\omega t) \\ \sinh(kz+kH)\sin(kx-\omega t) \end{pmatrix}$$

The velocity profile is plotted in Figure 21 for deep water waves (on the left) and for shallow water waves (on the right). In both cases, the velocity of the water is mostly

up/down, despite the fact that the wave travels to the right. In the trough of the wave, the water is moving up on the left and down on the right. In the peak of the wave, this is reversed: the water moves down on the left and up on the right. The net effect is that the wave travels to the right.

There's something misleading about the figure for deep water waves. In this case, $e^{-kH} \approx 0$ and the velocity profile is well approximated by

$$\begin{pmatrix} u_x \\ u_z \end{pmatrix} \approx \omega\eta_0 e^{kz} \begin{pmatrix} \cos(kx - \omega t) \\ \sin(kx - \omega t) \end{pmatrix} \tag{4.16}$$

We see that the magnitude of the velocity $|\mathbf{u}| \approx \omega\eta_0 e^{kz}$ decreases exponentially from its value at the surface $z = 0$. It means that all the action is really taking place within a depth of one wavelength or so from the surface. In contrast, for shallow water waves the speed does not vary greatly with height.

For deep water waves, the ratio of the fluid speed to the wave speed is $|\mathbf{u}|/c \approx k\eta_0 e^{kz}$. The condition (4.8) is tantamount to the requirement that $k\eta_0 \ll 1$. In other words, the wave travels much faster than the fluid from which it's made.

Particle Paths

Suppose that you drop a small ball into the flow that follows an element of fluid on its travels. What path does it take? As we described in Section 1.1, the trajectory $\mathbf{x}(t)$ is called a pathline and is governed by the equation (1.1)

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(t), t) \tag{4.17}$$

which we should solve given some initial starting point $\mathbf{x}(t = 0) = \mathbf{x}_0$.

To solve this, we will assume that the particle doesn't get far from its original starting position and approximate the velocity field $\mathbf{u}(\mathbf{x}, t)$ by its Taylor expansion about $\mathbf{x}_0$,

$$\mathbf{u}(\mathbf{x}, t) \approx \mathbf{u}(\mathbf{x}_0, t) + ((\mathbf{x} - \mathbf{x}_0) \cdot \nabla)\mathbf{u}(\mathbf{x}_0, t) + \ldots \tag{4.18}$$

If we keep just the first term, the equation for the pathline becomes

$$\frac{d\mathbf{x}}{dt} = \frac{\omega\eta_0}{\sinh(kH)} \begin{pmatrix} \cosh(kz_0 + kH)\cos(kx_0 - \omega t) \\ \sinh(kz_0 + kH)\sin(kx_0 - \omega t) \end{pmatrix}$$

$$\Rightarrow \qquad \mathbf{x}(t) = \mathbf{x}_0 + \frac{\eta_0}{\sinh(kH)} \begin{pmatrix} -\cosh(kz_0 + kH)\sin(kx_0 - \omega t) \\ \sinh(kz_0 + kH)\cos(kx_0 - \omega t) \end{pmatrix}$$

This is telling us that the particles travel in ellipses, squashed in the vertical direction. For deep water waves, these ellipses become circles with

$$\mathbf{x}(t) = \mathbf{x}_0 + \eta_0 e^{kz_0} \begin{pmatrix} -\sin(kx_0 - \omega t) \\ \cos(kx_0 - \omega t) \end{pmatrix} \tag{4.19}$$

The ellipses or circles becomes exponentially smaller as the depth increases. The vertical component of the velocity is in phase with the crests of the wave, $\eta \sim \cos(kx - \omega t)$. Meanwhile, the horizontal component ensures that the particle goes clockwise for waves that propagate to the right.

We can also look at the effect of the second term in (4.18). Things are simplest if we restrict attention to deep water waves, with velocity (4.16) and particle position (4.19). If we use our leading order expression (4.19) for $\mathbf{x}(t)$ we find, after a little algebra,

$$((\mathbf{x}(t) - \mathbf{x}) \cdot \nabla)\mathbf{u} = \omega k \eta_0^2 e^{2kz} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When substituted into (4.17), this has the interpretation of a constant, horizontal drift velocity for the particles, given by

$$v_{\text{drift}} = \omega k \eta_0^2 e^{2kz} = c(k\eta_0 e^{kz})^2$$

This is known as *Stokes' drift*. The ellipses traced by the particles don't quite close, but slowly inch their way in the direction in which the wave propagates. Note that there is a hierarchy of speeds,

$$v_{\text{drift}} \ll |\mathbf{u}| \ll c$$

with $k\eta_0 e^{kz} \ll 1$ the small, dimensionless number that governs successive ratios. The Stokes' drift v_{drift} is the speed at which matter bobbing in the waves moves.

4.1.3 Surface Tension

If you're a molecule, a liquid is a nice, comfortable place to spend your time. You're attracted to all your neighbouring molecules, but are afforded enough freedom to wander off on your own.

Things get more precarious at the surface of the liquid. There are now fewer neighbours to keep you company. As each neighbour offers a welcoming, attractive potential, the fact that you now find yourself a little isolated means that you are sitting in a higher energy state. This, in turn, means that, collectively, the molecules in a liquid can lower their energy by keeping the area of the surface as small as possible. This results in a force called *surface tension*. This force is the reason that droplets of water, or soap bubbles, are round: the sphere has the minimal surface area.

The existence of surface tension means that pressure need no longer be continuous across the surface. Instead, the surface can tolerate a local pressure difference by bending slightly and letting the surface tension push back. Said another way, the surface tension provides another restoring force for the wave motion.

This physics is captured by a change to the boundary condition (4.4). For a surface with embedding $z = \eta(x, y, t)$, the pressure difference should now be

$$P(x, y, \eta(x, y)) - P_0 = -\gamma \nabla^2 \eta \tag{4.20}$$

with γ the surface tension and $\nabla^2 \eta = \partial^2 \eta / \partial x^2 + \partial^2 \eta / \partial y^2$ the 2d Laplacian which is the appropriate characterisation of the curvature of the surface.

We would like to understand how the existence of surface tension affects the dynamics of waves. If we follow through our derivation of the time-dependent Bernoulli principle, equation (4.7) is replaced by

$$\rho \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right)_{z=\eta} + P_0 + \rho g \eta - \gamma \nabla^2 \eta = f(t)$$

After linearisation, the final condition in (4.9) becomes

$$\left. \frac{\partial \phi}{\partial t} \right|_{z=0} + g\eta - \frac{\gamma}{\rho} \nabla^2 \eta = \tilde{f}(t) \tag{4.21}$$

with $\tilde{f}(t)$ a function that can depend on time but, crucially, must be independent of space. We now make our usual ansatz for waves propagating in the x-direction,

$$\phi(x, z, t) = \phi_0(z)\, e^{ikx - i\omega t} \quad \text{and} \quad \eta(x, t) = \eta_0\, e^{ikx - i\omega t}$$

Much proceeds as before. In fact, we can see where how the surface tension affects the story just by staring at (4.21) where we see that it accompanies the gravitational acceleration: we just need to replace g with

$$g \longrightarrow g + \frac{\gamma k^2}{\rho} = g\left(1 + l_c^2 k^2\right) \tag{4.22}$$

in all our previous formulae. Here we've introduced the length scale

$$l_c = \sqrt{\frac{\gamma}{g\rho}} \tag{4.23}$$

This is known as the *capillary length*. From (4.22), we see that long wavelength modes with $\lambda \gg l_c$, so $l_c k \ll 1$, are pretty much unaffected by surface tension. In contrast, surface tension effects dominate when the wavelength becomes short, $\lambda \ll l_c$, so $l_c k \gg 1$. Waves with $\lambda \lesssim l_c$ are referred to *capillary waves*.

For water at room temperature, $l_c \approx 3$ mm. The capillary waves are little ripples on the water, up to a wavelength of 1 cm or so (with the factor of 2π in the definition of the wavelength raising us above l_c.)

The general dispersion relation is

$$\omega^2 = gk(1 + l_c^2 k^2) \tanh(kH) \tag{4.24}$$

while the phase velocity is

$$c = \sqrt{\frac{g}{k}(1 + l_c^2 k^2) \tanh(kH)}$$

For capillary waves, with $l_c k \gg 1$, in deep water, so $kH \gg 1$, we have

$$c \approx \sqrt{\frac{\gamma k}{\rho}}$$

In contrast to surface waves driven by gravity (4.12), the short wavelength modes now travel faster. Furthermore, the group velocity (4.15) is $v_g(k) = \frac{3}{2}c$. The wavepackets now travel faster than the individual Fourier modes.

4.2 Internal Gravity Waves

Gravitational waves are the ripples of the spacetime continuum that emerge from violent events such as the collision of two black holes. That, sadly, is not the topic of discussion here. Instead, "gravity waves" describe the disappointingly mundane phenomenon of fluids bobbing up and down due to gravity. If you want to learn more about gravitational waves, you'll need to open the lectures on General Relativity. Otherwise, read on.

Gravity waves are simply waves in fluids where the restoring force is provided by gravity. The surface waves above are examples (at least those with wavelength longer than the capillary length where surface tension is negligible). In this section we study gravity waves in the bulk of the fluid, as opposed to on the surface.

Stratified Flows and Buoyancy Frequency

A flow is said to be *stratified* if the density ρ varies from place to place. Typically this happens because of gravity and the density is a function of the vertical direction: $\rho = \rho(z)$.

Consider a small ball immersed in a stratified flow. If the ball has density $\rho_0 = \rho(z_0)$ for some height z_0 then, by Archimedes principle, it will naturally sit at height $z = z_0$. This is where the weight of water that it displaces is equal to its own weight.

Suppose now that we displace the ball upwards by some small amount δz. The density of the fluid there is

$$\rho(z_0 + \delta z) \approx \rho(z_0) + \left.\frac{\partial \rho}{\partial z}\right|_{z_0} \delta z$$

Now the weight of the displaced water differs from that of the ball, resulting in a net *upwards* force,

$$\text{Upwards Force} \approx g \left.\frac{\partial \rho}{\partial z}\right|_{z_0} \delta z$$

If $\partial\rho/\partial z > 0$ then the balls original position was unstable, and it flies upwards. But most stratified flows have density larger at the bottom than at the top, so $\partial\rho/\partial z < 0$. In this case, the ball oscillates about its equilibrium position, enacting simple harmonic motion with a frequency

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z} \tag{4.25}$$

This is called the *buoyancy frequency* or, sometimes, the *Brunt-Väisälä frequency*. In what follows, we'll look at similar motion but for the fluid itself.

Note that we haven't specified how $\rho(z)$ depends on the height z. Nor will we do this throughout the rest of this section. This follows only when we introduce an equation of state relating pressure and density. We'll meet this in Section 4.4.

Equations for Gravity Waves

Until now, the incompressibility condition was forced upon by the requirement that the density is constant. For stratified flows, this is no longer the case. Nonetheless, it is still physically sensible to insist on incompressibility (at least for speeds smaller than the sound speed)

$$\nabla \cdot \mathbf{u} = 0$$

With this, mass conservation becomes the requirement,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \Rightarrow \quad \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0$$

In addition, we will ignore viscosity and look at gravity waves in the Euler equation, now in the presence of gravity

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P - \rho(z)g\hat{\mathbf{z}}$$

We'll consider a boring background, with $\mathbf{u} = 0$ and the pressure $P_0(z)$ related to the density $\rho_0(z)$ through the Euler equation, by

$$\frac{dP_0}{dz} = -g\rho_0(z)$$

Now we look at small perturbations around this background. The gravity waves of interest travel in the horizontal x-direction, while bobbing up and down in the vertical z-direction. To this end, we look for solutions of the form

$$\mathbf{u}(\mathbf{x}, t) = (u_x, 0, u_z)e^{ik_x x + ik_z z - i\omega t}$$

with u_x and u_z constant. Both the density and pressure exhibit the same wavelike behaviour,

$$\rho(x, z, t) = \rho_0(z) + \tilde{\rho}\, e^{ik_x x + ik_z z - i\omega t} \quad \text{and} \quad P(x, z, t) = P_0(z) + \tilde{P}\, e^{ik_x x + ik_z z - i\omega t}$$

The incompressibility condition tells us that

$$k_x u_x + k_z u_z = 0 \tag{4.26}$$

Even before we proceed, this equation is telling us that $\mathbf{k}\cdot\mathbf{u} = 0$. In other words, the waves are transverse. This is like light waves (which have $\mathbf{E}\cdot\mathbf{k} = \mathbf{B}\cdot\mathbf{k} = 0$) but contrasts with the sound waves that we will meet in Section 4.4.

For the other equations, we linearise, throwing away any terms quadratic in perturbations. Mass conservation gives

$$-i\omega\tilde{\rho} + u_z\frac{d\rho_0}{dz} = 0$$

and the two components of the Euler equation are

$$-i\rho_0\omega u_x = -ik_x\tilde{P} \quad \text{and} \quad -i\rho_0\omega u_z = -ik_z\tilde{P} - g\tilde{\rho}$$

Solving these simultaneous equations gives us the dispersion relation for the frequency of gravity waves,

$$\omega = \pm N\frac{k_x}{\sqrt{k_x^2 + k_z^2}} \tag{4.27}$$

with N the buoyancy frequency (4.25). Note that we necessarily have $\omega \leq N$. Moreover, the frequency is non-vanishing only if $k_x \neq 0$. We can, however, consider the extreme example with $k_z = 0$. In this case $\omega = N$. The incompressibility condition then tells us that we must have $u_x = 0$. This, in turn, means that we have wave in the x-direction since $k_x \neq 0$ but with the motion of the fluid bobbing up and down with buoyancy frequency in the z-direction.

In general, the gravity wave propagates in the direction

$$\mathbf{k} = (k_x, 0, k_z)$$

The slight surprise comes when we compute the group velocity. For a one dimensional wave, this is $v_g = \partial\omega/\partial k$. For a higher dimensional waves, like we have here, the relevant definition is

$$\mathbf{v}_g = \frac{\partial\omega}{\partial k_x}\hat{\mathbf{x}} + \frac{\partial\omega}{\partial k_z}\hat{\mathbf{z}}$$

For the dispersion relation (4.27), this gives

$$\mathbf{v}_g = \frac{Nk_z}{(k_x^2 + k_z^2)^{3/2}}(k_z, 0, -k_x)$$

Strangely, group velocity is perpendicular to the direction of the wave, $\mathbf{v}_g \cdot \mathbf{k} = 0$. This means that both wavepackets and energy propagate in the direction $\mathbf{v}_g$, but this is orthogonal to the direction $\mathbf{k}$ of the wave itself! It is somewhat less surprising when you realise that $\mathbf{v}_g$ is parallel to the velocity $\mathbf{u}$ of the fluid.

4.3 Because the Earth Spins

In this section we take something of a diversion. We will explore some novel phenomena that arise when fluids rotate. The main motivation from this comes from the fact that Earth spins and this gives rise to some new types of waves with rather interesting properties.

Recall from the lectures on Dynamics and Relativity that if we sit in a reference frame that rotates with constant angular velocity $\boldsymbol{\Omega}$ then we experience two fictitious forces. These are the centrifugal force, proportional to $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})$ and the Coriolis force, proportional to $2\boldsymbol{\Omega} \times \mathbf{u}$. For fluids, these appear as forces on the right-hand side of the Navier-Stokes equation. Throughout this section, we will neglect viscosity and work with the Euler equation in a rotating frame, so we have

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u} = -\frac{1}{\rho}\nabla P + \mathbf{g} - 2\boldsymbol{\Omega} \times \mathbf{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) \tag{4.28}$$

The centrifugal force is not particularly interesting for our purposes. Locally, it simply redefines what we mean by "down" since, like gravity, it can be written as the gradient of a potential energy. We will simply ignore it. As we will see, all the interesting physics arises from the Coriolis force.

4.3.1 The Shallow Water Approximation

In what follows, we will make the so-called *shallow water* approximation. We will assume that the extent of the fluid in the horizontal directions, labelled by x and y, is much greater than the height of the fluid in the vertical z-direction. For our purposes, the Atlantic ocean counts as "shallow" since it is, on average, around 3.5 km deep but several thousand km wide. Similarly, the atmosphere also counts as "shallow" and the phenomena that we describe can be found in both.

Our choice of coordinates is shown in the figure to the right. Locally, "up" is in the z-direction, "North" is in the y-direction, and "East" is the x-direction. We define the *Coriolis parameter*

$$f = 2\boldsymbol{\Omega} \cdot \hat{\mathbf{z}} \tag{4.29}$$

If we're considering flows where we can neglect the curvature of the Earth, then we restrict attention to a given tangent plane as shown and take f to be constant. In contrast, if we need to take into account the curvature of the Earth, then f will be a function $f = f(y)$, reflecting the fact that as we move along the surface the local "up" direction $\hat{z}$ changes, while the spin $\boldsymbol{\Omega}$ remains fixed. In what follows, we will consider both situations in which f is taken to be constant and, in Section 4.3.5, situations in which f varies.

Our initial set-up will be similar to that of water waves described in Section 4.1. We'll take the average depth of the water to be H, with a flat, solid base at $z = -H$ and a varying surface at $z = \eta(x, y, t)$ with $|\eta| \ll H$ as shown in Figure 20. (Clearly the flat bottom is more appropriate for the ocean than the atmosphere!)

Next, we assume that the velocities in the horizontal direction are independent of the depth, so

$$\mathbf{u} = (u, v, w) \quad \text{with} \quad u = u(x, y, t), \quad v = v(x, y, t) \quad \text{and} \quad w = w(x, y, z, t)$$

Note that this is where our set-up starts to differ from the water waves of Section 4.1.

The vertical velocity can be eliminated in favour of the height fluctuation $\eta(x, y, t)$ by using the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

We integrate over the vertical z-direction, and use the free boundary condition (4.3), which tells us that $w(z = \eta) = D\eta/Dt$ and $w(z = -H) = 0$. We then have

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = -(H + \eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.30}$$

This is the first of our shallow water equations.

Next, we assume that the pressure in the vertical direction adapts to balance the gravitational force. This *hydrostatic approximation* is what led us to Archimedes principle in Section 2.1.3. We also need the boundary condition $P = P_0$ on the surface at $z = \eta$, meaning that we take the pressure to be

$$P = P_0 - \rho g(z - \eta) \tag{4.31}$$

In the Navier-Stokes equation (4.28), we can then replace $-\frac{1}{\rho}\nabla P + \mathbf{g} = -g\nabla\eta$.

With these pieces in place, the remaining two Navier-Stokes equations read

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = fv - g\frac{\partial \eta}{\partial x} \tag{4.32}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -fu - g\frac{\partial \eta}{\partial y} \tag{4.33}$$

As usual, we want an excuse to drop the non-linear terms to make life easy. If a flow has characteristic velocity U, changing over some length scale L then these non-linear terms scale as U^2/L. This should be compared with the Coriolis terms which scale as fU. We introduce a dimensionless number, this time called the *Rossby number* Ro,

$$Ro = \frac{U}{fL}$$

It's appropriate to drop the non-linear terms for flows with $Ro \ll 1$. The rotation of the Earth is $\Omega \approx 10^{-4}$ s^{-1} while typical atmospheric or oceanic speeds are around $U \sim 10\text{ms}^{-1}$. That means that

$$Ro \approx \frac{10^5 \text{ m}}{L}$$

We see that we can think about dropping the non-linear terms only for very long wavelength perturbations. For $L \sim 10^3$ km, we have $Ro \approx 0.1$ which, while admittedly < 1 is barely $\ll 1$. Nonetheless, this is the approximation that we will make. We further linearise the first equation (4.30), leaving us with our three linear shallow water equations

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x} - H\frac{\partial v}{\partial y} \tag{4.34}$$

$$\frac{\partial u}{\partial t} = fv - g\frac{\partial \eta}{\partial x} \tag{4.35}$$

$$\frac{\partial v}{\partial t} = -fu - g\frac{\partial \eta}{\partial y} \tag{4.36}$$

In the rest of this section, we will solve these equations in various scenarios for $u(x, y, t)$, $v(x, y, t)$ and $\eta(x, y, t)$.

4.3.2 Geostrophic Balance and Poincaré Waves

We're going to find a number of different solutions to the linearised shallow water equations (4.34), (4.35) and (4.36). Among these will be wave-like solutions. But, more surprisingly, we will also find some time independent solutions that are more interesting than just an ocean with a flat surface $h =$ constant.

It's simple to see the existence of time independent solutions by setting $\partial/\partial t = 0$ in (4.34), (4.35) and (4.36). Solutions can be built from any divergent free flow, with $\nabla \cdot \mathbf{u} = 0$, that obeys

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y} \quad \text{and} \quad v = +\frac{g}{f}\frac{\partial \eta}{\partial x} \tag{4.37}$$

Here the height η acts like a streamfunction of the kind we met in Section 1.1.4. Steady-state solutions of this form are said to be in *geostrophic balance*.

It's easy to understand the balance of forces underlying geostrophic balance. Suppose that there is some bump in the height of the fluid. Gravity, of course, wants to pull this down but, because the underlying fluid is incompressible, it results in a horizontal force in the direction $\nabla\eta$. The velocity in geostrophic balance is such that it gives rise to Coriolis force that balances gravity.

Flows in geostrophic balance (4.37) obey $\mathbf{u} \cdot \nabla\eta = 0$. In other words, the flow is along lines of constant height η. But, from hydrostatic balance (4.31), we know that the pressure in the fluid is proportional to the height. In other words, the flow is along

isobars. This is familiar from weather maps, where wind blows along lines of constant pressure, rather than from high to low pressure as one might naively expect. The large scale flow of both the ocean and atmosphere is largely in geostrophic balance.

Potential Vorticity

Our next task is to understand time-dependent solutions to the shallow water equations. To do this, it's best to first look more closely at the various conserved quantities.

In fact, it's best if we briefly return to the full non-linear equations (4.30), (4.32) and (4.33). These admit two conserved quantities. The first is simply the height, whose conservation follows from the underlying conservation of mass

$$\frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{u}h) = 0 \quad \text{with} \quad h = H + \eta \tag{4.38}$$

The second is conservation of vorticity. It can be checked that

$$\frac{\partial W}{\partial t} + \nabla \cdot (\mathbf{u}W) = 0 \quad \text{with} \quad W = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \tag{4.39}$$

In this equation, both ∇ and $\mathbf{u}$ are now 2d vectors, rather than 3d. Note that the vorticity includes the extra $+f$ contribution from the Coriolis force.

Both (4.38) and (4.39) are continuity equations, which is the the usual conservation law that we know and love. Elsewhere in these lectures, we've been able to use the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ to extract the velocity $\mathbf{u}$ from the clutches of the spatial derivative and write equations of this form as the vanishing of a material derivative. But we're not allowed to do this in the present context because the 2d velocity $\mathbf{u}$ does not necessarily obey $\nabla \cdot \mathbf{u} = 0$. The fluid is still incompressible of course, but the 2d velocity $\mathbf{u}$ can pile up at some point at the expense of increasing the height. Indeed, this is what our first equation (4.38) is telling us. Nonetheless, we can combine (4.38) and (4.39) to construct a quantity that has vanishing material derivative. This is

$$\frac{D\mathcal{Q}}{Dt} = \frac{\partial \mathcal{Q}}{\partial t} + \mathbf{u} \cdot \nabla \mathcal{Q} = 0 \quad \text{with} \quad \mathcal{Q} = \frac{W}{h} = \frac{1}{H+\eta}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right) \tag{4.40}$$

The quantity $\mathcal{Q}$ is called the *potential vorticity*. The equation $D\mathcal{Q}/Dt = 0$ is telling us that the value of the potential vorticity doesn't change as we follow the flow.

The discussion above is for the full non-linear equations. Something rather striking happens when we restrict to the linear equations. We linearise the conservations laws (4.38) and (4.39) about $h = H$ and $W = f$, to find

$$\frac{\partial h}{\partial t} + H\nabla\cdot\mathbf{u} = 0 \quad \text{and} \quad \frac{\partial W}{\partial t} + f\nabla\cdot\mathbf{u} = 0$$

The surprising fact is that these both have the same current: it is simply the velocity $\mathbf{u}$. This means that we can eliminate the current to find the linearised conservation law

$$\frac{\partial Q}{\partial t} = 0 \quad \text{with} \quad Q = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{f\eta}{H} \tag{4.41}$$

The quantity Q is (up to constant term and a scaling by H) the linearised potential vorticity. We see that Q is independent of time. That's a much stronger statement than our usual conservation laws. Usually when something is conserved, its value can change at some point in space by moving to a neighbouring point. That's the physics of the continuity equation. But now we learn that the function Q simply can't change at any point in space! That adds a rigidity to the system that will be responsible for some of the features we'll see below.

Poincaré Waves

With this understanding of potential vorticity in hand, we'll now turn to some wave solutions of the linearised shallow water equations (4.34), (4.35) and (4.36).

If there were no rotation, it's clear what would happen. With $f = 0$, it's simple to check that the equations (4.34), (4.35) and (4.36) become the wave equation $\ddot{\eta} = c^2\nabla^2\eta$ with $c^2 = gH$. This describes surface waves propagating with speed c and reproduces our previous result (4.13) for long wavelength waves.

The Coriolis force changes this. If we assume that $f =$ constant (which means that we are neglecting the effects of the curvature of the Earth), then the wave equation that we derive from (4.34), (4.35) and (4.36) is

$$\frac{\partial^2\eta}{\partial t^2} = c^2\nabla^2\eta - Hf\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad \text{with} \quad c^2 = gH$$

The additional terms can be rewritten in terms of the potential vorticity (4.41) to get

$$\frac{\partial^2\eta}{\partial t^2} - c^2\nabla^2\eta + f^2\eta = -HfQ \tag{4.42}$$

where Q is the potential vorticity which, as we have seen above, is a constant function that doesn't change with time. For a given problem, one might have to solve (4.42)

for some fixed Q. But, in addition, one can always add solutions to the complimentary solution which solves the homogeneous equation

$$\frac{\partial^2\eta}{\partial t^2} - c^2\nabla^2\eta + f^2\eta = 0 \tag{4.43}$$

where, as before, $c = \sqrt{gH}$. This is a rather famous equation that, in the world of Quantum Field Theory, it is known as the *Klein-Gordon equation*. It is a simple matter to find solutions by writing

$$\eta(\mathbf{x}, t) = \tilde{\eta}e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$$

with $\mathbf{x} = (x, y)$ and $\mathbf{k} = (k_x, k_y)$. This solves (4.43) provided that the frequency ω and wavevector $\mathbf{k}$ obey the dispersion relation

$$\omega^2 = c^2\mathbf{k}^2 + f^2 \tag{4.44}$$

These are known as *Poincaré waves*. They are a form of gravity wave, since gravity acts as the restoring force, as seen in the speed $c = \sqrt{gH}$. But their properties are affected by the Coriolis force. They are sometimes referred to as *inertia-gravity waves*.

For long wavelengths, $k \to 0$, Poincaré waves have a finite frequency, set by the Coriolis parameter $\omega \to f$. In the language of quantum mechanics, we say that the spectrum is *gapped*, the "gap" being the smallest frequency at which the system oscillates. (In quantum mechanics this translates into a gap in the energy spectrum because $E = \hbar\omega$.)

The cross-over from "short" to "long" wavelengths happens at the length scale

$$R = \frac{c}{f} = \frac{\sqrt{gH}}{f} \tag{4.45}$$

This is known as the *Rossby radius of deformation*. It is the characteristic length scale in the shallow water equations. (For the ocean at mid-latitudes, one has $R \approx 1000$ km. Short wavelength modes, with $k \gg R^{-1}$, act just like usual surface waves, with $\omega \approx ck$. It's the long wavelength modes, with $k \ll R^{-1}$, that feel the effect of the Coriolis force. In this limit, we can neglect the η-terms in (4.34) and (4.35) to find that the velocities obey $\dot{u} = fv$ and $\dot{v} = -fu$. This tells us that the wave velocity in the x and y-directions are $\pi/2$ out of phase.

In preparation for what follows, it's worth redoing the above calculation in a slightly different way. We write our three, linearised shallow water equations (4.34), (4.35) and (4.36) as a combined matrix eigenvalue equation

$$i\frac{\partial \Psi}{\partial t} = \begin{pmatrix} 0 & -ic\partial_x & -ic\partial_y \\ -ic\partial_x & 0 & if \\ -ic\partial_y & -if & 0 \end{pmatrix} \Psi \quad \text{with} \quad \Psi = \begin{pmatrix} \sqrt{g/H}\eta \\ u \\ v \end{pmatrix} \tag{4.46}$$

We've done some cosmetic manipulations to get the equation in this form. In addition to rescaling the η variable, we've also multiplied everything by a factor of i. This makes the resulting equation look very much like a time-dependent Schrödinger equation. In particular, the matrix is Hermitian. With our wave ansatz $\Psi = \tilde{\Psi} e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$, this becomes a standard eigenvalue problem

$$\begin{pmatrix} 0 & ck_x & ck_y \\ ck_x & 0 & -if \\ ck_y & if & 0 \end{pmatrix} \tilde{\Psi} = \omega \tilde{\Psi} \tag{4.47}$$

Because this is a Hermitian matrix, the eigenvalues are guaranteed to be real. They are

$$\omega = \pm\sqrt{c^2\mathbf{k}^2 + f^2} \quad \text{and} \quad \omega = 0 \tag{4.48}$$

We recognise the first of these as the dispersion relation for Poincaré waves (4.44). In addition, there are a collection of solutions with $\omega = 0$. In the context of condensed matter physics, this is known as a *flat band* (because if you plot ω vs k it is a flat plane.) The existence of the flat band follows from the functional conservation of the potential vorticity. It is telling us that there are additional, time independent equilibrium solutions. These are solutions like (4.45) that exhibit geostrophic balance.

4.3.3 We Need to Talk About Kelvin Waves

Everyone likes a trip to the coast. Now it's our turn. For the purposes of this course, the coast is not going to be very exciting. It's simply a boundary of our fluid, which we will take to run North/South. The fluid exists only in the $x \geq 0$ direction. For $x < 0$, there is only land.

Obviously we must put a boundary condition $u = 0$ at $x = 0$, ensuring that no flow passes the boundary. In fact, we'll do something more extreme than this. We will

search for solutions that have $u = 0$ everywhere. The linearised shallow water equation (4.35) then becomes

$$v = \frac{g}{f}\frac{\partial \eta}{\partial x} \tag{4.49}$$

This is telling us that the fluid lives in geostrophic balance in the x-direction, with the pressure gradient from $\partial\eta/\partial x$ pushing against the Coriolis force that arises because the fluid has velocity v in the y-direction. Meanwhile, the other two shallow water equations (4.34) and (4.35) become

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}$$

These are standard wave-like equations. If we make the usual ansatz that $v = v_0(x)\, e^{i\omega t - iky}$ and $\eta = \eta_0(x)\, e^{i\omega t - iky}$, these become

$$\omega\eta_0 = kHv_0 \quad \text{and} \quad \omega v_0 = gk\eta_0 \quad \Rightarrow \quad \omega^2 = c^2k^2$$

with the speed given by $c = \sqrt{gH}$ as for our previous examples. So far, things look fairly standard. But there's a slight twist in the tail. This arises when we return to (4.49) which tells us the profile of the water near the boundary. We have

$$\frac{\partial \eta_0}{\partial x} = \frac{f\omega}{kc^2}\eta_0$$

Our dispersion relation $\omega^2 = c^2k^2$ naively suggests that we have two options: $\omega = +ck$ or $\omega = -ck$. But that's not right. Suppose that we take $f > 0$, which is appropriate if we are in the Northern hemisphere. Then if we pick $\omega = +ck$ we're in trouble, because the height of the water will grow exponentially away from the boundary: $\eta_0(x) \sim e^{+fx/c}$. And that's bad. It means that we should throw away this solution. The only physical solution has

$$\omega = -ck$$

with the water profile decaying exponentially away from the boundary, $\eta_0(x) \sim e^{-fx/c}$. This means that the boundary waves propagate only in one direction which, in the current set-up, is the negative y-direction, also known as South. These are known as *Kelvin waves*.

Waves that propagate only in one direction are said to be *chiral*. In the Northern hemisphere, with $f > 0$, Kelvin waves propagate so that the land always sits to their right. (In other words, if these waves are propagating on the boundary of a lake, then they move in an anti-clockwise direction.) In the Southern hemisphere, where $f < 0$, the same argument tells us the we must have the $\omega = +ck$ solution, so Kelvin waves propagate with the land to its left as it moves.

Chiral waves also make an appearance in various condensed matter systems where, as here, they typically live at the edge of some system. In that context, there is often some deep topological reason for the emergence of such chiral waves. The same is also true here and we will elaborate on this further in Section 4.3.6.

4.3.4 Rossby Waves

As we've seen, the linearised shallow water equations admit time independent solutions in geostrophic balance, solving (4.37). But objects that are strictly unmoving are rare in Nature. One can ask: is there something that can coax flows to geostrophic balance to move? The answer, as we shall see, is yes. In this section, we will see that if we look at scales over which the Coriolis parameter f is no longer constant, then flows in geostrophic balance start to evolve in time. This is known as *quasi-geostrophic* balance.

Crucially, the evolution of flows in quasi-geostrophic balance happens much more slowly than the dynamics of Poincaré waves that we saw above. That means that it is this quasi-geostrophic flow governs the long-time dynamcis of the ocean and atmosphere. The purpose of this section is to construct the equations that describe this flow.

At different latitudes θ, the Coriolis parameter is given by $f = 2\Omega \sin\theta$, where $\Omega = 2\pi$ day^{-1}. To capture the variation of the Coriolis parameter, it will suffice to consider just the leading term in the Taylor expansion

$$f = f_0 + \beta y$$

with y the direction points North. Our strategy will be to turn again to the conservation of potential vorticity (4.40),

$$\frac{D\mathcal{Q}}{Dt} = 0 \quad \text{with} \quad \mathcal{Q} = \frac{1}{H+\eta}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right)$$

You can check that this equations remains valid even when $f = f(\mathbf{x})$. We will consider flows with Rossby number $Ro \ll 1$ that are very close to geostrophic balance (4.37). This means that we can replace the vorticity in the expression with,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{g}{f}\nabla^2\eta \approx \frac{g}{f_0}\nabla^2\eta$$

We further assume that variations in the height are small, so $\eta \ll H$, and the potential vorticity can be written solely in terms of the height fluctuations η. Ignoring an overall constant term, we have

$$\mathcal{Q} \approx \frac{f_0}{H^2}\left(\frac{c^2}{f_0^2}\nabla^2\eta - \eta + \frac{\beta H}{f_0}y\right) \ . \tag{4.50}$$

As we've seen, potential vorticity is materially conserved and, using the geostrophic balance condition (4.37), this too becomes an equation that can be written solely in terms of the height

$$\frac{D\mathcal{Q}}{Dt} = 0 \quad \Rightarrow \quad \dot{\mathcal{Q}} - \frac{g}{f_0}\frac{\partial \eta}{\partial y}\frac{\partial \mathcal{Q}}{\partial x} + \frac{g}{f_0}\frac{\partial \eta}{\partial x}\frac{\partial \mathcal{Q}}{\partial y} = 0 \tag{4.51}$$

This is now a dynamical equation for the height η. It is known as the shallow water *quasi-geostrophic* equation.

The quasi-geostrophic equation looks a little daunting. But we can easily extract some simple physics. We linearise about a flat surface with $\eta = 0$ and drop any term quadratic in η. The equation then becomes

$$\frac{\partial}{\partial t}\left(c^2\nabla^2\eta - f_0^2\eta\right) + c^2\beta\frac{\partial \eta}{\partial x} = 0$$

We see clearly that the term with β, which captures the variation of the Coriolis parameter, is driving the dynamics. If we look for plane wave solutions with $\eta = \eta_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}$, we find the dispersion relation

$$\omega = -\beta c^2 \frac{k_x}{c^2\mathbf{k}^2 + f_0^2} \tag{4.52}$$

When $\beta = 0$, this gives us the flat band $\omega = 0$ that corresponds to steady-state geostrophically balanced flows. But once we take into account the variation of the Coriolis parameter, these flows start to move. The resulting waves are called *Rossby waves*. The minus sign in (4.52) is important. It is telling us that long wavelength (small k) waves travel in a westward direction. This is indeed the dominant motion of the ocean seen in satellite images. These images clearly reveal Rossby waves that take months, or even years, to cross the Pacific ocean.

It's useful to summarise what we've seen here. The shallow water equations admit two classes of solutions: fast-moving Poincaré waves and slow-moving quasi-geostrophic flows, including Rossby waves. The magic of the quasi-geostrophic equation (4.51) is that it has successfully filtered out the fast-moving Poincaré waves, leaving us just with the slow-moving modes. It is what is referred to in other areas of physics as the "low energy (or frequency) effective field theory". Historically, it the development of the quasi-geostrophic equation was crucial in developing successful weather prediction.

4.3.5 Equatorial Waves

We now ask: what happens when we sit at the equator. Here the Coriolis parameter (4.29) vanishes,

$$f = 2\boldsymbol{\Omega}\cdot\hat{\mathbf{z}} = 0$$

and one might naively think that there can't be any interesting physics due to the Coriolis force. In fact, things are more subtle and more interesting.

To find the more interesting physics, we look a little away from the equator. If we Taylor expand, Coriolis parameter becomes position dependent

$$f(y) = \beta y$$

Here the y-direction is North, and $y = 0$ corresponds to the equator. The parameter β has dimension $[\beta] = L^{-1}T^{-1}$. We can form a distance scale

$$L_{\rm eq} = \sqrt{\frac{c}{\beta}}$$

For the Earth's oceans, this is around $L_{\rm eq} \approx 250$ km. It is somewhat larger for the atmosphere.

We again arrange the height perturbation $\eta(x, y, t)$ and the velocities $u(x, y, t)$ and $v(x, y, t)$ as a vector $\Psi(x, y, t)$ as in (4.46). This time we will look for solutions that are localised near the equator but propagate as waves in the x-direction (i.e. East/West),

$$\Psi(x, y, t) = \tilde{\Psi}(y)e^{i\omega t - ikx} \tag{4.53}$$

The shallow water equations now become

$$\begin{pmatrix} 0 & ck & ic\partial_y \\ ck & 0 & -i\beta y \\ ic\partial_y & i\beta y & 0 \end{pmatrix} \tilde{\Psi} = \omega\tilde{\Psi}$$

Again, we're looking for eigenmodes of this equation. As in the case when f was constant, we expect different branches.

Equatorial Kelvin Waves

To kick us off, there is a special solution to (4.53). This occurs when $v = 0$, so there is no velocity in the y-direction. The equations coming from the first two components of (4.53) are simply algebraic. They relate $\tilde{u} = (\omega/kH)\tilde{\eta}$ and result in the dispersion relation

$$\omega^2 = c^2k^2 \quad \Rightarrow \quad \omega = \pm ck \tag{4.54}$$

We're left just with the third component of (4.53), which governs the profile of $\tilde{\eta}(y)$ and $\tilde{u}(y)$ in the y-direction,

$$\frac{c^2}{H}\frac{\partial\tilde{\eta}}{\partial y} = -\beta y\tilde{u} \quad \Rightarrow \quad \frac{\partial\tilde{\eta}}{\partial y} = -\frac{\omega}{ck}\frac{y}{L_{\rm eq}^2}\tilde{\eta}$$

The key feature of the solution comes from that factor of ω/ck on the right-hand side. From the dispersion relation (4.54), this is either ± 1. However, the resulting solution is only normalisable, and localised around the equator, if we take the positive sign

$$\omega = +ck \quad \Rightarrow \quad \tilde{\eta} = \eta_0 e^{-y^2/2L_{\rm eq}^2}$$

The other choice of sign, with $\omega = -ck$, leads to a divergent solution $\tilde{\eta} \sim e^{+y^2}$ which is not physically permissible. The upshot is rather nice: we have waves at the equator that only travel in the positive x-direction. In other words, they only go east. In analogy with the coastal waves that we met in Section 4.3.3, these are known as *equatorial Kelvin waves*.

Rossby, Poincaré and Yanai Waves

Let's now return to the general problem of equatorial waves, given by the Schrödinger-like equation (4.53). The second component of (4.53) is algebraic and allows us to eliminate $\tilde{u}$ in favour of $\tilde{v}$ and $\tilde{\eta}$. This results in a pair of coupled, first order differential equations

$$\begin{aligned} i\left(\frac{\partial}{\partial y} - \frac{\beta k y}{\omega}\right)\tilde{v} &= \frac{1}{H}\left(\omega - \frac{c^2k^2}{\omega}\right)\tilde{\eta} \\ i\left(\frac{\partial}{\partial y} + \frac{\beta k y}{\omega}\right)\tilde{\eta} &= \frac{H}{c^2}\left(\omega - \frac{\beta^2 y^2}{\omega}\right)\tilde{v} \end{aligned} \tag{4.55}$$

We can eliminate $\tilde{\eta}$ to manipulate this into a second order differential equation for $\tilde{v}$ alone. After a little bit of algebra, this is

$$\left(-c^2\frac{\partial^2}{\partial y^2} + \beta^2 y^2\right)\tilde{v} = \left(\omega^2 - c^2k^2 - \frac{\beta c^2 k}{\omega}\right)\tilde{v}$$

But this is an equation that we've seen elsewhere: it is the Schrödinger equation for the harmonic oscillator that we met in our first course in Quantum Mechanics. In that context, we write

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\bar{\omega}^2 y^2\right)\tilde{v} = E_n\tilde{v}$$

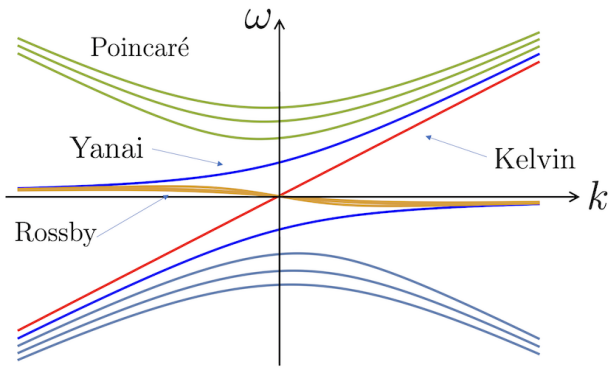


Figure 22. Chiral Kelvin (in red) and Yanai (in dark blue) waves, together with a discretum of Poincaré and Rossby waves.

where m is the mass of the particle and $\bar{\omega}$ is the frequency of the harmonic oscillator (not to be confused with ω, the frequency of our waves the we're trying to determine). Because we are again interested in normalisable solutions, we can simply import our results from quantum mechanics. The velocity $\tilde{v}(y)$ is given by Hermite polynomials. More importantly, the energies of the harmonic oscillator are, famously,

$$E_n = \hbar\bar{\omega}\left(\frac{1}{2} + n\right) \quad \text{with} \quad n = 0, 1, \ldots$$

Translating back into the variables of our equatorial waves, the dispersion relation is given by

$$\omega^3 - \omega\left(c^2k^2 + \beta c(1+2n)\right) - \beta c^2 k = 0 \quad \text{with} \quad n = 0, 1, 2, \ldots \tag{4.56}$$

We'll now look at these for different n. We'll see, the $n = 0$ waves are somewhat different from the $n \geq 1$ waves.

Let's start with the $n = 0$ waves. First note that in this case (4.56) has a root $\omega = -ck$. Naively, this looks like a wave moving in the opposite direction to the Kelvin wave. But it is a spurious solution. This is because although $\tilde{v}(y)$ is normalisable, when we plug this solution into (4.55) we find that $\tilde{\eta}(y)$ is non-normalisable: it has a piece that diverges as $\tilde{\eta} \sim e^{+y^2/2L_{\rm eq}^2}$. So this solution should be thrown out. It turns out that it's the only spurious solution and all others are fine.

If we factor out the spurious $\omega = -ck$ solution, then we find a single $n = 0$ wave, with dispersion relation

$$\omega = \frac{ck}{2} \pm \frac{1}{2}\sqrt{c^2k^2 + 4\beta c}$$

This too is a chiral wave. At large wavenumber, it has the same dispersion relation $\omega \sim +ck$ as the Kelvin wave. However, it differs at small wavenumber, with the dispersion relation affected by the Coriolis force. These are known as *Yanai* waves. (They are also sometimes called mixed Rossby-gravity waves.) The velocity profile is Gaussian around the equator, with $\tilde{v} \sim e^{-y^2/2L_{\rm eq}^2}$.

For $n \geq 1$, the general shape of the dispersion relation takes the same form. There are three branches of modes, which are modified versions of the dispersion relations (4.48) that we saw when f is constant. We again see the dispersion relations corresponding to Poincaré waves, with their characteristic gapped spectrum, asymptoting to $\omega \to \pm ck$.

In addition, we see that our flat band, which previously had $\omega = 0$, is also deformed. Now, it is no longer flat, but asymptotes to $\omega \to -\beta/k$ for large $|k|$. These are equatorial *Rossby waves*. The various modes for $n = 0, 1, 2, 3$, together with the Kelvin wave, are shown in Figure 22. Note that the dispersion relation for the Rossby waves is much flatter than those of the Poincaré waves. Correspondingly, the group velocity of the Rossby waves will be much slower.

4.3.6 Chiral Waves are Topologically Protected

As we mentioned previously, chiral waves appear in various condensed matter systems. The most familiar example is the Quantum Hall Effect, where a sample of electrons in a magnetic field has chiral modes propagating on its edge.

In the context of condensed matter, it turns out that the presence of chiral edge modes can be traced to some interesting topological features of the system, an observation that led to many new developments in the field. The purpose of this section is to point out that, rather wonderfully, the same is true for chiral waves in fluids. I should warn you that this section is something of a departure from the rest of the notes and the motivation is, in part, simply to illustrate the unity of physics.

We will describe the topology associated to equatorial chiral modes. (There is a similar, but more complicated, story for coastal Kelvin waves.) The idea is that the existence of the two chiral modes – Kelvin and Yanai – is a direct consequence of topology in momentum space.

To set the scene, we will return to the case of constant Coriolis parameter f. As we've seen in (4.48), there are three bands with dispersion

$$\omega = \pm\sqrt{c^2\mathbf{k}^2 + f^2} \quad \text{and} \quad \omega = 0 \tag{4.57}$$

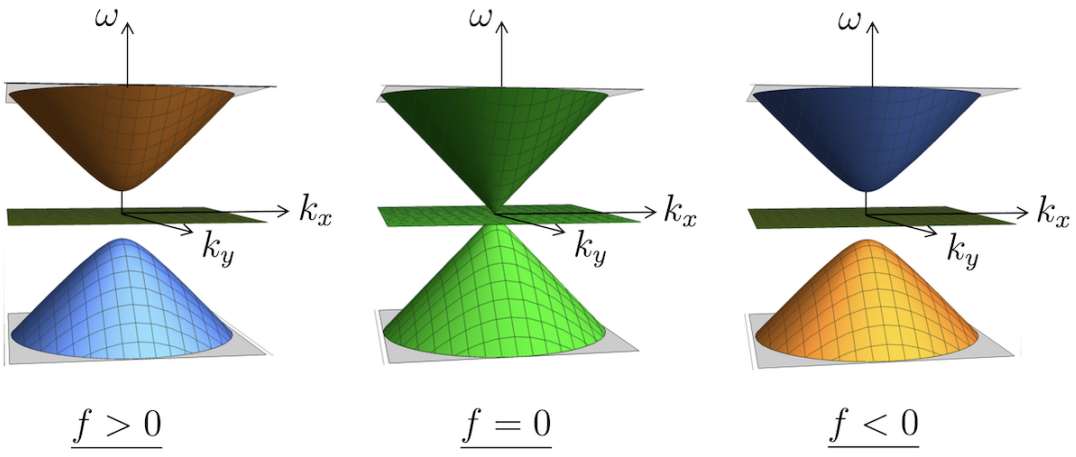


Figure 23. The band structure as a function of constant Coriolis parameter f.

The resulting bands are shown in Figure 23 for three cases: $f > 0$, $f = 0$ and $f < 0$. For $f \neq 0$, there is a gap between the geostrophic flat band and the Poincaré waves. This gap closes when $f = 0$. The fact that the gap closes at $f = 0$ is closely related to the existence of the chiral equatorial waves.

The question that the topological approach addresses is: how robust is this situation? Could we, for example, add some further parameters to the problem so that, as we vary f from positive to negative, the gap never closes? Topology tells us that the answer to this is: no. There must always be some point that looks like the $f = 0$ figure where the gap closes.

The reason for this is that there is a subtle difference between the $f > 0$ and $f < 0$ situations. This difference doesn't show up in dispersion relations (4.57) which are clearly symmetric under $f \to -f$. Instead, we have to look more closely at what's going on in each band.

Recall from (4.47) that the frequencies arise as the solution to the following eigenvalue problem

$$\begin{pmatrix} 0 & ck_x & ck_y \\ ck_x & 0 & -if \\ ck_y & if & 0 \end{pmatrix} \tilde{\Psi} = \omega \tilde{\Psi}$$

We will focus on the positive frequency band of Poincaré waves, with

$$\omega(\mathbf{k}) = +\sqrt{c^2\mathbf{k}^2 + f^2}$$

As we've already mentioned, the eigenvalues are clearly invariant under $f \to -f$. To see the difference between $+f$ and $-f$ we need to look at the eigenvector. This is given by

$$\tilde{\Psi}_+(\mathbf{k}, f) = \frac{1}{\sqrt{2\omega^2\mathbf{k}^2}} \begin{pmatrix} c\mathbf{k}^2 \\ k_x\omega - ifk_y \\ k_y\omega + ifk_x \end{pmatrix}$$

Obviously, the eigenvector depends on the wavenumber $\mathbf{k}$. This means that as we move around momentum space, labelled by $\mathbf{k} \in \mathbb{R}^2$, the eigenvector Ψ_0 evolves in $\mathbb{C}^3$. The key idea is that as we explore all of momentum space, the eigenvector may twist within the larger space $\mathbb{C}^3$. This twist is where topology enters the story.

The fact that eigenvectors twist and turn in a larger space is more familiar in the context of quantum mechanics where it goes by the name of *Berry phase*. (You can read about this both in the lectures on Topics in Quantum Mechanics and in the lectures on the Quantum Hall Effect.) We will not review this in detail, but simply state how to characterise the topology of the eigenvector. First, given an eigenvector Ψ_0 we define the *Berry connection*,

$$A_i(\mathbf{k}) = -i\tilde{\Psi}_+^\dagger \frac{\partial \tilde{\Psi}_+}{\partial k^i} \qquad i = 1, 2$$

A short calculation shows that

$$A_i = \frac{f}{\omega\mathbf{k}^2}(k_y, -k_x)$$

The Berry connection has the same mathematical structure as the gauge potential in electromagnetism. In particular, as the next step we compute something akin the magnetic field,

$$B = \partial_1 A_2 - \partial_2 A_1 = \frac{c^2 f}{(f^2 + c^2\mathbf{k}^2)^{3/2}}$$

This is known as the *Berry curvature*. Finally, we integrate this curvature over momentum space to get an object known as the *Chern number*, which we calculate to be

$$C = \frac{1}{2\pi}\int_{\mathbb{R}^2} d^2k \; B = \text{sign}[f] \tag{4.58}$$

Note that, as promised, the Chern number distinguishes between f positive and f negative: we have $C = +1$ for $f > 0$ and $C = -1$ for $f < 0$.

At this stage the argument becomes slightly delicate. When the Chern number is computed by integrating over a compact space (i.e. one which doesn't stretch to infinity), then there is a mathematical result that says

$$C \in \mathbb{Z}$$

(In physics, this is usually referred to as Dirac quantisation.) The fact that C is integer valued is important. It is telling us that we have some discrete way of characterising the system, even though the underlying fluids are continuous. This is the essence of topology.

However, things are not so straightforward for our fluids because the integral (4.58) is not over a compact space but instead over $\mathbb{R}^2$. (This is not a problem in condensed matter systems because the underlying spatial lattice means that momentum lives in a compact Brillouin zone.) And there's no mathematical theorem that says such an integral should be integer valued. Indeed, if you integrate the magnetic flux through a solenoid then you can get anything at all. There are a couple of ways around this and we will take the cheapest. Note that asymptotically, as $|\mathbf{k}| \to \infty$, we have $A_i \to 0$. In fact, more importantly, we have $\oint A_i dk^i \to 0$ as the integration curve is taken out to infinity. This is a property of short wavelength modes and so should hold for regardless of any deformation of the system which doesn't affect arbitrarily short wavelengths. So we insist that A_i is trivial asymptotically and this allows us to effectively compactify the problem, by adding a point at infinity and viewing $\mathbb{R}^2 + \{\infty\} = \mathbf{S}^2$. Correspondingly, we learn that the Chern number C – which is clearly an integer in (4.58) – should remain an integer no matter how we deform the system.

Now we're in business. For $f > 0$, we have $C = 1$. This is the yellowy-brown, upper band on the left-hand side of Figure 23. But because C is restricted to be an integer, it can't change as we vary parameters. The only exception to this is if the gap to some other band closes, because then the eigenvector Ψ_+ becomes degenerate with another eigenvector and the calculation above breaks down. But the Chern number for $f < 0$ is $C = -1$, depicted in blue in Figure 23, so we learn that here is no path in any enlarged parameter space that takes us from $f > 0$ to $f < 0$ without closing the gap.

In fact, there's more to learn from this. The number of chiral edge modes that appear as we vary from $f > 0$ to $f < 0$ is given by the difference in the Chern numbers. In other words, the number of chiral waves is necessarily $C[f > 0] - C[f < 0] = 2$. And this is indeed what we find, with the Kelvin and Yanai waves appearing at the equator.

As we've mentioned previously, the kind of calculation that we've performed above underlies various properties of materials, notably quantum Hall states and topological insulators. (See, for example, Section 2.3 of the lectures on the Quantum Hall Effect for a closely related computation for the Chern insulator.) In that context, the use of topology blossomed, revealing many deep and new ideas about exotic materials. So far, in fluid mechanics, topology seems to be little more than a curiosity. Hopefully, that will change in the future.

4.4 Sound Waves

Throughout these lectures, we've focussed on incompressible fluids, obeying $\nabla \cdot \mathbf{u} = 0$. It is now time to abandon this assumption. Instead, we want to ask: what new physics arises when the density of a fluid changes? The answer, as we shall see, is sound.

Apart from its inherent interest, this question forces us to re-examine the fundamental equations of fluid mechanics. So far, we have mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{4.59}$$

and the Navier-Stokes equation

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P + \mu \nabla^2 \mathbf{u} + \left(\frac{\mu}{3} + \zeta \right) \nabla (\nabla \cdot \mathbf{u})$$

where, because the fluid is now compressible, there's a second viscosity term that can appear on the right-hand side. This comes with a new coefficient ζ, the *bulk viscosity* in addition to μ, the *dynamic shear viscosity*. (In fact, we'll largely ignore the effects of both viscosities in this section, but it's worth keeping it in play while we return to the fundamentals.)

When the density was constant, these equations were all we needed. They are four equations that govern four independent, dynamical fields, $\mathbf{u}(\mathbf{x}, t)$ and $P(\mathbf{x}, t)$. However, when $\rho = \rho(\mathbf{x}, t)$ is also a dynamical field we need a further equation before we can get going. As we now explain, this additional equation is dictated by thermodynamics and forces us to think about the temperature of the fluid.

4.4.1 Compressible Fluids and the Equation of State

The *equation of state* is a relationship between the pressure P, the volume V and the temperature T of a fluid. The simplest such example is very familiar: it is the ideal gas law

$$PV = Nk_B T$$

Here N is the number of particles in the gas and k_B is Boltzmann's constant. The ideal gas equation describes a gas of non-interacting particles. (Note that the "ideal" in "ideal gas" means something different from the "ideal" in "ideal fluid"! The latter just refers to something that obeys the Euler equation with no viscosity.) Other equations of state can be more complicated, capturing some internal interactions between the constituent molecules. For example, a simple generalisation of the idea gas law is the van der Waals equation

$$\frac{PV}{Nk_BT} = 1 - \frac{N}{V}\left(\frac{a}{k_BT} - b\right)$$

where a and b are two constants that characterise the interactions of the gas. You can find derivations of both these equations of state in the lectures on Statistical Physics.

We can make contact with our previous equation if we replace the volume with the density ρ of the fluid,

$$\rho = \frac{Nm}{V}$$

where m is the mass of the each individual particle in the fluid. Then the ideal gas law becomes

$$P = \frac{\rho k_B T}{m} \tag{4.60}$$

When we first meet the equation of state, we think of P, $\rho \sim 1/V$ and T as numbers that describe the global, equilibrium properties of the system. However, the whole point of fluid mechanics is that we can understand what happens as we move away from equilibrium. To achieve this, we assume that locally the system is still described by P, ρ and T but these are now dynamical fields whose values can vary in space and in time. The equation of state now gives a *local* relationship between these quantities, for example

$$P(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t) k_B T(\mathbf{x}, t)}{m}$$

The existence of the equation of state tells us why we need to start thinking about temperature. If the pressure P and density ρ are changing, then so too is T. Indeed, this is true even when ρ is constant but throughout these lectures we have implicitly assumed that $T(\mathbf{x}, t)$ simply tracks the pressure $P(\mathbf{x}, t)$. Now, however, we need to think more carefully about how T changes.

4.4.2 Some Thermodynamics

The correct way to proceed is to derive an equation of motion for the temperature $T(\mathbf{x}, t)$. For now, however, we'll take something of a shortcut. For completeness, we will then describe the better approach in Section 4.4.3.

The shortcut that we have in mind is called the *adiabatic approximation*. Heuristically, this means that we assume that the timescale over which the fluid moves is much shorter than the timescale of heat diffusion within the fluid. Mathematically, it means that we assume a quantity called *entropy* is conserved. The purpose of this section is to review some basic facts about thermodynamics, the purpose of which is to lead us to the following, simple result: under the adiabatic approximation

$$\frac{P}{\rho^\gamma} = \text{constant} \tag{4.61}$$

where γ is the ration of heat capacities $\gamma = c_P/c_V$ and will be defined below. For air, $\gamma \approx 1.4$. Starting in Section 4.4.4 we'll make use of this result to study the properties of sound waves.

For now, we'll revert to the older setting where P, V and T as just numbers that characterise the global property of an equilibrium system. We then need to turn to the laws of thermodynamics. (A much fuller discussion of this material can be found in Section 4 of the lectures on Statistical Physics.).

The *first law of thermodynamics* says that the energy E of a system can change in one of two ways: either by adding heat δQ, or by adding work, δW

$$dE = \delta Q + \delta W$$

The energy is a function of the system, but both heat and work are things that you do to the system. There's no sense in which we can talk about the "work" of a gas or the "heat" of a gas; only the heat added to a gas. Roughly speaking, this is the reason that we write the terms on the right-hand side as δQ and δW instead of dQ and dW.

However, it should be possible to describe the effect of both the work done and the heat added in terms of changes to the state of the system. For the work done, this is straightforward. If the fluid has pressure P and we squeeze it by changing its volume, then the infinitesimal work done is

$$\delta W = -P dV$$

To write a similar statement for the heat added to a gas we need to turn to the second law of thermodynamics. This is the statement that the state of the system in equilibrium can be characterised by a function $S(T, P)$ known as *entropy*. Furthermore, for a reversible change we have

$$\delta Q = TdS$$

This definition, relating entropy to heat, is due to Clausius. Subsequently, Boltzmann understood entropy in terms of counting microscopic arrangements of atoms. A large part of the course on Statistical Physics is to understand why these two definitions are actually equivalent. For our purposes, we'll only need the definition above.

Adiabatic processes, of which sound waves are an example, have $\delta Q = 0$. You might think that this means we can simply ignore the heat term. Sadly, that's not quite true! We need to understand a little better what heat actually is before we can discard it.

Next, we need the idea of a heat capacity. This is straightforward: it measures how much the temperature of a system rises if you add some heat. (Actually, it's defined to be the inverse of this.) The subtle point is that you must specify what you are holding fixed when you do this experiment. You could, for example, hold the volume fixed. The corresponding heat capacity C_V is defined by

$$C_V = T\left.\frac{\partial S}{\partial T}\right|_V = \left.\frac{\partial E}{\partial T}\right|_V$$

where, in the second equality, we've used the first law of thermodynamics $dE = TdS - PdV$ where the $-PdV$ term doesn't contribute precisely because we're holding the volume fixed. Alternatively, we can add heat keeping the pressure fixed, rather than the volume. Again, using the first law, we have

$$C_P = T\left.\frac{\partial S}{\partial T}\right|_P = \left.\frac{\partial E}{\partial T}\right|_P + P\left.\frac{\partial V}{\partial T}\right|_P \tag{4.62}$$

In this case, the temperature is expected to rise less because the energy from the heat must now also do work expanding the volume of the gas. Correspondingly, we expect $C_P > C_V$. We often talk about the *specific heats*, which is the heat capacity per unit volume: $c_V = C_V/V$ and $c_P = C_P/V$.

The Ideal Gas

So far, our discussion has been general. To make progress, we now focus on a specific system: the ideal gas, with the familiar equation of state

$$PV = Nk_BT$$

This is a good approximation for dilute gases, like the air in this room. It's not a good approximation for liquids.

The final fact that we need is known as *equipartition*. It is the statement that, at temperature T, the energy of each microscopic degree of freedom is given by $\frac{1}{2}k_BT$. This means that if we have a gas of N "monatomic" particles, meaning that each particle is itself a structureless object, then

$$E = \frac{3}{2}Nk_BT \quad \text{for monatomic gases}$$

where the $\frac{3}{2}$ comes because each particle can move in three dimensions. However, if the particles comprising the gas have additional internal degrees of freedom then equipartition ensures that these too contribute to the energy. For example, a diatomic molecule can be viewed as a dumbbell-like object. It has three translational degrees of freedom, but also two rotational degrees of freedom. (The rotation about the axis doesn't count[8].) This means that the energy is

$$E = \frac{5}{2}Nk_BT \quad \text{for diatomic gases}$$

Air is mostly N_2 and O_2, both of which are diatomic molecules, so this is the energy of air.

We can now compute the heat capacities for the different ideal gases. We have

$$C_V = \frac{3}{2}Nk_B \quad \text{and} \quad C_P = \frac{5}{2}Nk_B \quad \text{for monatomic gases}$$

and

$$C_V = \frac{5}{2}Nk_B \quad \text{and} \quad C_P = \frac{7}{2}Nk_B \quad \text{for diatomic gases}$$

Note that, in both cases, $C_P - C_V = Nk_B$, which follows from (4.62), together with the equation of state. It will be useful to define the ratio of the heat capacities

$$\gamma = \frac{C_P}{C_V} = \begin{cases} 5/3 & \text{for monatomic gases} \\ 7/5 & \text{for diatomic gases} \end{cases}$$

This is where we get the statement that $\gamma \approx 1.4$ for air.

[8]Rather wonderfully, this is a quantum mechanical effect! Both the rotation about the axis and the vibrational mode of the dumbbell have a minimum energy required to excite them due to quantum mechanics, and this energy is higher than k_BT at room temperatures. The same is true of the rotational mode of a monatomic gas.

Finally, we can use the technology above to compute the entropy of an ideal gas. We start from the first law, now written as

$$dS = \frac{1}{T}dE + \frac{P}{T}dV = \frac{C_V}{T}dT + \frac{Nk_B}{V}dV$$

We now replace $Nk_B = C_P - C_V$ and integrate to get

$$\begin{aligned} S &= C_V \log\left(\frac{T}{T_0}\right) + (C_P - C_V)\log\left(\frac{V}{V_0}\right) \\ &= C_V \log\left(\frac{T}{T_0}\left(\frac{V}{V_0}\right)^{\gamma-1}\right) \\ &= C_V \log\left(\frac{P}{P_0}\left(\frac{V}{V_0}\right)^{\gamma}\right) \end{aligned} \tag{4.63}$$

This means that if entropy is to remain constant under some change, the pressure and volume must scale so that PV^γ is constant. Or, written in terms of the density $\rho \sim 1/V$,

$$\frac{P}{\rho^\gamma} = \text{constant} \tag{4.64}$$

This is the result (4.61) that we advertised at the beginning of this section.

4.4.3 Briefly, Heat Transport

There's a more sophisticated way of stating the result above, in which we focus directly on the dynamics of the temperature field $T(\mathbf{x}, t)$. For an ideal fluid, the temperature is governed by the transport equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)T + (\gamma - 1)T\,\nabla\cdot\mathbf{u} = 0 \tag{4.65}$$

We won't derive this here. (You can find the derivation in the lectures on Kinetic Theory.) But we can at least see how it reproduces the result above. To see this, note that the requirement of constant entropy can also be written as $TV^{\gamma-1} \sim T\rho^{1-\gamma}$ is constant, as in (4.63). But with T and ρ both fields, we can see how these evolve within the fluid. The appropriate meaning of "constant" is that the material derivative vanishes. We have

$$\begin{aligned} \frac{D}{Dt}(T\rho^{1-\gamma}) &= \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)(T\rho^{1-\gamma}) \\ &= (1-\gamma)\rho^{1-\gamma}T\nabla\cdot\mathbf{u} + T\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\rho^{1-\gamma} \end{aligned}$$

Here the first term follows from (4.65). We can evaluate the second term using the conservation of mass (4.59). We find

$$\frac{D}{Dt}(T\rho^{1-\gamma}) = 0$$

as expected for adiabatic evolution.

This language has the advantage that it allows us to go beyond ideal fluids. In fact, the heat transport equation (4.65) should be viewed as analogous to the Euler equation for the velocity: both are missing the effect of dissipation. For the velocity field, this is captured by viscosity. For the temperature field, it is captured by heat conductivity. κ. This appears as an additional term in the heat equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right) T + (\gamma-1)T\,\nabla\cdot\mathbf{u} = \frac{\gamma-1}{\gamma}\frac{\kappa m}{\rho k_B}\nabla^2 T \tag{4.66}$$

where the strange collection of coefficients on the right-hand side means that the coefficient multiplying $\nabla^2 T$ can be identified as κ/c_P where $c_P = \rho k_B \gamma/(\gamma-1)m$ is the specific heat at constant pressure. These terms tell us how heat diffuses in the fluid. Indeed, in the absence of any flow, so $\mathbf{u} = 0$, it reduces to the heat equation

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c_P}\nabla^2 T$$

The adiabatic approximation that we invoked above is essentially the statement that the diffusion of heat can be neglected in the problem of interest. And that problem is, of course, sound waves

4.4.4 The Equations for Sound Waves

Finally, after that long preamble, we can turn to the subject of interest: sound waves. We will initially ignore viscosity (remedying this in Section 4.4.5) and work with the Euler equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P$$

Our starting point is the simplest possible solution to the Euler equation: a stationary fluid, with constant density and pressure

$$\mathbf{u} = 0 \quad , \quad \rho = \rho_0 \quad , \quad P = P_0$$

We then study small perturbations about this background. We will take

$$\rho = \rho_0 + \tilde{\rho} \quad \text{and} \quad P = P_0 + \tilde{P}$$

with the perturbations small, meaning $\tilde{\rho} \ll \rho_0$ and $\tilde{P} \ll P_0$. We'll also take $\mathbf{u}$ to be small, in the sense that we keep terms only linear in $\mathbf{u}$, $\tilde{\rho}$ and $\tilde{P}$. The linearised Euler equation then becomes

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla \tilde{P} \tag{4.67}$$

We augment this with the equation of mass conservation,

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = 0 \tag{4.68}$$

We can combine these by taking the gradient ∇ of the first and the time derivative of the second. This gives

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - \nabla^2 \tilde{P} = 0 \tag{4.69}$$

At this point, we need to invoke the adiabatic approximation (4.64) which, after linearising, becomes

$$\frac{(P_0 + \tilde{P})}{(\rho_0 + \tilde{\rho})^\gamma} = \text{constant} \quad \Rightarrow \quad \tilde{P} - \frac{P_0}{\rho_0}\gamma\tilde{\rho} = 0$$

The equation (4.69) then becomes

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - c_s^2 \nabla^2 \tilde{\rho} = 0 \tag{4.70}$$

This is the wave equation. The speed of sound is given by

$$c_s = \sqrt{\frac{\gamma P_0}{\rho_0}} \tag{4.71}$$

For an ideal gas, the equation of state (4.60) relates this to the temperature T_0 of the background fluid, and the mass m of the constituent particles,

$$c_s = \sqrt{\frac{\gamma k_B T_0}{m}} \tag{4.72}$$

We see that the speed of sound depends on the temperature. For the air at 20°, the speed is $c_s \approx 340$ ms^{-1}. This was first measured by Newton by clapping his hands in Nevile's court, Trinity College. (He got a value around 300 ms^{-1}.)

A General Fluid

The equation (4.69) holds for any fluid while, the subsequent derivation of the wave equation, we restricted to the ideal gas. But we get the same wave equation for any equation of state; it's just the speed of sound that changes.

It's useful to think of the pressure as a function of

$$P = P(\rho, S)$$

(It's perhaps more natural to think of $P = P(V, T)$. The density is trivially related to the volume by $\rho \sim 1/V$. But the entropy S is a conjugate variable to the temperature T and it is also possible to think of pressure as a function of entropy. This kind of "what function depends on what variable" is a large part of the game of thermodynamics.) Taylor expanding, the fluctuations in pressure and density are then related by

$$\tilde{P} = \left.\frac{\partial P}{\partial \rho}\right|_S \tilde{\rho}$$

In general, the speed of sound is then given by

$$c_s = \sqrt{\left.\frac{\partial P}{\partial \rho}\right|_S} \tag{4.73}$$

Measurements of this derivative are usually given in terms of the *bulk modulus*, defined to be $K = \rho\, \partial P/\partial \rho$. For water at 20°, this is $K \approx 200$ Nm^{-2}. It's much higher than the corresponding value for gases, reflecting the fact that it is more difficult to squeeze water than air. The density of water is $\rho_0 \approx 10^3$kg m^{-3}. The speed of sound in water is then much higher than in air, $c_s \approx 1500$ ms^{-1}.

Sound Waves are Longitudinal

The wave equation (4.70) is solved by any Fourier mode

$$\tilde{\rho}(\mathbf{x}, t) = \hat{\rho}\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \tag{4.74}$$

Here $\hat{\rho}$ is the constant amplitude of the wave. In the exponent, ω is the frequency and $\mathbf{k}$ is the wavevector which points in the direction of propagation. The two are related by the dispersion relation

$$\omega = c_s |\mathbf{k}|$$

This is now a dispersion relation that doesn't disperse, in the sense that all wavelengths propagate with the same speed. As we've seen, this contrasts with the surface waves of

Section 4.1. Because the wave equation is linear, we can combine many Fourier modes to make a wavepacket. If this is made from wavevectors $\mathbf{k}$ that all point in the same direction, then the wavepacket will keep its shape as it moves. We can also see this directly from the wave equation. If the wave is moving in the x-direction, then the wave equation is solved by any function of the form

$$\tilde{\rho} = F(t - x/c_s) + G(t + x/c_s) \tag{4.75}$$

Here F and G are the profiles of two wave packets, moving to the right and left respectively.

We can reconstruct the pressure and velocity oscillations from our original, first order equations. The pressure perturbations are simply given by $\tilde{P} = c_s^2\tilde{\rho}$. From (4.68) we have

$$\mathbf{u}(\mathbf{x}, t) = \frac{\hat{\mathbf{k}}}{\rho_0 c_s}\tilde{P}(\mathbf{x}, t) = \frac{c_s\hat{\mathbf{k}}}{\rho_0}\,\tilde{\rho}(\mathbf{x}, t)$$

The oscillations of the fluid velocity and the pressure are all in phase with the density The velocity oscillations are also parallel to the direction $\mathbf{k}$ in which the wave travels. Such waves are called *longitudinal*.

Spherically Symmetric Waves

Although we can construct any solution from the Fourier modes (4.74), that's often not the best way to proceed. For example, if we have some localised source which, for convenience, we will assume is spherically symmetric then it's clear that we are best served by working in spherical polar coordinates. Ignoring the angular directions, the wave equation becomes

$$\frac{\partial^2\tilde{\rho}}{\partial t^2} - c_s^2\nabla^2\tilde{\rho} = 0 \quad \Rightarrow \quad \frac{\partial^2(r\tilde{\rho})}{\partial t^2} - c_s^2\frac{\partial^2(r\tilde{\rho})}{\partial r^2} = 0$$

This is now a 1d wave equation. It is solved, analogously to (4.75) by any two functions

$$\tilde{\rho}(r, t) = \frac{1}{4\pi r}\left[F(t - r/c_s) + G(t + r/c_s)\right]$$

The factor of 4π is just for convenience. The function F describes the outgoing wave, while G describes the incoming wave. In many situations, there's no wave coming in from infinity so we set $G = 0$. This is the choice we make here.

The associated velocity field is most simply computed from (4.67) using $\tilde{P} = c_s^2\tilde{\rho}$. To write down the solution, we need to integrate the wave profile. We write

$$F(t - r/c_s) = \dot{Q}(t - r/c_s)$$

In spherical polars, we then have

$$\nabla\tilde{P} = -\frac{c_s^2}{4\pi}\left[\frac{\dot{Q}(t-r/c_s)}{r^2} + \frac{\ddot{Q}(t-r/c_s)}{c_s r}\right]\hat{\mathbf{r}}$$

and, comparing to (4.67), the velocity field is radial, with

$$\mathbf{u}(r,t) = \frac{c_s^2}{4\pi\rho_0}\left[\frac{Q(t-r/c_s)}{r^2} + \frac{\dot{Q}(t-r/c_s)}{c_s r}\right]\hat{\mathbf{r}} \tag{4.76}$$

Close to the source, the first term dominates; far away the second term dominates.

As an example, consider the sound waves generated by a pulsating sphere of radius a. We'll take this sphere to beat in and out, with frequency ω and amplitude ϵ, so the radius changes with time as

$$R(t) = a + \epsilon e^{i\omega t} \quad \Rightarrow \quad \dot{R} = i\omega\epsilon e^{i\omega t}$$

The solution must take the form (4.76) for some $Q(t) = Ae^{i\omega t}$. This means that

$$\mathbf{u}(r,t) = \frac{Ac_s^2}{4\pi\rho_0}\left[\frac{1}{r^2} + \frac{i\omega}{c_s r}\right]e^{i\omega(t-r/c_s)}\hat{\mathbf{r}}$$

This is subject to the requirement that the fluid velocity matches that of the sphere on its surface, i.e.

$$\mathbf{u}(R(t),t) = \dot{R}\,\hat{\mathbf{r}} \quad \Rightarrow \quad \mathbf{u}(a,t) + \frac{\partial\mathbf{u}}{\partial r}\epsilon e^{i\omega t} + \ldots = i\omega\epsilon e^{i\omega t}\hat{\mathbf{r}}$$

Since $\mathbf{u} \sim \mathcal{O}(\omega\epsilon)$, the second term in the above expression is lower order and it will suffice to set

$$\mathbf{u}(a,t) = i\omega\epsilon e^{i\omega t}\hat{\mathbf{r}} \quad \Rightarrow \quad \frac{Ac_s^2}{4\pi\rho_0 a^2}\left[1 + \frac{i\omega a}{c_s}\right]e^{-i\omega a/c_s} = i\omega\epsilon$$

which fixes the overall coefficient A.

4.4.5 Viscosity and Damping

It is natural to ask: how does viscosity affect the propagation of sound? Because viscosity is dissipative, any process will necessarily increase the entropy and so is no longer adiabatic. This means that we can't just use the simple relation $P\rho^{-\gamma}$ and must instead turn to the more sophisticated description in terms of the temperature field.

We met the heat transport equation in (4.66)

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right) T + (\gamma-1)T\,\nabla\cdot\mathbf{u} = \frac{\gamma-1}{\gamma}\frac{\kappa m}{\rho k_B}\nabla^2 T$$

This should be augmented with the Navier-Stokes equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P + \mu\nabla^2\mathbf{u} + \left(\frac{\mu}{3}+\zeta\right)\nabla(\nabla\cdot\mathbf{u})$$

together with mass conservation and an appropriate equation of state that relates P, ρ and T. We'll stick with the ideal gas equation of state, so

$$P = \frac{k_B T\rho}{m}$$

and we substitute this into the Navier-Stokes equation. For dilute gases, it turns out that $\zeta \approx 0$ so we choose to set it to zero. (It doesn't qualitatively change the physics because, as you can see, the shear viscosity μ already appears in the relevant term.) Our goal is to reproduce our previous results about sound waves in this framework, and then to understand how these results are affected by the viscosity μ and the heat conductivity κ.

As before, we start with a stationary fluid but now also include the fact that it has constant temperature

$$\mathbf{u} = 0 \quad , \quad \rho = \rho_0 \quad , \quad T = T_0$$

We then consider time-dependent perturbations,

$$\begin{aligned}\mathbf{u} &= \hat{u}\hat{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\\ \rho &= \rho_0 + \hat{\rho}\, e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\\ T &= T_0 + \hat{T}\, e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\end{aligned}$$

Note that we're looking for longitudinal waves, with $\mathbf{u}$ parallel to $\mathbf{k}$. Linearising, the mass conservation equation tells us that

$$\omega\hat{\rho} = \rho_0 k\hat{u}$$

The linearised heat transport equation is

$$-i\omega\hat{T} + i(\gamma-1)T_0 k\hat{u} = -\frac{\kappa}{c_P}k^2\hat{T}$$

where $c_P = \rho_0 k_B \gamma/(\gamma-1)m$. Last, the linearised Navier-Stokes equation is

$$-i\rho_0\omega\hat{u} = -\frac{ik_Bk}{m}\left(T_0\hat{\rho} + \rho_0\hat{T}\right) - \frac{4}{3}\mu k^2\hat{u}$$

We can write these simultaneous equations as a matrix,

$$M\begin{pmatrix}\hat{\rho}\\ \hat{u}\\ \hat{T}\end{pmatrix} = \omega\begin{pmatrix}\hat{\rho}\\ \hat{u}\\ \hat{T}\end{pmatrix} \quad \text{with} \quad M = \begin{pmatrix}0 & \rho_0 k & 0\\ k_Bk T_0/m\rho_0 & -\frac{4i}{3}\mu k^2/\rho_0 & k_Bk/m\\ 0 & (\gamma-1)T_0 k & -i\kappa k^2/c_V\end{pmatrix} \tag{4.77}$$

The frequencies of the perturbations ω are given by the eigenvalues of the matrix M. As we will see, this will give the dispersion relation between ω and k. Note, moreover, that the elements of the matrix are real except for those that multiply the dissipative coefficients μ and κ. We'll see what this means for the physics shortly.

First let's look at what happens when $\mu = \kappa = 0$. There are solutions

$$\begin{pmatrix}\hat{\rho}\\ \hat{u}\\ \hat{T}\end{pmatrix} = \epsilon\begin{pmatrix}\rho_0\\ \omega/k\\ (\gamma-1)T_0\end{pmatrix}$$

with ϵ some small, dimensionless parameter needed for the linearised approximation to be valid. This immediately solves the first two and third equations, while the second requires

$$m\omega^2 = \gamma k_B T_0 k^2 \quad \Rightarrow \quad \omega = \pm\sqrt{\frac{\gamma k_B T_0}{m}}k = \pm c_s k$$

But this is just our previous result (4.72) for the speed of sound. Moreover, we see that this perturbation has $(\gamma-1)T_0\tilde{\rho} - \rho_0\tilde{T} = 0$ which means that $T/\rho^{\gamma-1}$ is constant. But this is the expected behaviour (4.63) for an adiabatic deformation of the fluid. So, in the limit that the dissipative effects vanish, we do indeed recover the adiabatic sound waves of the previous section.

There is also a novel solution to the equation (4.77) with $\mu = \kappa = 0$ that we haven't seen previously. This has $\hat{u} = 0$ and

$$T_0\hat{\rho} + \rho_0\hat{T} = 0$$

a combination that ensures that $P \sim \rho T$ is constant in this perturbation. It solves the matrix equation above only when $\omega = 0$. Because the pressure is constant, there is no restoring force for this perturbation.

Having made contact with our previous result, we can now see how it's changed when we turn on viscosity μ and heat conductivity κ. Rather than directly finding the eigenvectors, we can take a bit of a shortcut to extract just the eigenvalues ω. First note that the determinant and trace of M are given by

$$\det M = i\frac{\kappa c_s^2}{\gamma c_V}k^4 \quad \text{and} \quad \text{Tr}\, M = -i\left(\frac{\kappa}{c_V} + \frac{4}{3}\frac{\mu}{\rho_0}\right)k^2$$

The product of the three eigenvalues must be equal to $\det M$. When $\mu = \kappa = 0$, we know that one of the eigenvalues vanishes and the other two were $\pm c_s k$. But now we see that the three must multiply to give something proportional to κ. This means that, to leading order in κ, the zero eigenvalue that arose from perturbations of constant pressure must change to

$$-c_s^2 k^2 \omega = \det M \quad \Rightarrow \quad \omega = -i\frac{\kappa}{\gamma c_V}k^2$$

The frequency is imaginary and negative. This is telling us that the modes decay exponentially quickly. To see this, write $\omega = -i\Gamma$. Then the behaviour of all modes goes as $e^{-i\omega t} = e^{-\Gamma t}$. The behaviour that we find above scales as $\omega \sim -ik^2$. This is characteristic of diffusion. It is the kind of behaviour that we get from the heat equation.

The two remaining modes are what becomes of sound waves. These too are expected to get a dissipative contribution. If we anticipate that they take the form

$$\omega = \pm c_s k - i\tilde{\Gamma}$$

possibly with some change to the sound speed, then we can compute $\tilde{\Gamma}$ by noting that the trace must equal the sum of all three eigenvalues, so

$$-2i\tilde{\Gamma} - i\frac{\kappa}{\gamma c_V}k^2 = \text{Tr}\, M \quad \Rightarrow \quad \tilde{\Gamma} = \frac{1}{2}\left(\frac{4}{3}\frac{\mu}{m\rho_0} + \frac{\gamma - 1}{\gamma}\frac{\kappa}{c_V}\right)k^2$$

We see that the effect of viscosity and of heat conduction is similar: the sound waves diffuse and decay over time, with their lifetime set by $1/\tilde{\Gamma}$.

In addition, we can ask about velocity perturbations that are transverse to the wave, so that $\mathbf{k} \cdot \mathbf{u} = 0$. These are known as *shear perturbations*. It's straightforward to see that mass conservation and heat transport require $\tilde{\rho} = \tilde{T} = 0$, while the linearised Navier-Stokes equation gives the dispersion relation

$$\omega = -i\frac{\mu}{\rho_0}k^2$$

We see that these modes also behave diffusively.

4.5 Non-Linear Sound Waves

So far, throughout this section we've only considered linear wave equations. For surface waves we went to some lengths to pick an approximation which made our equations linear and for sound wave we dropped the $\mathbf{u} \cdot \nabla \mathbf{u}$ term in the Navier-Stokes equation. This is a good first step since linear equations are significantly easier to solve than non-linear equations. But it's natural to wonder: under what circumstances are the non-linearities important? And what effect do they have? Here we start to address such questions, albeit in the somewhat restricted context of waves propagating in one dimension.

We'll revisit our analysis of sound waves, but now restricted to 1d. Our defining equations are the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \tag{4.78}$$

and the Euler equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{4.79}$$

Previously we dropped the $u\partial u/\partial x$ term. Our goal now is to understand what role it plays.

So far we have two equations for three variables: u, P and ρ. As we stressed previously in Section 4.4, we must add one further equation. Rather than getting all hot and bothered by introducing temperature, we will instead work directly with an adiabatic equation of state that relates the pressure to the density,

$$P = P(\rho)$$

For example, for the ideal gas undergoing adiabatic deformations, we showed that the relevant equation is $P\rho^{-\gamma} = \text{constant}$ with $\gamma = c_P/c_V$ the ratio of specific heats. (See equation (4.64).) We'll turn to this example later but for now we keep things general. We also saw that in the previous section that, in the linearised approximation, the speed of sound is given by (4.73),

$$c_s^2(\rho) = \frac{dP}{d\rho}$$

(Previously we wrote this as a partial derivative, keeping entropy S fixed. In this section we assume that entropy is fixed and view P only as a function of ρ.) One of the things we would like to learn is the sense in which c_s retains its interpretation as the speed of sound waves beyond the linearised approximation.

4.5.1 The Method of Characteristics

From the definition of c_s^2, together with (4.78), we have

$$\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} = c_s^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right) = -\rho c_s^2 \frac{\partial u}{\partial x} \tag{4.80}$$

To make progress, we're going to rewrite the Euler equation (4.79) and our equation for pressure (4.80) in a clever way. Starting from (4.79), we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} + (u - c_s)\frac{\partial}{\partial x}\right) u &= -\frac{1}{\rho}\frac{\partial P}{\partial x} - c_s\frac{\partial u}{\partial x} \\ &= \frac{1}{\rho c_s}\left(\frac{\partial}{\partial t} + (u - c_s)\frac{\partial}{\partial x}\right) P\end{aligned} \tag{4.81}$$

where, to get to the second line, we've used (4.80). There's a nice symmetry between the left- and right-hand side of this equation, with the same differential operator appearing in both. The only difference between them is that extra function $1/\rho c_s$ sitting on the right-hand side. To make things look even more symmetric, we define the new variable,

$$Q(\rho) = \int_{\rho_0}^{\rho} d\rho' \, \frac{c_s(\rho')}{\rho'} \tag{4.82}$$

with ρ_0 some useful fiducial, constant density such as the asymptotic value of the density if such a thing exists. This has the property that

$$\frac{\partial Q}{\partial t} = \frac{c_s}{\rho}\frac{\partial \rho}{\partial t} = \frac{1}{\rho c_s}\frac{\partial P}{\partial t} \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{1}{\rho c_s}\frac{\partial P}{\partial x}$$

This means that we can write (4.81) as

$$\left(\frac{\partial}{\partial t} + (u - c_s)\frac{\partial}{\partial x}\right)(u - Q) = 0$$

The same argument, with some minus signs flipped, also gives

$$\left(\frac{\partial}{\partial t} + (u + c_s)\frac{\partial}{\partial x}\right)(u + Q) = 0$$

We introduce the *Riemann invariants*

$$R_\pm = u \pm Q \tag{4.83}$$

These obey the *Riemann wave equation*

$$\left(\frac{\partial}{\partial t} + (u \pm c_s)\frac{\partial}{\partial x}\right) R_\pm = 0 \tag{4.84}$$

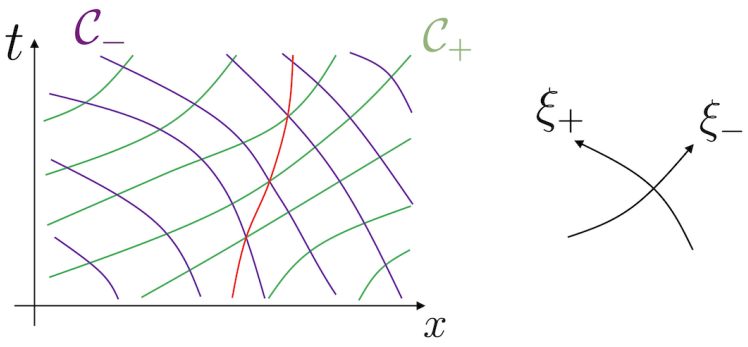


Figure 24. The characteristic curves $\mathcal{C}_+$, in green, run from bottom left to top right, and the curves $\mathcal{C}_-$, in purple, run bottom right to top left. They curves depend locally on both the flow $u(x,t)$ and, through $c_s(\rho)$, the density $\rho(x,t)$. The red line depicts one integral curve of the fluid flow $u(x,t)$. The coordinates $\xi_\pm$ are constant on $\mathcal{C}_\pm$ respectively. This means that, as shown in the axes on the right, ξ_+ increases as we move in the up-left direction, while ξ_- increases as we move in the up-right direction.

We next want to understand what this equation is telling us. To this end, for a given flow $u(x,t)$ with density $\rho(x,t)$, we construct two collections of *characteristic curves*, $\mathcal{C}_\pm$. These are worldlines in the spacetime parameterised by (x,t), defined by

$$\mathcal{C}_\pm : \quad \frac{dx}{dt} = u(x,t) \pm c_s(x,t) \tag{4.85}$$

We introduce two new coordinates in spacetime: ξ_+ and ξ_-. These have the property that $\xi_\pm$ are constant on the characteristic curves $\mathcal{C}_\pm$ respectively. Then the meaning of (4.84) is:

Claim: $R_\pm$ is constant on characteristic curves $\mathcal{C}_\pm$.

Proof: To show this, we just need to think carefully about what depends on what. Suppose that we vary both ξ_+ and ξ_- a tiny tiny bit. Then we move in the t direction an infinitesimal amount

$$dt = \left.\frac{\partial t}{\partial \xi_+}\right|_{\xi_-} d\xi_+ + \left.\frac{\partial t}{\partial \xi_-}\right|_{\xi_+} d\xi_-$$

and we move in the x direction an infinitesimal amount

$$dx = \left.\frac{\partial x}{\partial \xi_+}\right|_{\xi_-} d\xi_+ + \left.\frac{\partial x}{\partial \xi_-}\right|_{\xi_+} d\xi_-$$

On the characteristic curves $\mathcal{C}_+$ we know that ξ_+ is constant, so we have

$$\mathcal{C}_+ : \quad d\xi_+ = 0 \quad \Rightarrow \quad \left.\frac{\partial x}{\partial \xi_-}\right|_{\xi_+} = \frac{dx}{dt}\left.\frac{\partial t}{\partial \xi_-}\right|_{\xi_+} = (u + c_s)\left.\frac{\partial t}{\partial \xi_-}\right|_{\xi_+}$$

Now, if we view $R_+(x,t)$ as a function $R_+(\xi_+, \xi_-)$, then

$$\begin{aligned}\left.\frac{\partial R_+}{\partial \xi_-}\right|_{\xi_+} &= \frac{\partial R_+}{\partial t}\left.\frac{\partial t}{\partial \xi_-}\right|_{\xi_+} + \frac{\partial R_+}{\partial x}\left.\frac{\partial x}{\partial \xi_-}\right|_{\xi_+} \\ &= \left(\frac{\partial R_+}{\partial t} + (u + c_s)\frac{\partial R_+}{\partial x}\right)\left.\frac{\partial t}{\partial \xi_-}\right|_{\xi_+} = 0\end{aligned}$$

In other words, $R_+(\xi_-, \xi_+)$ is really just a function of a single variable, $R_+(\xi_+)$. The same argument also tells us that $R_- = R_-(\xi_-)$. So if we move along a characteristic curve $\mathcal{C}_+$, where ξ_+ is constant, then R_+ doesn't change. Similarly, R_- doesn't change if we move along a characteristic curve $\mathcal{C}_-$. □

It's worth taking stock of what we've achieved. Our goal is to solve for the flow $u(x,t)$ and the density $\rho(x,t)$. We haven't done this yet! However, we have showed that, *if* we can solve it, then we can construct characteristic curves $\mathcal{C}_\pm$ on which the variables $R_\pm$ are constant. And $R_\pm$, in turn, depends on u and ρ that we are trying to figure out. All of which means that the Riemann invariants don't immediately solve our problem, but they should contain some information that we can exploit.

Furthermore, if it's possible to somehow figure out $R_\pm(x,t)$ then it's straightforward to reconstruct the velocity field which, from (4.83), is given by

$$u(x,t) = \frac{1}{2}(R_+(\xi_+) + R_-(\xi_-))$$

This is the generalisation of the more familiar solution to the linearised wave equation (4.75),

$$u(x,t) = F(x - c_s t) + G(x + c_s t)$$

which describes wave packets moving left and right at a constant speed c_s.

4.5.2 Soundcones

The equations (4.84) are telling us that the something is propagating in the fluid with speed c_s relative to the flow.

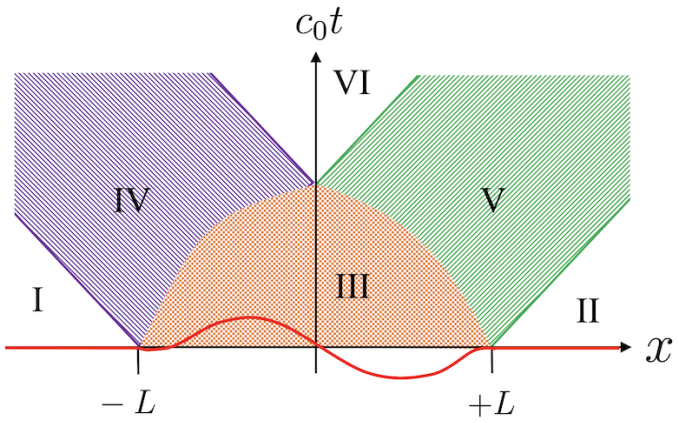


Figure 25. An initial disturbance in a region $|x| < L$ propagates to the left and to the right. Regions I,II and VI are undisturbed; regions IV and V have only left-moving and right-moving waves respectively, and anything can happen in region III.

To see this more clearly, consider some initial disturbance with $u(x,0) \neq 0$ for $|x| < L$ as shown in Figure 25. We'll also assume that the density $\rho(x,t)$ differs from some asymptotic value ρ_0 only within this same region. From (4.82), this ensures that $Q = 0$ outside of this region so so $R_\pm(x,0) = 0$ for $|x| > L$.

We can draw this on a spacetime diagram, with the vertical axis labelled by $c_0 t$ where $c_0 = c_s(\rho_0)$ is the asymptotic sound speed. This ensures that linearised sound waves travel at $\pm 45°$ in the diagram, rather like light rays in Minkowski space. In analogy with special relativity, we will say that the pair of characteristic curves $\mathcal{C}_\pm$ emerging from any point form a *soundcones*. (In fact, the analogy works better with general relativity where the lightcones depend on the curvature of spacetime, just like the soundcones depend on the background flow $u(x,t)$.)

Consider the soundcones emerging from the points $x = \pm L$ at time $t = 0$. These are shown in Figure 25 They divide spacetime for $t > 0$ into six distinct regions with the following properties

- Regions I and II have the property that both $\mathcal{C}_+$ and $\mathcal{C}_-$ characteristic curves pass through the x-axis in the region with $R_\pm(x,0) = 0$. Because $R_\pm$ are constant on $\mathcal{C}_\pm$ respectively, this means that we must have $R_+ = R_- = 0$ throughout the regions I and II. This makes sense: the initial disturbance takes time to propagate and, just as signals can't travel faster than the speed of light in special relativity, here they can't travel faster than the (local) speed of sound. Hence the regions

I and II know nothing about the disturbance. Both characteristic curves are straight lines at $\pm 45^\circ$ in these regions.

- Region III has a complicated flow, with both left- and right-travelling waves. Here we would expect both R_+ and R_- to be non-vanishing and there is no reason to think that the characteristic curves $\mathcal{C}_\pm$ will be straight.

- In Region IV, we can trace the $\mathcal{C}_+$ curves back to the region in which $R_+ = 0$. So we know that

$$R_+ = u + Q = 0 \quad \Rightarrow \quad u = -Q$$

Correspondingly, $R_- = u - Q = 2u$. We know that R_- is constant on characteristic curves $\mathcal{C}_-$. So this tells us that both u and Q, and hence ρ and $c_s(\rho)$, must also be constant on $\mathcal{C}_-$. In other words, all of these are functions only of ξ_-,

$$u(x,t) = u(\xi_-) \quad , \quad \rho(x,t) = \rho(\xi_-) \quad , \quad c_s(x,t) = c_s(\xi_-)$$

These are purely left-moving waves. Now, the defining equation for the characteristic curve $\mathcal{C}_-$ is

$$\frac{dx}{dt} = u(\xi_-) - c_s(\xi_-)$$

But for a given $\mathcal{C}_-$ curve, defined by some fixed value of ξ_-, the right-hand side is obviously constant. This means that the $\mathcal{C}_-$ characteristic curves are straight lines in region IV. Although these curves are all straight lines in region IV, they need not necessarily be parallel: the gradient $u(\xi_-) - c_s(\xi_-)$ can depend, as advertised, on ξ_- and typically will. Of course, if they're not parallel then there is the possibility that they will converge and cross at some point. That's somewhat confusing because $R_-(\xi_-)$ is expected to have different values on different curves and if two curves collide then it looks like, say, the velocity $u(x,t)$ will have two different values at that point! We will learn how to think about this shortly.

Region V has similar properties to Region IV, now with the characteristic curves $\mathcal{C}_+$ straight lines. This is a purely right-moving wave. In general, situations where one of the Riemann invariants vanish (or, indeed, is constant) over a region of space are referred to as *simple waves* and are associated to waves that are either purely left-moving or purely right-moving.

- In Region VI, the same arguments apply as for Region I and II: you can trace back both $\mathcal{C}_+$ (to the left) and $\mathcal{C}_-$ (to the right) to regions where $R_+ = 0$ and

$R_- = 0$ respectively. This means that this is once again a region of calm, with $u = 0$ and $\rho = \rho_0$. The disturbance has passed. This is because there is no option to dawdle in this system: all modes must travel at the speed of sound. The only question is how that speed of sound changes.

4.5.3 Wave Steepening and a Hint of Shock

We can illustrate these ideas further. We'll first give a heuristic discussion of the physics and then fill in some of the details. To do this, it's useful to pick a concrete example and we'll choose to look at ideal gas obeying $P\rho^{-\gamma} = \text{constant}$.

The speed of sound for the ideal gas is (4.71)

$$c_s^2(\rho) = \frac{\gamma P}{\rho} \tag{4.86}$$

The function $Q(\rho)$ defined in (4.82) is

$$Q(\rho) = \int_{\rho_0}^{\rho} d\rho' \, \frac{c_s(\rho')}{\rho'} = \frac{2}{\gamma - 1}\left(c_s(\rho) - c_0\right)$$

where $c_0 = c_s(\rho_0)$. The Riemann invariants are then

$$R_\pm = u \pm \frac{2}{\gamma - 1}(c_s - c_0)$$

These are normalised so that in a boring, static flow with $u = 0$ and $\rho = \rho_0$, both Riemann invariants vanish: $R_\pm = 0$.

Now consider a simple, right-moving flow in which $R_- = 0$ everywhere. This means that

$$R_- = u - \frac{2}{\gamma - 1}(c_s - c_0) = 0 \quad \Rightarrow \quad c_s = c_0 + \frac{1}{2}(\gamma - 1)u \tag{4.87}$$

This gives us a relation between the velocity field and the speed of sound. In fact, this contains the key bit of physics. Any region of fluid with $u > 0$ has a speed of sound $c_s > c_0$. And any region of fluid with $u < 0$ has a speed of sound $c_s < c_0$. (Assuming $\gamma > 1$.)

The wave propagates along the characteristic curves $\mathcal{C}_+$, on which u and c_s are both constant. These curves are given by (4.85)

$$\mathcal{C}_+ : \quad \frac{dx}{dt} = u(\xi_+) + c_s(\xi_+) = c_0 + \frac{1}{2}(\gamma + 1)u(\xi_+) \tag{4.88}$$

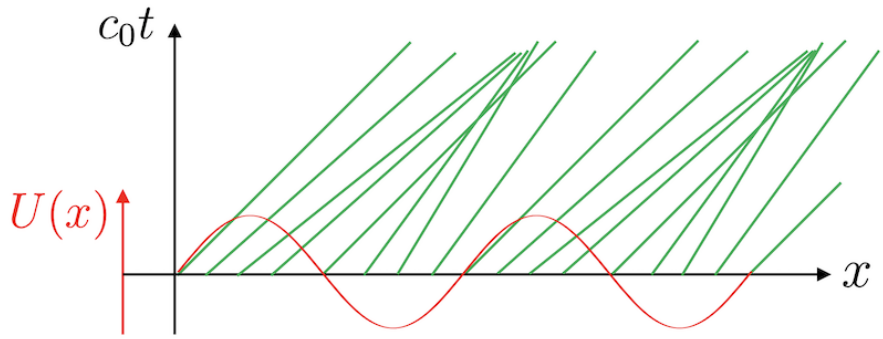


Figure 26. An initial, purely right-moving, sine wave has characteristic curves $\mathcal{C}_+$ that intersect.

Suppose that we're given some initial data at time $t = 0$. We're told that

$$u(x, 0) = U(x) \tag{4.89}$$

The requirement that we've got a purely right-moving wave then fixes the density (or, equivalently, the speed of sound) along this slice using (4.87). We know that ξ_+ labels the different $\mathcal{C}_+$ curves, but we haven't yet got a natural way to fix the normalisation of this coordinate. We'll resolve this by choosing ξ_+ to be the value of x where a given curve $\mathcal{C}_+$ intersects the $t = 0$ axis. Then the characteristic curves (4.88) are simply the straight lines

$$\mathcal{C}_+ : \quad x(t; \xi_+) = \xi_+ + \left[c_0 + \frac{1}{2}(\gamma + 1) U(\xi_+) \right] t \tag{4.90}$$

We can see the slope of the characteristic curves in the square bracket. Those parts of the fluid that had an initial velocity $U > 0$ travel along characteristic curves that have an angle greater than 45° (as measured from the vertical axis). And those parts of the fluid with $U < 0$ travel along lines that sit at less than 45°.

The resulting characteristic curves are shown in green in Figure 26 for the simple initial data where $U(x)$ is a sine wave. Importantly, we see that, as we anticipated previously, the characteristic curves meet. But this is very confusing! On a given characteristic curve, the velocity of the fluid is fixed as $U(x)$. Wherever two curves intersect, the velocity must be multi-valued. In other words, the non-linearities have pushed our nice, simple initial sine wave into a solution that is discontinuous in the velocity field!

In fact, there's a straightforward interpretation of this. As the nonlinearities cause the peak of the wave, where $U > 0$, to move faster than the trough of the wave, where $U < 0$. This means that the wave will become skewed, and increasingly sawtooth-like. This is known as *wave steepening* and is shown in Figure 27 for an initial Gaussian wavepacket rather than the sin wave considered above (because it's easier to draw). Eventually, the peak will catch up with the trough, at which point the velocity field is no longer single-valued. This is the reason that the characteristic curves cross. It turns out that this is telling us that a *shock* forms. We'll understand more about what this means in Section 4.6. For now, we will simply adopt the terminology and say that a shock is tantamount to two characteristic curves intersecting.

We can ask: how long does it take for the shock to form? Two curves intersect whenever a shift to an adjacent curve doesn't require a shift in space. Or, in equations,

$$\left.\frac{\partial x}{\partial \xi_+}\right|_t = 0$$

From (4.90), this gives the requirement

$$t = -\frac{2}{\gamma+1}\frac{1}{U'(\xi_+)} \tag{4.91}$$

We want the value of ξ_+ that minimises $t > 0$ since this is when two of the characteristic curves first cross. We take, as our initial conditions, a sine wave as shown in Figure 26

$$u(x,0) = U(x) = U\sin kx$$

with U the overall amplitude of the wave. The first shock then forms when

$$t_{\rm shock} = \frac{2}{\gamma+1}\frac{1}{kU}$$

It's useful to compare this time scale with the period T of the wave itself. Recall from Section 4.4 that sound waves have the simple dispersion relation $\omega = c_s|k|$ and the period is $T = 2\pi/\omega \approx 2\pi/c_0|k|$. This then gives

$$\frac{t_{\rm shock}}{T} = \frac{1}{\pi(\gamma+1)}\frac{c_0}{U} \tag{4.92}$$

This is our first sign of an important dimensionless quantity called the *Mach number*, the ratio of the velocity of the fluid flow to the speed of sound

$$M = \frac{U}{c_0} \tag{4.93}$$

The time to shock formation is roughly $t_{\rm shock} \sim T/M$.

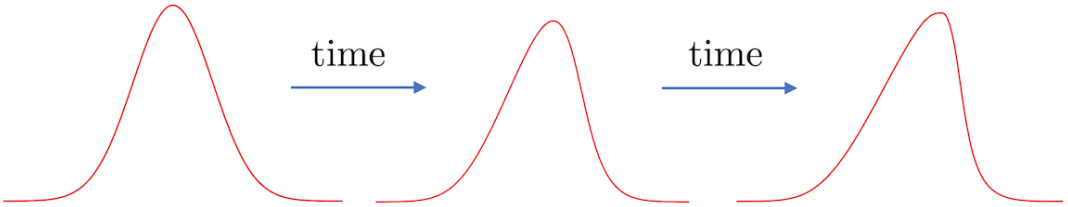


Figure 27. The steepening of an initial Gaussian wavepacket over time.

We can put some numbers into this. The decibel scale is the familiar scale used to measure how loud a sound is. It's a $\log_{10}$ scale such that if the amplitude of the wave U changes by a factor of 10 then the decibels increase additively by 10. A quiet chat down the pub with a friend will involve sound waves with frequency around $\omega \sim 1000s^{-1}$ and a corresponding period of $T \sim 3 \times 10^{-3}$ s. The volume is around 60 dB and this corresponds to a Mach number $M \sim 10^{-13}$. Apparently you need to wait about 1000 years for a shock wave to form! We can instead crank up the volume. If you stand next to a rocket at take off, you will suffer around 180 dB. (And get permanent ear damage.) Now $M \sim 10^{-1}$. Perhaps unsurprisingly, you can expect a shock wave to form in a very short time.

4.5.4 Burgers' Equation

We can elucidate the analysis above a little further with a simple change of variables. We continue to study a purely right-moving wave for which $R_+ = 2u$ and, from (4.87), the local speed of sound c_s is related to the fluid velocity by

$$c_s(u) = c_0 + \frac{1}{2}(\gamma - 1)u$$

The Riemann wave equation (4.84) is now a non-linear equation for u

$$\left(\frac{\partial}{\partial t} + \left[c_0 + \frac{1}{2}(\gamma + 1)u\right]\frac{\partial}{\partial x}\right)u = 0$$

We introduce the co-moving coordinate

$$X = x - c_0 t$$

This would travel along with the wave if the wave were travelling at a constant speed c_0. Of course, it's not, which is where much of the fun lies. We also introduce the rescaled velocity field

$$v = \frac{1}{2}(\gamma + 1)u$$

and we think of $v = v(X,t)$. Then the Riemann wave equation becomes

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial X} = 0 \tag{4.94}$$

where the partial time derivative is now taken with X held fixed, rather than x held fixed as before. This is the *inviscid Burgers' equation*. (Inviscid because throughout this section we've neglected viscosity.)

The Burgers' equation (4.94) takes a particularly simple form. We'll again take the initial data to be

$$v(X,0) = V(X)$$

and identify X with ξ_+ when $t = 0$. (The function $V(X)$ differs from our previous $U(x)$ defined in (4.89) only by the constant factor $2/(\gamma+1)$.) The characteristic curves $\mathcal{C}_+$ given in (4.90) are then just

$$\mathcal{C}_+ : \quad X(t;\xi_+) = \xi_+ + V(\xi_+)t \tag{4.95}$$

along which v is constant. This means that $v(X,t) = v(\xi_+,0)$. Or, substituting the expression for X above, we have the solution

$$v(X,t) = v(X - V(\xi_+)t, 0) \tag{4.96}$$

It's worth pausing to parse what this solution means. First, for a given X and t, we need to use (4.95) to figure out what ξ_+ is. In other words, what characteristic curve $\mathcal{C}_+$ the point (X,t) lies on. Clearly this depends on the initial data $v(\xi_+,0)$ that we're given. It's an algebraic computation and, for a given initial condition, the answer may not be available in closed form. Nonetheless, it's doable numerically. With this in hand, the solution (4.96) then tells us how the initial data evolves. Indeed, it's just a rewriting of what we saw previously: the points with higher initial velocity propagate at a faster speed.

Steepening Again

The coordinate X has the advantage that it keeps up with the propagating wave, at least on average. The slope of the wave is

$$\left.\frac{\partial v}{\partial X}\right|_t = \frac{\partial v(\xi_+,0)}{\partial \xi_+}\left.\frac{\partial \xi_+}{\partial X}\right|_t = \frac{V'(\xi_+)}{1+V'(\xi_+)t}$$

where, in the second equality, we've used (4.95). This now gives us a better handle on the phenomenon of wave steepening. Those parts of the wave with $V'(\xi_+) < 0$ get

steeper over time; those parts of the wave with $V'(\xi_+) > 0$ become flatter. The shock occurs when the wave becomes infinitely steep, so

$$\left.\frac{\partial v}{\partial X}\right|_t = \infty \quad \Rightarrow \quad t = -\frac{1}{V'(\xi_+)}$$

This agrees with our earlier condition (4.91).

Very Briefly, the Effect of Viscosity

So far our discussion has neglected viscosity. But we can see at the level of equations what it changes. If we go back to the 1d Euler equation (4.79) and trace through various change of variables, we find that (4.94) is replaced by

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial X} = \nu\frac{\partial^2 v}{\partial X^2}$$

where ν is, as always, the kinematic viscosity and we've inadvertently stumbled upon the rather unfortunate situation of having both ν and v in the same equation. (It won't be for long and we can live with it.) This is the full *Burgers equation*.

We know that viscosity causes the velocity to diffuse, and this mitigates large velocity gradients. This can be shown to remove the formation of the discontinuous shock.

4.6 Shocks

> "This is a manifest absurdity. No step, however, of the reasoning by which this result has been obtained can be controverted. What then is the meaning of it?"

The Rev. James Challis, in 1848, expressing shock on first discovering that initial data in the Euler equations gives rise to discontinuities in the velocity.

We got our first hints of shock waves in the last section where a discontinuity in the velocity field arises as the flow evolves over time. Although this discontinuity is expected to be smoothed out by the effects of viscosity, if we zoom out and look at suitably coarse-grained scales then the discontinuous flow is a good approximation to what actually happens. In this section we'll explore some of the properties of these shocks. We'll see that the discontinuities have a remarkably simple and constrained structure, all of which follows from conservation laws (together with a little bit of thermodynamics).

We're going to upgrade from one dimension to two. We'll consider flows of a compressible fluid in the (x, y)-plane, with a shock that sits at some specific point in the x-direction and extends along the y-direction. The flows themselves will be invariant under translations in the y-direction. This means that we will restrict our attention to flows of the form

$$\mathbf{u}(\mathbf{x}, t) = (u(x, t), v(x, t))$$

We have mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{4.97}$$

and the non-linear Euler equations

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial P}{\partial x} = 0 \tag{4.98}$$

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} = 0 \tag{4.99}$$

where to get the second set of equations we use the continuity equation (4.97) associated to mass conservation. We've rewritten the Euler equations in this way to stress that they too are continuity equations, describing the conservation of momentum.

We've got three equations for four variables, ρ, u, v and P. We need a fourth. This is a road that we've been down before: for adiabatic variations, the fourth condition is a relation

$$P = P(\rho)$$

and our standard example is the ideal gas with $P/\rho^\gamma = \text{constant}$.

As we will see, the physics of the shock is all about understanding conserved quantities. The final conservation law that we need is for energy. But here there's a subtlety because there are additional contributions to the energy for a compressible fluid. As we will now show, these additional contributions are fully determined by the relation $P = P(\rho)$.

We start by introducing the energy density of the fluid

$$\text{Energy} = \frac{1}{2}\rho(u^2 + v^2) + \rho e(\rho)$$

where $e(\rho)$ is some internal energy (per unit mass) of the fluid that we will determine below. In addition, we introduce the energy current

$$\text{Energy Current} = \frac{1}{2}\rho u(u^2 + v^2) + \rho u h(\rho)$$

where the addition quantity $h(\rho)$ is known as the *enthalpy*. Again, we'll figure out what this is shortly. Then using the same kind of manipulations that we saw when first deriving Bernoulli's principle in Section 2.1.4, we can write the conservation of energy as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho(u^2 + v^2) + \rho e(\rho)\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho u(u^2 + v^2) + \rho u h(\rho)\right) = 0 \tag{4.100}$$

A little algebra shows that this equation holds provided that the internal energy $e(\rho)$ and enthalpy $h(\rho)$ obey the equations

$$\frac{d(\rho e)}{d\rho} = h \quad \text{and} \quad \frac{d(\rho h)}{d\rho} = h + \frac{dP}{d\rho}$$

These equations can then be solved given the relation $P = P(\rho)$. For example, for the ideal gas with $P = A\rho^\gamma$ for some constant A, we can solve these to get

$$e = \frac{A\rho^\gamma}{\gamma - 1} = \frac{1}{\gamma - 1}\frac{P}{\rho} \quad \text{and} \quad h = \frac{A\gamma}{\gamma - 1}\rho^{\gamma-1} = \frac{\gamma}{\gamma - 1}\frac{P}{\rho} \tag{4.101}$$

Note that $h = e + P/\rho$.

Equations (4.97), (4.98), (4.99) and (4.100) are our starting point. Our goal now is to search for discontinuous solutions to these equations describing shock waves.

4.6.1 Jump Conditions

A shock is a discontinuous flow. But we don't allow any old discontinuity. Instead, the discontinuity itself has certain properties. And these are derived from the conservation laws described above.

The discontinuity splits the flow into two, as shown in Figure 28. On the left, the flow has values $\mathbf{u}_1$, ρ_1 and P_1; on the right values $\mathbf{u}_2$, ρ_2 and P_2. To make life particularly simple, we'll assume that each of these flows is constant in space and time. All of the physics arises from the discontinuity.

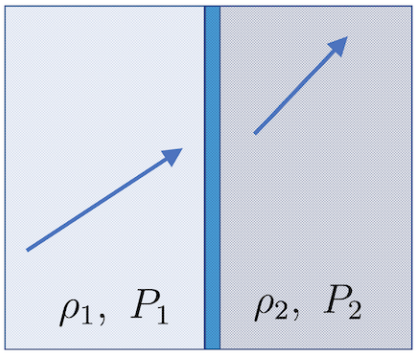


Figure 28. Two flows separated by a shock wave.

We'll assume that the shock itself is not propagating, but is fixed at some position, say $x = 0$. We model the discontinuity as in infinitely thin surface and, as such, it can't carry any conserved charge density. Any mass that enters from one side must exit through the other. The same holds for momentum and energy. This means that each of the currents in (4.97) through (4.100) must coincide on the left and right. Mass conservation (4.97) tells us

$$\rho_1 u_1 = \rho_2 u_2 \tag{4.102}$$

Momentum conservation tells us

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \tag{4.103}$$

and

$$\rho_1 u_1 v_1 = \rho_2 u_2 v_2 \quad \Rightarrow \quad v_1 = v_2 \tag{4.104}$$

where the second equation follows from (4.102). This tells us that the velocity tangent to the shock remains constant. Finally, energy conservation gives

$$\begin{aligned} & \rho_1 u_1 \left(\frac{1}{2}(u_1^2 + v_1^2) + h_1 \right) = \rho_2 u_2 \left(\frac{1}{2}(u_2^2 + v_2^2) + h_2 \right) \\ \Rightarrow \quad & \frac{1}{2}u_1^2 + h_1 = \frac{1}{2}u_2^2 + h_2 \end{aligned} \tag{4.105}$$

where now the second equation follows from (4.102) and (4.104). This is Bernoulli's theorem applied to the shock. Equations (4.102), (4.103), (4.104) and (4.105) are called the *Rankine-Hugoniot jump conditions*.

It's simplest to transform to a frame in which $v_1 = v_2 = 0$, so that all the action is taking place transverse to to shock wave. We're then left with three conditions which fix u_2, ρ_2 and P_2 in terms of the initial flow data. To see this in more detail, we first use (4.102) and (4.103) to derive the relation

$$\rho_1 u_1^2 = \left(1 - \frac{\rho_1}{\rho_2}\right)^{-1} (P_2 - P_1) \tag{4.106}$$

Since the left-hand-side is positive, the right-hand side must also be positive. That gives us two possibilities: either pressure and density both increase across the shock

$$P_2 > P_1 \quad \text{and} \quad \rho_2 > \rho_1$$

or the opposite happens. Clearly these two options are related by a parity flip, so we'll assume that the above occurs and the pressure is greater on the right of the shock. Then, from (4.102), we have

$$u_2 = \frac{\rho_1}{\rho_2} u_1 \tag{4.107}$$

This tells us that $|u_2| < |u_1|$, so the *speed* of the flow is smaller on the right of the shock. Note, however, that we haven't yet said anything about the sign of u_1 and u_2, i.e. is the flow left-to-right or right-to-left? We'll come to this shortly.

The Size of the Shock

The *shock compression ratio* is defined to be

$$r = \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \tag{4.108}$$

Obviously it's a measure of how big the discontinuity is. We can also get an expression for r in terms of the pressure difference on each side but, for this, we need our final matching condition associated to consevation of energy (4.105). For an ideal gas, this reads

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma - 1}\frac{P_2}{\rho_2}$$

We can use (4.106) and (4.107) to write this as

$$\frac{1}{2}\frac{\rho_1 + \rho_2}{\rho_1 \rho_2}(P_2 - P_1) = \frac{\gamma}{\gamma - 1}\left(\frac{P_2}{\rho_2} - \frac{P_1}{\rho_1}\right)$$

Now substituting $\rho_2 = r\rho_1$, we find

$$r = \frac{(\gamma-1)P_1 + (\gamma+1)P_2}{(\gamma+1)P_1 + (\gamma-1)P_2} \tag{4.109}$$

This form of r puts some bounds on the strength of the shock. For a *strong shock*, $P_2 \gg P_1$. In the limit $P_2/P_1 \to \infty$, we have

$$r \to r_{\max} = \frac{\gamma+1}{\gamma-1}$$

This is the largest compression factor that we can have. For a monatomic gas, with $\gamma = 5/3$, we have $r_{\max} = 4$. Note that the discontinuity is very different for pressure and speed: if the pressure changes by an infinite amount, the speed changes only by a factor of 4.

There's also something familiar hiding in this unfamiliar setting. Suppose that we have a *weak shock*, meaning $P_2 = P_1 + \Delta P$ with $\Delta P \ll P_1$. Then we have $r \approx 1 + \Delta P/\gamma P_1$. We can also write, $\rho_2 = \rho_1 + \Delta\rho$ and this gives $r \approx 1 + \Delta\rho/\rho_1$. Equating these, we have

$$\frac{\Delta P}{\Delta\rho} = \frac{\gamma P_1}{\rho_1}$$

But this is the equation for the speed of sound in an ideal gas (see, for example, (4.71) and (4.73))

$$\frac{dP}{d\rho} = c_s^2 = \frac{\gamma P}{\rho} \tag{4.110}$$

We previously derived this result for linearised (i.e. small) sound waves. Here we make contact with the shock waves. A very weak shock wave can be viewed as the limit of a very strong sound wave.

The Entropy Jump

There's a very basic question that we haven't yet addressed. Which way is the flow is going? Is the fluid moving left-to-right, so $u_1, u_2 > 0$ as shown in Figure 28? Or is it moving right-to-left, with $u_1, u_2 < 0$. In other words, does the pressure increase in the direction of the flow or decrease in the direction of the flow? It turns out that the answer to this question lies in the second law of thermodynamics.

The entropy density $s = S/V$ for an ideal gas computed in (4.63): it is given by

$$s = c_V \log\left(\frac{P}{P_0}\left(\frac{\rho_0}{\rho}\right)^\gamma\right)$$

with P_0 and ρ_0 some fiducial values. Indeed, this is where we got the now-familiar equation of state $P\rho^{-\gamma} =$ constant for adiabatic processes which have constant entropy.

We'd like to understand how the entropy changes across the shock,

$$\Delta s = s_2 - s_1 = c_V \log\left(\frac{P_2}{P_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right)$$

The second law of thermodynamics means that entropy must increase. But is the change of entropy Δs or is it $-\Delta s$? In other words, does the flow from region 1 to region 2, in which case Δs is the change in entropy. Or does it go from region 2 to region 1 in which case it's $-\Delta s$?

This is straightforward to answer using the expression for the compression ratio (4.108) and (4.109). The entropy jump Δs is then

$$\Delta s = c_V \log\left(\frac{r(\gamma+1)-(\gamma-1)}{(\gamma+1)-r(\gamma-1)}r^\gamma\right)$$

For $r < r_{\rm max}$, we have $\Delta s > 0$. But the second law of thermodynamics then gives us a direction for the shock: the flow must propagate from left to right, as anticipated in Figure 28, so that Δs is the change of entropy. Correspondingly, the speed of the flow decreases and the density increases. We say that the shock is *compressive*.

Although we've studied the shock only for an ideal gas, it turns out that the result above is general: shocks are always compressive for any equation of state $P(\rho)$, with the velocity decreasing after the shock.

The fact that the entropy is not constant across the discontinuity means that shocks are necessarily dissipative. There's something a little surprising about this. We've worked with the Euler equation which, as mentioned previously, enjoys the symmetry of time reversal. Moreover, we've also used the adiabatic condition $P\rho^{-\gamma} =$ constant for an ideal gas. Nonetheless, the discontinuity is a violent event and allows dissipative behaviour to be hidden in the singularity, even though the underlying equations did not themselves have dissipation.

Physically, we've captured the dissipation by allowing the internal, heat energy $e(\rho)$ to increase downstream. A fuller understanding of the dissipation mechanism would need us to look more closely at the shock wave by understanding the role that viscosity plays in thickening the discontinuity. But the results above tell us that, ultimately, fact these microscopic details don't affect the amount of dissipation: that's fully determined by the properties of the initial flow and some basic conservation laws.

4.6.2 Shocks Start Supersonic

There is more physics to extract from our expressions for the compression ratio. We define the *normal Mach number*

$$\mathcal{M} = \frac{u}{c_s}$$

where the speed of sound is (4.110)

$$c_s^2 = \frac{\gamma P}{\rho}$$

This is not quite the same thing as the mach number (4.93) because we've ignored the tangential velocity v. Each side of the flow has a normal Mach number, $\mathcal{M}_1$ and $\mathcal{M}_2$. Note, in particular, the speed of sound also differs on either side of the shock. We'll now show that we can express the normal Mach numbers $\mathcal{M}_1$ and $\mathcal{M}_2$ on either side of the flow directly in terms of the compression factor r.

Lemma: The algebra is a little fiddly so we'll tread slowly. To begin, we'll need:

$$\begin{aligned} \rho_1 u_1^2 &= \frac{1}{2}\left[(\gamma-1)P_1 + (\gamma+1)P_2\right] \\ \rho_2 u_2^2 &= \frac{1}{2}\left[(\gamma+1)P_1 + (\gamma-1)P_2\right] \end{aligned} \tag{4.111}$$

Proof: We start with two expressions for the compression factor r, the first following from (4.108) and the second (4.109),

$$r = \frac{\rho_1 u_1^2}{\rho_2 u_2^2} = \frac{(\gamma-1)P_1 + (\gamma+1)P_2}{(\gamma+1)P_1 + (\gamma-1)P_2} \tag{4.112}$$

Note that if the first equation in (4.111) is true, then (4.112) immediately implies that the second is also true. So we just need to prove the first. This follows from (4.103) which reads $\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$. If we divide through by $\rho_1 u_1^2$ then, after a little rearranging, we find

$$\rho_1 u_1^2 = \frac{r}{r-1}(P_2 - P_1)$$

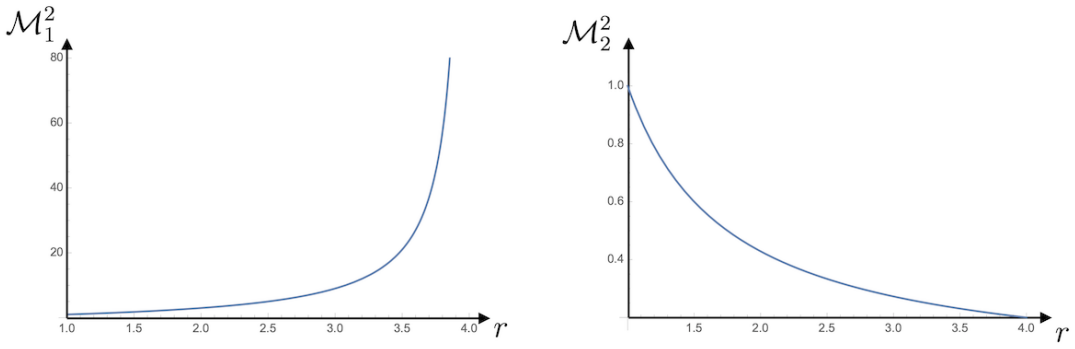


Figure 29. The initial and final Mach numbers as a function of the compression ratio r, plotted for $\gamma = 5/3$. We have $\mathcal{M}_1 \geq 1$ and $\mathcal{M}_2 \leq 1$ for all values of r.

Now compute $r/(r-1)$ using the expression in (4.112) involving pressure. This will give the result (4.111) that we want. □

Now we've done the hard work. We use the expression for the speed of sound $c_s^2 = \gamma P/\rho$ to write the two equations in (4.111) as

$$\gamma \mathcal{M}_1^2 = \frac{1}{2}\left[(\gamma-1) + (\gamma+1)\frac{P_2}{P_1}\right]$$
$$\gamma \mathcal{M}_2^2 = \frac{1}{2}\left[(\gamma+1)\frac{P_1}{P_2} + (\gamma-1)\right]$$

To finish, we just need an expression for the pressure ratios. We can easily get this from (4.112). It is

$$\frac{P_2}{P_1} = \frac{r(\gamma+1)-(\gamma-1)}{(\gamma+1)-r(\gamma-1)} \tag{4.113}$$

Finally we get the results that we wanted: the normal Mach numbers before and after the shock are

$$\mathcal{M}_1^2 = \frac{2r}{(\gamma+1)-r(\gamma-1)}$$
$$\mathcal{M}_2^2 = \frac{2}{r(\gamma+1)-(\gamma-1)} \tag{4.114}$$

The key takeaway from these equations is that, for $1 < r \leq r_{\rm max}$, we always have $\mathcal{M}_1 > 1$ and $\mathcal{M}_2 < 1$, as shown in Figure 29. This means that shocks only form in supersonic flows, where the speed of the fluid exceeds the speed of sound. After the

shock, the speed of the fluid is reduced below the sound speed. (Although, as the reduction of the fluid speed is limited by a factor of $r < r_{\rm max}$, for very fast flows this is achieved by increasing the pressure, and hence increasing the sound speed, rather than by reducing the flow speed.)

From a physical perspective, the equations (4.114) are kind of backwards: the compression factor r doesn't determine the initial speed $\mathcal{M}_1$. It's the other way round! We can easily invert these equations to get the compression factor in terms of the initial Mach number,

$$r = \frac{(\gamma+1)\mathcal{M}_1^2}{2+(\gamma-1)\mathcal{M}_1^2}$$

Similarly, the jump in pressure, given in (4.113), is also determined by the initial speed

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(\mathcal{M}_1^2 - 1)$$

These equations are known as the *Rankine-Hugoniot relations*

4.6.3 On Singularities and Physics

There is a general expectation that non-linear, partial differential equations will develop singularities in a finite time. When those non-linear equations describe something physical, these singularities are particularly interesting. A singularity is telling us the equations are no longer sufficient to capture the underlying physics and must be replaced by something more fundamental. This suggests that singularities may offer a window into the microscopic realm.

Within classical physics, there are two pre-eminent sets of non-linear equations. These are the Navier-Stokes equation (or its baby brother, the Euler equation) for fluids, and the Einstein equations for gravity. As we now explain, both these equations are rather special and the way in which singularities form, or fail to form, is surprising and poorly understood.

For fluids, it's useful to distinguish between the compressible and non-compressible cases. As we've seen above, the compressible Euler equation readily develops singularities in finite time. These are the shock waves that we've explored in this section, characterised by a discontinuity in the density ρ and other dynamical variables. As we anticipated above, the presence of the shock does mean that we have to introduce new physics. But the surprise of this section is that this new physics is the most minimal imaginable: it is just the second law of thermodynamics. Once we accept that entropy

must increase when the shock develops, we have all we need to tell us what happens to the subsequent evolution. We certainly don't need to resort to any detailed microscopic description involving atoms and quantum world. This is rather remarkable. Shocks may be singular but, from a physical perspective, the singularity is very mild.

It is natural to ask: is this same property shared by all singularities of the compressible Euler equation? Or, indeed singularities of the compressible Navier-Stokes equation? The answer is: we don't know. For example, what happens when many shocks collide and start to interact with each other? Is it still the case that we can track the singular evolution of the Euler equations using only the second law as our guide? This situation is complicated and we don't know the answer. Moreover, one may worry that there are singularities worse than shocks that can arise in the compressible Euler equation. For example, it may be possible that $\rho(\mathbf{x}, t) \to \infty$ in some finite time. This kind of singularity would surely need some detailed understanding of the underlying atoms to resolve. But does such a singularity actually arise? Again, the answer is: we don't know. It can be shown that such singularities occur for very special initial data, but to be physically relevant it should happen for generic initial conditions, meaning initial conditions that lie within some open ball rather than at specific points. And it remains an open problem to show whether or not this occurs.

The situation for the incompressible Euler and Navier-Stokes equations is somewhat simpler to state, but still not well understood. Here there is a conjecture that no singularities occur in a finite time. No counter example is known, but a mathematical proof appears challenging to say the least. Indeed, proving the existence and smoothness of solutions to the Navier-Stokes equation is one of the Millennium Prize problems with a $1 million dollar prize attached. (If you're genuinely motivated by the money then I would suggest that mathematics may not be your true calling. There are easier ways to be both happy and rich.)

Finally, that leaves us with the Einstein equations of General Relativity. Here the situation is most intriguing of all. It is straightforward to show that singularities do develop in finite time (at least with a suitable definition of "time"!). This arises when matter collapses to form a black hole, with a singularity forming in the centre where the curvature of spacetime becomes infinite. The presence of such a singularity is telling us that the laws of classical gravity are breaking down and must be replaced by something quantum. This means that singularities provide a wonderful opportunity to teach us something new about the "atoms of spacetime", whatever that means. Sadly, however, nature has made these singularities very difficult to access experimentally. It appears that they are generically shielded by an event horizon, so that they can't be

seen by anyone sensible who chooses not to jump into the black hole. The idea that singularities necessarily sit behind an event horizon goes by the name of the *cosmic censorship conjecture*. From a mathematical perspective, it appears utterly miraculous and a proof is generally thought to be even more challenging than the Navier-Stokes existence and smoothness conjecture.

The upshot is that the laws of physics appear to be surprisingly robust against the formation of singularities. Even when singularities do arise – as in the compressible Euler equation and the Einstein equations – some poorly understood feature of the equations means that they are more innocuous than we would have naively thought. They are either hidden behind horizons, or neatly resolved by the second law. In both cases, we can largely carry on with our lives without worrying too much about what microscopic physics lurks inside the singularity.

It feels like there is an important lesson hiding within this story. The refusal of both the Navier-Stokes and the Einstein equations to develop readily accessible singularities, that require something atomic or quantum to fully understand, is a striking mathematical fact. It should have a striking physical reason behind it. But I don't know what it is.

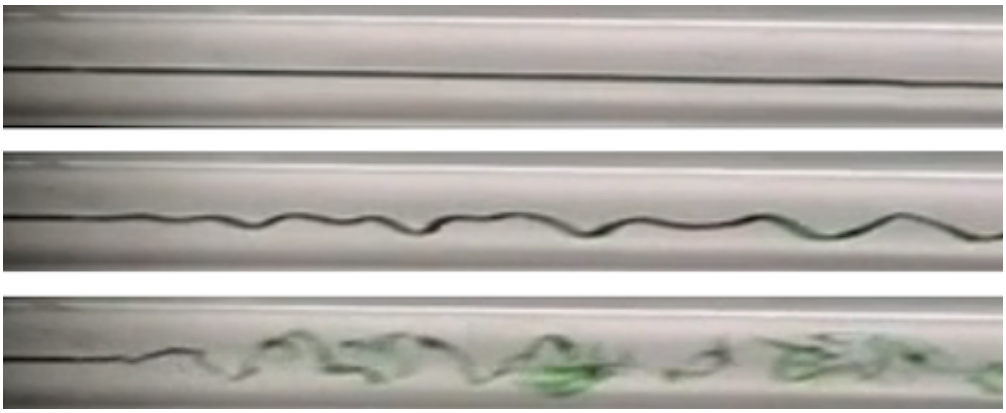


Figure 30. Poiseuille flow for low, medium, and high Reynolds number.

5 Instabilities

We started these lectures by studying some simple, laminar flows in which the fluid moves smoothly through space. One important question that we haven't yet addressed is: are these flows stable? If we perturb them in some way, does the flow persist or does it get driven to something more complicated?

The answer is that many flows, even very simple ones, can be unstable. This statement is especially true at high Reynolds number, where viscosity fails to dampen the perturbations. You can see an example in the photographs[9] of Poiseuille flow, describing fluid flowing down a pipe, shown in Figure 30. The fluid in the pictures is transparent and the flow is moving from left to right, but some black dye is released from the same point on the left-hand edge and traces out what is known as a *streakline*. (For time independent flows, it coincides with both the path line and the streamline that we introduced previously. For time dependent flows, it differs.)

The top picture shows the flow when it is slow, so that $Re \lesssim 1$ and the fluid is well described by the solution that we already met in Section 3.2.3. The flow is laminar, with the velocity field lying parallel to the pipe and the picture looks boring:

But as the flow speeds up, so that $Re \sim 100$ or so, things become more interesting. This is shown in the middle picture where the streakline starts to wobble in places,

[9]These pictures are snapshots from this Youtube video. You can also find remarkably similar pictures drawn by Reynolds in his original paper from 1883.

with the wobbles dragged along by the flow. This is the manifestation of instabilities of the simple solution.

Finally, by the time we are in the regime $Re \sim 10^3$ or 10^4 (the exact number depends on how rough the boundary of the pipe is), the flow appears qualitatively different yet again. This is shown in the bottom figure. Any clearly defined wobbles have vanished. The flow just looks messy. This is the turbulent regime.

Our goal in this section is to start to understand these kinds of instabilities. Although Poiseuille and Couette are the simplest examples of laminar flows, it turns out that understanding their instabilities is not so straightforward. For this reason, we start by looking at specific instabilities in other contexts. (For what it's worth, our instabilities of choice are the Kelvin-Helmholtz instability, the Plateau-Rayleigh instability, and the Rayleigh-Bénard instability. We'll see what these names mean as we proceed.)

Our method to analyse instabilities will mirror the analysis of waves in Section 4. That is: we start with some background flow and perturb it. The perturbations that we call waves oscillate back and forth about the original flow. In contrast, the unstable perturbations that we will meet here grow without bound. These are known as *linear instabilities*. Our linear analysis only shows the beginning of the instability, rather than the end point of the flow, but with some imaginative thinking (and the help of experiment!) we can figure out the qualitative form of the final flow.

With these successes in hand, we then turn to understand the stability of seemingly simpler flows, like Couette flow and Poisseuille flow that we met in Section 3.2. As we mentioned above, it turns out that understanding the fate of these flows is somewhat harder. We will succeed in giving some general results about when flows of this kind are stable against linear perturbations and when they are not. However, as we explain, these results do not stand up particularly well against either experiment or numerical simulation. This is because, ultimately, these flows have more complicated *non-linear instabilities* where an arbitrarily small perturbation is innocuous, but the instability manifests itself only when the perturbation grows to a certain size. We will not study these non-linear instabilities in these lectures.

There are a number of other topics that we won't touch upon. These include transitory phenomenon, and the transition to turbulence, with features characteristic of chaos such as period doubling and the appearance of strange attractors. We will however describe some aspects of fully developed turbulence in Section 6.

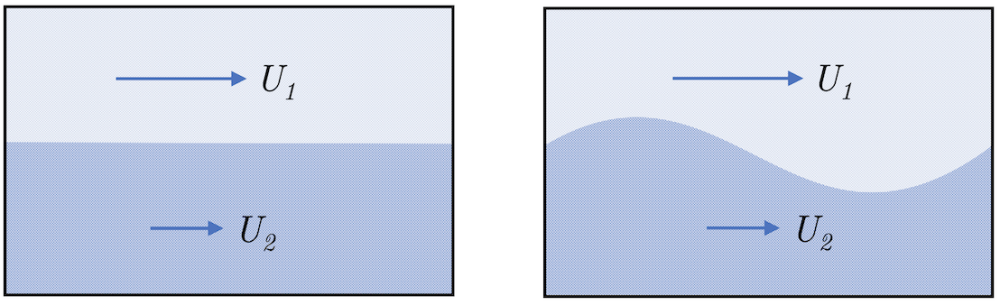


Figure 31. The Kelvin-Helmholtz instability between two moving fluids. The left-hand figure is unstable to turn into the right.

5.1 Kelvin-Helmholtz Instability

The set-up is straightforward. We have two fluids, with densities $\rho_1 \leq \rho_2$. The lighter fluid sits on top and travels with constant speed U_1 in the x-direction. The heavier fluid sits at the bottom and travels with constant speed $U_2 \neq U_1$ in the same direction. Initially, there is an interface between the two at $z = 0$ as shown in Figure 31.

The initial velocity profile has a discontinuity at $z = 0$. In reality, this will be smoothed out by a thin layer in which viscosity is important. For our purposes, we'll neglect this and study the flow using the Euler equation for each fluid,

$$\rho_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \right) = -\nabla P_i + \rho_i \mathbf{g} \quad i = 1, 2$$

Note that we've included the effect of gravity on the right-hand side. It turns out that this won't be necessary to exhibit the physics that we're interested in, but it does have an interesting role to play.

We want to understand whether the initial flow is stable. (Spoiler: it won't be!) In particular, we'll look at perturbations in which the interface is perturbed to

$$z = \eta(x, t) = \eta_0 e^{ikx - i\omega t} \tag{5.1}$$

In other words, we're looking for an instability in which the interface develops waves.

The analysis is almost identical to that of Section 4.1 where we first met surface waves. We look for 2d flows, with $\mathbf{u} = \mathbf{u}(x, z, t)$. We will assume that the flow remains irrotational after the perturbation, so we introduce velocity potentials and write

$$\mathbf{u} = \begin{cases} \mathbf{U}_1 + \nabla \phi_1 & z > \eta \\ \mathbf{U}_2 + \nabla \phi_2 & z < \eta \end{cases}$$

where $\mathbf{U}_i = U_i\hat{\mathbf{x}}$. The requirement that the fluid is incompressible then means that we must, once again, solve the Laplace equation

$$\nabla^2\phi_1 = \nabla^2\phi_2 = 0$$

As usual, all the subtleties lie in the boundary conditions. We require that the fluid returns to its initial state asymptotically, so $\phi_1 \to 0$ as $z \to \infty$ and $\phi_2 \to 0$ as $z \to -\infty$. On the interface, we impose the free boundary condition that we described previously in (4.6)

$$u_z = \frac{D\eta}{Dt}$$

This now reads

$$\left.\frac{\partial\phi_i}{\partial z}\right|_{z=\eta} = \frac{\partial\eta}{\partial t} + \left(U_i + \left.\frac{\partial\phi_i}{\partial x}\right|_{z=\eta}\right)\frac{\partial\eta}{\partial x} \quad i = 1, 2 \tag{5.2}$$

Finally, the pressure forces on either side of the interface must balance the surface tension,

$$P_2(x,\eta) - P_1(x,\eta) = -\gamma\frac{\partial^2\eta}{\partial x^2} \tag{5.3}$$

with γ the surface tension. Rather like gravity, it will turn out that the presence of surface tension is unnecessary to explain the main physics point that we're interested in, but has an interesting implication later on.

To implement the pressure difference equation, we follow the analysis of Section 4.1.2 and use the Bernoulli principle. The analog of our earlier result (4.7) is now

$$\rho_i\frac{\partial\phi_i}{\partial t} + \frac{1}{2}\rho_i|\mathbf{U}_i + \nabla\phi_i|^2 + P_i + \rho_i g z = f_i(t) \quad i = 1, 2$$

with $f_i(t)$ independent of space. The condition (5.3) then becomes

$$\begin{aligned}\rho_1\left(\frac{\partial\phi_1}{\partial t} + \frac{1}{2}|\mathbf{U}_1 + \nabla\phi_1|^2 + gz\right)_{z=\eta} \\ - \rho_2\left(\frac{\partial\phi_2}{\partial t} + \frac{1}{2}|\mathbf{U}_2 + \nabla\phi_2|^2 + gz\right)_{z=\eta} = \tilde{f}(t) - \gamma\frac{\partial^2\eta}{\partial x^2}\end{aligned} \tag{5.4}$$

for some spatially-independent function $\tilde{f}(t)$. This, then, is our goal. Solve the Laplace equations subject to (5.2) and (5.4).

The Linearised Approximation

As for surface waves, we make progress by assuming that the amplitude of the perturbation is small. For us, this means

$$k\eta_0 \ll 1$$

This allows us to linearise the boundary conditions (5.2) and (5.4). The first (5.2) becomes

$$\left.\frac{\partial\phi_i}{\partial z}\right|_{z=0} = \frac{\partial\eta}{\partial t} + U_i\left.\frac{\partial\eta}{\partial x}\right|_{z=0} \qquad i = 1, 2 \tag{5.5}$$

while (5.4) becomes

$$\rho_1\left(\frac{\partial\phi_1}{\partial t} + U_1\frac{\partial\phi_1}{\partial x}\right)_{z=0} - \rho_2\left(\frac{\partial\phi_2}{\partial t} + U_2\frac{\partial\phi_2}{\partial x}\right)_{z=0} + g(\rho_1 - \rho_2)\eta = \tilde{f}(t) - \gamma\frac{\partial^2\eta}{\partial x^2} \tag{5.6}$$

Now we have something eminently more achievable on our hands: solve the Laplace equations subject to (5.5) and (5.6).

5.1.1 The Simplest Instability

To illustrate the key idea, we first ignore both gravity and surface tension. This means that we set $g = \gamma = 0$ in (5.6). We've already shown our hand for the kind of perturbation (5.1) of the interface that we're looking for. We augment this with a commensurate wavey solution for the velocity perturbations,

$$\phi_i(x, z, t) = \hat{\phi}_i(z)e^{ikx-i\omega t} \quad \text{and} \quad \eta(x, t) = \eta_0 e^{ikx-i\omega t}$$

The Laplace equations tell us that

$$\frac{d^2\hat{\phi}_i}{dz^2} = k^2\hat{\phi}_i \quad \Rightarrow \quad \begin{cases} \hat{\phi}_1 = A_1 e^{-kz} \\ \hat{\phi}_2 = A_2 e^{+kz} \end{cases}$$

where the solutions have been chosen so that $\hat{\phi}_1 \to 0$ as $z \to +\infty$ and $\hat{\phi}_2 \to 0$ as $z \to -\infty$. The two boundary conditions in (5.5) then tell us that

$$-kA_1 = (-i\omega + ikU_1)\,\eta_0 \quad \text{and} \quad +kA_2 = (-i\omega + ikU_2)\,\eta_0$$

while the fact that $\tilde{f}(t)$ in (5.6) is independent of x and z means that

$$\rho_1(-i\omega + ikU_1)A_1 = \rho_2\,(-i\omega + ikU_2)\,A_2$$

We can eliminate A_1, A_2 and η_0 to get the quadratic in ω,

$$(\rho_1 + \rho_2)\omega^2 - 2k\omega(\rho_1 U_1 + \rho_2 U_2) + k^2(\rho_1 U_1^2 + \rho_2 U_2^2) = 0 \tag{5.7}$$

The roots of this quadratic give us the dispersion relation,

$$\omega = \frac{k}{\rho_1 + \rho_2}\Big[(\rho_1 U_1 + \rho_2 U_2) \pm i\sqrt{\rho_1\rho_2}|U_1 - U_2|\Big] \tag{5.8}$$

The key piece of physics is sitting in that factor of i. This is the telltale sign of an instability. To see this, we substitute this frequency into the expression (5.1) for the interface, to learn that the perturbations of the interface evolve as

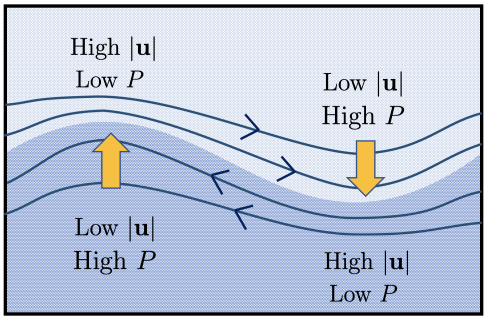
$$z = \eta(x,t) = \eta_0 \exp\left(ik\left(x - \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2}t\right) \pm \frac{\sqrt{\rho_1\rho_2}}{\rho_1 + \rho_2}|U_1 - U_2|kt\right) \tag{5.9}$$

The first term comes with an oscillatory factor of i and tells us that the disturbance propagates with velocity

$$v = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2}$$

But our real interest lies in the second term. Here there's no factor of i: the oscillatory time dependent behaviour $e^{-i\omega t}$ becomes exponential growth or decay when ω is complex. We learn that the perturbation grows exponentially with time. Or, more precisely, one perturbation grows and the other decays. The existence of the growing mode means that the original flow is unstable. This is the *Kevin-Helmholtz instability*.

The next question that we can ask is: why does this happen? Some intuition comes from boosting to a frame where the disturbance is stationary and then thinking about the streamlines, as shown in the figure. The streamlines of the upper fluid are more clustered together near the peaks and so, because of mass conservation, the fluid must travel faster there. Bernoulli's theorem then tells us that the pressure is smaller. But, in the trough, the streamlines are less closely packed, the fluid moves slower and so the pressure is greater.



This story is reversed for the lower fluid. The pressure is now greatest in the peak and lowest in the trough. Of course, the pressure is continuous across the interface itself — this was one of our initial boundary conditions (5.3) — but there is a pressure difference over the whole perturbation and this drives the crest upwards and the trough downwards. This is the reason for the instability.

In fact, this same effect also explains the existence of the decaying mode. If we set up a perturbation where the crest is travelling downwards, and the trough travelling up, then the pressure differences now act to decelerate the amplitude and the disturbance grows smaller over time.

Viscosity Acts as a UV Cut-Off

For a fixed background fluid flow, the instability (5.9) grows as $\sim e^{Ukt}$. This means that the small wavelength modes, with k large, are more unstable. In fact, taken at face value the instability for very small wavelengths (i.e. very large k) grows without bound. What should we make of this? Are the equations telling us that the continuum description of fluids will ultimately break down, and the interface should be thought of in terms of individual atoms?

Thankfully, no. There are (at least) two mechanisms within the continuum description that halt the runaway behaviour for large k. One of these is surface tension, and we will describe this shortly. But even in the absence of surface tension, viscosity does the job. We should replace the discontinuity in the velocity profile with an appropriate boundary layer. Our analysis above is then valid only for wavenumbers $k \lesssim |U_1 - U_2|/\nu$. For wavelengths smaller than this, the perturbation is stabilised by the effects of viscosity. This is a common theme in fluid dynamics, and one that we will meet again in Section 6 when we discuss turbulence: viscosity acts as a UV cut-off.

5.1.2 Rolling Up The Vortex Sheet

There is another way to think about the instability, this time in terms of vorticity. If we integrate around a rectangular contour in the (x, z)-plane that crosses the interface, with sides of length L in the x direction, then the initial flow have vorticity

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{x} = (U_1 - U_2)L$$

The flows on either side of the interface are irrotational, so this vorticity must be localised at the interface itself. In the initial flow, the interface has constant vorticity per unit length. For this reason, the interface is referred to as a *vortex sheet*.

The vorticity points in the direction $\boldsymbol{\omega} = |\boldsymbol{\omega}|\hat{\mathbf{y}}$, where $\hat{\mathbf{y}}$ points out of the page in the previous figures. (Note: we'll refer to the magnitude of vorticity as $|\boldsymbol{\omega}|$ rather than ω to avoid confusion with the frequency) and is given by

$$|\boldsymbol{\omega}| = (U_1 - U_2)\delta(z)$$

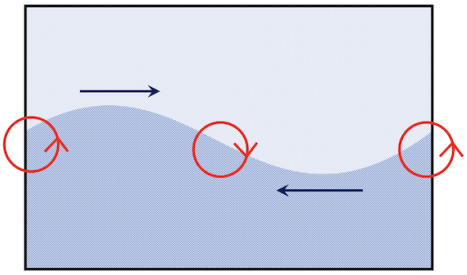
We can view the instability as a deformation of this vortex sheet. Evaluated on the linearised solution, we have

$$\begin{aligned} |\boldsymbol{\omega}| &= \left[(U_1 - U_2) + \left.\frac{\partial \phi_1}{\partial x}\right|_{z=0^+} - \left.\frac{\partial \phi_2}{\partial x}\right|_{z=0^-}\right]\delta(z-\eta) \\ &= \left[(U_1 - U_2) + (-2\omega + k(U_1+U_2))\eta_0 e^{ikx-i\omega t}\right]\delta(z-\eta) \end{aligned}$$

It's simplest to illustrate the physics if we restrict to the case $\rho_1 = \rho_2$, so that we have two identical fluids travelling at different speeds. Then, using the dispersion relation (5.8), this becomes

$$|\boldsymbol{\omega}| = \Big[(U_1 - U_2) - i|U_1 - U_2|k\eta\Big]\delta(z-\eta)$$

with $\eta = \eta_0 e^{ikx-i\omega t}$ the unstable boundary. We see that, as the perturbation grows, the vorticity is no longer constant in space. It tracks the development of the interface but, because of that factor if i, is $\pi/2$ out of phase, corresponding to 1/4 of a wavelength. This means that there is no change to the vorticity at the maxima and minima, but the vorticity alternates clockwise/anti-clockwise at the midway points of the perturbation, as shown in the figure. This acts to push the crests up, and the troughs down, again



Our analysis throughout this section has been done in the linearised approximation, where the perturbation is small. However, the vortex picture gives us a good sense for what happens as the perturbation grows. If we sit in the frame where the disturbance doesn't propagate, then the upper fluid moves to the right, and the lower fluid moves to the left. The vorticity is advected in this direction, and so accumulates at the midpoint points between the peaks and troughs, shown in red in the figure. As the perturbation grows, so too does the vorticity density in this region with the result that the perturbation starts to curl around, or "roll up". The end result is the distinctive

Figure 32. When non-linear effects are taken into account, the Kelvin-Helmholtz instability rolls up to produce, among other things, these beautiful cloud formations.

rolling-wave feature of the Kelvin-Helmholtz instability, as shown in the cloud formation in Figure 32[10].

5.1.3 Gravity Helps. Surface Tension Helps Too.

With our basic understanding of the instability, we can now repeat the analysis with gravity and surface tension turned on. Both manifest themselves in the boundary condition (5.6). After some algebra, the steps that previously led us to (5.7) now give

$$(\rho_1+\rho_2)\omega^2 - 2\omega k(\rho_1 U_1+\rho_2 U_2) + k^2(\rho_1 U_1^2+\rho_2 U_2^2) = \gamma k^3 - gk(\rho_1-\rho_2)$$

Solving this as a quadratic in ω gives the dispersion relation

$$\omega = \frac{1}{\rho_1+\rho_2}\left[k(\rho_1 U_1+\rho_2 U_2) \pm \sqrt{(\rho_1+\rho_2)(\gamma k^3 + gk(\rho_2-\rho_1)) - \rho_1\rho_2 k^2 (U_1-U_2)^2}\right]$$

This reduces to (5.8) when $g=\gamma=0$. The presence of gravity and surface tension means that the function under the square root is no longer negative for all k. This is telling us that some wavelengths are no longer unstable, while others are.

First, suppose that have gravity $g \neq 0$ but negligible surface tension $\gamma = 0$. In this case, there is an instability only if

$$k > \frac{\rho_2^2-\rho_1^2}{\rho_1\rho_2}\frac{g}{(U_1-U_2)^2} \tag{5.10}$$

We see that the long wavelength (small k) modes are no longer unstable. Heuristically, these modes are heavier and so the effect of gravity pulling them down wins over the runaway instability.

[10]and also in this wonderful Sixty Symbols video.

Next suppose that we have surface tension $\gamma \neq 0$, but gravity is negligible so $g = 0$. Then there is an instability only if

$$k < \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} \frac{(U_1 - U_2)^2}{\gamma} \tag{5.11}$$

This time the short wavelength (large k) modes are stabilised. This too makes intuitive sense: the surface tension means that you pay a large cost in energy when the interface has large gradients, so small wavelength perturbations are rescued from the instability.

Alternatively, we can view (5.10) and (5.11) as conditions on how large the velocity difference $|U_1 - U_2|$ must be to initiate an instability for a fixed k. If we have both $g, \gamma \neq 0$ then there is no instability at all for small velocity differences. The instability only kicks in when there is some value of k such that the frequency ω has an imaginary part. This arises when

$$(U_1 - U_2)^2 > 2 \frac{\rho_1 + \rho_2}{\rho_1 \rho_2} \sqrt{(\rho_2 - \rho_1)\gamma g}$$

Moreover, there is maximally unstable mode for which ω has the largest imaginary value. This is the wavelength at which the instability will tend to develop.

We can do some order of magnitude estimates. For wind blowing above water, we have $\rho_1 = \rho_{\text{air}} \approx 1 \text{ kg m}^{-3}$ and $\rho_2 = \rho_{\text{water}} \approx 10^3 \text{ kg m}^{-3}$. The surface tension turns out to be given by $\gamma \approx 0.07 \text{ J m}^{-2}$. This gives a critical wind speed of $U_{\text{wind}} \approx 7 \text{ m s}^{-1}$. This is the minimum speed needed to before the wind becomes responsible for making waves. It is, it turns out, in the ballpark of the average speed across the ocean. The corresponding unstable mode number is around $k \approx 400 \text{ m}^{-1}$, corresponding to a wavelength of a couple of centimetres. In other words, the Kelvin-Helmholtz instability due to wind causes only tiny capillary waves. The Kelvin-Helmholtz instability is not responsible for the great rolling waves on the ocean.

5.1.4 The Rayleigh-Taylor Instability

There's a particularly simple application of the machinery above. We consider two fluids, now both stationary, but with the heavier one on top. It's unlikely to come as a surprise to learn that this situation is unstable. But how does the heavy fluid succeed in moving past the lighter fluid? If they're truly immiscible, then they can't just swap places. Instead, something more complicated must be going on[11].

[11] Again, there is a nice demonstration on Youtube.

What happens is a simple form of the Kelvin-Helmholtz instability. We can trivially import our previous dispersion relation, now with $U_1 = U_2 = 0$ and $\rho_1 > \rho_2$,

$$\omega = \pm\sqrt{\frac{\gamma k^3 - gk(\rho_1 - \rho_2)}{\rho_1 + \rho_2}}$$

We see that the long wavelength modes, with $k^2 < g(\rho_1 - \rho_2)/\gamma$ are unstable. If introduce the generalisation of the *capillary length* that we met earlier in (4.23),

$$l_c = \sqrt{\frac{\gamma}{g(\rho_1 - \rho_2)}}$$

The most unstable mode occurs for $3k^2 = 1/l_c^2$. This determines the size of the perturbation at which the upper layer descends into the lower. This is the *Rayleigh-Taylor instability*.

Taylor's contribution to the story above was to invoke the equivalence principle. The mathematics remains unchanged when a heavier fluid is accelerated through a lighter fluid, rather than pulled down due to gravity. This is actually a more common phenomenon, and the Rayleigh-Taylor instability describes, among other things, the mushroom clouds that appear from volcanic eruptions and nuclear explosions.

5.2 A Piece of Piss

We've all seen it. The stream of liquid starts out unblemished and pure. But, as it falls, ripples start to appear on the surface and these grow until the stream ultimately disintegrates into individual droplets[12]. This is known as the Plateau-Rayleigh instability and our goal in this section is to understand it. Fluid dynamicists refer to the stream of liquid as a *jet* and we'll adopt this terminology (although, in some contexts, that seems overly praiseworthy).

In most situations the column of liquid forms in the first place because it falls under gravity. But it will turn out that the most prominent role in the development of the

[12]The picture is a snapshot from this Youtube video.

instability is played by surface tension. For this reason, we'll start by ignoring gravity and assuming that we are simply given a cylindrical jet, meaning a column of flowing liquid. Then, in Section 5.2.1, we'll see how gravity changes things. We ignore viscosity throughout this section.

We can go to a frame where the fluid is stationary, $\mathbf{u} = 0$, and with constant pressure P_0. The first question that we have to ask is: what keeps the jet together? Why doesn't the liquid just fly off in random directions? The answer is surface tension.

As we learned in Section 4.1.3, the surface tension allows for a pressure difference between the liquid and the outside air. For simplicity, we'll assume that the external pressure is vanishing. We take the jet to lie in the z-direction and have radius R_0. We use the radial coordinate $r = \sqrt{x^2 + y^2}$ so, following (4.2), the surface is given by $F(r) = r - R_0 = 0$. The curvature of a circle of radius R_0 is simply $1/R_0$ so, from (4.20), the surface tension γ balances the pressure inside the jet when

$$P_0 = \frac{\gamma}{R_0} \tag{5.12}$$

Now we perturb. We will consider capililary surface waves that run along the jet, so that the radius is displaced by

$$R(z,t) = R_0 + \tilde{R}e^{ikz-i\omega t}$$

This means that we're anticipating wave-like behaviour. But, as in the previous section, this will be unstable if ω is imaginary. We look for solutions in which the velocity profile and pressure have the same behaviour,

$$\mathbf{u}(\mathbf{x},t) = [u_r(r)\hat{\mathbf{r}} + u_z(r)\hat{\mathbf{z}}]\, e^{ikz-i\omega t} \quad \text{and} \quad P(\mathbf{x},t) = P_0 + \tilde{P}(r)e^{ikz-i\omega t}$$

Note that each of the perturbations is a function of the radial direction r, as well as exhibiting wave-like behaviour in the z direction.

The incompressibility condition $\nabla \cdot \mathbf{u} = 0$ tells us that

$$\frac{1}{r}\frac{d(ru_r)}{dr} + iku_z = 0$$

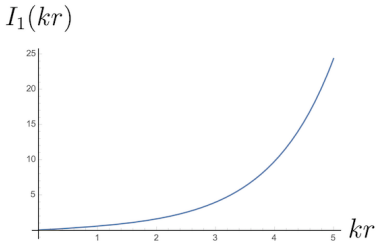
After linearising, the two components of the Euler equation are

$$-i\omega u_r = -\frac{1}{\rho}\frac{d\tilde{P}}{dr} \quad \text{and} \quad -i\omega u_z = -\frac{ik}{\rho}\tilde{P} \tag{5.13}$$

We can combine our three equations into a single, second order equation for u_r,

$$r^2\frac{d^2u_r}{dr^2} + r\frac{du_r}{dr} - (1 + k^2r^2)u_r = 0$$

This is a standard differential equation, with solutions given by the modified Bessel functions $I_1(kr)$ and $K_1(kr)$. Of these, $K_1(x)$ diverges as $x \to 0$ and so is not appropriate for our needs. Meanwhile, $I_1(x) \to 0$ as $x \to 0$ but grows exponentially for large x, as shown in the figure. This is the solution for us. We learn that the radial velocity profile is given by



$$u_r(r) = VI_1(kr)$$

for some constant V. We still have two boundary conditions to impose on the surface of the jet, which is now defined by the constraint

$$F(r, z, t) = r - R(z, t) = 0$$

The first is the usual boundary condition (4.3) for a free surface, which tells us that the velocity of the fluid normal to the surface must track the surface itself. After linearisation, this gives

$$\frac{DF}{Dt} \approx \mathbf{u} \cdot \hat{\mathbf{r}} - \frac{\partial R}{\partial t} \quad \Rightarrow \quad V \approx -\frac{i\omega \tilde{R}}{I_1(kR_0)} \tag{5.14}$$

and determines the unknown constant V in terms of ω and k. The second boundary condition is the requirement that the pressure is balanced by the surface tension. But now there are two contributions to the curvature of the surface. The first is because the surface is a curved in the (x, y)-plane, and gives a contribution like (5.12). The second comes from the additional curvature of the form (4.20) arising from waves in the z-direction. Combining these, we have

$$P(R(t)) = P_0 + \tilde{P}(R(t))e^{ikz-i\omega t} = \frac{\gamma}{R(t)} + \gamma k^2 \tilde{R} e^{ikz-i\omega t}$$

We can linearise the $R(t)$ terms in the second equation. We have $\tilde{P}(R(t)) \approx \tilde{P}(R_0)$ and, by Taylor expanding, $1/R(t) \approx 1/R_0 - (\tilde{R}/R_0^2)e^{ikz-i\omega t}$. The terms involving just the background variables P_0 and R_0 cancel by virtue of (5.12), and we're left with a condition that relates the perturbation of the pressure $\tilde{P}$ to the perturbation of the radius $\tilde{R}$,

$$\tilde{P}(R_0) = \frac{\tilde{R}\gamma}{R_0^2}(k^2 R_0^2 - 1) \tag{5.15}$$

Already here we can see that there is a difference between short wavelength and long wavelength perturbations. For short wavelengths, with $kR_0 > 1$, the pressure increases at the surface. But for long wavelengths, with $kR_0 < 1$, the pressure decreases at the surface. This will turn out to be important.

We can get another expression for the pressure perturbation everywhere within the jet from the radial component of the Euler equation (5.13). This tells us that

$$\frac{d\tilde{P}}{dr} = i\omega\rho u_r = i\omega\rho V I_1(kr) \quad \Rightarrow \quad \tilde{P}(r) = \frac{i\omega\rho V}{k} I_0(kr) \tag{5.16}$$

where we've made use of the Bessel function identity $I_1 = dI_0/dx$ to integrate the first expression. (The function $I_0(x)$ also grows exponentially as $x \to \infty$, but has the value $I_0(0) = 1$ at the origin.) Equating (5.15) and (5.16), and using our expression (5.14) for the constant V, we find

$$\omega^2 = \frac{k\gamma}{\rho R_0^2} \frac{I_1(kR_0)}{I_0(kR_0)} (k^2 R_0^2 - 1)$$

This is the dispersion relation that we were looking for. For large k, the ratio of Bessel functions is $I_1(x)/I_0(x) \to 1$ as $x \to \infty$ and so we get waves propagating with the dispersion $\omega \sim k^{3/2}$. This is the same dispersion relation that we found for surface capillary waves. (See the large k limit of (4.24).) This is to be expected: waves with a small wavelength have no knowledge of their global surrounding. They're unaware if they're sitting on a stream of liquid or on a rolling ocean.

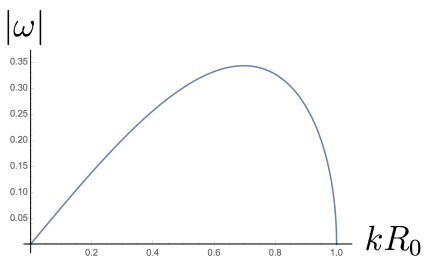
More interesting for our purposes is the long wavelength behaviour of the dispersion relation. The frequency is imaginary when

$$kR_0 < 1$$

This means that perturbations with wavelength bigger than the radius R_0 of the jet will grow exponentially quickly.

The plot to right shows the function $|\omega| \sim \sqrt{x(1-x^2)I_1(x)/I_0(x)}$ in the unstable regime, corresponding to $0 \leq x \equiv kR_0 \leq 1$. We see that there is a maximally unstable mode at around $kR_0 \approx 0.7$, corresponding to a wavelength

$$\lambda_{\rm max} \approx 9R_0$$



The associated time that it takes the perturbation to grow is

$$T \sim \frac{1}{|\omega|} \approx \frac{1}{0.34}\sqrt{\frac{\rho R_0^3}{\gamma}}$$

where the 0.34 is seen to be roughly the maximum of the graph. For water, with $\rho \approx 10^3$ kg m^{-3} and $\gamma \approx 0.07$ J m^{-2}. A stream of water from a tap has a radius of roughly $\frac{1}{2}$ cm, and, from the analysis above, is expected to decay in around 0.1 seconds.

5.2.1 Gravity Makes the Flow Thinner

We've neglected gravity in the above analysis, despite the fact that gravity is clearly responsible for the formation of the most familiar streams of liquid. We now remedy this. In fact, the most important ramification of gravity is not in the evolution of the perturbations, but in the form of the stream itself. That's what we will focus on here.

Our result is a simple application of Bernoulli's theorem that we first met in Section 2.1.4. This says that, in a gravitational field with potential $\Phi = -\rho g z$, the quantity

$$H = \frac{1}{2}\rho|\mathbf{u}|^2 + P - \rho g z$$

is constant. (Here we're measuring z so that it increases in a downwards direction.) Clearly we expect the component of the velocity $U = u_z$ to increase with z simply because the fluid is being accelerated downwards. But now we have to reconcile this with the fact that the pressure must balance the surface tension as in (5.12)

$$P = \frac{\gamma}{R}$$

As we will see, this requires that both the radius $R(z)$ and the velocity $U(z)$ depend on z. This follows because of mass conservation, which tells us that the flux of fluid along the stream is

$$\text{Flux} = \pi R^2 U$$

and this too must be constant.

Suppose that the fluid has velocity U_0 and radius R_0 at $z = 0$. Then flux conservation gives $R_0^2 U_0 = R(z)^2 U(z)$ and Bernoulli's principle becomes

$$\frac{1}{2}\rho U_0^2 + \frac{\gamma}{R_0} = \frac{1}{2}\rho\frac{R_0^4 U_0^2}{R(z)^4} + \frac{\gamma}{R(z)} - \rho g z$$

On rearranging,

$$R(z)^4\left(\frac{1}{2}\rho U_0^2+\frac{\gamma}{R_0}+\rho g z\right)-\gamma R(z)^3-\frac{1}{2}\rho R_0^4 U_0^2=0 \tag{5.17}$$

In general, this has no closed form solution, although the physics is clear: as z increases, $R(z)$ must decrease roughly as $z^{-1/4}$, at least for large z. There is also a limit in which we can make more progress. We define the dimensionless *Weber number* as

$$We=\frac{\rho R_0 U_0^2}{\gamma}$$

The limit of $We\to\infty$ is where surface tension effects are no longer important. In this limit we can drop the γ terms in (5.17) and we have

$$R(z)=R_0\left(1+\frac{2gz}{U_0^2}\right)^{-1/4}$$

where we see very clearly the advertised $z^{-1/4}$ behaviour for large z. From our analysis of the Plateau-Rayleigh instability, we know that as the radius of the stream gets thinner, the wavelength at which the instability kicks in gets shorter.

5.3 Rayleigh-Bénard Convection

In this section, we will discuss fluid flows due to temperature differences. This is known as *convection*.

To start, we discuss some basics of heat. Consider a system with an energy density $\mathcal{E}(\mathbf{x},t)$. We'll assume, for now, that there is no net fluid flow of the system. Nonetheless, energy can be transported from one place to another by means of *heat*. Microscopically, this describes the kinetic energy of the underlying atoms, which can be flying around in random directions even when the macroscopic fluid appears to be stationary. When energy is transported in this way, the associated continuity equation, telling us that energy is conserved, is

$$\frac{\partial\mathcal{E}}{\partial t}+\nabla\cdot\mathbf{q}=0$$

where $\mathbf{q}$ is the *heat flow*.

To make progress, we need a couple of results. First, at constant pressure, the change in the energy density is related to the change in the temperature $T(\mathbf{x},t)$ by

$$\frac{\partial\mathcal{E}}{\partial t}=c_P\frac{\partial T}{\partial t}$$

where c_P is the specific heat capacity at constant pressure. (We derived an expression for this in Section 4.4.) Second, the heat flow is related temperature difference,

$$\mathbf{q} = -\kappa \nabla T$$

with the minus sign telling us that heat flows from high temperatures to small. The coefficient κ is called the *thermal conductivity*. Combining these with the continuity equation, we get

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c_P} \nabla^2 T \tag{5.18}$$

This is the *heat equation*.

Now we want to upgrade this description to include a background motion $\mathbf{u}(\mathbf{x}, t)$ of the fluid. We'll focus on a thin layer of fluid, sandwiched between two plates held at different temperatures, the bottom hot and the top cold. Our expectation is that this temperature difference will drive the flow upwards and our goal is to understand how this happens.

We assume that the flow is incompressible, so

$$\nabla \cdot \mathbf{u} = 0$$

As we'll see shortly, this assumption isn't entirely innocent when we have heat conduction and we will induce a small amount of handwringing below. The generalisation of the heat equation is then straightforward: it should now be applied to parcels of fluid that are swept along with the flow, meaning

$$\frac{DT}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) T = \frac{\kappa}{c_P} \nabla^2 T \tag{5.19}$$

We already met this equation in (4.66) when discussing sound waves. There was an additional $\nabla \cdot \mathbf{u}$ term in (4.66) but we've set this to zero on the grounds that we're dealing with an incompressible fluid.

Our other equation is, of course, Navier-Stokes.

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P - \rho(\mathbf{x}, t) g \hat{\mathbf{z}} + \mu \nabla^2 \mathbf{u} \tag{5.20}$$

We've included a gravitational force and viscosity. Both will be important for what is to come. We now have five equations (two scalar and one vector) for six unknowns: $\mathbf{u}$, P, ρ and T. We must augment these with an equation of state, relating ρ, P and T. When discussing sound waves we relied heavily on the ideal gas equation of state. Here we are dealing with a thin layer of liquid and our equation of state will be different.

5.3.1 The Boussinesq Approximation

As always, we will invoke some approximation to allow us to solve the equations. This one is more subtle than most.

We start by assuming that the temperature difference between the upper and lower plate is small. Mathematically, this means that if we think of $\rho = \rho(T)$, then the difference in density is well approximated by a Taylor expansion

$$\rho(\mathbf{x}, t) \approx \rho_0 \left(1 - \alpha(T(\mathbf{x}, t) - T_0)\right) \tag{5.21}$$

with α constant. We take T_0 and ρ_0 to be the temperature and density at the bottom plate. Above the plate, we expect $T < T_0$ and so if $\alpha > 0$, the density will get greater as we go up. This is what we expect: increased temperature causes fluids to expand. In addition, the form (5.21) assumes that the density is independent of pressure.

At this point, it turns out that not all of our assumptions are mutually compatible! In particular, the assumption of an incompressible flow, $\nabla \cdot \mathbf{u} = 0$, means that mass conservation reduces to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \Rightarrow \quad \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0$$

We saw the same equation in Section 4.2 when discussing gravity waves in stratified flows. But if ρ and T are linearly related, as in (5.21), then they should solve the same equation. And the corresponding equation for temperature (5.19) has the diffusion term, proportional to κ/c_P on the right-hand side. In other words, incompressible flows are incompatible with heat diffusion.

Our strategy is to just bluff our way through this impasse. We will have our cake and eat it by pretending that the density ρ is actually constant. Except when it's not constant. More precisely, we will assume that the changes in density due to temperature are much smaller than the initial density ρ_0. We will make the substitution $\rho = \rho_0$ in all places except one: the exception is the term $g\rho$ in the Navier-Stokes equation. This is the term that will ultimately govern the physics for the simple reason that we have orchestrated a situation in which the denser, colder fluid sits on top. This means that there's a balance of forces at play in the problem, in which the heavier fluid wants to sink, but is kept afloat by the temperature gradient. It is this buoyancy force that will be important in driving the instability.

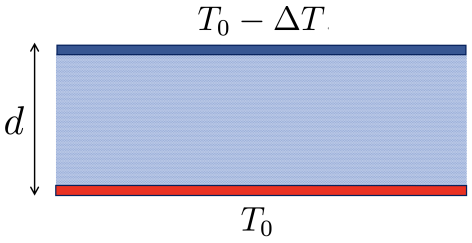
Another, perhaps more palatable, way of saying this is that we're in a situation in which the changes in the density are small, but the gravitational acceleration is large enough to make these changes important in the buoyancy force. (In addition, we use this to justify treating κ/c_P as constant. Recall from Section 4.4 that c_P depends on ρ, at least for an ideal gas.) This collection of ideas is known as the *Boussinesq approximation*.

Practically, this means that we work with an incompressible fluid $\nabla \cdot \mathbf{u} = 0$ and the temperature is governed by the heat diffusion equation (5.19), while the Navier-Stokes equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0}\nabla P - g\left(1 - \alpha(T - T_0)\right)\hat{\mathbf{z}} + \nu \nabla^2 \mathbf{u} \tag{5.22}$$

where the kinematic viscosity is $\nu = \mu/\rho_0$. These are now five equations (two scalar, one vector) for five unknowns: $\mathbf{u}$, P and T.

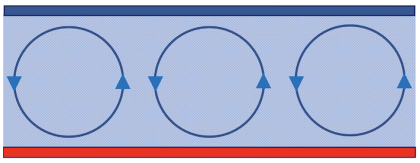
We now start by solving these equations for the heat flow between two plates. The lower plate sits at $z = 0$ and has temperature T_0; the upper plate sits at $z = d$ and has temperature $T_0 - \Delta T$. If the plates are separated by a distance d, then there is a simple, time independent, solution to our equations with $\mathbf{u} = 0$ and



$$T = T_0 - \Delta T\,\frac{z}{d} \quad \text{and} \quad P = P_0 - \rho_0 g\left(z + \frac{\alpha \Delta T}{2d} z^2\right) \tag{5.23}$$

In this solution, the heat transfer is not due to the motion of the fluid itself. In other words, there is conduction but no convection. We would like to understand if this situation is stable against perturbations.

It turns out, unsurprisingly given the general theme of this section, that it won't be. The solution (5.23) is unstable to the fluid moving, so that heat is transferred by convection as well as conduction. The end point of this instability looks something like the situation on the right, with the fluid arranging itself into blocks known as *Bénard cells*, within each of which the fluid rises and falls[13].



[13] A nice demonstration can be seen in this Youtube video.

Before doing the full analysis, we can get some sense for why this happens. The key, as so often, is to look at the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. The physics is actually clearest before we make the Boussinesq approximation. We take the curl of the Navier-Stokes equation (5.20) (after dividing by the density ρ) to get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \frac{1}{\rho^2} \nabla \rho \times \nabla P + \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega} \tag{5.24}$$

The novelty is the first term on the right-hand side. (A similar term survives after the Boussinesq approximation if we take the curl of (5.22).) This is telling us that if the gradient of the density lies in a different direction to the gradient of pressure then it will drive a creation of vorticity. As we will see below, the perturbations that have this property give rise to the convection cells.

We can also try to figure out under what circumstances the instability will occur. Let's suppose that the width of each cell is d, the same as the height of the fluid. (At this point we're just guessing this on the grounds, but it will turn out that the size of the cell is indeed set by the depth.) We've already noted that the heavier fluid sits on top. This means that the potential energy gained if the fluid can somehow right itself is

$$\text{P.E.} \sim gd(d^3 \Delta\rho) \sim gd^4 \rho_0 \alpha \Delta T$$

But to flip over, the fluid has to overcome some viscous forces. If it flips over at speed U, then the viscous force per volume is $\mu \nabla^2 \mathbf{u} \sim \mu U/d^2$. This means that the actual force is $\mu U d$ and the work done, which is $\int F \, dz$, scales as

$$\text{Work Done} \sim \mu U d^2$$

Comparing these two expressions, the gain in potential energy is sufficient to overcome the work done against viscous forces when

$$\frac{gd^2 \alpha \Delta T}{\nu U} \gg 1$$

We can also get an order of magnitude estimate for the speed U at which this takes place. From the heat equation (5.18), the time scale at which heat diffuses is $\tau_{\text{diff}} \sim d^2 c_P/\kappa$. Meanwhile, if the fluid flips over at speed U then it takes a time $\tau \sim d/U$. If we equate these we get $U \sim \kappa/c_P d$. The inequality above then becomes a condition on a collection of dimensionless constants

$$Ra = \frac{g\alpha c_P d^3 \Delta T}{\nu \kappa} \gg 1 \tag{5.25}$$

This is called the *Rayleigh number*. This back of the envelope analysis above suggests that we will find an instability when $Ra \gg 1$. We'll see that this is confirmed in our subsequent analysis.

5.3.2 Perturbation Analysis

We will perturb the background solution (5.23), turning on a velocity field $\mathbf{u}$ together with temperature and pressure perturbations that we denote as $\delta T(\mathbf{x}, t)$ and $\delta P(\mathbf{x}, t)$ respectively. These will all be considered small in the sense that we will linearise the equations of motion. The temperature equation (5.19) becomes

$$\frac{\partial \delta T}{\partial t} - \frac{\Delta T}{d} u_z = \frac{\kappa}{c_P} \nabla^2 \delta T \tag{5.26}$$

and the Navier-Stokes equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla \delta P + g\alpha\, \delta T\, \hat{\mathbf{z}} + \nu \nabla^2 \mathbf{u} \tag{5.27}$$

Meanwhile, the velocity perturbations remain incompressible, $\nabla \cdot \mathbf{u} = 0$.

Dimensional Analysis

We can illustrate the physics in these equations if we write them in terms of dimensionless variables. For this, we'll need to rescale both space and time as

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{d} \quad \text{and} \quad \tilde{t} = \frac{\kappa}{c_P d^2} t$$

This means, in particular, that our lower and upper boundaries are at $\tilde{z} = 0$ and $\tilde{z} = 1$ respectively. We also rescale our dynamical variables

$$\tilde{T} = \frac{\delta T}{\Delta T} \quad , \quad \tilde{P} = \frac{c_P^2 d^2}{\kappa^2 \rho_0} \delta P \quad , \quad \tilde{\mathbf{u}} = \frac{c_P d}{\kappa} \mathbf{u}$$

Then the temperature perturbation equation (5.26) becomes

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} - \tilde{u}_z = \tilde{\nabla}^2 \tilde{T} \tag{5.28}$$

while the perturbed Navier-Stokes equation (5.27) exhibits two dimensionless coefficients.

$$\frac{\partial \tilde{\mathbf{u}}}{\partial \tilde{t}} = -\tilde{\nabla} \tilde{P} + Ra\, Pr\, \tilde{T} \hat{\mathbf{z}} + Pr\, \tilde{\nabla}^2 \tilde{\mathbf{u}} \tag{5.29}$$

The first of these is the Rayleigh number Ra that we already met in (5.25). The second is the *Prandtl number*,

$$Pr = \frac{c_P \nu}{\kappa}$$

Unlike most of our other dimensionless numbers, the Prandtl number is a property only of the liquid, not of the flow. It has the value $Pr \approx 0.7$ for air and for most gases. It is $Pr \approx 7.5$ for water at room temperature.

Our goal is to solve (5.28) and (5.29) subject to suitable boundary conditions. Clearly we require $\tilde{u}_z = 0$ at $\tilde{z} = 0$ and $\tilde{z} = 1$ so that nothing flows into either plate. Also $\tilde{T} = 0$ at $\tilde{z} = 0$ and $\tilde{z} = 1$ as the plates are at a fixed temperature.

In addition, it is natural to impose the no-slip boundary condition on both plates, since our liquid is viscous. It may be natural, but we're not going to do it. The reason is simply laziness! The calculation below is challenging enough and by the time we get to solve the equations life is much easier if we impose the somewhat unphysical requirement that there is no stress on the plate, meaning

$$\frac{\partial \tilde{u}_x}{\partial \tilde{z}} = \frac{\partial \tilde{u}_y}{\partial \tilde{z}} = 0 \quad \text{at} \quad \tilde{z} = 0, 1$$

In fact it turns out that this is the correct boundary condition to impose on a free surface, rather than a rigid plate. As we'll see (in the discussion following (5.37)), to do the calculation we really need a boundary condition on $\tilde{u}_z$. We get this by writing the boundary condition above as

$$\frac{\partial}{\partial \tilde{z}}\left(\frac{\partial \tilde{u}_x}{\partial \tilde{x}} + \frac{\partial \tilde{u}_y}{\partial \tilde{y}}\right) = 0 \quad \Rightarrow \quad \frac{\partial^2 \tilde{u}_z}{\partial \tilde{z}^2} = 0 \quad \text{at} \quad \tilde{z} = 0, 1 \tag{5.30}$$

where we've invoked the incompressibility condition $\tilde{\nabla} \cdot \tilde{\mathbf{u}} = 0$.

You May Wish To Roll Up Your Sleeves

To proceed, we first take the curl of (5.29) to get

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial \tilde{t}} = Ra\, Pr\, \tilde{\nabla}\tilde{T} \times \hat{\mathbf{z}} + Pr\, \tilde{\nabla}^2 \tilde{\boldsymbol{\omega}}$$

where $\tilde{\boldsymbol{\omega}} = \tilde{\nabla} \times \tilde{\mathbf{u}}$. Curiously, it turns out that the best way to proceed is to take yet another curl. This then gives

$$\frac{\partial \tilde{\nabla}^2 \tilde{\mathbf{u}}}{\partial \tilde{t}} = Ra\, Pr \left(\hat{\mathbf{z}}\, \tilde{\nabla}^2 \tilde{T} - \tilde{\nabla}\left(\frac{\partial \tilde{T}}{\partial \tilde{z}}\right)\right) + Pr\, \tilde{\nabla}^4 \tilde{\mathbf{u}}$$

where we've used $\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{\mathbf{u}}) = -\tilde{\nabla}^2 \tilde{\mathbf{u}}$ for an incompressible flow with $\tilde{\nabla} \cdot \tilde{\mathbf{u}} = 0$. This is now fourth order in spatial derivatives. We'll focus on the z-component, which is

$$\frac{\partial \tilde{\nabla}^2 \tilde{u}_z}{\partial \tilde{t}} = Ra\, Pr \left(\frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{T}}{\partial \tilde{y}^2}\right) + Pr\, \tilde{\nabla}^4 \tilde{u}_z \tag{5.31}$$

This is to be solved in conjunction with the temperature equation (5.28).

We'll look for solutions using separation of variables. Furthermore, we'll anticipate the instability by looking for solutions that grow exponentially in time as $e^{\Gamma\tilde{t}}$,

$$\tilde{u}_z(\mathbf{x},t) = V(\tilde{z})\,X(\tilde{x},\tilde{y})e^{\Gamma\tilde{t}} \quad \text{and} \quad \tilde{T}(\mathbf{x},t) = \theta(\tilde{z})\;X(\tilde{x},\tilde{y})e^{\Gamma\tilde{t}} \tag{5.32}$$

Note that we've assumed that both the velocity field and temperature have the same spatial dependence $X(x,y)$ in the x and y directions. With this ansatz, the temperature equation (5.28) becomes

$$\frac{1}{\theta}\left[\frac{d^2\theta}{d\tilde{z}^2} - \Gamma\theta + V\right] = -\frac{1}{X}\left[\frac{\partial^2 X}{\partial\tilde{x}^2} + \frac{\partial^2 X}{\partial\tilde{y}^2}\right] \tag{5.33}$$

Now, the left-hand side depends only on $\tilde{z}$ and the right-hand side depends only on $\tilde{x}$ and $\tilde{y}$, which means that both sides must actually be constant. For the solution to be bounded in the x and y directions, this constant should be positive. We'll call it K^2. We then have

$$\tilde{\nabla}_2^2 X \equiv \frac{\partial^2 X}{\partial\tilde{x}^2} + \frac{\partial^2 X}{\partial\tilde{y}^2} = -K^2 X \quad \Rightarrow \quad X(\tilde{x},\tilde{y}) \sim e^{iK_x\tilde{x}+iK_y\tilde{y}} \tag{5.34}$$

where $K^2 = K_x^2 + K_y^2$. We already start to see the ripples forming in the (x,y)-plane. Reverting back to dimensionful coordinates, we see that these ripples will have wavelength $\lambda = 2\pi d/K$ and so the constant K is simply the (magnitude of the) dimensionless wavenumber.

Our next task is to relate K to the instability constant Γ. This ultimately comes from equating the left-hand side of (5.33) to the same constant,

$$\left(\frac{d^2}{d\tilde{z}^2} - \Gamma - K^2\right)\theta = -V \tag{5.35}$$

This equation relates the temperature profile $\theta(\tilde{z})$ to the velocity profile $V(\tilde{z})$.

We can get another equation relating these variables by substituing the same ansatz (5.32) into the Navier-Stokes equation (5.31). We replace $\tilde{\nabla}_2^2 X$ with the expression (5.34) to get

$$\Gamma\left(\frac{d^2}{d\tilde{z}^2} - K^2\right)V = -Ra\,Pr\,K^2\theta + Pr\left(\frac{d^2}{dz^2} - K^2\right)^2 V \tag{5.36}$$

We can eliminate θ in this equation using (5.35). To do this, we have to act with $(d^2/d\tilde{z}^2 - \Gamma - K^2)$. We have

$$\left(\frac{d^2}{d\tilde{z}^2} - K^2\right)\left(\frac{d^2}{d\tilde{z}^2} - \Gamma - K^2\right)\left(\Gamma - Pr\left(\frac{d^2}{d\tilde{z}^2} - K^2\right)\right)V = Ra\,Pr\,K^2 V \tag{5.37}$$

Good? Good. After all of this, we're left with a sixth order differential equation. Sometimes, physics just isn't pretty.

We want to solve this subject to the boundary condition $V = 0$ and, from (5.30), $d^2V/d\tilde{z}^2 = 0$ at the two boundaries $\tilde{z} = 0, 1$. We also require $\theta = 0$ at $\tilde{z} = 0, 1$ and, from (5.36), we can see that this is only consistent if, in addition, we require $d^4V/d\tilde{z}^4 = 0$ at the boundaries. This means that we must solve (5.37) subject to the six boundary conditions

$$V = \frac{d^2V}{d\tilde{z}^2} = \frac{d^4V}{d\tilde{z}^4} = 0 \quad \text{at} \quad \tilde{z} = 0, 1 \tag{5.38}$$

This, as we shall now see, isn't so bad.

Before we do the thing that isn't so bad, it's worth pausing to reconsider these boundary conditions. Recall that the condition (5.30) was born more out of sloth than physical necessity. With a rigid boundary, it would have been more appropriate to impose the no-slip condition which leads to the requirement $dV/d\tilde{z} = 0$ at the boundaries rather than $d^2V/d\tilde{z}^2 = 0$. It turns out that we have to work harder to find such solutions.

Returning to the boundary conditions (5.38), the solutions are simply sine functions

$$V(\tilde{z}) = \sin(n\pi\tilde{z})$$

with $n \in \mathbb{Z}$. The somewhat daunting looking equation (5.37) then becomes the algebraic condition

$$\left(n^2\pi^2 + K^2\right)\left(n^2\pi^2 + \Gamma + K^2\right)\left(n^2\pi^2 + K^2 + \frac{\Gamma}{Pr}\right) = Ra\, K^2 \tag{5.39}$$

This is the analog of our previous dispersion relations, now telling us how the characteristic (inverse) time of the instability, Γ, relates to the wavelength. For some fixed $n \in \mathbb{Z}$, there is clearly a solution to this equation with $\Gamma > 0$ when the Rayleigh number is sufficiently large,

$$Ra > \frac{(n^2\pi^2 + K^2)^3}{K^2} \tag{5.40}$$

This confirms that intuition that we saw previously that the instability should only kick in when Ra is big enough.

The smallest value of Ra for which there is a solution occurs when $n = 1$. In this case, we minimise the function above

$$\frac{d}{dK^2}\left[\frac{(n^2\pi^2 + K^2)^3}{K^2}\right] = 0 \quad \Rightarrow \quad K^2 = \frac{\pi^2}{2}$$

From our previous discussion, this corresponds to the formation of a cell with size $\sim 2\sqrt{2}d$. Plugging this value back into (5.40), we have an instability provided that

$$Ra > \frac{27\pi^4}{4} \approx 660$$

For values of Ra just above this number, only the $n = 1$ mode is unstable. As Ra increases, more and more K modes become unstable as well as mode of higher n. As usual, the most unstable mode is that with largest Γ and this determines the preferred choice of K.

5.4 Instabilities of Inviscid Shear Flows

In this section, we turn our attention to instabilities of shear flows. That is, flows that takes the form

$$\mathbf{u}(\mathbf{x}, t) = U(z)\hat{\mathbf{x}}$$

together with some pressure field $P(\mathbf{x})$. We'll consider this velocity field over some finite width,

$$-h \leq z \leq +h$$

with suitable boundary conditions imposed on the edges $z = \pm h$. Both Couette flow and Poiseuille flow described in Section 3.2 are of this form. Our goal is to understand when instabilities of such flows may arise. We'll make progress, albeit limited. In particular, it turns out that we will not be able to demonstrate instability of either of these basic flows, even though they both unstable at suitably high Reynolds number. We will, however, be able to understand why this is a difficult problem!

We give the shear flow a nudge with perturbations of the form

$$\mathbf{u}(\mathbf{x}, t) = U(z)\hat{\mathbf{x}} + \tilde{\mathbf{u}}(z)e^{ik_x x + ik_y y - i\omega t}$$

together with some perturbation of the pressure

$$\delta P(\mathbf{x}) = \rho\tilde{P}(z)e^{ik_x x + ik_y y - i\omega t}$$

Note that we've included a factor of ρ on the right-hand side to ensure that later equations are a little simpler. It does mean, however, that $\tilde{P}$ does not have the dimensions of pressure.

As in our previous examples, we take the wavenumbers k_x and k_y to be some fixed real numbers and we choose them to be positive, $k_x, k_y > 0$. We then use the equations of motion to determine ω. In general, ω can be complex (as, indeed, can $\tilde{u}_z$ and $\tilde{P}_z$,) and an instability occurs when we find $\mathrm{Im}\,\omega > 0$. We will now derive some general statements about when the original flow is unstable.

We first met shear flows when we introduced viscosity in Section 3, with the shear induced by the no-slop boundary condition. Nonetheless, we will first study instabilities in the context of the Euler equation,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P$$

We'll then add viscosity to the discussion in Section 5.5. The linearised perturbation equations read

$$i(k_x U - \omega)\tilde{u}_x + \tilde{u}_z \frac{dU}{dz} = -ik_x\tilde{P} \tag{5.41}$$

$$i(k_x U - \omega)\tilde{u}_y = -ik_y\tilde{P} \tag{5.42}$$

$$i(k_x U - \omega)\tilde{u}_z = -\frac{d\tilde{P}}{dz} \tag{5.43}$$

These should be augmented with the incompressibility condition

$$ik_x\tilde{u}_x + ik_y\tilde{u}_y + \frac{d\tilde{u}_z}{dz} = 0 \tag{5.44}$$

We've allowed for wavelike perturbations in both the x and y-directions and a general perturbation in the z-direction. In fact, if we're looking for the onset of instabilities then we can ignore motion in the y-direction and focus just on the two-dimensional setting. This follows from:

Squire's Theorem: Perturbations in the y-direction do not induce further instabilities of the flow.

Proof: Define a velocity in the diagonal direction in the (x, y)-plane by

$$v = \frac{1}{K}\left(k_x\tilde{u}_x + k_y\tilde{u}_y\right) \quad \text{with} \quad K^2 = k_x^2 + k_y^2$$

We can combine (5.41) and (5.42) into the equation

$$i(KU - \omega')v + \tilde{u}_z\frac{dU}{dz} = -iK\tilde{P}' \tag{5.45}$$

where the primes do not denote derivatives but, instead, are a rescaled frequency and a rescaled pressure perturbation,

$$\omega' = \frac{K}{k_x}\omega \quad \text{and} \quad \tilde{P}' = \frac{K}{k_x}\tilde{P}$$

In terms of these rescaled variables, the perturbation equation (5.43) in the z-direction and the incompressibility condition (5.44) become

$$i(KU - \omega')\tilde{u}_z = -\frac{d\tilde{P}'}{dz} \quad \text{and} \quad iKv + \frac{d\tilde{u}_z}{dz} = 0 \tag{5.46}$$

Equations (5.45) and (5.46) describe an effective 2d system, with various quantities rescaled as above. In particular, the frequency is rescaled so that $|\omega'| > |\omega|$. But, as in previous examples, the question of whether the flow is unstable will boil down to whether the imaginary part of the frequency is non-zero. Clearly, we have $\text{Im}\,\omega' \neq 0$ whenever $\text{Im}\,\omega \neq 0$. The fact that $|\omega'| > |\omega|$ tells us that the growth of instability is faster in the v direction, but doesn't change when the instability kicks in. □

5.4.1 Rayleigh's Criterion

Because our interest will lie in identifying the conditions under which flows are unstable, Squire's theorem affords us the option of simplifying the situation by setting $\tilde{u}_y = k_y = 0$. We'll also take this opportunity to write $k_x = k$. This leaves us with the perturbation equations (5.41), (5.43) and (5.44). We can eliminate $\tilde{u}_x$ using the incompressibility condition (5.44), leaving us with two equations

$$-\left(U - \frac{\omega}{k}\right)\frac{d\tilde{u}_z}{dz} + \tilde{u}_z\frac{dU}{dz} = -ik\tilde{P}$$
$$i(kU - \omega)\tilde{u}_z = -\frac{d\tilde{P}}{dz}$$

Next, we can eliminate the pressure perturbation at the expense of differentiating the first of the equations above. In doing so, there are two terms of the form $(dU/dz)(d\tilde{u}_z/dz)$ that cancel and we are left with a simple, second order differential equation for $\tilde{u}_z$,

$$\left[\left(U - \frac{\omega}{k}\right)\left(\frac{d^2}{dz^2} - k^2\right) - \frac{d^2U}{dz^2}\right]\tilde{u}_z = 0 \tag{5.47}$$

This is the *Rayleigh equation*. Our goal is to solve this equation for a given background flow $U(z)$ and wavenumber k, subject to the boundary condition $\tilde{u}_z = 0$ at $z = \pm h$, which ensures that the perturbations don't flow into the edges of the channel. The solution contains both the profile of the boundary perturbation $\tilde{u}_z$ as well as the corresponding frequency ω.

We'd like to know when solutions to the Rayleigh equation (5.47) have $\mathrm{Im}(\omega) \neq 0$, signalling an instability. One possibility is to simply pick a choice of initial flow $U(z)$ and solve the eigenvalue equation (5.47). That is often hard. Here, instead, we derive some simple, general conditions that any flow must obey if it is to have a linear instability.

We write the Rayleigh equation as

$$\frac{d^2\tilde{u}_z}{dz^2} - k^2\tilde{u}_z - \frac{U''}{U-\omega/k}\tilde{u}_z = 0$$

We now multiply both sides by the complex conjugate of the velocity perturbation, $\tilde{u}_z^\star$ and integrate across the width of flow in the z-direction. After integrating by parts, and using the boundary condition $\tilde{u}_z = 0$ at $z = \pm h$, we have

$$\int_{-h}^{+h} dz \ \left(\left| \frac{d\tilde{u}_z}{dz} \right|^2 + k^2|\tilde{u}_z|^2 \right) = -\int_{-h}^{+h} dz \ \frac{U''}{U-\omega/k}|\tilde{u}_z|^2 \tag{5.48}$$

The left-hand side is real and positive. This means that the imaginary part of the right-hand side must vanish. The only complex quantity on the right-hand side is the frequency ω, so this tells us

$$\mathrm{Im}\left[\int_{-h}^{+h} dz \ \frac{U''}{U-\omega/k}|\tilde{u}_z|^2 \right] = \frac{\mathrm{Im}(\omega)}{k} \int_{-h}^{+h} dz \ \frac{U''}{|U-\omega/k|^2}|\tilde{u}_z|^2 = 0$$

How can this equation be satisfied? One obvious way is if $\mathrm{Im}(\omega) = 0$, in which case the flow is stable. But they're not the perturbations that we care about. Instead, we're interested in situation in which $\mathrm{Im}(\omega) \neq 0$ and the flow is unstable. This can only happen if

$$\int_{-h}^{+h} dz \ \frac{U''}{|U-\omega/k|^2}|\tilde{u}_z|^2 = 0 \tag{5.49}$$

But both $|\tilde{u}_z|^2$ and the denominator are manifestly positive. The only possible way that the equality can hold is if U'' changes sign at some value of $-h < z < h$, so that it is positive in some region of the flow and negative in others. This means that there must be an inflection point,

$$\frac{d^2U}{dz^2} = 0 \quad \text{for some } -h < z < h$$

This is the *Rayleigh criterion* for linear instability of an inviscid flow. It is not a sufficient criterion. But it is necessary.

5.4.2 Fjortoft's Criterion

There is more that we can say about the condition for instability, at least under certain circumstances. Our equation (5.48) tells us that

$$\text{Re}\left[\int_{-h}^{+h} dz\ \frac{U''}{U-\omega/k}|\tilde{u}_z|^2\right] < 0 \quad \Rightarrow \quad \int_{-h}^{+h} dz\ \frac{U''(U-\text{Re}(\omega)/k)}{|U-\omega/k|^2}|\tilde{u}_z|^2 < 0$$

If there is an instability, then (5.49) must hold and this means that the $\text{Re}(\omega)$ term in the above expression actually vanishes when integrated over the width of the channel. So we actually have the requirement

$$\int_{-h}^{+h} dz\ \frac{U''U}{|U-\omega/k|^2}|\tilde{u}_z|^2 < 0 \tag{5.50}$$

It's not immediately obvious that this buys us anything beyond the Rayleigh criterion. This is because we have little information about most of the variables in the equation (5.50). We assume that we're given some starting flow $U(z)$ and we're searching for a perturbation with some fixed wavenumber k. But both ω and $\tilde{u}_z$ are fixed by the Rayleigh equation (5.47) that we haven't yet solved. In particular, for a given k it may well be that $\tilde{u}_z$ is peaked at some value of z, but we have no idea where. So it's difficult to see what new information can be buried in the inequality (5.50)

Nonetheless, there's a clever trick that does allow us to extract more information from (5.50). The equation (5.48) that previously allowed us to drop the $\text{Re}(\omega)$ term from the integral also allows us to add any other constant to the numerator, so we can equally well write (5.50) as

$$\int_{-h}^{+h} dz\ \frac{U''(U-U_\star)}{|U-\omega/k|^2}|\tilde{u}_z|^2 < 0 \tag{5.51}$$

for any constant $U_\star$ of our choice. At this point, we need two further assumptions to make progress. We assume that:

- There is just a single Rayleigh inflection point at $z = z_\star$.
- The background flow $U(z)$ is monotonically increasing, so $U'(z) \geq 0$ for all z.

With these two assumptions, we take $U_\star = U(z_\star)$. Together with the assumption of monotonicity of U, this ensures that the factor $(U - U_\star)$ flips sign at $z = z_\star$. This, of course, is the same place that $U''(z)$ flips sign so, for this particular choice, the product $U''(U - U_\star)$ has the same sign for all values of z. The inequality (5.50) tells us that this sign must be negative:

$$U''(U-U_\star) \leq 0 \quad \forall\ z$$

with equality only when $z = z_\star$. This is the *Fjortoft criterion*.

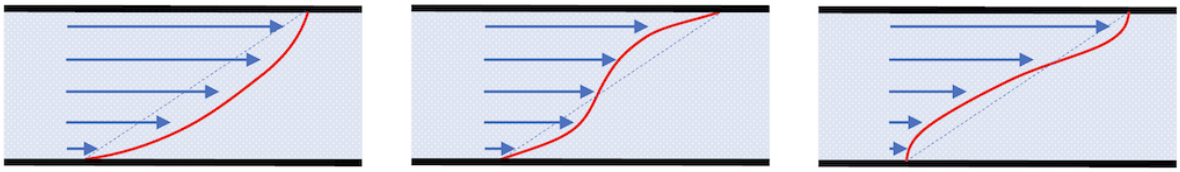


Figure 33. The first flow is stable by Rayleigh, the second by Fjortoft. Only the third may be unstable.

We can also phrase this criterion in terms of the vorticity of the background flow, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, whose magnitude is

$$|\boldsymbol{\omega}| = \Omega = \frac{dU}{dz}$$

The Rayleigh criterion tells us that there is a point in the flow where the vorticity is stationary, with $d\Omega/dz = 0$. The Fjortoft criterion tells us that, away from this inflection point,

$$\frac{d\Omega}{dz}(U - U_\star) < 0 \tag{5.52}$$

Suppose that, away from this point, $U'(z) > 0$ so that $\Omega > 0$. Then for $z < z_\star$ we have $U - U_\star < 0$ and (5.52) tells us that Ω is increasing. Similarly, for $z > z_\star$, (5.52) says that the vorticity Ω must be decreasing. In other words, the Fjortoft criterion is telling us that, for a flow to be unstable, the vorticity Ω must have a *maximum* at $z = z_\star$. The same conclusion is reached if $U'(z) < 0$: the magnitude of the vorticity $|\Omega|$ must have a maximum for the flow to be unstable.

Three examples of shear flows are shown in Figure 33. The first has no point where $U''(z) = 0$ and so is stable by the Rayleigh criterion. The second has an inflection point but the vorticity is a minimum there: it is stable by the Fjortoft criterion. The third has a point where the vorticity is maximum and obeys both criterion. Only this one may have a linear instability. (And even then, it is not guaranteed. Our conditions are sufficient, not necessary.)

You might have noticed that our original Poiseuille flow of Section 3.2 is stable according to the Rayleigh criterion, with $U''(z) < 0$ everywhere. That's something of a surprise because we showed photographs in Figure 30 of Poiseuille flow disintegrating as the Reynolds number is cranked up. But the Rayleigh and Fjortoft criteria apply only to inviscid flows and it turns out that to fully understand instabilities we need to

include the effects of viscosity. We'll turn to this in Section 5.5. (The Couette flow is something of a special case since it has $U''(z) = 0$ everywhere and we'll consider it separately in Section 5.4.4.)

5.4.3 Howard's Semi-circle Theorem

There's yet more information to be extracted from the Rayleigh equation. To do this, we make the change of variables that combines the perturbation $\tilde{u}_z$ with the background flow U,

$$V = \frac{\tilde{u}_z}{U - \omega/k}$$

We should be a little careful because it may be that the new variable V is ill-defined if, for some value of z, we have $U(z) = \omega/k$. Clearly, however, this will not be a problem if we have an unstable perturbation with ω complex because $U(z)$ is real.

The Rayleigh equation (5.47) is

$$\left[(U - \omega/k)\left(\frac{d^2}{dz^2} - k^2\right) - \frac{d^2U}{dz^2}\right](U - \omega/k)V = 0$$

After expanding out, this becomes

$$(U - \omega/k)^2\frac{d^2V}{dz^2} + 2(U - \omega/k)\frac{dU}{dz}\frac{dV}{dz} - k^2(U - \omega/k)^2V = 0$$

which we can then write as

$$\frac{d}{dz}\left[\left(U - \frac{\omega}{k}\right)^2\frac{dV}{dz}\right] - k^2\left(U - \frac{\omega}{k}\right)^2 V = 0$$

Now we play the same trick as before: we multiply by the complex conjugate $V^\star$ and integrate over the width of the flow. After integrating by parts, and using the boundary condition $\tilde{u}_z = 0$ at $z = \pm h$, we have

$$\int_{-h}^{+h} dz\ \left(U - \frac{\omega}{k}\right)^2 Q = 0 \quad \text{with} \quad Q = \left|\frac{dV}{dz}\right|^2 + k^2|V|^2 \tag{5.53}$$

Clearly the quantity Q is non-negative for all z. If the flow is unstable then $\mathrm{Im}(\omega) \neq 0$, and the imaginary part of the equation is

$$\frac{2\mathrm{Im}(\omega)}{k}\int_{-h}^{+h} dz\ \left(U - \frac{\mathrm{Re}(\omega)}{k}\right) Q = 0 \tag{5.54}$$

So the quantity in brackets must change sign. This is only possible if

$$kU_{\min} < \mathrm{Re}(\omega) \leq kU_{\max}$$

This can also be viewed as a bound on the phase velocity of the disturbance, $c = \mathrm{Re}(\omega)/k$. The phase velocity is bounded by the velocities of the background flow: $U_{\min} < c < U_{\max}$.

The real part of (5.53) reads

$$\int_{-h}^{+h} dz \, \Big((Uk - \mathrm{Re}(\omega))^2 - \mathrm{Im}(\omega)^2 \Big) Q = 0 \tag{5.55}$$

There's a way to massage this into something more useful by removing the explicit factors of U in the integrand. To remove the term quadratic in U, we make use of the trivial fact that $(U - U_{\min})(U_{\max} - U) \geq 0$, so we have

$$\int_{-h}^{+h} dz \, (U - U_{\min})(U_{\max} - U) k^2 Q \geq 0 \tag{5.56}$$

Adding (5.55) and (5.56), and using (5.57), we have

$$\int_{-h}^{+h} dz \, \Big(Uk \Big(-2\mathrm{Re}(\omega) + k(U_{\max} - U_{\min}) \Big) + \mathrm{Re}(\omega)^2 - \mathrm{Im}(\omega)^2 - U_{\max} U_{\min} k^2 \Big) Q \geq 0$$

But we can remove terms linear in U since, by (5.54), we have

$$\int_{-h}^{+h} dz \, Uk \, Q = \int_{-h}^{+h} dz \, \mathrm{Re}(\omega) Q \tag{5.57}$$

The upshot is that we get the inequality

$$\int_{-h}^{+h} dz \, \Big(\mathrm{Re}(\omega)^2 + \mathrm{Im}(\omega)^2 - \mathrm{Re}(\omega)(U_{\max} - U_{\min})k + U_{\max} U_{\min} k^2 \Big) Q \leq 0$$

Clearly the object in brackets must be negative, meaning that the real and imaginary parts of the frequency are bounded by

$$\left(\mathrm{Re}(\omega) - \frac{k}{2}(U_{\max} - U_{\min}) \right)^2 + \mathrm{Im}(\omega)^2 \leq \frac{k^2}{4} (U_{\max} + U_{\min})^2$$

This is *Howard's semi-circle theorem*. It puts a bound on the instability in the complex ω plane in the form of a semi-circle as shown in Figure 34. In particular, this condition gives a bound on the growth rate $\mathrm{Im}(\omega)$ of the instability.

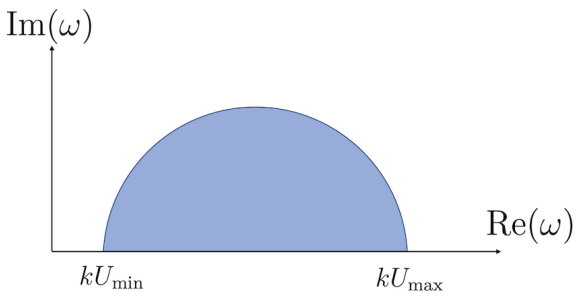


Figure 34. The range of allowed unstable behaviour in the complex ω plane.

5.4.4 Couette Flow Revisited

Couette flow is special because, as we saw in Section 3.2, the velocity field is linear: $U(z) \sim z$. This means that $U''(z) = 0$ for all z and the Rayleigh criterion (5.49) appears to be toothless. Which leads us to ask: is the flow actually unstable?

In this case, the background flow is simple enough that we can just go ahead and solve the equations. The Rayleigh equation (5.47) reads

$$\left(\alpha z - \frac{\omega}{k}\right)\left(\frac{d^2}{dz^2} - k^2\right)\tilde{u}_z = 0 \tag{5.58}$$

where we've taken the background flow to be $U(z) = \alpha z$ for constant α. If we look for smooth solutions then the background perturbation must be given by

$$\left(\frac{d^2}{dz^2} - k^2\right)\tilde{u}_z = 0 \quad \Rightarrow \tilde{u}_z = A\sinh k(z - z_0)$$

But we still need to impose the boundary condition $\tilde{u}_z = 0$ at $z = \pm h$. And that can only hold if $A = 0$.

This is a little confusing! It seems to be telling us that there can be no perturbations of Couette flow, stable or otherwise. But that doesn't sound right. What if you splash it?! Surely something must happen. What?

In fact, to find perturbations we must relax the condition that $\tilde{u}_z$ is smooth. Suppose that we admit the solution

$$\tilde{u}_z = \begin{cases} A_+ \sinh k(z-h) & z > z_c \\ A_- \sinh k(z+h) & z < z_c \end{cases}$$

This immediately satisfies the boundary condition requirement $\tilde{u}_z = 0$. We still need the velocity perturbation to be continuous, even if it's not differentiable, and this imposes the condition

$$A_+ \sinh k(z_c - h) = A_- \sinh k(z_c + h)$$

This should be thought of as fixing, say A_+, in terms of A_- given the position z_c of the jump.

How worried should we be about the discontinuity? The derivative in the Rayleigh equation (5.58) will certainly hit it, giving infinity. But we can rescue the situation if the pre-factor vanishes. This holds if the frequency ω is given by

$$\omega = \alpha z_c k$$

This is rather nice. We find that we have a continuous spectrum of stable, wavelike modes, with the discontinuity propagating at the same phase velocity as the background flow at that point, $\omega/k = \alpha z_c = U(z_c)$.

The modes that we have found are marginal, in the sense that they neither grow nor decay with time. But we've only looked at first order in perturbation theory and it may well be that higher order corrections change the story. Although we will not show it here, it turns out that these marginal modes are not rendered unstable by higher order terms. Nor are they unstable when we include viscosity. Couette flow is stable to arbitrarily small perturbations.

However, that doesn't mean that Couette flow actually stable! Experiment and numerics show that the Couette flow is actually unstable, and develops turbulence for $Re \gtrsim 400$. The instability arises due to perturbations of some finite amplitude.

5.5 Instabilities of Viscous Shear Flows

In this section, we'll repeat the analysis of the stability of shear flows, this time with viscosity added.

The set-up is the same as in Section 5.4, with a flow over a finite width $|z| \leq h$, with suitable boundary conditions (such as no-slip) imposed on the edges. The initial flow $\mathbf{U}$, together with the perturbations $\tilde{\mathbf{u}}$, take the form

$$\mathbf{u}(\mathbf{x}, t) = U(z)\hat{\mathbf{x}} + \tilde{\mathbf{u}}(z)e^{ik_x x + ik_y y - i\omega t}$$

In addition, there is a perturbation of the pressure field $P(\mathbf{x})$.

$$\delta P(\mathbf{x}) = \rho \tilde{P}(z)e^{ik_x x + ik_y y - i\omega t}$$

where we've again included a factor of ρ on the right-hand side to simplify subsequent equations.

This time we will perturb the full Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{u}$$

The linearised perturbation equations are

$$i(k_x U - \omega)\tilde{u}_x + \tilde{u}_z\frac{dU}{dz} = -ik_x\tilde{P} + \nu\left(\frac{d^2}{dz^2} - k_x^2 - k_y^2\right)\tilde{u}_x \tag{5.59}$$

$$i(k_x U - \omega)\tilde{u}_y = -ik_y\tilde{P} + \nu\left(\frac{d^2}{dz^2} - k_x^2 - k_y^2\right)\tilde{u}_y \tag{5.60}$$

$$i(k_x U - \omega)\tilde{u}_z = -\frac{d\tilde{P}}{dz} + \nu\left(\frac{d^2}{dz^2} - k_x^2 - k_y^2\right)\tilde{u}_z \tag{5.61}$$

and the incompressibility condition is, as before,

$$ik_x\tilde{u}_x + ik_y\tilde{u}_y + \frac{d\tilde{u}_z}{dz} = 0 \tag{5.62}$$

We would like to derive a generalisation of the Rayleigh equation (5.47), now including the effects of viscosity. This is an equation just for $\tilde{u}_z$.

To achieve this, we first take the divergence of the Navier-Stokes equation which, after using the incompressibility condition $\nabla\cdot\mathbf{u} = 0$, gives

$$\frac{1}{\rho}\nabla^2 P = -2\frac{dU}{dz}\frac{\partial u_z}{\partial x} \quad \Rightarrow \quad \left(\frac{d^2}{dz^2} - k_x^2 - k_y^2\right)\tilde{P} = -2ik_x\frac{dU}{dz}\tilde{u}_z$$

If we now take ∇^2 of the third equation (5.61) (which really means multiplying by $(d^2/dz^2 - k_x^2 - k_y^2)$, then we get a fourth order differential equation for $\tilde{u}_z$,

$$\left[i(k_x U - \omega)\left(\frac{d^2}{dz^2} - k_x^2 - k_y^2\right) - ik_x\frac{d^2U}{dz^2} - \nu\left(\frac{d^2}{dz^2} - k_x^2 - k_y^2\right)^2\right]\tilde{u}_z = 0 \tag{5.63}$$

This is the *Orr-Sommerfeld equation*. If we set $k_y = 0$ then, in the limit $\nu \to 0$ (which, as always, should really be thought of as the limit $Re \to \infty$) it reduces to the Rayleigh equation (5.47).

We want to impose the no-penetration boundary condition $\tilde{u}_z = 0$ and also the no-slip boundary condition $\tilde{u}_x = \tilde{u}_y = 0$ at the edges of the flow. From (5.62), no-slip implies that $\tilde{u}'_z = 0$. But that still leaves the other linear combination of $\tilde{u}_x$ and $\tilde{u}_y$ which we recognise as the vorticity normal to the edge,

$$\zeta := \boldsymbol{\omega} \cdot \hat{\mathbf{z}} = ik_x\tilde{u}_y - ik_y\tilde{u}_x$$

We'll impose $\zeta = 0$ at the boundary. But we can also look at the dynamics of this component of vorticity in the interior of the flow. By taking suitable linear combinations of (5.59) and (5.60) (k_y of the first minus k_x of the second) we get

$$\left[i(k_xU - \omega) - \nu\left(\frac{d^2}{dz^2} - k_x^2 - k_y^2\right)\right]\zeta = ik_y\frac{dU}{dz}\tilde{u}_z \tag{5.64}$$

This is the *Squire equation*. We should solve it in conjunction with the Orr-Sommerfeld equation (5.63), with the velocity perturbations $\tilde{u}_z$ acting as a source for the vorticity perturbations ζ.

In the limit $\nu = 0$ we have no possibility of interesting vorticity fluctuations since ζ is fully determined by the velocity perturbation $\tilde{u}_z$. However, the presence of viscosity turns the algebraic relation into a differential equation and this has more interesting solutions. In particular, we can have vorticity perturbations $\zeta \neq 0$ even when $\tilde{u}_z = 0$. These are known as *Squire modes*.

Claim: Squire modes are always damped. This means that the vorticity doesn't lead to an instability unless driven by the velocity perturbation $\tilde{u}_z$.

Proof: The proof uses the same kind of trick that we saw when deriving the Rayleigh criterion. We multiply the homogeneous Squire equation by the complex conjugate $\zeta^\star$ and integrate over the width of the flow. After integrating by parts and imposing the boundary condition $\zeta = 0$ at $z = \pm h$, we have

$$\nu\int_{-h}^{+h} dz\ \left(\left|\frac{d\zeta}{dz}\right|^2 + (k_x^2 + k_y^2)|\zeta|^2\right) = -\int_{-h}^{+h} dz\ i(k_xU - \omega)|\zeta|^2$$

Taking the real part gives

$$\nu\int_{-h}^{+h} dz\ \left(\left|\frac{d\zeta}{dz}\right|^2 + (k_x^2 + k_y^2)|\zeta|^2\right) = -\mathrm{Im}(\omega)\int_{-h}^{+h} dz\ |\zeta|^2$$

Clearly the left-hand side is positive if $\zeta \neq 0$. So the right-hand side must also be positive which means that $\mathrm{Im}(\omega) < 0$. But the fluctuations evolve as $e^{-i\omega t} \sim e^{\mathrm{Im}(\omega)t}$ so for $\mathrm{Im}(\omega) < 0$ these are decaying modes. □

5.5.1 Poiseuille Flow Revisited

To finish this section, we return to Poiseuille flow, restricted to a plane. Recall from Section 3.2 that Poiseuille flow is driven by a constant pressure gradient, with the background velocity field taking the parabolic form

$$U(z) = \alpha(h^2 - z^2)$$

for some constant α. We define the Reynolds number for this flow to be

$$Re = \frac{U_{\max} h}{\nu} = \frac{\alpha h^3}{\nu}$$

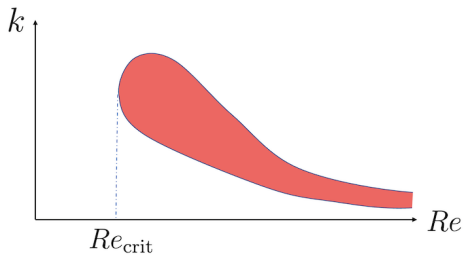
To determine the stability of Poiseuille flow, our task is clear. We look for perturbations with $k_y = 0$ and $k_x = k$ and try to solve the Orr-Sommerfeld equation

$$\left[i(kU - \omega)\left(\frac{d^2}{dz^2} - k^2\right) - ik\frac{d^2U}{dz^2} - \nu\left(\frac{d^2}{dz^2} - k^2\right)^2 \right] \tilde{u}_z = 0$$

We view this as an eigenvalue equation, with the both the eigenfunction $\tilde{u}_z$ and the eigenvalue ω to be determined. The flow is unstable if there exists solutions with $\mathrm{Im}(\omega) > 0$.

This is not a computation that can be done analytically and the equation must be solved numerically[14]. One finds that there is no instability for low Reynolds number, but Pouiseuille flow becomes unstable for any Reynolds number beyond the critical value

$$Re > Re_{\text{crit}} \approx 5772$$



The range of unstable k-values for a given Reynolds number takes the form of the red region shown in the figure.

There's something rather surprising about this result. By now we're used to viscosity acting as a dampening force, causing perturbations to die away. But here it plays the opposite role! Without viscosity, Poiseuille flow is linearly stable. Viscosity causes it to become unstable at high Reynolds number.

[14] This was done by Orszag in 1971 in this paper where the critical Reynolds number was first calculated.

This is far from the last word on instabilities. The linear analysis sketched above suggests that Couette flow should be stable for all Re while Poiseuille flow should break down only for $Re \approx 6000$. Neither of these agrees well with experiment. Instead, Couette flow is turbulent by $Re \approx 400$ while Poiseuille flow is typically turbulent by $Re \approx 2000$. This shows up the failure of our simple linear analysis. While the flows are stable against infinitesimally small perturbations, fluctuations whose amplitude grows beyond some critical value can push them over the edge. The instability is akin to a first order phase transition in thermodynamics, or to a tunnelling event in quantum mechanics, with some finite barrier to cross.

6 Turbulence

When the speed of fluid flows increases beyond some critical value, things have a tendency to go a bit squirly. The calm, serene laminar flows that we've seen in earlier chapters become unstable and are replaced by something messy and dirty, with the fluid moving in seemingly random directions, eddies forming and stretching, before disintegrating into smaller eddies. This is turbulent flow.

There is every reason to believe that turbulent flow is correctly described by the Navier-Stokes equation, not least computer simulations which, in this context, go by the name of 'DNS, standing for "direct numerical simulation". But understanding the full details of turbulent flow remains, to put it mildly, a formidable problem. Turbulence kicks in when the Reynolds number is greater than some critical value $Re > Re_{\rm crit}$. The exact number depends on the kind of flow we're looking at, but a ballpark figure is

$$Re_{\rm crit} \sim 10^3$$

At these speeds, the advective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ in the Navier-Stokes equation is important. This is the only non-linear term in the equation and it drives the system to a chaotic state, with the motion wildly dependent on the initial conditions. The challenge is to understand this motion.

This challenge is, it turns out, hard. Despite many decades of study, turbulence remains poorly understood. It is clear that it is not feasible to find explicit solutions exhibiting turbulence. Instead, we will retreat and look at averaged properties of flows. This might seem like a strange thing to do. After all, the Navier-Stokes equation is, at the end of a day, just a differential equation and, as such, its behaviour is entirely deterministic. Nonetheless, turbulent motion appears to be random. This can be traced to the sensitive dependence on initial conditions that characterises chaotic systems. To proceed, we will embrace this randomness and work in a statistical sense. Rather than trying to analyse any specific solution, we will instead try to extract properties of appropriately averaged solutions.

Our goals in this section will be limited. We won't look at any specific turbulent flows, such as boundary layers or wakes. Instead, we will just try to understand some very general properties that are shared by all turbulent flows, at least in some regime. Nor will we study the interesting behaviour that happens for flows around the critical Reynolds number $Re_{\rm crit}$, where instabilities develop. Instead we focus on what happens with $Re \gg Re_{\rm crit}$, a regime known as *fully developed turbulence*.

6.1 Mean Flow

As we mentioned above, to understand turbulence it's necessary to think on a more probabilistic level about the Navier-Stokes equation. But given that the Navier-Stokes equation is purely deterministic, it's not obvious what this means. If we're going to think about averaged properties, the first question we should ask is: what are we actually averaging over?

There are different answers that we could give to this. One way to proceed is to average over different initial conditions to the Navier-Stokes equation. We could pick some collection of initial conditions, all of which look similar. Because of the chaotic nature of the equation, each will give rise to very different solutions. We could then try to figure out average properties of these solutions. This is known as the *ensemble average* and is similar to the philosophy underlying Kinetic Theory and Statistical Mechanics.

Alternatively, we could do something that feels more physical. A turbulent velocity field $\mathbf{u}(\mathbf{x}, t)$ varies rapidly in both space and time and we could choose to average over either of these. There is a general expectation (although no proof) that, for a typical flow, it doesn't matter which average we choose: all should give the same answer. This goes by the name of the *ergodic hypothesis*.

Here we will average over time (because it turns out to be the simplest). We decompose the complicated turbulent flow $\mathbf{u}(\mathbf{x}, t)$ into an averaged, mean flow $\mathbf{U}(\mathbf{x}, t)$ together with some fluctuations $\delta\mathbf{u}(\mathbf{x}, t)$,

$$\mathbf{u} = \mathbf{U} + \delta\mathbf{u} \tag{6.1}$$

where to define the mean flow we average over some time scale T,

$$\mathbf{U}(\mathbf{x}, t) = \langle \mathbf{u}(\mathbf{x}, t) \rangle := \frac{1}{T} \int_t^{t+T} dt' \; \mathbf{u}(\mathbf{x}, t')$$

This is called *Reynolds averaging*. There are two options for how to think about the time scale T,

- We could simply take $T \to \infty$. In this case, we have a steady mean flow $\mathbf{U}(\mathbf{x})$.
- Alternatively, we may have a situation in which there are two different time scales in the flow. The turbulent fluctuations occur over some short time scale τ_{short}, which is superimposed on some averaged flow which takes place over some much longer time scale τ_{long}. In this case we could take $\tau_{\text{short}} \ll T \ll \tau_{\text{long}}$ to get a mean velocity field $\mathbf{U}(\mathbf{x}, t)$ which varies only over the long time scale.

In what follows, we'll adopt the second of these. This isn't for any particularly well-motivated physical reason, but simply because it's not much more effort to do this and it obviously includes the $T \to \infty$ situation as a special case in which $\mathbf{U}(\mathbf{x})$ is stationary.

Since our mean flow is $\mathbf{U} = \langle \mathbf{u} \rangle$, the complicated velocity fluctuations are $\delta\mathbf{u} = \mathbf{u} - \langle \mathbf{u} \rangle$. By construction, this means that the average of the fluctuations vanishes:

$$\langle \delta\mathbf{u} \rangle = 0 \tag{6.2}$$

Importantly the Reynolds averaging commutes with spatial differentiation so if our fluid is incompressible then both the mean flow and the fluctuations must be separately incompressible,

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \langle \nabla \cdot \mathbf{u} \rangle = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{U} = 0 \quad \Rightarrow \quad \nabla \cdot \delta\mathbf{u} = 0$$

We do a similar averaging for other fields, including the pressure which we write as

$$P = \langle P \rangle + \delta P \tag{6.3}$$

with $\langle P \rangle$ defined by a time-average in the same way as (6.1). Again, we have designed things so that the average fluctuation necessarily vanishes: $\langle \delta P \rangle = 0$.

We will actually explore the averaged properties of the Navier-Stokes equation twice in these lectures. Our focus in this section will on deriving an equation for the mean flow $\mathbf{U}$ after integrating out the fluctuations. This won't take us particularly far, not least because it feels like we're throwing out the baby and keeping the bath water since, for many situations, the fluctuations are much more interesting than the mean flow! Nonetheless, we include this approach because it gives some intuition for the difficulties involved. Moreover, this is a popular approach when modelling turbulence in situations where there is clearly some overarching mean flow, with turbulence bubbling away underneath (it turns out that this takes often place in a regime of Reynolds numbers $10^3 \lesssim Re \lesssim 10^5$) and will allow us to define some commonly used concepts such as "Reynolds stress" and "eddy viscosity". Then, in Section 6.3, we will retrace the same steps, this time focussing on the fluctuations themselves. It's only in this second approach that we'll start to make some real progress.

6.1.1 The Reynolds Averaged Navier-Stokes Equation

If we substitute the decomposition (6.1) and (6.3) into the Navier-Stokes equation, we have

$$\rho\left(\frac{\partial(\mathbf{U}+\delta\mathbf{u})}{\partial t} + (\mathbf{U}+\delta\mathbf{u})\cdot\nabla(\mathbf{U}+\delta\mathbf{u})\right) = -\nabla(\langle P\rangle + \delta P) + \mu\nabla^2(\mathbf{U}+\delta\mathbf{u}) \tag{6.4}$$

We've neglected any further forces acting on the fluid, such as the forcing term needed to drive turbulence, but they can be added as needed. Now we average this equation and use the fact that $\langle \delta\mathbf{u}\rangle = \langle \tilde{P}\rangle = 0$. We need to be a little careful with the time derivative term: we have

$$\left\langle \frac{\partial(\delta\mathbf{u})}{\partial t} \right\rangle = \frac{1}{T}\int_t^{t+T} dt' \; \frac{\partial(\delta\mathbf{u})}{\partial t'} = \frac{1}{T}\left[\delta\mathbf{u}(\mathbf{x}, t+T) - \delta\mathbf{u}(\mathbf{x}, t)\right]$$
$$= \frac{1}{T}\frac{\partial}{\partial t}\int_t^{t+T} dt' \; \delta\mathbf{u}(\mathbf{x}, t') = \frac{\partial\langle\delta\mathbf{u}\rangle}{\partial t} = 0$$

where, again, we've used the fact that $\langle\delta\mathbf{u}\rangle = 0$. We're then left with

$$\rho\left(\frac{\partial\mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U} + \langle\delta\mathbf{u}\cdot\nabla\delta\mathbf{u}\rangle\right) = -\nabla\langle P\rangle + \mu\nabla^2\mathbf{U}$$

This is *almost* the Navier-Stokes equation for the averaged velocity $\mathbf{U}$. The only difference is the term $\delta\mathbf{u}\cdot\nabla\delta\mathbf{u}$, quadratic in the fluctuations, that wasn't killed by averaging. We take this over to the right-hand side and treat it as part of the stress tensor, writing the Navier-Stokes equation for the averaged flow in the form (3.7)

$$\rho\left(\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x^j}\right) = \frac{\partial\sigma_{ij}}{\partial x^j} \tag{6.5}$$

This is the *Reynolds' averaged Navier-Stokes* equation. The stress tensor on the right-hand side is

$$\sigma_{ij} = -\langle P\rangle\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i}\right) - \rho\langle\delta u_i\delta u_j\rangle \tag{6.6}$$

where we've used the fact that $\nabla\cdot\delta\mathbf{u} = 0$ in writing it in this form. We see that, in this approximation, the role of the fluctuations is to guide the mean flow through the additional term

$$R_{ij} = \rho\langle\delta u_i\delta u_j\rangle$$

in the stress tensor. This is known as the *Reynolds stress* or, sometimes, the *turbulent stress*. (Actually, more often the Reynolds stress is defined without the factor of ρ, even though that isn't, strictly, a stress.) So if we want to understand how the mean flow flows, we need to understand something about the variance of the fluctuations $\langle\delta u_i\delta u_j\rangle$.

Finding Closure

Our next task is to get an expression for this extra contribution to the stress tensor R_{ij}. To this end, if we subtract the averaged Navier-Stokes equation (6.5) from our starting point (6.4), we have

$$\rho\left(\frac{\partial(\delta\mathbf{u})}{\partial t}+(\mathbf{U}\cdot\nabla)\delta\mathbf{u}+(\delta\mathbf{u}\cdot\nabla)\mathbf{U}+(\delta\mathbf{u}\cdot\nabla)\delta\mathbf{u}-\langle\delta\mathbf{u}\cdot\nabla\delta\mathbf{u}\rangle\right)=-\nabla\delta P+\mu\nabla^2\delta\mathbf{u}$$

If we multiply this by $\delta\mathbf{u}$, we get the following expression for the tensor $\delta u_i\delta u_j$

$$\begin{aligned}\rho\Bigg(\frac{\partial(\delta u_i\delta u_j)}{\partial t}+U_l\frac{\partial(\delta u_i\delta u_j)}{\partial x^l}+\delta u_i\delta u_l\frac{\partial U_j}{\partial x^l}+\delta u_j\delta u_l\frac{\partial U_i}{\partial x^l}+\frac{\partial}{\partial x^l}(\delta u_l\delta u_i\delta u_j)\\ +\delta u_i\frac{\partial R_{lj}}{\partial x^l}+\delta u_j\frac{\partial R_{li}}{\partial x^l}\Bigg)=-\delta u_i\frac{\partial(\delta P)}{\partial x^j}-\delta u_j\frac{\partial(\delta P)}{\partial x^i}+\mu^2\left(\delta u_i\nabla^2\delta u_j+\delta u_j\nabla^2\delta u_i\right)\end{aligned}$$

We now take the average to get an equation for the Reynolds' stress tensor that, schematically, takes the form

$$\frac{\partial R_{ij}}{\partial t}+(\mathbf{U}\cdot\nabla)R_{ij}=-\rho\frac{\partial}{\partial x^l}\langle\delta u_l\delta u_i\delta u_j\rangle+\text{other stuff}$$

where the other stuff includes other averages such as $\langle\delta P\,\delta\mathbf{u}\rangle$. The key point is that we can get ourselves an equation for R_{ij}, but it involves a 3-point average $\langle\delta\mathbf{u}^3\rangle$. And if we try to get an equation for $\langle\delta\mathbf{u}^3\rangle$ then you probably won't be surprised to hear that it involves $\langle\delta\mathbf{u}^4\rangle$, and so on. We find that we have an infinite hierarchy of equations. This is not unusual in physics when doing this kind of analysis. (An analogous situation arises in Kinetic Theory when deriving the Boltzmann equation where it is called the BBGKY hierarchy.) Within the context of turbulence, this is known as the *closure problem*: the set of equations don't close and keep forcing you to look at the next order in fluctuations.

What to do about it? Well, there is no mathematically well-defined way to truncate this infinite series of equations. Nor is there a physical reason to expect some simplification to occur. Turbulence is a strongly coupled problem and to do things properly, you really need to worry about this infinite series of equations. Of course, that's not particularly practical. So to proceed, the usual strategy is just to make something up. This made-up thing is unlikely to have any real justification behind it for the simple reason that no such justification exists. But these made-up approaches to have a name: they are collectively called "closure models". There are many.

Here's the simplest example of a made-up thing, due to Boussinesq. Suppose that, for some reason, the three-point averages $\langle \delta u^3 \rangle$ and higher are actually unimportant. Then we can look for an expression for the Reynolds' stress R_{ij} that depends only on the mean flow $\mathbf{U}$. One, particularly simple option is to postulate that it takes the form

$$R_{ij} = -\mu_T \left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i} \right) + \frac{2}{3} K \delta_{ij} \tag{6.7}$$

which depends on two, unknown constant μ_T and K. The latter has a nice physical interpretation: it is the kinetic energy in the fluctuations $K = \frac{1}{2}\rho\langle \delta\mathbf{u} \cdot \delta\mathbf{u} \rangle$. The former has a nice name: it is called the *turbulent viscosity*, or sometimes the *eddy viscosity*. This guess for the Reynolds' stress has the nice effect of simply renormalising the stress tensor σ_{ij} on the right-hand side of the averaged Navier-Stokes equation which, from (6.6), becomes

$$\sigma_{ij} = -\left(\langle P \rangle + \frac{2}{3} K \right) \delta_{ij} + (\mu + \mu_T) \left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x^i} \right)$$

This takes the same form as the usual stress tensor, but with an effective pressure $P_{\text{eff}} = \langle P \rangle + \frac{2}{3}K$ and an effective viscosity $\mu_{\text{eff}} = \mu + \mu_T$. The end result is that this guess has led us back to the original Navier-Stokes equation, but with an extra contribution to the pressure and a shifted value of the viscosity.

There are many other, more sophisticated closure models, in which one tries to incorporate $\langle \delta \mathbf{u}^3 \rangle$ corrections and so on, and then gives up at some higher order. They may be more sophisticated, but it's not obvious that they are more right and we won't discuss them here. Instead, we will reset and go in a different direction.

6.2 Some Dimensional Analysis

> "Big whorls have little whorls
> Which feed on their velocity,
> And little whorls have lesser whorls
> And so on to viscosity."
>
> *Lewis Fry Richardson*

Turbulence is one of the great problems in physics. To make progress it's clear that we're going to have to break out some pretty powerful machinery. And things don't get more powerful than dimensional analysis. In this section, we will use dimensional analysis to get a handle on one very specific property of turbulence: what happens to the energy?

The set of ideas described here is due to Richardson, Taylor and Kolmogorov. These ideas culminated in a series of papers by Kolmogorov in 1941 and fluid dynamicists often refer to this argument, rather elliptically, as K41. It is, I think, one of the greatest applications of dimensional analysis in all of physics.

To get started, we need a few facts about turbulent flows. First is the observation that turbulence is very much a dissipative phenomenon: if you leave a turbulent fluid alone, it will quickly relax back to equilibrium with the turbulent properties dying away. This means that something must be feeding the turbulence to keep it alive. In other words, there has to be some injection of energy into the system. This could be due to some external pressure difference, some shear effect due to gravity, or some teaspoon stirring the fluid. The details won't concern us and we'll model this energy injection by some external force density $\mathbf{f}(\mathbf{x}, \mathbf{t})$, as in our original Navier-Stokes equation (3.2). The work done by this force is

$$\text{Work Done} = \int d^3x \; \mathbf{f} \cdot \mathbf{u}$$

Although we're doing work on the system, the turbulent flow doesn't speed up over time. Or, said more precisely, we're not interested in situations where the fluid speeds up over time. Instead, this energy drains away through dissipation. We've already seen that dissipation occurs due to viscosity. The kinetic energy of the fluid is K.E. $= \frac{1}{2}\rho \int d^3x \, \mathbf{u}^2$ and, from (3.8), the energy lost is

$$\frac{\partial(\text{K.E.})}{\partial t} = -\rho\nu \int d^3x \; \left|\frac{\partial u_i}{\partial x^j}\right|^2$$

In a steady state, we simply equate the work done with the lost energy

$$\text{Work Done} = \rho\nu \int d^3x \; \left|\frac{\partial u_i}{\partial x^j}\right|^2 \tag{6.8}$$

The key to understanding the physics is to appreciate that the processes on either side of this equation take place at very different scales. On the left-hand side, the driving force is a macroscopic phenomenon, typically comparable to the size of the entire system. Meanwhile, on the right-hand side dissipation is a phenomenon that occurs at a much smaller level. This shows up in the equation above because dissipation is greatest when there are large gradients in velocity. So energy goes in at the largest scale, and out at the smallest. We would like to put some equations to these words to make them more concrete.

We start by quantifying the work done. Suppose that the driving force takes place over some large length scale L. This is sometimes called the *outer scale*. Over this scale, the mean velocity field will vary with some magnitude ΔU. The Reynolds number (3.17) for the flow is roughly

$$Re \sim \frac{\Delta U\, L}{\nu}$$

and, by assumption, we have $Re \gg 1$.

The turbulent flow is not laminar, but swirling in many directions. It's useful to think of this in terms of vorticity, with the different swirling referred to as eddies. In this somewhat cartoon picture, the flow on the large, outer scale consists of eddies of size L.

Now our first stab at some dimensional analysis. We will focus our attention on ϵ, the work done per unit mass, defined by

$$\epsilon = \frac{\text{Work Done}}{\rho V} \tag{6.9}$$

where V is the volume of the system. This has dimension $[\epsilon] = L^2T^{-3}$. Since this energy is injected on the large outer scale L, we expect that it manifests itself in terms of the macroscopic quantities ΔU and L that we introduced above. Dimensional analysis means that there is a unique possibility, namely

$$\epsilon \sim \frac{(\Delta U)^3}{L} \tag{6.10}$$

What about the energy dissipated? The relation (6.8) equates the work done with the energy lost, so

$$\frac{(\Delta U)^3}{L} \sim \frac{\nu}{V}\int d^3x\ |\nabla \mathbf{u}|^2 \tag{6.11}$$

Suppose that the dissipation also comes from these same, macroscopically large velocity gradients. Then we would have $|\nabla \mathbf{u}| \sim \Delta U/L$, which would give a dissipation rate

$$|\nabla \mathbf{u}| \sim \frac{\Delta U}{L} \quad \Rightarrow \quad \frac{\nu}{V}\int d^3x\ |\nabla \mathbf{u}|^2 \sim \frac{\nu(\Delta U)^2}{L^2} \sim \frac{1}{Re}\frac{(\Delta U)^3}{L} \tag{6.12}$$

But that's nowhere near enough! It's less dissipation than we need by a factor of Re and, as we have stressed, turbulent flow takes place at values of the Reynolds number $Re \gg 1$. It must be the case that dissipation takes place with a larger $|\nabla \mathbf{u}|$ than that caused by the driving force. Which means that there must larger gradients of $\mathbf{u}$ and so physics taking place on some smaller scale. That, of course, agrees with experimental observations of turbulence, where there are features on many different scales. We would like to construct a simple model of this.

6.2.1 Scale Invariance

Energy is injected at some length scale L where it causes eddies of size L. But, as we have seen, these eddies don't dissipate enough energy and structures on smaller scales must form. It's useful to think of these new length scales emerging as the original, large eddies break up into smaller ones.

There's nothing clean and simple going on here where, for example, the initial eddy neatly splits in two. Instead, as we have stressed, turbulence is a messy and complicated phenomenon and it consists of eddies of all possible sizes, at least within a range, bounded above by the outer scale L and, as we will see shortly, bounded below by a much smaller length scale l_0.

As the larger eddies break up, they lose energy which is fed into the smaller eddies below them. The eddies of size l have some velocity difference Δu_l. These eddies are being fed some energy by the bigger boys above them but, at the same time, they're losing energy as they themselves decay into the smaller eddies below. The key assumption here is that, at least in some regime of scales, this process takes place with no dissipation at all. Indeed, we've seen that the dissipation due to the very largest eddies (6.12) is suppressed by $1/Re$, and we make the approximation that this can be ignored completely. This is essentially the statement that viscosity is irrelevant for this aspect of turbulence. This assumption means that the eddies at size l receive some energy (per time per unit mass) ϵ and promptly pass it down to smaller scales. This process is known as the *energy cascade* and was first proposed by Richardson.

The kinetic energy of eddies at scale l and below is $\sim (\Delta u_l)^2$. (Note: rather unusually, we're talking about the energy in vortices of size $\lesssim l$ rather than the more usual formulation of vortices between size, say l and $l + dl$.) Suppose that these eddies hold on to the energy for some "cascade time" τ_l which, as the notation suggests, also depends on l. The energy passing through is equal to (6.10) and, at a given scale l, is

$$\frac{(\Delta u_l)^2}{\tau_l} \sim \frac{(\Delta U)^3}{L} \sim \epsilon \tag{6.13}$$

The next question is: what is the cascade time τ_l? On dimensional grounds, there's only one possibility: this it must be

$$\tau_l \sim \frac{l}{\Delta u_l} \tag{6.14}$$

Note that it's important that we don't allow the viscosity ν to sneak into this formula since that carries dimensions $[\nu] = L^2T^{-1}$ and messes up the dimensional analysis. This

is a reiteration of a point that we made above: the viscosity is irrelevant for the energy cascade since no dissipation is taking place. If we now substitute (6.14) into (6.13), we find that there is a scale invariance in the energy cascade, with the velocities of eddies of size l obeying the same formula independent of l,

$$\frac{(\Delta u_l)^3}{l} \sim \frac{(\Delta U)^3}{L} \sim \epsilon$$

Rearranging this, we find that the velocities of eddies of size l scale as

$$\Delta u_l \sim (\epsilon l)^{1/3} \tag{6.15}$$

This is known as the *Kolmogorov-Obhukov* law. (Later, in 6.3, we will give a more precise formulation of this law and see that it is better viewed as originally written as $(\Delta u)^3 \sim \epsilon l$.)

Viscosity Brings the Cascade to a Halt

The energy cascade does not involve dissipation, merely a transfer of energy from large scales to small. But at some point this energy cascade should come to a halt and the energy ϵ should be dissipated into heat. To understand when this happens, we return to the statement (6.11) which says that the energy in is equal to the energy out. We saw that this certainly wasn't satisfied by the dissipation from large eddies. But now we can ask: for what scale l_0 does this energy balance hold?

The eddies of size l_0 have velocity differentials Δu_0 and (6.11) holds if

$$\frac{\nu(\Delta u_0)^2}{l_0^2} \sim \epsilon$$

But we also know from (6.15) how Δu_0 and l_0 are related. This gives us the *Kolmogorov scale*, also known as the *inner scale*,

$$l_0 \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \sim \left(\frac{\nu^3 L}{(\Delta U)^3}\right)^{1/4} \sim \frac{L}{(Re)^{3/4}}$$

Clearly $l_0 \ll L$ since $Re \gg 1$. This is where energy is finally dissipated to heat. Note that the Kolmogorov scale can also be determined by dimensional analysis: it is the unique length scale that can be formed from the energy dissipation rate ϵ and the viscosity ν.

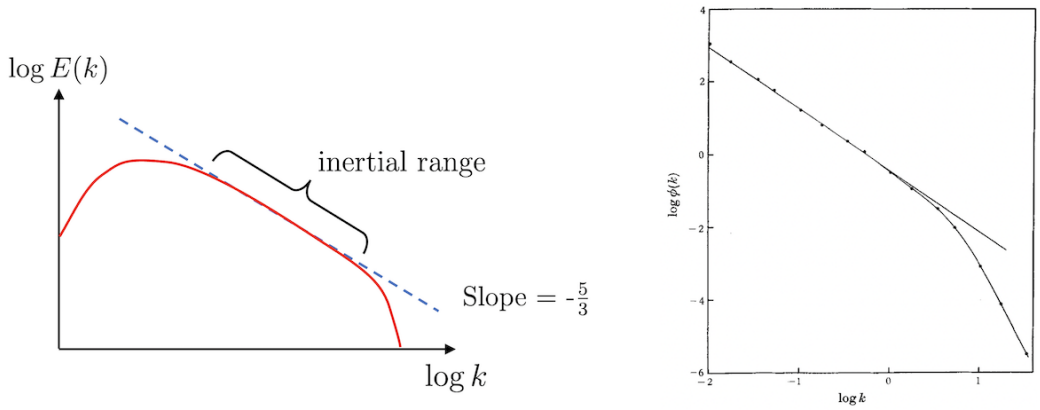


Figure 35. On the left, a sketch of the expected behaviour of $E(k)$ based on dimensional analysis. The energy is injected at small k and dissipated at large k, with the characteristic $E(k) \sim k^{-5/3}$ in the inertial range. On the right, data.

This finishes our crude, dimensional analysis approach to turbulence. The energy is injected at some scale L and dissipated at the much smaller scale $l_0 = L/(Re)^{3/4}$. The scales in-between, $l_0 \ll l \ll L$ are called the *inertial range* and exhibit a scale invariant energy cascade. Note that we're not really used the Navier-Stokes equation at any point in the analysis. Everything follows from the hypothesis that, in the inertial range, big eddies cascade down into smaller eddies in a way that does not involve any dissipation.

Wavenumbers

We can also phrase the energy cascade in terms of wavenumbers $k \sim 1/l$. Let $E(k)\,dk$ be the kinetic energy per unit mass storied in eddies with wavenumber between k and $k + dk$. Then $E(k)$ has dimension $[E] = L^3T^{-2}$. On dimensional grounds, we must have

$$E(k) \sim \epsilon^{2/3} k^{-5/3} \tag{6.16}$$

The expected behaviour is sketched on the left of figure 35. This $k^{-5/3}$ behaviour matches well with experiment. The first test was done with a probe attached to a ship which sailed back and forth in a tidal channel just off Vancouver Island. The data[15] is shown in the right-hand side of Figure 35, with the straight line having slope $-5/3$.

[15]This is taken from Grant, Stewart and Moilliet, "Turbulence Spectra from a Tidal Channel". The function $\phi(k)$ shown on the vertical axis is closely related to $E(k)$ described above: it is $2E = k^2\partial^2\phi/\partial k^2 - k\partial\phi/\partial k$.

We can reconcile the result $E(k) \sim k^{-5/3}$ with our previous analysis. If we integrate over all wavenumbers larger than k, we have

$$\int_k^\infty dk' \; E(k') \sim \epsilon^{2/3} k^{-2/3} \sim (\epsilon l)^{2/3} \sim (\Delta u_l)^2$$

This is the kinetic energy $(\Delta u_l)^2$ which, from (6.15), we see should indeed scale as $(\Delta u_l)^2 \sim (\epsilon l)^{2/3}$, in agreement with (6.15).

Briefly, Intermittency

There is a slight problem with the dimensional analysis that we described above. It's not really correct. The subtlety comes because we assume that the energy cascade retains no memory of the outer scale L on which we initially inject energy. If this scale could sneak into the energy cascade, then it would infect our dimensional analysis and the result (6.16) could be corrected to

$$E(k) \sim \epsilon^{2/3} k^{-5/3} (kL)^\zeta$$

for some ζ known as the *intermittency exponent*, the name arising because the experimental manifestation is that turbulent flows have periods in which the velocity fluctuations are weak, interspersed with intermittent bursts in which the fluctuations are much larger. In many situations, this exponent seems to be small. But it is not known how to calculate it.

There are close similarities between this story and what's seen in so-called critical points in phase transitions. There too one sees scale invariance and a naive dimensional analysis argument (known in that context as "mean field theory") suggests a particular value for certain exponents. But that's not the value that is seen experimentally. The flaw in that context is that the short distance UV cut-off (i.e. the atomic scale) unexpectedly sneaks in to the dimensional analysis and contributes what's known as an "anomalous critical exponent". It is entirely analogous to the intermittency exponent in turbulence, except that now this involve the long distance IR cut-off. You can read more about critical points and how to compute anomalous exponents in the lectures on Statistical Field Theory.

6.3 Velocity Correlations

In this section, we put a little more meat on the dimensional analysis argument above. In particular, we will derive a more rigorous version of the Kolmogorov-Obhukov law (6.15).

We will do this by returning to the averaging procedure that we introduced in Section 6.1, but now with a focus on the fluctuations $\delta\mathbf{u}$ rather than the mean flow $\mathbf{U}$. To do this, it's simplest if we assume that there are only fluctuations with a vanishing mean flow $\mathbf{U} = 0$. This means, of course, that $\delta\mathbf{u} = \mathbf{u}$ and we can drop the δ's. With no background flow governing the fluctuations, all points in space and time are, statistically at least, the same. This is known as homogeneous and isotropic turbulence and it offers the simplest setting where we may hope to understand a little of what's going on.

(An aside: if the restriction to vanishing $\mathbf{U}$ seems to restrictive then there is another way to think about things. We could, alternatively, zoom into some patch where $\mathbf{U}$ is approximately constant and then boost to a frame in which it vanishes. In this way, our analysis should hold locally even for general mean flows $\mathbf{U}(\mathbf{x})$.)

The averages that we encountered in the previous section involved fluctuations at the same point in space. For example, the Reynolds' stress tensor is

$$R_{ij} = \rho\langle\delta u_i(\mathbf{x})\delta u_j(\mathbf{x})\rangle$$

with the two velocities evaluated at the same point. Because we no longer have a background flow $\mathbf{U}(\mathbf{x})$, the system has translational invariance. This means, among other things, that R_{ij} doesn't depend on the point $\mathbf{x}$ at which it's evaluated.

There is now an obvious generalisation in which the correlation between velocity fields is computed at different points,

$$C_{ij}(\mathbf{x}_1, \mathbf{x}_2) = \langle u_i(\mathbf{x}_1)u_j(\mathbf{x}_2)\rangle$$

A number of constraints on this correlation function follow simply from the symmetries of our problem which are enhanced because we're assuming $\mathbf{U} = 0$. First, translational invariance means that it is only a function of the difference $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ and we write $C_{ij}(\mathbf{x}_1, \mathbf{x}_2) = C_{ij}(\mathbf{r})$. Second, isotropy, together with parity invariance, means that the six components of a general symmetric tensor are reduced to just two,

$$C_{ij}(\mathbf{r}) = C_{TT}(r)\,(\delta_{ij} - \hat{r}_i\hat{r}_j) + C_{LL}(r)\hat{r}_i\hat{r}_j \tag{6.17}$$

Here $C_{TT}(r)$ is the transverse correlation function and $C_{LL}(r)$ the longitudinal correlation function. Note, in particular, that $C_{ij}(\mathbf{r}) = C_{ij}(-\mathbf{r})$, which follows by parity invariance.

When $r = 0$, so the two points in the correlation function coincide, we have a handle on the correlation function: it must take the form

$$C_{ij}(0) = \frac{2}{3}\mathcal{E}\delta_{ij} \tag{6.18}$$

where $\mathcal{E} = \frac{1}{2}\langle \mathbf{u}\cdot\mathbf{u}\rangle$ is the kinetic energy (divided by the density ρ). This coincides with the expression (6.7) when $\mathbf{U} = 0$ (with $\mathcal{E} = K/\rho$). But while (6.7) was pulled out of thin air, here (6.18) follows because, in the absence of a background mean flow, the only symmetric two-tensor that we have at our disposal is δ_{ij}.

There's one last constraint that comes from the fact that the fluid is incompressible, $\nabla\cdot\mathbf{u} = 0$, which means that

$$\frac{\partial C_{ij}}{\partial r^i} = 0$$

We can use this to relate $C_{TT}(r)$ and $C_{LL}(r)$. To do this, we write $\hat{r}_k = r_k/r$ and make use of the identities

$$\frac{\partial \hat{r}_k}{\partial r^i} = \frac{\delta_{ik}}{r} - \frac{r_i r_k}{r^3} \quad\Rightarrow\quad \frac{\partial(\hat{r}_i\hat{r}_j)}{\partial r^i} = 2\frac{\hat{r}_j}{r}$$

Then

$$\frac{\partial C_{ij}}{\partial r^i} = \frac{dC_{TT}}{dr}\hat{r}_i\left(\delta_{ij} - \hat{r}_i\hat{r}_j\right) + \frac{dC_{LL}}{dr}\hat{r}_i\hat{r}_i\hat{r}_j + 2(C_{LL} - C_{TT})\frac{\hat{r}_j}{r}$$

The first term vanishes and we're left with the simple expression

$$C_{TT}(r) = C_{LL} + \frac{r}{2}\frac{dC_{LL}}{dr} \tag{6.19}$$

This is known as the *von Kármán relation*.

In what follows, we'll also have need for the closely related *structure function*. This looks at the correlation between the difference in the velocity fluctuations between two points,

$$S_{ij}(\mathbf{r}) = \langle (u_i(\mathbf{x}_1) - u_i(\mathbf{x}_2))(u_j(\mathbf{x}_1) - u_j(\mathbf{x}_2))\rangle \tag{6.20}$$

Expanding out the four terms, the structure function can be trivially expressed in terms of the correlation function as

$$S_{ij}(\mathbf{r}) = 2C_{ij}(0) - 2C_{ij}(\mathbf{r}) = \frac{4}{3}\mathcal{E}\delta_{ij} - 2C_{ij}(\mathbf{r})$$

Note, in particular, that $S_{ij}(0) = 0$. As with the correlation function, we can decompose the structure function into transverse and longitudinal pieces

$$S_{ij}(\mathbf{r}) = S_{TT}(r)\,(\delta_{ij} - \hat{r}_i\hat{r}_j) + S_{LL}(r)\hat{r}_i\hat{r}_j$$

Comparing to the components of the correlation function, we have

$$S_{LL}(r) = \frac{4}{3}\mathcal{E} - 2C_{LL}(r) \tag{6.21}$$

with a similar expression for the transverse component: $S_{TT}(r) = \frac{4}{3}\mathcal{E} - 2C_{TT}(r)$.

One advantage of working with the structure function is that we can make contact with the simple dimensional analysis arguments of Section 6.2. In particular, if we take the Kolmogorov-Obhukov law (6.15) at face value then it should apply to the structure function, telling us to expect

$$S_{ij}(r) \sim r^{2/3} \quad \text{when} \quad l_0 \ll r \ll L \tag{6.22}$$

This should hold only in the inertial range, as shown. Since the von Kármán relation (6.19) also holds for the structure function, it tells us that, in the inertial range, $S_{TT} = \frac{4}{3}S_{LL}$. (Actually, as part of our analysis we'll get a better understanding of the Kolmogorov-Obhukov law and see that the result $S_{ij} \sim r^{2/3}$ is not exact: nonetheless, we may hope that it's not a wildly inaccurate expectation.)

6.3.1 Navier-Stokes for Correlation Functions

We'll attempt to compute the correlation functions using the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{1}{\rho}(\nabla P - \mathbf{f}) + \nu\nabla^2\mathbf{u} \tag{6.23}$$

We've included the driving force $\mathbf{f}$ which, as in Section 6.2, will be responsible for the injection of energy. Because averaging commutes with differentiation (both with respect to time and space), we have

$$\begin{aligned}
\frac{\partial C_{ij}(\mathbf{r};t)}{\partial t} &= \langle \partial_t u_i(\mathbf{x}_1)\, u_j(\mathbf{x}_2)\rangle + \langle u_i(\mathbf{x}_1)\,\partial_t u_j(\mathbf{x}_2)\rangle \\
&= -\frac{\partial}{\partial x_1^k}\langle u_k(\mathbf{x}_1)u_i(\mathbf{x}_1)\,u_j(\mathbf{x}_2)\rangle - \frac{\partial}{\partial x_2^k}\langle u_i(\mathbf{x}_1)u_k(\mathbf{x}_2)u_j(\mathbf{x}_2)\rangle \\
&\quad - \frac{1}{\rho}\langle \partial_i P(\mathbf{x}_1)\, u_j(\mathbf{x}_2)\rangle - \frac{1}{\rho}\langle u_i(\mathbf{x}_1)\partial_j P(\mathbf{x}_2)\rangle \\
&\quad + \frac{1}{\rho}\langle f_i(\mathbf{x}_1)u_j(\mathbf{x}_2)\rangle + \frac{1}{\rho}\langle u_i(\mathbf{x}_1)f_j(\mathbf{x}_2)\rangle \\
&\quad + \nu\langle \nabla^2 u_i(\mathbf{x}_1)\, u_j(\mathbf{x}_2)\rangle + \nu\langle u_i(\mathbf{x}_1)\nabla^2 u_j(\mathbf{x}_2)\rangle
\end{aligned} \tag{6.24}$$

In the first line we have used the incompressibility of the fluid, $\nabla \cdot \mathbf{u} = 0$, to take the derivative outside the average. We can see immediately from this that, as in Section 6.1, to get an equation for the two-point correlation function $C_{ij}(\mathbf{r})$, we need to know something about the three-point function $\langle \delta u^3 \rangle$. If we tried to get an equation for $\langle \delta u^3 \rangle$ then we would, as before, find that it pushes us towards the four-point function $\langle \delta u^4 \rangle$ and so on. This is the same closure problem that we met previously.

This time, however, there is something that we can say without going down the rabbit hole. First we'll sort out some of the terms in (6.24), and then return to the 3-point function.

Claim: The pressure terms vanish: $\langle u_i(\mathbf{x}_1) P(\mathbf{x}_2) \rangle = 0$.

Proof: Using homogeneity and isotropy, we must have $\langle P(\mathbf{x}_1) u_i(\mathbf{x}_2) \rangle = f(r)\hat{r}_i$ for some function $f(r)$ where, as before, $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. But incompressibility tells us that

$$0 = \frac{\partial}{\partial x_1^i} \langle u_i(\mathbf{x}_1) P(\mathbf{x}_2) \rangle = f'(r) + \frac{2}{r} f(r) \quad \Rightarrow \quad f(r) = \frac{\alpha}{r^2}$$

for some constant α. But the correlation $\langle u_i(\mathbf{x}_1) P(\mathbf{x}_2) \rangle$ should be finite as $r \to 0$. Which means that we must have $\alpha = 0$ and $\langle u_i(\mathbf{x}_1) P(\mathbf{x}_2) \rangle = 0$. □

Next, we turn our attention to the energy injection terms involving the correlation $\langle u_i(\mathbf{x}_1) f_j(\mathbf{x}_2) \rangle$. To make sense of this, we need to specify the form of the forcing term although, as explained in Section 6.2, the expectation is that the energy will cascade down to smaller scales in a way that is ultimately independent of the forcing term we choose. It turns out that things are particularly simple if we pick a random forcing term that takes the form of Gaussian white noise, meaning that

$$\langle f_i(\mathbf{x}_1, t_1) f_j(\mathbf{x}_2, t_2) \rangle = \delta(t_1 - t_2) \rho^2 \epsilon_{ij}(\mathbf{r}) \tag{6.25}$$

for some choice of function $\epsilon_{ij}(\mathbf{r})$ which we get to decide. We'll take it to be symmetric, so $\epsilon_{ij} = \epsilon_{ji}$ and isotropic so $\epsilon_{ij}(\mathbf{r}) = \epsilon_{ij}(-\mathbf{r})$. We'll shortly see how this tensor ϵ_{ij} is related to the work done per unit mass ϵ that played such a key role in our dimensional analysis argument of Section 6.2. One important property of white noise is that the value of the force at any time t is completely uncorrelated with its value at any earlier time t'. This will be important shortly.

Claim: With the force given by the Gaussian white noise (6.25), the correlation between the force and velocity is give by

$$\frac{1}{\rho}\Big(\langle f_i(\mathbf{x}_1) u_j(\mathbf{x}_2) \rangle + \langle u_i(\mathbf{x}_1) f_j(\mathbf{x}_2) \rangle \Big) = \epsilon_{ij}(\mathbf{r}) \tag{6.26}$$

Proof: We integrate up the Navier-Stokes equation (6.23) to get the expression for the velocity field $\mathbf{u}(\mathbf{x}, t)$,

$$\mathbf{u}(\mathbf{x}, t) = \int_0^t dt' \left[-(\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{1}{\rho}(\nabla P - \mathbf{f}) + \nu\nabla^2\mathbf{u} \right]$$

with $t \geq t' \geq 0$. Substituting this into the correlation function then gives

$$\langle f_i(\mathbf{x}_1, t) u_j(\mathbf{x}_2, t)\rangle = \int_0^t dt' \ \langle f_i(\mathbf{x}_1, t) \left[-u_k \frac{\partial u_j}{\partial x^k} - \frac{1}{\rho}\left(\frac{\partial P}{\partial x^j} - f_j \right) + \nu\nabla^2 u_j \right] (\mathbf{x}_2, t')\rangle$$

All the fields $\mathbf{u}(\mathbf{x}, t')$ and $P(\mathbf{x}, t')$ are uncorrelated with the force $\mathbf{f}(\mathbf{x}, t)$ at a later time $t > t'$ because the force is taken to be white noise. The only contribution comes from in this correlation function therefore comes from

$$\langle f_i(\mathbf{x}_1, t) u_j(\mathbf{x}_2, t)\rangle = \frac{1}{\rho}\int_0^t dt' \ \langle f_i(\mathbf{x}_1, t) f_j(\mathbf{x}_2, t')\rangle = \rho \int_0^t dt' \ \delta(t - t')\epsilon_{ij}(\mathbf{r})$$

Now, if we integrate $\int dt' \ \delta(t - t')$ over a range that includes the point $t' = t$ then the integral clearly gives 1. Here, however, the point $t' = t$ sits right at the end of the integral range. This is a bit ambiguous but there's a sensible way to think about it. If we were to extend the integral a little further beyond t, the integral clearly gives 1. But this gets a contribution both from our original integral and from the extension. It seems fair to share these. This then gives

$$\int_0^t dt' \ \delta(t - t') = \frac{1}{2} \quad \Rightarrow \quad \langle f_i(\mathbf{x}_1, t) u_j(\mathbf{x}_2, t)\rangle = \frac{\rho}{2}\epsilon_{ij}(\mathbf{r})$$

Adding the second contribution then gives the claimed result (6.26). □

From (6.26), we see that if we take the trace of the tensor $\epsilon_{ij}(\mathbf{r})$ and evaluate it at $\mathbf{r} = 0$ we have the work done (divided by the density),

$$\epsilon_{ii}(0) = \frac{2}{\rho}\langle \mathbf{f}\cdot\mathbf{u}\rangle := 2\epsilon \tag{6.27}$$

On the right-hand side, we have the same ϵ that we met in Kolmogorov's dimensional analysis argument in Section 6.2. We'll make contact with these ideas later in this section. For now, note that we can make the same tensor decomposition as in (6.17) and write

$$\epsilon_{ij}(\mathbf{r}) = \epsilon_{TT}(r)\left(\delta_{ij} - \hat{r}_i\hat{r}_j\right) + \epsilon_{LL}\hat{r}_i\hat{r}_j$$

This gives $\epsilon_{LL}(r) = \hat{r}_i\hat{r}_j\epsilon_{ij}(\mathbf{r})$. If we evaluate this tensor at $\mathbf{r} = 0$ then we must have $\epsilon_{ij}(0) = \epsilon_{LL}(0)\delta_{ij}$ as there is no other tensor in the game. This then gives

$$\epsilon_{LL}(0) = \frac{2}{3}\epsilon \tag{6.28}$$

Let's pause to take stock. Our equation (6.24) for the correlation function has now become

$$\begin{aligned}\frac{\partial C_{ij}(\mathbf{r};t)}{\partial t} = &-\frac{\partial}{\partial x_1^k}\langle u_k(\mathbf{x}_1)u_i(\mathbf{x}_1)\,u_j(\mathbf{x}_2)\rangle - \frac{\partial}{\partial x_2^k}\langle u_i(\mathbf{x}_1)u_k(\mathbf{x}_2)u_j(\mathbf{x}_2)\rangle \\ &+ \epsilon_{ij}(\mathbf{r}) + 2\nu\nabla^2 C_{ij}(\mathbf{r})\end{aligned} \tag{6.29}$$

which is starting to look a little simpler. Our next task is to better understand the structure of the three-point functions.

6.3.2 The Structure of the Three-Point Function

We write the three-point function as

$$C_{ij,k}(\mathbf{x}_1, \mathbf{x}_2) = \langle u_i(\mathbf{x}_1)u_j(\mathbf{x}_1)u_k(\mathbf{x}_2)\rangle$$

The comma is there to remind us that two of the velocities are evaluated at $\mathbf{x}_1$ and the third at $\mathbf{x}_2$. (The comma doesn't mean differentiation. This isn't general relativity!) By isotropy, we must have

$$C_{ij,k}(\mathbf{x}_1, \mathbf{x}_2) = C_{ij,k}(\mathbf{r}) = C_{ji,k}(\mathbf{r})$$

and by parity invariance,

$$C_{ij,k}(\mathbf{r}) = -C_{ij,k}(-\mathbf{r}) = -\langle u_i(-\mathbf{x}_1)u_j(-\mathbf{x}_1)u_k(-\mathbf{x}_2)\rangle = -\langle u_i(\mathbf{x}_2)u_j(\mathbf{x}_2)u_k(\mathbf{x}_1)\rangle \tag{6.30}$$

with the overall minus sign arising because the correlation function involves an odd number of velocities. In the final equality, we've invoked translational invariance and shifted all arguments by $\mathbf{x}_1 + \mathbf{x}_2$.

In fact, the tensor structure means that we can reduce the correlation function to just three function of $r = |\mathbf{r}|$,

$$C_{ij,k}(\mathbf{r}) = A(r)\delta_{ij}\hat{r}_k + B(r)\,(\delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i) + D(r)\hat{r}_i\hat{r}_j\hat{r}_k$$

These different functions are further related by the incompressibility of the flow. For the two-point function, this gave us the von Kármán relation (6.19). For the three-point function, we have

Claim: Incompressibility gives the relations

$$B = -\frac{1}{2r}\frac{d(r^2 A)}{dr} \tag{6.31}$$

and

$$3A + 2B + D = 0 \tag{6.32}$$

Proof: Incompressibility $\nabla \cdot \mathbf{u} = 0$ means that

$$\frac{\partial C_{ij,k}}{\partial x_2^k} = -\frac{\partial C_{ij,k}}{\partial r^k} = 0$$

We use $\partial \hat{r}_k / \partial r^i = \delta_{ik}/r - \hat{r}_i \hat{r}_k / r$ and, after a line or two of algebra, we get

$$\frac{\partial C_{ij,k}}{\partial r^k} = \delta_{ij}\left(A' + \frac{2A}{r} + \frac{2B}{r}\right) + \hat{r}_i \hat{r}_j \left(2B' - \frac{2B}{r} + D' + \frac{2D}{r}\right)$$

Each of these tensor structures must individually vanish. The first gives the relation (6.31). The vanishing of the second term can be written as

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2(D + 2B)\right] = \frac{6B}{r}$$

If we substitute in our expression for B in the recently proved (6.31), this becomes

$$\frac{d}{dr}\left[r^2(D + 2B + 3A)\right] = 0 \quad \Rightarrow \quad 3A + 2B + D = \frac{\text{constant}}{r^2} \tag{6.33}$$

We can fix the constant by looking at $C_{ij,k}(0)$ which must take the value $C_{ij,k}(0) = 0$ for the simple reason that there's no invariant 3-tensor with the right symmetry properties that it can equal. This means that the constant in (6.33) is actually zero and so we get (6.32). □

We can combine the two relations (6.31) and (6.32) to give $D = rA' - A$. The upshot is that the three-point correlation function actually depends on just a single function $A(r)$,

$$C_{ij,k}(\mathbf{r}) = A\delta_{ij}\hat{r}_k - \frac{1}{2}\left(rA' + 2A\right)\left(\delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i\right) + (rA' - A)\hat{r}_i\hat{r}_j\hat{r}_k \tag{6.34}$$

Next, it will also be useful to introduce the three-point structure function, which is the obvious generalisation of (6.20),

$$S_{ijk}(\mathbf{r}) = \langle (u_i(\mathbf{x}_1) - u_i(\mathbf{x}_2))(u_j(\mathbf{x}_1) - u_j(\mathbf{x}_2))(u_k(\mathbf{x}_1) - u_k(\mathbf{x}_2))\rangle$$

This is completely symmetric in all three indices. If we expand and cancel terms (remembering that we have translational invariance so $\langle \mathbf{u}(\mathbf{x}_1)^3 \rangle = \langle \mathbf{u}(\mathbf{x}_2)^3 \rangle$) then we can relate the structure function to the correlation function

$$S_{ijk}(\mathbf{r}) = -2(C_{ij,k} + C_{ik,j} + C_{jk,i})$$

If we substitute in the expression (6.34) we get

$$S_{ijk}(\mathbf{r}) = 2(A + rA')(\delta_{ij}\hat{r}_k + \delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i) - 6(rA' - A)\hat{r}_i\hat{r}_j\hat{r}_k \tag{6.35}$$

The fully longitudinal part of the structure function is defined to be

$$S_{LLL}(r) = S_{ijk}(\mathbf{r})\hat{r}_i\hat{r}_j\hat{r}_k$$

From (6.35), we see that this is the same thing as the function $A(r)$, up to an overall constant,

$$S_{LLL}(r) = 12A(r)$$

In what follows, we'll work with $S_{LLL}(r)$ as the object that describes the three-point function.

6.3.3 The von Kármán-Howarth Equation

Now we can return to our expression (6.29) for the dynamics of the correlation function. This reads

$$\frac{\partial C_{ij}}{\partial t} = -\frac{\partial C_{ik,j}}{\partial r^k} - \frac{\partial C_{jk,i}}{\partial r^k} + \epsilon_{ij}(\mathbf{r}) + 2\nu\nabla^2 C_{ij}(\mathbf{r})$$

where the indices in the second $\partial C/\partial r$ term have rearranged themselves courtesy of (6.30). We focus on the longitudinal component of the two-point correlator, $C_{LL} = C_{ij}\hat{r}_i\hat{r}_j$ which obeys,

$$\frac{\partial C_{LL}}{\partial t} - \epsilon_{LL} - 2\nu\hat{r}_i\hat{r}_j\nabla^2 C_{ij} = -2\hat{r}_i\hat{r}_j\frac{\partial C_{ik,j}}{\partial r^k} \tag{6.36}$$

We have a little bit of work to do to move those $\hat{r}_i\hat{r}_j$ terms inside the derivatives. On the right-hand side, we have

$$\begin{aligned}\hat{r}_i\hat{r}_j\frac{\partial C_{ik,j}}{\partial r^k} &= \frac{\partial}{\partial r^k}(C_{ik,j}\hat{r}_i\hat{r}_j) - C_{ik,j}\frac{\partial(\hat{r}_i\hat{r}_j)}{\partial r^k} \\ &= \frac{\partial}{\partial r^k}(C_{ik,j}\hat{r}_i\hat{r}_j) - \frac{1}{r}\left(C_{ii,j}\hat{r}_j + C_{ik,k}\hat{r}_i - 2C_{ij,k}\hat{r}_i\hat{r}_j\hat{r}_k\right)\end{aligned} \tag{6.37}$$

where we've again made use of the identity $\partial\hat{r}_i/\partial r^k = \delta_{ik}/r - \hat{r}_i\hat{r}_k/r$. Now, from (6.34), we can compute the various contractions of $C_{ij,k}$ with $\hat{r}$,

$$\begin{aligned} C_{ik,j}\hat{r}_i\hat{r}_j &= A\hat{r}_k - (rA' + 2A)\,\hat{r}_k + (rA' - A)\hat{r}_k = -2A\hat{r}_k \\ C_{ii,j}\hat{r}_j &= 3A - (rA' + 2A) + (rA' - A) = 0 \\ C_{ik,k}\hat{r}_i &= A - 2\,(rA' + 2A) + (rA' - A) = -rA' - 4A \\ C_{ij,k}\hat{r}_i\hat{r}_j\hat{r}_k &= A - (rA' + 2A) + (rA' - A) = -2A \end{aligned}$$

So (6.37) becomes

$$\hat{r}_i\hat{r}_j\frac{\partial C_{ik,j}}{\partial r^k} = -2A\frac{\partial}{\partial r^k}(A\hat{r}_k) + A' = -\frac{4A}{r} - A'$$

We have a similar task for the $\nu\nabla^2 C_{ij}$ term in (6.36). For this, it's best to return to the expression (6.17). A slightly tedious exercise in algebra gives

$$\begin{aligned} \nabla^2 C_{ij} = \nabla^2 C_{TT}\,(\delta_{ij} - \hat{r}_i\hat{r}_j) + \nabla^2 C_{LL}\hat{r}_i\hat{r}_j + \frac{C_{LL} - C_{TT}}{r^2}\,(2\delta_{ij} - 6\hat{r}_i\hat{r}_j) \\ + \frac{1}{r}\left(\frac{\partial C_{LL}}{\partial r^k} - \frac{\partial C_{TT}}{\partial r^k}\right)(\delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i - 2\hat{r}_i\hat{r}_j\hat{r}_k) \end{aligned}$$

If we now contract with $\hat{r}_i\hat{r}_j$, the second line disappears and we're left with

$$\hat{r}_i\hat{r}_j\nabla^2 C_{ij} = \nabla^2 C_{LL} - \frac{4}{r^2}(C_{LL} - C_{TT})$$

But now we can use the von Kármán relation (6.19) to write this purely in terms of C_{LL},

$$\begin{aligned} \hat{r}_i\hat{r}_j\nabla^2 C_{ij} &= \nabla^2 C_{LL} + \frac{2}{r}\frac{\partial C_{LL}}{\partial r} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial C_{LL}}{\partial r}\right) + \frac{2}{r}\frac{\partial C_{LL}}{\partial r} \\ &= \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial C_{LL}}{\partial r}\right) \end{aligned}$$

Now we can put these pieces back into the expression (6.36), which becomes an equation that relates the longitudinal two-point function C_{LL} with the three-point function $S_{LLL} = 12A$,

$$\frac{\partial C_{LL}}{\partial t} - \epsilon_{LL} - \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial C_{LL}}{\partial r}\right) = \frac{1}{6}\left(S'_{LLL} + \frac{4S_{LLL}}{r}\right)$$

We can express everything in terms of the structure function using the relation (6.21) which relates $S_{LL} = \frac{4}{3}\mathcal{E} - 2C_{LL}$ with $\mathcal{E} = \frac{1}{2}\langle \mathbf{u}\cdot\mathbf{u}\rangle$ the average kinetic energy (divided by the density). The end result is:

$$\frac{\partial S_{LL}}{\partial t} = \frac{4}{3}\frac{\partial \mathcal{E}}{\partial t} - 2\epsilon_{LL} + \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_{LL}}{\partial r}\right) - \frac{1}{3r^4}\frac{\partial}{\partial r}\left(r^4 S_{LLL}\right) \tag{6.38}$$

This is the *von Kármán-Howarth equation*. It tells us how the two-point correlations of the velocity evolve with time. In the limit $\mathbf{r} \to 0$, it reduces to the equation describing energy balance.

6.3.4 Kolmogorov's 4/5

it's a short step from the von Kármán-Howarth equation to what we want. We'll focus on the static case where we've reached a kind of equilibrium where, as described in Section 6.2, all the energy fed into the system at large scales is lost to viscosity at small scales. This allows us to drop the time derivatives in (6.38) and we have

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\left(2\nu\frac{\partial S_{LL}}{\partial r} - \frac{S_{LLL}}{3}\right)\right) = 2\epsilon_{LL}(r)$$

We will further assume that the energy is injected on large scales. Following (6.28), we interpret this as the statement

$$\epsilon_{LL}(\mathbf{r}) = \frac{1}{3}\epsilon_{ii}(\mathbf{r}) \approx \frac{2}{3}\epsilon + \ldots$$

where ϵ is the work done per unit mass that we met in the dimensional analysis argument of Section 6.2. We then integrate our differential equation to get

$$2\nu\frac{\partial S_{LL}}{\partial r} - \frac{1}{3}S_{LLL} = \frac{4}{15}\epsilon r$$

where the constants of integration have been put to zero using the fact that $S_{LL}(0) = S_{LLL}(0) = 0$. Rearranging, we have an expression for the three-point correlations,

$$S_{LLL}(r) = -\frac{4}{5}\epsilon r + 6\nu\frac{\partial S_{LL}}{\partial r} \tag{6.39}$$

This is *Kolmogorov's* $4/5^{\text{th}}$*'s law*. It's important because the number of exact results about turbulence can be counted on one finger. This is the one. We recognise the first term as a more rigorous version of the Kolmogorov-Obhukov law (6.15) that we derived using dimensional analysis. In fact, this result tells us how to think of the Kolmogorov-Obhukov law: it holds for three-point functions.

The second term in (6.39) is a correction. A naive application of Kolmogorov-Obhukov suggests that the two-point structure function scales as $S_{LL}(r) \sim r^{2/3}$. (We already mentioned this is (6.22).) But that's not the way correlation functions work: just because $\langle u^3 \rangle \sim r$ doesn't mean that $\langle u^p \rangle \sim r^{p/3}$. Nonetheless, if we take this as a ballpark guess for the behaviour of the correlation function then the second term is much smaller than the first if we focus on distance scales that are much larger than the Kolmogorov viscosity scale, $r \gg l_0 \sim (\nu^3/\epsilon)^{1/4}$. This is because if $S_{LL} \sim (\epsilon r)^{2/3}$ then $S'_{LL} \sim \epsilon^{2/3} r^{-1/3}$ and hence $\nu S'_{LL} \sim (l_0/r)^{4/3} \epsilon r \ll \epsilon r$. So, in the inertial range we have the Kolmogorov-Obhukov result

$$S_{LLL}(r) \approx -\frac{4}{5}\epsilon r$$

Except now we know the prefactor. It is $-4/5$.