3 The Navier-Stokes Equation

So far we've built up some intuition for how fluids flow. But we've missed an ingredient and it turns out that this ingredient is rather important. This is the internal friction experienced by a fluid, known as *viscosity*. We'll give a more careful discussion of viscosity shortly in Section 3.1. But first we give a quick, slightly handwavy derivation of the relevant equation.

The Euler equation describing fluid motion is, as we've seen, just the continuum version of "$F = ma$". It is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mathbf{f} \tag{3.1}$$

where $\mathbf{f}$ describes any other forces experienced by the fluid. We want to write down the force that arises from friction.

We already met friction briefly in our introduction to Newtonian Mechanics. There we explained that friction is not a fundamental force, but one that arises from the underlying interactions of many (say, 10^{23}) particles. We stated, without proof, that particles moving slowly with speed v through very viscous fluids experience linear drag, with $F \sim -v$, while particles moving more quickly through less viscous fluids experience quadratic drag $F \sim -v^2$. (For what it's worth, we'll recover the equations for linear drag in Section 3.4, while quadratic drag involves turbulent flows and is not so easy to derive from first principles.) The first question that we want to ask in these lectures is: what kind of friction force does the fluid experience when it rubs against itself?

The answer is that the viscosity is *linear* in the fluid velocity $\mathbf{u}(\mathbf{x}, t)$. At heart, this statement follows simply from Taylor's theorem. The underlying atomic interactions are complicated and they surely result in a friction force that is some arbitrarily complicated function of $\mathbf{u}$. But, for suitably small velocities, the linear term is always larger than the quadratic term. This simple but powerful argument is known as *linear response*. It's the same argument that leads to Ohm's law with the current given by $I = V/R$ rather than some more complicated function of voltage. (The lecture notes on Kinetic Theory contain a section devoted to linear response in more general settings.) Those fluids for which the linear approximation is a good one are called *Newtonian*.

So the viscosity should be a force $\mathbf{f}$ that is linear in the fluid velocity $\mathbf{u}$. What can we write down? First, the force can't be proportional to $\mathbf{u}$ itself: that's in contradiction with Gallilean relativity which says that the equations of fluid mechanics must be invariant under a boost of the whole fluid (and any container) by a constant $\mathbf{u}$. This tallies with the idea that viscosity is associated to the friction force experienced by the fluid when one part rubs up against another. That means that the different parts of the fluid should be moving at different speeds for viscosity to kick in. Or, in other words, the friction force must depend on spatial changes of $\mathbf{u}$.

The simplest possibility is that the friction force depends on the first derivative of $\mathbf{u}$, but there is only one vector that we can form in this way and that's the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. But this isn't a good candidate for a force. This is because of the symmetry of *parity* or reflections, $\mathbf{x} \rightarrow -\mathbf{x}$. Under parity $\mathbf{u} \rightarrow -\mathbf{u}$ and each term in the Euler equation is odd. This should continue to be true of any force that acts on the fluid. But vorticity is even: $\boldsymbol{\omega} \rightarrow \boldsymbol{\omega}$. It cannot arise as a force.

This means that we must look to two-derivative terms if we are to build a force linear in velocity. There are two possibilities: $\nabla^2\mathbf{u}$ and $\nabla(\nabla \cdot \mathbf{u})$. But, given that we are dealing with an incompressible fluid, the second of these necessarily vanishes. We're left with just one choice for our friction force,

$$\mathbf{f}_{\text{viscous}} = \mu \, \nabla^2 \mathbf{u}$$

The coefficient μ is a constant known as the *dynamic (shear) viscosity*[2]. The resulting equation describing the motion of fluids is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{u} + \mathbf{f} \tag{3.2}$$

where, in $\mathbf{f}$, we retain the option to include any other external force such as gravity. This is the famous *Navier-Stokes equation*. Its solutions will occupy us for the rest of this course.

In what follows, we will also frequently come across the ratio

$$\nu = \frac{\mu}{\rho}$$

This arises so frequently that it is given its own name: it is called the *kinematic viscosity*.

Simple as the Navier-Stokes equation is, there is a great deal about it that remains to be understood. This includes the most basic questions about the existence and uniqueness of solutions. We will touch upon some of these issues in these lectures although much our focus will be on the things that are known rather than those that are not.

[2]Annoyingly, the shear viscosity was denoted as η in the lectures on Kinetic Theory. Sorry.

Viscosity Causes Diffusion of Momentum

As these lectures progress, we'll develop some intuition for what viscosity does. But even before we get going, we can gain some insight through analogy. If we focus on two terms in the Navier-Stokes equation, ignoring the others (and also ignoring why we might be allowed to ignore them!) we have

$$\rho\frac{\partial\mathbf{u}}{\partial t} \sim \mu\nabla^2\mathbf{u}$$

It's useful to compare this to the heat equation which describes how temperature changes

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c_V}\nabla^2 T$$

where (for what it's worth) κ is the thermal conductivity and c_V the specific heat. Solutions to the heat equation are well studied: if we start with a hot spot somewhere, a place where there is a localised increase in temperature, then it will spread out increasing in size as $L \sim \sqrt{t}$. This kind of behaviour is called *diffusion*. It is the characteristic behaviour of any conserved quantity when undergoing random bombardment by some microscopic process.

This also tells us how to think of the viscosity term in the Navier-Stokes equation. It is causing momentum $\rho\mathbf{u}$ to diffuse. It doesn't cause the momentum to change direction. But if there was some place in the fluid where the momentum density was higher then the viscosity term will cause this to spread out, much like temperature in the heat equation. This also makes physical sense: if the momentum density is higher in one region of the fluid then that region will be rubbing against its neighbouring regions. Viscosity is the friction force induced by this rubbing and results in a transfer of momentum from one region into neighbouring regions.

Viscosity Breaks Time Reversal

More intuition comes from the observation that the additional viscosity term breaks the symmetry of *time reversal*. This acts as

$$T : t \to -t \quad , \quad T : \mathbf{x} \to \mathbf{x} \quad , \quad T : \mathbf{u} \to -\mathbf{u}$$

You can check that all the terms in the Euler equation are invariant under this transformation (at least if the external force is time-reversal invariant, so $T : \mathbf{f} \to \mathbf{f}$). But the extra viscosity term is not invariant under time-reversal: it transforms as

$$T : \nabla^2\mathbf{u} \to -\nabla^2\mathbf{u}$$

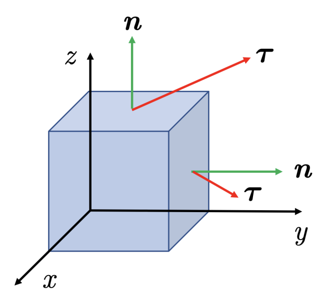
This is important. If we're given any solution to the Euler equation, then we can always run it backwards in time and we will get another solution. This is not true of solutions to the Navier-Stokes equation which exhibits an arrow of time. This is because, as advertised above, the viscosity is a friction force which, like other friction forces in classical mechanics, causes the system to lose energy. We'll see this explicitly in Section 3.1.2 where we compute the energy lost due to viscosity.

3.1 Stress, Strain and Viscosity

We'll now give a slightly more involved derivation of the Navier-Stokes equation which will allow us to unpick the meaning of viscosity. To do this, we go back to basics and start thinking afresh about the kinds of forces that act on a fluid. Recall from (2.5) that, for a volume V of fluid, the equation $F = ma$ becomes

$$\rho \int_V \frac{D\mathbf{u}}{Dt}\, dV = -\int_S P\, d\mathbf{S} + \text{Other Forces} \tag{3.3}$$

with $S = \partial V$ the surface of the volume. Importantly, the pressure P acts on the surface of the volume and this ensures that it appears in the Navier-Stokes equation as the gradient ∇P as the surface integral is converted to a volume integral by the divergence theorem. Similarly, the friction forces also naturally act on the surface of the volume as a neighbouring piece of fluid brushes past. So our first task is to better understand what the general kind of force acting on a surface looks like.



Consider a small cubic volume V as shown in the figure. Obviously there are six sides, and there may be a force acting on each. The pressure force is special because it acts parallel to the normal on each side. But that not necessarily the case for all forces. In general, the force might act in any direction. Moreover, the direction of the force will generally depend on the orientation of the surface. For example, this is obviously true of pressure which is parallel to the normal. The figure to the right shows the normals to two faces in green, while the force acting on those faces is shown in red.

These considerations mean that to specify the force that acts on a surface, we first have to specify the orientation to the surface. This is achieved through the introduction of the *stress tensor*, σ_{ij}. It is defined so that the force $\mathbf{F}$ acting on a small surface of

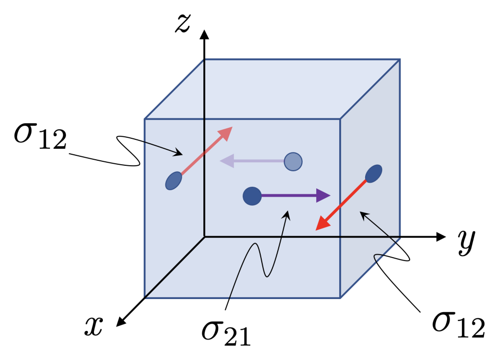


Figure 9. The torque experienced by a small parcel of fluid. The red arrows depict σ_{12}, the purple arrows σ_{21}. (The purple arrow on the furthest face is unlabelled to avoid clutter.)

area δA and normal $\mathbf{n}$ is given by

$$\mathbf{F} := \mathbf{f}\,\delta A = \sigma\mathbf{n}\,\delta A$$

Here $\mathbf{f}$ is the force per unit area. (Like pressure). In index notation, we have

$$f_i = \sigma_{ij} n_j \tag{3.4}$$

For pressure, the stress tensor takes the simple diagonal form

$$\sigma_{ij} = -P\,\delta_{ij}$$

But, in general, it may take a more complicated form. Furthermore, for a fluid the stress tensor is itself a field $\sigma_{ij}(\mathbf{x}, t)$ that may vary in both space and time. This means that the forces acting on various parts of the fluid depend both on their position in the fluid and on the orientation of the surface that is considered.

The stress tensor has an important property: it is symmetric

$$\sigma_{ij} = \sigma_{ji}$$

We will now show this. Consider the (slightly messy) Figure 9 depicting a small cube with side lengths L. The two red lines depict the force (per unit area) in the x direction on the faces that are normal to the y direction. From (3.4) this force is σ_{12}. Meanwhile, the two purple lines depict the force in the y-direction on the faces that are normal to the x direction. This force is σ_{21}.

These four forces give rise to a torque. Each σ_{ij} is a force-per-unit-area, so the actual force is $L^2\sigma_{ij}$. Furthermore, the moment of each force about the centre of the cube is $L/2$. This means that the total torque around a line parallel to the z-axis, through the centre of the cube, is

$$\tau = L^3(\sigma_{12} - \sigma_{21}) + \mathcal{O}\left(\frac{\partial \sigma_{12}}{\partial y}L^4, \frac{\partial \sigma_{21}}{\partial x}L^4\right)$$

The leading term comes from the difference between σ_{12} and σ_{21}. The sub-leading terms come from the difference of, say, σ_{12} on the left and right-hand faces. (The statement that the cube is small is the assumption that σ_{ij} does not vary much over the length scale L.)

Further torque may come from bulk forces whose strength varies over the inside of the cube. But this torque will always be of order L^4 (times some suitable dimensionful parameter) so, for small cubes, the leading contribution to the torque is proportional to the difference $(\sigma_{12} - \sigma_{21})$ and scales as L^3.

But torques that scale as L^3 are bad. To see this, recall that the angular acceleration is given by $\dot{\omega} = \tau/I$ where I is the moment of inertia. But the moment of inertia of any object always scales as L^5 (which is mass $\times$ $L^2 = \rho L^5$) and so $\dot{\omega} \sim 1/L^2$. The actual speed of the object is $v \sim \omega L$ so if the torque scales as L^3, the acceleration will diverge as $\dot{v} \sim 1/L$ for small L. That makes no sense. To avoid this we must have

$$\sigma_{12} = \sigma_{21}$$

Obviously the same argument works for all other components: $\sigma_{ij} = \sigma_{ji}$. The stress tensor is necessarily symmetric.

3.1.1 Newtonian Fluids

With the technology of the stress tensor, it is straightforward to describe the effect of friction. A *Newtonian fluid* is one where the friction forces are linear in velocity. If we assume that the fluid is isotropic then the form of the force is pretty much fixed by rotational invariance: it must be a symmetric tensor constructed from ∇ and $\mathbf{u}$ and the only option is $\partial_i u_j + \partial_j u_i$. In fact, a symmetric tensor can be decomposed into its trace and a traceless piece (see the lectures on Vector Calculus) so in general we have, including the pressure term,

$$\sigma_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - \frac{2}{3}\nabla\cdot\mathbf{u}\,\delta_{ij}\right) + \zeta\nabla\cdot\mathbf{u}\,\delta_{ij}$$

where, as we saw previously, μ is the *dynamical shear viscosity*. This time we've included the extra term proportional to $\nabla \cdot \mathbf{u}$ with a coefficient ζ known as the *bulk viscosity* or sometimes the *volume viscosity*. Importantly, it can be shown that each of these coefficients is necessarily positive. For μ, this follows from energy dissipation and we will give the argument shortly. For ζ it turns out that this follows from considerations of entropy. However, in this course we are dealing only with incompressible fluids with $\nabla \cdot \mathbf{u} = 0$ which means that we can forget all about the bulk viscosity. We have

$$\sigma_{ij} = -P\delta_{ij} + 2\mu E_{ij} \tag{3.5}$$

where E_{ij} is the rate of strain tensor that we met previously (2.14)

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) \tag{3.6}$$

We can now use this form of the stress tensor in the equation of motion for the fluid. With a general surface force, captured by σ_{ij}, the equation of motion for a fluid (3.3) become

$$\rho \int_V \frac{Du_i}{Dt}\, dV = \int_S \sigma_{ij}\, dS^j$$

where we have neglected other forces such as gravity. We use the divergence theorem to change the surface integral into a volume integral

$$\rho \int_V \frac{Du_i}{Dt}\, dV = \int_V \frac{\partial \sigma_{ij}}{\partial x^j}\, dV$$

This formula holds for arbitrary volume V, so the equation of motion is

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x^j} \tag{3.7}$$

From our equation (3.5), the right-hand side becomes

$$\frac{\partial \sigma_{ij}}{\partial x^j} = -\frac{\partial P}{\partial x^i} + \mu\left(\frac{\partial^2 u_i}{\partial x^j \partial x^j} + \frac{\partial^2 u_j}{\partial x^j \partial x^i}\right)$$

The second of these vanishes, again using our incompressibility condition $\nabla \cdot \mathbf{u} = 0$, and we're left with promised Navier-Stokes equation,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{u}$$

In what follows, we'll often divide by the density to write the Navier-Stokes equation as

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{u}$$

where, as defined earlier, $\nu = \mu/\rho$ is the kinematic viscosity. We can also add further forces on the right-hand side to taste.

The derivation of the Navier-Stokes equation that we described above sits entirely within the continuum language that underlies this course. There is another remarkable, and ultimately better, derivation that really goes back to basics. This is due to Boltzmann. The derivation starts with the underlying $\sim 10^{23}$ atoms and tracks their interactions, albeit in a statistical way. It explains why the variables of the Navier-Stokes equation are the right thing to focus on if you care only about long-time physics and gives a microscopic explanation of the various terms. You can find this derivation in the lectures on Kinetic Theory.

3.1.2 Momentum and Energy Conservation Revisited

For inviscid fluids, the Euler equation is simply the statement that momentum is conserved, while energy conservation (2.11) led to Bernoulli's principle. What becomes of these in the presence of viscosity?

First momentum. Here there is no problem: in the absence of external forces, we can write the Navier-Stokes equation in the form of a continuity equation, telling us that momentum is conserved. The only difference from the Euler equation is that we get an extra term in the momentum current, proportional to the viscosity

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x^j} = 0 \quad \text{with} \quad \Pi_{ij} = \rho u_i u_j + P\delta_{ij} - 2\mu E_{ij}$$

As before, we've used the fact that $\nabla \cdot \mathbf{u} = 0$ for incompressible fluids. In particular, we've used this to keep Π_{ij} symmetric by taking the extra term proportional to the rate of strain tensor (3.6) rather than just $\partial_j u_i$.

This gives us another perspective on the Navier-Stokes equation: it is, like the Euler equation, simply conservation of momentum, but with an additional term in the momentum current coming from gradients of the velocity. The idea that gradients drive currents is something that also occurs in other, perhaps more familiar, contexts where it goes by the name of *Fick's law*. For example, differences in temperature result in a heat current $\mathbf{J} \sim \nabla T$.

What about energy? We will ignore other bulk forces for now. (We've already seen in Section 2.1.4 that conservative forces don't spoil conservation of energy.) However, it's useful to briefly return to the form of the Navier-Stokes equation (3.7) in which we allow for general stress forces σ_{ij}. Taking the inner product with the velocity $\mathbf{u}$, the matter derivative becomes

$$\mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} = u_i \cdot \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x^j} \right) = \frac{1}{2} \frac{\partial |\mathbf{u}|^2}{\partial t} + \frac{1}{2} \mathbf{u} \cdot \nabla |\mathbf{u}|^2$$

and our proto-Navier-Stokes equation (3.7) becomes

$$\frac{\rho}{2}\left(\frac{\partial|\mathbf{u}|^2}{\partial t} + \mathbf{u}\cdot\nabla|\mathbf{u}|^2\right) = u_i\frac{\partial\sigma_{ij}}{\partial x^j}$$

Remember the game that we're playing: we'd like to massage this into the continuity equation to see the conservation of energy. Using the fact the fluid is incompressible, so $\nabla\cdot\mathbf{u} = 0$, we have

$$\frac{\rho}{2}\frac{\partial|\mathbf{u}|^2}{\partial t} + \frac{\partial}{\partial x^j}\left(\frac{\rho}{2}|\mathbf{u}|^2 u_j - u_i\sigma_{ij}\right) = -\sigma_{ij}\frac{\partial u_i}{\partial x^j} = -\sigma_{ij}E_{ij}$$

where, in the second equality, we've used the fact that $\sigma_{ij} = \sigma_{ji}$ so the contraction picks out the symmetric part of $\partial u_i/\partial x^j$ which is E_{ij}, the rate of strain tensor defined in (3.6). The two terms on the left-hand side take the form of a continuity equation. But now the right-hand side is not zero. This tells us that, in contrast to the Euler equation, energy is *not* conserved in the Navier-Stokes equation.

We can get an expression for how energy is low. If we integrate over some fixed volume V then we have

$$\frac{\rho}{2}\frac{\partial}{\partial t}\int_V |\mathbf{u}|^2 dV + \int_S \left(\frac{\rho}{2}|\mathbf{u}|^2 u_j - u_i\sigma_{ij}\right) dS^j = -\int_V \sigma_{ij}E_{ij}\; dV \tag{3.8}$$

with $S = \partial V$. The volume term on the left-hand side is clearly the change in the kinetic energy in V. The surface term accounts for (some of) this change: the $|\mathbf{u}|^2 u_j$ term captures the energy that flows out through the surface, while the $\sigma_{ij}u^j$ is the work done by the surface forces on the fluid contained in V. This includes the work done by the both the pressure and by the viscous forces. All of this is consistent with the conservation of energy. However, because the right-hand side of (3.8) doesn't vanish is telling us that energy is, in fact, no longer conserved. Instead, the right-hand side tells us the rate at which energy is dissipated.

$$\text{Dissipation} = \int_V \sigma_{ij}E_{ij}\; dV = 2\mu\int_V E_{ij}E_{ij}\; dV \tag{3.9}$$

where, in the second equality, we've used the explicit form of the stress tensor (3.5). We see that the pressure doesn't contribute to energy dissipation (because $\delta_{ij}E_{ij} = \nabla\cdot\mathbf{u} = 0$). This, of course, is something that we found when studying the Euler equation. But we now see that one important consequence of viscosity is we no longer have energy conservation. Correspondingly, the Bernoulli's principle no longer holds when the effects of viscosity are important

The dissipation is the integral of a total square, so it clearly positive provided that $\mu > 0$. And the minus sign on the right-hand side of (3.8) is telling us that energy is lost to friction, rather than gained. This is reason why we should take $\mu > 0$.

It is natural to ask: where did the energy go?! After all, energy is certainly conserved at a fundamental level. The answer is that it went into heat. The dissipation (3.9) is a transfer of energy from the macroscopic, coherent kinetic energy of the fluid, captured by the coarse-grained velocity field $\mathbf{u}$, to some microscopic, incoherent internal motion of the underlying atoms. This internal motion is still kinetic energy, but not with any overall preferred direction. To properly account for this, we should understand how the temperature and entropy of the fluid changes due to these dissipative effects. As with friction forces in classical mechanics, we won't attempt to do this here: we will simply count this as lost energy. (We will, however, return to the interplay of heat and energy in Section 4.4 when we discuss sound waves.)

3.2 Some Simple Viscous Flows

Our first task is to explore some very simple solutions to the Navier-Stokes equation (3.2). This will allow us to build some intuition for the role that viscosity plays.

3.2.1 The No-Slip Boundary Condition

We've already seen the importance of boundary conditions in constructing fluid flows. For an inviscid flow, we introduced the obvious "you shall not pass" condition in Section 2.3

$$\mathbf{n} \cdot \mathbf{u} = 0 \tag{3.10}$$

where $\mathbf{n}$ is the normal to a solid surface. This solid surface might be the walls of the container, or an obstacle sitting in the fluid like the sphere and cylinder we studied previously. If the solid object is itself moving with some velocity $\mathbf{U}$ then this condition becomes

$$\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \mathbf{U}$$

For viscous fluids, we introduce a further boundary condition that restricts the flow *tangent* to a solid. This is the *no-slip* condition that states

$$\mathbf{t} \cdot \mathbf{u} = \mathbf{t} \cdot \mathbf{U} \tag{3.11}$$

where $\mathbf{t}$ is now the vector tangent to the boundary. This states that the velocity of the fluid along the boundary must match the velocity of the boundary itself. It is sometimes written as the requirement that $\mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$ is continuous at the boundary.

The no-slip condition (3.11) doesn't follow from the Navier-Stokes equation. Instead, it is something additional that we assert. It is, however, physically sensible and arises from the friction forces between the fluid and the boundary. Importantly, it is also the boundary condition that is observed to be correct for most experiments.

Note that the flows that we met in Section 2 describing fluids moving around spheres and cylinders do *not* obey the no-slip condition. Of course, they also failed miserably in explaining drag forces. This is our first hint that we should do a better job of describing the flows close to the boundary of an object. You might wonder why we just don't search for other solutions to the Euler equations that include the no-slip condition. The reason is that there simply aren't any such solutions. This is because the Euler equation is first order in spatial derivatives and we only get to impose one boundary condition, namely the inpenetrability condition (3.10). In contrast, the Navier-Stokes equation is second order. This means that we must impose an additional boundary condition when solving the equation. The no-slip condition is the boundary condition of choice.

3.2.2 Couette Flow

Take two, infinite parallel plates lying in the (x, y) plane and separated by some distance h in the z-direction. The bottom plate is stationary while the top plate moves with a constant speed U in the x-direction. What happens to fluid trapped between them?

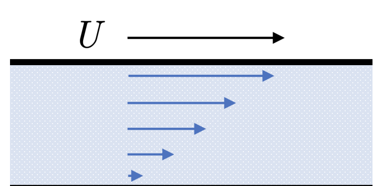
We will look for a steady flow with $\partial \mathbf{u}/\partial t$ with the velocity lying solely in the x-direction. The speed of the fluid depends only on the z direction, meaning

$$\mathbf{u}(\mathbf{x}, t) = (u(z), 0, 0)$$

With this ansatz $(\mathbf{u} \cdot \nabla)\mathbf{u} = 0$ so the material time derivative vanishes: $D\mathbf{u}/Dt = 0$. There are no pressure gradients in the fluid, so the only surviving term in the Navier-Stokes equation comes from the viscosity,

$$\mu \frac{d^2 u}{dz^2} = 0$$

The boundary conditions are $u(0) = 0$ and $u(h) = U$. This is an easy equation to solve and the velocity profile must increase linearly to match the speeds of the two plates,



$$u(z) = \frac{Uz}{h}$$

The result is known as *Couette flow* and is shown in the figure. Flows of this kind, in which adjacent layers of fluids move at different speeds, are collectively referred to as *shear flows*.

Couette flow is not a potential flow. The simplest way to see this is to note that, even though the flow doesn't look like its rotating, it has vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = (0, U/h, 0)$$

This vorticity arises because we've implemented the no-slip boundary condition, ensuring that the upper plate drags the fluid along with it. This suggests that the no-slip boundary condition may be a way to generate vorticity. We will see later that this is an important observation.

It is a simple matter to compute the stress extered on the fluid using (3.5),

$$\sigma = \begin{pmatrix} -P & 0 & \mu U/h \\ 0 & -P & 0 \\ \mu U/h & 0 & -P \end{pmatrix}$$

This tells us that the force per unit area exerted by the top plate with $\mathbf{n} = \hat{\mathbf{z}}$ is

$$\mathbf{f} = (\mu U/h, 0, -P)$$

while the bottom plate, with $\mathbf{n} = -\hat{\mathbf{z}}$ exerts an equal and opposite force. We usually think of the bottom plate, and the distance h between the plates, as fixed externally. We then ask what force we have to exert on the upper plate to keep it moving if it has (large) area A. The answer is

$$\frac{F}{A} = \frac{\mu U}{h}$$

This is operational definition of viscosity μ that we met in our first course on Newtonian Mechanics. The work done by this pushing (again, per unit area) is just $\mu U^2/h$. You can check that this agrees with the more formal definition of dissipation given in (3.9).

Circular Couette Flow

The same basic idea arises in different geometries. Consider, for example, two concentric, infinite cylinders, aligned along the z-direction. The inner cylinder has radius R_1 and rotates with angular velocity Ω_1. The outer cylinder has radius R_2 and rotates with angular velocity Ω_2.

From the geometry, we see that the flow should be rotationally invariant, meaning that it takes the form

$$\mathbf{u} = \Omega(r)\,(y, -x, 0)$$

where $r^2 = x^2 + y^2$ and $\Omega(r)$ is the angular velocity of the fluid. The no-slip condition implements the boundary conditions $\Omega(R_1) = \Omega_1$ and $\Omega(R_2) = \Omega_2$.

This time the story is a little different because we can no longer ignore the non-linear term in the Navier-Stokes equation,

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -r\Omega^2 \hat{\mathbf{r}}$$

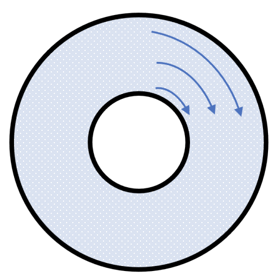
But this is something familiar, it is just the outward pointing centrifugal force that comes from the rotation of the fluid. It gives rise to a pressure gradient in the fluid, with the radial pressure $P(r)$ obeying

$$\frac{\partial P}{\partial r} = r\Omega^2 \quad \Rightarrow \quad \frac{D\mathbf{u}}{Dt} = -\nabla P$$

Such a flow obeys the Euler equation for any choice of $\Omega(r)$. But to satisfy the Navier-Stokes equation we must have, in addition,

$$\mu \nabla^2 \mathbf{u} = 0$$

A quick calculation shows that $\nabla^2\mathbf{u} = (3\Omega'/r + \Omega'')(y, -x, 0)$ so the angular velocity of the flow must take the form



$$\Omega'' = -\frac{3\Omega'}{r} \quad \Rightarrow \quad \Omega = A + \frac{B}{r^2}$$

The first term is just a constant rotation, while the second term corresponds to the irrotational line vortex that we met in Section 2.2. The no-slip boundary conditions fix these coefficients to be

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2} \quad \text{and} \quad B = (\Omega_1 - \Omega_2)\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}$$

This circular Couette flow is also known as *Taylor-Couette flow*. (Taylor gets his name attached because he discovered certain instabilities in the flow.)

3.2.3 Poiseuille Flow

Here's another simple example. Again, take a fluid sitting between two, infinite parallel plates lying in the (x, y) plane. This time is will be slight more convenient if we separate them by distance $2h$ in the z-direction. We take them to sit at $z = \pm h$.

In contrast to Couette flow, both plates are now stationary. However, this time we induce a constant pressure gradient through the fluid

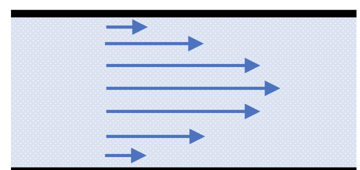
$$\frac{dP}{dx} = \text{constant}$$

We again look for a steady, shear flow of the form $\mathbf{u} = (u(z), 0, 0)$. The Navier-Stokes equation is now

$$\mu \frac{d^2u}{dz^2} = \frac{dP}{dx} = \text{constant}$$

With the no-slip boundary conditions $u(z = \pm h) = 0$, the solution is

$$u(z) = -\frac{1}{2\mu}\frac{dP}{dx}(h^2 - z^2) \qquad (3.12)$$



This is known as *Poiseuille flow*. The minus sign is sensible: it tells us that if the pressure is greatest to the left, so $dP/dx < 0$, then the fluid moves to the right. Clearly the speed increases as we move away from the edges and is maximum in the middle where $z = 0$. Again, the flow has vorticity induced by the no-slip boundary condition.

The stress (3.5) is

$$\sigma = \begin{pmatrix} -P(x) & 0 & z\, dP/dx \\ 0 & -P(x) & 0 \\ z\, dP/dx & 0 & -P(x) \end{pmatrix}$$

and, perhaps surprisingly, is independent of the viscosity. The top and bottom plates have normal $\mathbf{n} = \pm\hat{\mathbf{z}}$ and sits at $z = \pm h$, giving a force per unit area

$$\mathbf{f} = (h\frac{dP}{dx}, 0, \mp P(x))$$

The force exerted by each plate is now in the negative x direction, as it should be.

Circular Poiseuille Flow

A simple generalisation of this story describes flow down a circular pipe of radius R with a constant pressure gradient. We work in cylindrical polar coordinates, (r, θ, x) with

$$\frac{dP}{dx} \neq 0$$

The velocity takes the form

$$\mathbf{u} = u(r)\hat{\mathbf{x}}$$

The Navier-Stokes equation is

$$\mu\nabla^2 u = \frac{dP}{dx} \quad \Rightarrow \quad \frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{dP}{dx}$$

The solution with the appropriate boundary conditions is

$$u(r) = -\frac{1}{4\mu}\frac{dP}{dx}(R^2 - r^2)$$

This is known as *Hagen-Poiseuille flow*.

3.2.4 Vorticity Revisited and the Burgers Vortex

As our final example of a flow, we will look at something that swirls. This gives us the opportunity to revisit vorticity in the presence of viscosity.

Previously we derived the vorticity equation (2.20) from the Euler equation. We can follow the same steps, now taking the curl of the Navier-Stokes equation to find

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega} \tag{3.13}$$

This is the *vorticity equation* for a viscous fluid. The term due to viscosity, naturally written in terms of $\nu = \mu/\rho$, should be viewed as analogous to the diffusion term in the heat equation. Just as viscosity gives rise to diffusion of momentum, so it gives rise to diffusion of vorticity too. It is telling us that if there is some vorticity localised in some region of space, the viscosity will tend to make it diffuse into neighbouring regions. For example, if you blow a smoke ring then the size of the ring will grow over time as the vorticity diffuses into neighbouring regions.

For inviscid fluids, the Kelvin circulation theorem told us that $\Gamma = \oint_{C(t)} \mathbf{u} \cdot d\mathbf{x}$ doesn't change for curves $C(t)$ that move with the fluid. You can check that the addition of the viscosity term means that the circulation is no longer conserved in the full Navier-Stokes equations.

Burgers Vortex

To highlight how viscosity changes the physics, we can return to the vortex solution that we saw back in Section 2.2. There we looked at a combination of a strain and rotation,

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_{\text{strain}}(\mathbf{x}) + \mathbf{u}_{\text{rot}}(\mathbf{x},t) \quad \text{with} \quad \begin{cases} \mathbf{u}_{\text{strain}} = \alpha(-x,-y,2z) \\ \mathbf{u}_{\text{rot}} = f(r,t)(-y,x,0) \end{cases}$$

The strain part of the flow stretches the fluid in the z-direction, while squeezing in the (x,y)-plane; the rotational flow clearly rotates in the (x,y)-plane, giving rise to a vorticity $\boldsymbol{\omega} = (0,0,\omega)$ with ω given by (2.16),

$$\omega = \frac{1}{r}\frac{d}{dr}(r^2 f) \tag{3.14}$$

The vorticity equation (3.13) is a partial differential equation for ω,

$$\frac{\partial\omega}{\partial t} - \alpha r\frac{\partial\omega}{\partial r} - 2\alpha\omega = \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\omega}{\partial r}\right) \tag{3.15}$$

Previously we solved this equation when $\nu = 0$ to find an example of vortex stretching (2.22). The solution we found was time dependent, with $\omega(r,t) = e^{2\alpha t}W(re^{\alpha t})$ and shows the magnitude of vorticity increasing exponentially, while being squeezed in the (x,y) plane so that the overall flux is conserved, in a way that is consistent with the circulation theorem.

Now we want to solve the vorticity equation with $\nu \neq 0$ to include the effect of viscosity. We already noted that the contribution $\nu\nabla^2\boldsymbol{\omega}$ to the vorticity equation looks like a diffusion term. This suggests that we might be able to find a time independent solution in which the squeezing of vorticity is balanced by an outward diffusion caused by the viscosity. For steady solutions, the equation (3.15) becomes

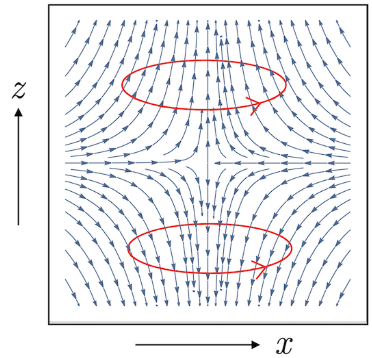
$$\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha r^2\omega + \nu r\frac{\partial\omega}{\partial r}\right) = 0$$

We can integrate once to get

$$\frac{\partial\omega}{\partial r} = -\frac{\alpha r}{\nu}\omega$$

where we've set the integration constant to zero by requiring that ω and ω' decay suitably quickly asymptotically. This equation gives an exponentially localised vorticity

$$\omega(r) = \frac{\Gamma\alpha}{2\pi\nu}\, e^{-\alpha r^2/2\nu}$$



Here Γ is a constant that the determines the overall magnitude of vorticity. The slightly strange combination of constants that accompany it ensure that Γ can also be identified with the asymptotic circulation of the flow,

$$\Gamma = \int_S \boldsymbol{\omega}\cdot d\mathbf{S} = 2\pi\int_0^\infty dr\; r\omega(r)$$

We can then solve (3.14) to get the associated profile function for the angular velocity,

$$f(r) = \frac{\Gamma}{2\pi r^2}\left(1 - e^{-\alpha r^2/2\nu}\right)$$

This is *Burgers vortex solution*. It is the simplest model for a hurricane.

We can compute the dissipation due to the vortex. We first rewrite our previous formula (3.9) as

$$\text{Dissipation} = 2\mu\int d^3x\; E_{ij}E_{ij} = \mu\int d^3x\; |\boldsymbol{\omega}|^2$$

where some simple algebra shows that the difference is a boundary term which vanishes at infinity. It is now a simple computation to get the dissipation per unit length

$$\text{Dissipation per unit length} = 2\pi\mu\int_0^\infty dr\; r\omega^2 = \frac{\Gamma^2\alpha\rho}{4\pi}$$

where the density ρ has made an appearance through $\mu = \nu\rho$. Curiously, for fixed circulation Γ, the dissipation is independent of the viscosity ν.

3.3 Dimensional Analysis

The Navier-Stokes equation is

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{u} \tag{3.16}$$

Each term has dimension LT^{-2}. This means that the dimension of the kinematic viscosity ν is

$$[\nu] = L^2T^{-1}$$

Fluid	Kinematic Viscosity (m^2s^{-1})	Dynamic Viscosity (Nsm^{-2})
Air	1.5×10^{-5}	1.8×10^{-5}
Water	10^{-6}	10^{-3}
Honey	$\sim 10^{-3}$	~ 10
Pitch	$\sim 10^{5}$	$\sim 10^{8}$

Table 1. The viscosities of some substances at room temperature.

Meanwhile, the dimension of dynamic viscosity $\mu = \rho\nu$ is

$$[\mu] = ML^{-1}T^{-1}$$

Values of these viscosities for various fluids are shown in Table 1. To get a sense of the scales involved, we can do some further dimensional analysis. The kinematic viscosity has dimension of velocity times distance. For a fluid, the relevant internal velocity (as opposed to the velocity of some flow) is the speed of sound, c_s. On dimensional grounds, this is given by

$$c_s \sim \sqrt{\frac{k_B T}{m}}$$

where T is the temperature, k_B is Boltzmann's constant and m is the mass of the constituent atom or molecule. (We'll derive this formula, together with the overall coefficient, in Section 4.4. You can also find a derivation in the lectures on Kinetic Theory.) Meanwhile, the relevant distance scale is the average separation a of atoms in the fluid. This suggests that the viscosity should be of order

$$\nu \sim c_s a$$

For water, $c_s \sim 1000\ \mathrm{ms}^{-1}$, with a characteristic separation between molecules of $a \sim 10^{-9}$ m. This gives $\nu \sim 10^{-6}\ \mathrm{m}^2\mathrm{s}^{-1}$ which is, indeed, in the right ballpark.

For some fluids, the internal molecular forces are strong, resulting in a much higher viscosity. Honey is a particularly familiar example. One of the most viscous fluids is pitch, also known as tar, which has a viscosity many orders of magnitude higher than water[3].

[3]A pitch drop experiment was set up in the University of Queensland, Australia in 1927. The flow of the pitch is ten times slower than continental drift. To date, nine drops have fallen. None have been witnessed. A webcam was set up in the 1990s, but was offline when the eighth drop fell in 2000. The ninth drop was accidently broken off before it fell in 2014. You can watch and wait for the tenth drop here although there's likely to be another eight years or so before anything actually happens.

At the other end of the spectrum, superfluids, such as Helium-4 at low temperatures, have strictly zero viscosity. This is very much a quantum mechanical effect and a proper description requires us to leave the comfortable classical realm of these lectures.

3.3.1 The Reynolds Number

Solving the Navier-Stokes equation (3.16) in full generality is, to put it mildly, a challenging problem. We make progress only by making some approximation. This involves deciding which terms, if any, can be ignored in any given situation. The obvious thing to do is to ask whether the viscosity is small or large. But this question in itself doesn't make any sense. Viscosity is dimensionful. There's no meaning to it being absolutely small or absolutely large. It can only be small or large relative to something else.

That something else depends on the flow. Suppose that the flow has a characteristic speed U and length L. Here U could be the speed of the fluid relative to some boundary, or the rotational speed of the fluid. Similarly L could be some geometrical distance over which the flow changes. From this we can construct a dimensionless ratio called the *Reynolds number*

$$Re = \frac{UL}{\nu} \tag{3.17}$$

Roughly speaking, this captures the relative importance of the inertial term $\mathbf{u}\cdot\nabla\mathbf{u}$ and the viscosity term $\nu\nabla^2\nu$,

$$\frac{\text{intertial term}}{\text{viscosity term}} = \frac{|\mathbf{u}\cdot\nabla\mathbf{u}|}{|\nu\nabla^2\mathbf{u}|} \sim \frac{U^2/L}{\nu U/L^2} \sim Re$$

With very broad brush, fluid flows can be characterised in one of two different types:

- <u>High Reynolds Number, $Re \gg 1$:</u> In this case, the flow is inertia dominated. In many cases, we can drop the viscosity term and return to the Euler equation that we studied in Section 2. Flows at high Reynolds number have an associated time scale that comes from equating the kinetic term $\partial\mathbf{u}/\partial t$ with the inertial term. This time scale is simply the time it takes the fluid to move some distance: $T \sim L/U$.

  For example, for the flow past an aircraft wing, $U \sim 100\ \mathrm{ms}^{-1}$ while $L \sim 1\mathrm{m}$ is the width of the wing. Using the value $\nu \sim 10^{-5}$, we have $Re \sim 10^7 \gg 1$ which suggests that the viscosity term is unimportant for such flows.

  However, this example also suggests that we should be nervous about such simple arguments. If we can really neglect viscosity at high Reynolds number then we run smack into the d'Alembert paradox that we met previously because, as we saw in Section 2.3, the Euler equation doesn't correctly capture the drag force that a fluid exerts on an object. Indeed, the argument that we can ignore the viscosity term is precisely what led to physicists being unable to understand how planes fly! We'll resolve these issues in Section 3.5 where we will see that, even at high Reynolds number, there can be situations where the viscosity term is important after all because it qualitatively changes certain aspects of the flow, in particular through the introduction of a so-called "boundary layer".

- Low Reynolds Number, $Re \ll 1$: In this case, the flow is dominated by viscosity. If we ignore both the inertial term and the pressure term, then the Navier-Stokes equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \nabla^2 \mathbf{u} \tag{3.18}$$

  As we've seen previously, this is heat equation that describes diffusion. It's telling us that flows at low Reynolds number exhibit diffusive transport of momentum, with the kinematic viscosity understood as a measure of momentum diffusivity. In this regime, the time scale associated to the flow is $T \sim L^2/\nu$.

  For example, consider a bug of size $L \sim 10^{-5}$ m moving in water. It could be bombing along at a whopping $U \sim 10^{-5}$ ms^{-1} – that's one body length every second – but the associated Reynolds number is still $Re \sim 10^{-4}$. The bug's world, viscosity rules. We'll explore the low Reynolds world further in Section 3.4

Other Dimensionless Ratios

In different circumstances, there are further dimensionless ratios that we can form to characterise a flow and help us formulate good approximations to the equations. For example, if there is some characteristic time scale T to the flow – perhaps because the flow is being forced in some way – then we can form the *Strouhal number*

$$Sr = \frac{L}{UT}$$

This is also written as $Sr = L\omega/U$, with ω the frequency of oscillation. The Strouhal number tells us the relative importance of the acceleration term $\partial \mathbf{u}/\partial t \sim U/T$ and the inertial term $\mathbf{u} \cdot \nabla \mathbf{u} \sim U^2/L$, with the acceleration term dominant when $Sr \gg 1$.

We get further dimensionless numbers when we add further forces. For example

- The *Euler number* captures the relative importance of pressure gradients to the inertial term
$$Eu = \frac{\Delta P}{\rho U^2}$$
- The *Froude number* captures the relative importance of the inertial term $\sim U^2/L$ to the gravitational force $\sim g$
$$Fr = \frac{U}{\sqrt{gL}}$$

We'll meet other dimensionless quantities as these lectures progress. In Section 4.6, we'll come across the *Mach number* which measures how fast the flow is compared to the speed of sound, and in Section 5.3 the *Rayleigh number* and *Prandtl number*, both of which play a role when temperature differences are important.

3.3.2 Scaling

We upgraded ourselves from the Euler equation to the Navier-Stokes equation by adding a higher derivative term $\nabla^2\mathbf{u}$. But if we're happy to add a term with two derivative, why not further terms with four derivatives? Or sixteen derivatives? Why should we stop here?

In fact there is a reason why higher derivative terms are irrelevant, at least if we look on suitably large distance scales. (The term "irrelevant" has a technical meaning in the language of physics, but happily it coincides with the usual meaning in this context!) To see this note that, in the absence of any external force, the Navier-Stokes equation
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{u}$$
has a novel scaling symmetry
$$t \to \lambda^2 t \quad , \quad \mathbf{x} \to \lambda\mathbf{x} \quad , \quad \mathbf{u} \to \lambda^{-1}\mathbf{u} \quad , \quad P \to \lambda^{-2}P \tag{3.19}$$
The whole Navier-Stokes equation scales with an overall factor of λ^{-3} under this scaling. But, crucially, all terms scale in the same way. This means that if we find one solution to the Navier-Stokes equations, then we can always rescale by some factor λ and get another solution. Because the spatial coordinate scales as $\mathbf{x} \to \lambda\mathbf{x}$, as we increase λ any features in the flow – for example, vortices – will clearly get bigger. Note that the Reynolds number (3.17) is invariant under this scaling: $Re \to Re$. This, in large part, is why it's important: the Reynolds number is a scale-invariant way of characterising a flow.

Now suppose that, in a fit of excitement, you decide that you'd like to add further terms to the Navier-Stokes equation. You should retain rotational symmetries and Galilean boosts (i.e. constant shifts of $\mathbf{u}$) but otherwise you can write down anything you like. The terms that you add will contain some number of time derivatives, spatial derivatives and factors of the fields $\mathbf{u}$ and P. Schematically, we might have

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{u} + \mathcal{O}\left(\partial_t^{n_1}, \nabla^{n_2}, \mathbf{u}^{n_3}, P^{n_4}\right)$$

where the integers n_i, with $i = 1, 2, 3, 4$, tell us the number of the various objects that appear. We can ask how this new term fares under the scaling symmetry (3.19). We have

$$\mathcal{O}\left(\partial_t^{n_1}, \nabla^{n_2}, \mathbf{u}^{n_3}, P^{n_4}\right) \to \lambda^{-(2n_1+n_2+n_3+2n_4)}\,\mathcal{O}\left(\partial_t^{n_1}, \nabla^{n_2}, \mathbf{u}^{n_3}, P^{n_4}\right)$$

The key point is that the Navier-Stokes equation already contains the leading terms, each of which scales as λ^{-3}. Any new term that you try to construct scales away more quickly than the λ^{-3}. This means that if you try to scale a flow to larger length scales, then these additional terms play an increasingly diminished role in determining the form of the solution. In particular, on suitably large length scales they will always be less important than those that appear in the Navier-Stokes equation. This is what we mean when we say that they are irrelevant.

This isn't to say that higher derivative terms are never important under any circumstances. If the gradients of fields are large enough, then higher derivative terms will surely compete with the others. But how large do they need to be? The answer to that is governed by the coefficients of these higher derivative terms which characterise the fluid. On dimensional grounds, these coefficients must have certain length or time dimensions, with the relevant scale set by some microscopic interactions. But these new scales are then likely to be set by microscopic physics – say the mean free path of the underlying molecules – and we certainly don't expect fluid mechanics to be relevant if we have large changes on such scales.

The upshot of this discussion is that the Navier-Stokes equation is special because each of the terms scales as λ^{-3} and any other term is always more irrelevant. For this reason, we never add any higher derivative terms. Instead we go the other way! Much of our work in the remainder of these lectures will be in figuring out what terms in the Navier-Stokes equation we get to drop in certain circumstances, in the hope that the equations may actually become easy enough to solve.

3.4 Stokes Flow

At low Reynolds number, $Re \ll 1$, the flow is dominated by viscosity. In many situations, we can ignore the matter derivative $D\mathbf{u}/Dt$ completely: it is unimportant for the physics of interest. What remains are the *Stokes equations*

$$\nabla P = \mu \nabla^2 \mathbf{u} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \tag{3.20}$$

We view these as four equations in four unknowns: $\mathbf{u}$ and P. They should certainly be augmented with the no-slip boundary condition, as appropriate for our uber-viscous Stokesian world. In some circumstances, we may wish to add further external forces $\mathbf{f}$ to the first of the equations.

Solutions to the Stokes equations are known as *Stokes flows*, or sometimes *creeping flows*. They describe, among many other things, micro-organisms swimming in water.

The lack of time derivatives is unusual when solving dynamical equations. It means that the fluid reacts instantaneously to any imposed force. In some sense, the fluid has no life of its own as there are no propagating waves. Instead, it just does what it's told.

More surprising, the lack of any time derivatives means that any flow is reversible. Act with an external force $\mathbf{F}$ for some time and the fluid will evolve. Then act with the opposite force $-\mathbf{F}$ for an equal amount of time and it will evolve back again, returning to its original state. There are dramatic demonstrations of this in which some ink is dropped in a fluid at low Reynolds number. The ink doesn't disperse, but just sits there. The fluid is then stirred, and the ink swirls and mixes with the fluid as expected. But when the stirring is reversed, so too is the mixing until the ink returns to its original starting point[4]. It's the kind of behaviour that the second law of thermodynamics usually prohibits. But life is different at low Reynolds number.

There's something a little disconcerting about this reversible behaviour. Not least because, as we've seen above, dissipation in fluids only arises because of viscosity. This means that the increase of entropy in a fluid is also due to viscosity. Yet, when viscosity completely dominates, the dynamics becomes reversible and there is no increase in entropy! Or, said better, there is no dynamics since we have neglected the time derivative term.

[4]Here's a rather wonderful video demonstrating this effect. It can also be seen in this old school documentary, with the great fluid dynamicist G.I.Taylor doing the mixing.

Solving the Stokes Equations

In the remainder of this section, we'll explore Stokes flow in a number of different settings. There are some simple manipulations that we can make to highlight the mathematical structure of the Stokes equations. Taking the divergence of both sides of $\nabla P = \mu \nabla^2 \mathbf{u}$, and using the fact that $\nabla \cdot \mathbf{u} = 0$, tells us that the pressure is necessarily a harmonic function

$$\nabla^2 P = 0 \tag{3.21}$$

Meanwhile, taking the curl of both sides tells us that the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is also harmonic

$$\nabla^2 \boldsymbol{\omega} = 0 \tag{3.22}$$

Finally, acting with ∇^2 on both sides, and using the fact that $\nabla^2 P = 0$, tells us that the velocity itself is "biharmonic", meaning that

$$\nabla^4 \mathbf{u} := \nabla^2 \nabla^2 \mathbf{u} = 0$$

In some situations, this is a useful starting point for solving the equations. But, for our first application, we'll take a different route.

3.4.1 Flow Around a Sphere

We want to repeat the calculation that we did for an inviscid fluid in Section 2.3 for the flow around a sphere. In that case, we found the solution by superimposing a constant flow with a dipole flow, and then hiding the singularity behind the sphere. Because the Stokes equations are linear, it's perfectly possible that a similar strategy will again work, now for very viscous fluids. We'll see that this is indeed the case, albeit with some of the details changed.

To kick things off, we'll look for the Green's function to the equations (3.20). This is a velocity field $\mathbf{u}$ and a pressure P obeying

$$\mu \nabla^2 \mathbf{u} - \nabla P = -\mathbf{a}\, \delta^3(\mathbf{x}) \tag{3.23}$$

together with the requirement that $\nabla \cdot \mathbf{u} = 0$. The right-hand-side of (3.23) includes an arbitrary constant vector $\mathbf{a}$.

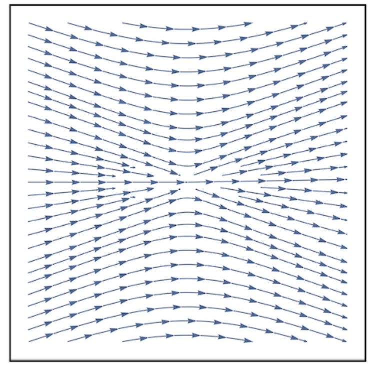
Claim: The Green's function for the Stokes equations is

$$\mathbf{u} = G\mathbf{a} \quad \text{and} \quad P = \frac{\mathbf{x} \cdot \mathbf{a}}{4\pi r^3} \tag{3.24}$$

where G is the matrix

$$G_{ij} = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right)$$

The tensor G is known as the *Stokeslet*. The two terms in G conspire to ensure that $\nabla\cdot\mathbf{u} = 0$. The Stokeslet flow is shown in the figure, with $\mathbf{a} = \hat{\mathbf{z}}$ pointing to the right.



Proof: First, we'll check that the solution (3.24) obeys $\mu\nabla^2\mathbf{u} = \nabla P$ everywhere except for $r = 0$. Then we'll check that the coefficient of the delta function works out. First look at the velocity term. We have

$$\mu\nabla^2 u_i = \frac{1}{8\pi}\nabla^2\left(\frac{a_i}{r} + \frac{x_i x_j a_j}{r^3}\right)$$

We recognise the first $1/r$ term as the Green's function for ∇^2 (we already met this interpretation when we discussed potential flows in Section 2.3), with $\nabla^2(1/r) = -4\pi\delta(\mathbf{x})$. Clearly this contributes to the delta function on the right-hand side of (3.23), but only with a coefficient of $\frac{1}{2}$. We'll see that another $\frac{1}{2}$ comes from the other terms. Staying away from $r = 0$ for now, a little bit of algebra is needed to differentiate the second term twice. We have

$$\mu\nabla^2 u_i = \frac{1}{8\pi}\partial_k\partial_k\left(\frac{x_i x_j a_j}{r^3}\right) = \frac{1}{4\pi}\left(\frac{a_i}{r^3} - 3\frac{x_i x_j a_j}{r^5}\right) \qquad \text{for } r \neq 0 \tag{3.25}$$

But now it's simple to check that this is cancelled by the pressure

$$(\nabla P)_i = \frac{1}{4\pi}\partial_i\left(\frac{x_j a_j}{r^3}\right) = \frac{1}{4\pi}\left(\frac{a_i}{r^3} - 3\frac{x_i x_j a_j}{r^5}\right) \qquad \text{for } r \neq 0$$

So we do indeed have a solution to (3.23) away from the origin. Now we just need to check that the $1/8\pi$ normalisation of G gives the correct strength for the delta function. For this we integrate over a ball of radius R centred at the origin, and use the divergence theorem to convert this into an integral over the sphere S^2_R of radius R,

$$\begin{aligned}
\int d^3x\ (\mu\nabla^2 u_k - \partial_k P) &= \int_{S^2_R} d^2S_i\ \left(\mu\partial_i G_{kj}a_j - \frac{1}{4\pi}\delta_{ik}\frac{x_j a_j}{r^3}\right) \\
&= \frac{a_j}{8\pi}\int_{S^2_R} d^2S_i\left(\partial_i\left(\frac{\delta_{kj}}{r} + \frac{x_k x_j}{r^3}\right) - 2\frac{\delta_{ik}x_j}{r^3}\right) \\
&= \frac{a_j}{8\pi}\int_{S^2_R} d^2S_i\ \left(-\frac{\delta_{jk}x_i}{r^3} - \frac{\delta_{ik}x_j}{r^3} + \frac{\delta_{ij}x_k}{r^3} - 3\frac{x_i x_j x_k}{r^5}\right)
\end{aligned}$$

At this stage, it's all about the placement of indices. The first term is straightforward: it is the usual integral of a radial field over a sphere and gives

$$\frac{a_j}{8\pi}\int_{S^2_R} d^2S_i\,\left(-\frac{\delta_{jk}x_i}{r^3}\right) = -\frac{1}{2}a_k$$

This is the same factor of $\frac{1}{2}$ contribution that we noted above. The remaining three terms in the integral must, ultimately, be proportional to δ_{kj} because that's the only invariant tensor available. A standard trick (see, for example, the lectures on Vector Calculus) is to take the trace over k and j indices and evaluate the integral: this then gives $3\times$ the coefficient in front of δ_{kj}. If we do this, we find that the second and third terms cancel, while the final term is

$$\frac{a_j}{8\pi}\int_{S^2_R} d^2S_i\,\left(-3\frac{x_ix_jx_k}{r^5}\right) = -\frac{1}{2}a_k$$

That's the extra factor of $\frac{1}{2}$ that we were looking for. We learn that our flow and pressure do indeed satisfy (3.23). $\square$

Given a basic solution like (3.23), we can always generate further solutions by differentiating. These solutions will be more singular at the origin, but drop off quicker asymptotically. This is how the dipole solution is generated for potential flow (and, in fact, for electromagnetism). And it turns out to be what we need to solve our problem of the sphere. The relevant flow is again referred to as a dipole and is given by

$$\mathbf{u}_{\text{dipole}} = (\nabla^2 G)\mathbf{a} \quad \text{with} \quad (\nabla^2 G)_{ij} = \frac{1}{4\pi\mu}\left(\frac{\delta_{ij}}{r^3} - 3\frac{x_ix_j}{r^5}\right)$$

where we computed $\nabla^2 G$ previously in (3.25). The associated pressure field is simply $P_{\text{dipole}} = 0$ because, as we saw in (3.21), the original pressure (3.23) is necessarily a harmonic function.

We now have all the ingredients to solve our problem of interest: a Stokes flow around a sphere of radius R. Importantly, this flow must satisfy the no-slip condition which means that $\mathbf{u} = 0$ for all $|\mathbf{x}| = R$.

We start with a superposition of the different flows that we've found. We take a constant flow $\mathbf{u} = \mathbf{U}$, together with some combination of the Stokeslet and dipole flows. Both the latter flows involve some constant vector $\mathbf{a}$ and, on symmetry grounds, this must be proportional to the asymptotic velocity $\mathbf{U}$. We're left with

$$\begin{aligned}\mathbf{u} &= \mathbf{U} + 4\pi\mu\alpha\left(G + \beta\nabla^2 G\right)\mathbf{U} \\ &= \mathbf{U}\left(1 + \frac{\alpha}{2r} + \frac{\alpha\beta}{r^3}\right) + (\mathbf{U}\cdot\mathbf{x})\mathbf{x}\left(\frac{\alpha}{2r^3} - \frac{3\alpha\beta}{r^5}\right)\end{aligned}$$

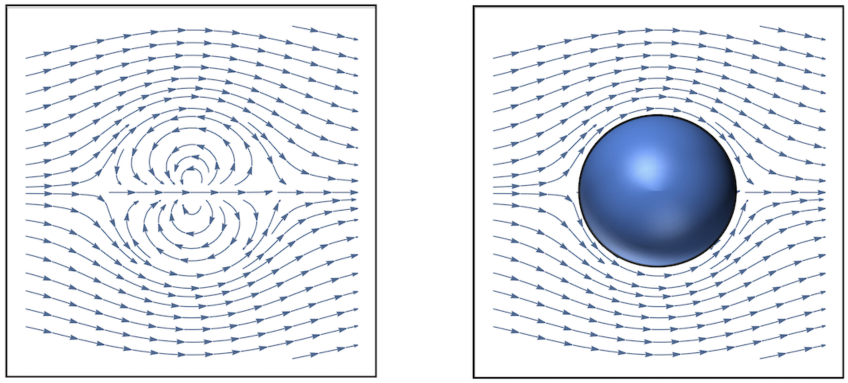


Figure 10. Stokes flow around a sphere. The left-hand figure shows the flow in the middle is in the opposite direction to the flow outside. This means that it must vanish on some surface. This surface is that of the solid sphere, as shown on the right.

where α and β are constants that are fixed by the boundary condition on the sphere. As we've seen, this requires that $\mathbf{u} = 0$ when $|\mathbf{x}| = R$, which is achieved only if both terms are individually vanishing. So we must have

$$\beta = \frac{R^2}{6} \quad \text{and} \quad \alpha = -\frac{3R}{2}$$

so our final flow for a very viscous fluid around a sphere is

$$\mathbf{u} = \mathbf{U}\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right) + (\mathbf{U}\cdot\mathbf{x})\mathbf{x}\left(-\frac{3R}{4r^3} + \frac{3R^3}{4r^5}\right) \tag{3.26}$$

This is shown in Figure 10. By eye, the flow outside the sphere doesn't look wildly different from the potential flow that we saw in Section 2.3. But there is a key difference that is clear if you look closely at the left-hand figure, before we placed the sphere over it. The fluid inside is moving in the opposite direction to the flow outside. (In contrast, for the potential flow shown in Figure 7, the fluid inside moves in the same direction as the fluid outside.) This is what ensures the existence of a surface $r = R$ for which the flow is strictly vanishing, as befits the no-slip boundary condition. This, it turns out, makes a big difference.

The difference first manifests itself in the pressure field, which is

$$P = P_\infty - \frac{3}{2}R\mu\frac{\mathbf{U}\cdot\mathbf{x}}{r^3}$$

If we take $\mathbf{U} = U\hat{\mathbf{z}}$ and work in spherical polar coordinates, the pressure on the surface of the sphere is

$$P = P_\infty - \frac{3U\mu\cos\theta}{2R}$$

This means that the pressure is bigger than P_∞ on the front of the sphere (the left in the figure) where $\pi/2 < \theta \leq \pi$ and $\cos\theta < 0$. The pressure is less than P_∞ at the back of the sphere where $0 \leq \theta < \pi/2$. This, of course, sounds very reasonable: it's simply because the flow is exerting pressure on the sphere. But it was this simple physics that was noticeably absent in the potential flow of Section 2.3 (see equation (2.31) for the analogous equation in that case). This is the first hint that we may be on the way to finally understand the drag force.

Such a Drag

The pressure is not the only force that the sphere experiences. The technology to compute the drag force comes from the stress tensor (3.5)

$$\sigma_{ij} = -P\delta_{ij} + 2\mu E_{ij}$$

where E_{ij} is the rate of strain tensor. We can compute this for the flow (3.26). It simplifies somewhat when evaluated on the surface of the sphere:

$$E_{ij}(|\mathbf{x}| = R) = \frac{3}{4R^2}(U_i x_j + U_j x_i) - \frac{3}{2R^4}(\mathbf{U}\cdot\mathbf{x})x_i x_j$$

To compute the force experienced by any point on the sphere, we consider $\sigma_{ij}n_j = \sigma_{ij}(x_j/R)$ where $\mathbf{n} = \mathbf{x}/R$ is the unit normal to the surface of the sphere. Using our expressions above, we have (ignoring the asymptotic pressure P_∞ which has no net effect on the sphere),

$$\sigma_{ij}n^j = \frac{3\mu(\mathbf{U}\cdot\mathbf{x})x_i}{2R^3} + 2\mu\left(\frac{3U_i}{4R} - \frac{3(\mathbf{U}\cdot\mathbf{x})x_i}{4R^3}\right)$$

We see that, rather nicely, the first term from the pressure cancels the final term from the strain. This means that the force acting on any point of the sphere is constant, and in the direction $\mathbf{U}$ of the asymptotic flow

$$\sigma\mathbf{n} = \frac{3\mu}{2R}\mathbf{U}$$

It's now very easy to compute the drag force: we just integrate this over the whole sphere, getting an additional factor of the surface area $4\pi R^2$. The total drag force acting on the sphere is

$$\text{Drag Force} = 6\pi\mu R\mathbf{U} \tag{3.27}$$

This is known as *Stokes' law.* It is the drag experienced by a sphere moving at very low Reynolds number.

3.4.2 Uniqueness and the Minimum Dissipation Theorem

We found a solution for the flow around the sphere. But it turns out that it is *the* solution: there is no other with the same boundary conditions. This follows from a uniqueness theorem that is proven in the same way as the uniqueness of solutions to the Laplace equation (see the lectures on Vector Calculus).

Suppose that we have two solutions, $\mathbf{u}_1$ and $\mathbf{u}_2$, both obeying non-slip boundary conditions on the surface. Then the difference $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ necessarily vanishes on the boundary. With $\tilde{P} = P_1 - P_2$ the difference in the pressure fields, we have

$$0 = \int_V \mathbf{v} \cdot (\mu \nabla^2 \mathbf{v} - \nabla \tilde{P})\, dV = \int_V \partial_i \left(\mu v_j \partial_i v_j - v_i \tilde{P} \right)\, dV - \int_V (\partial_i v_j)^2\, dV$$

The first term on the right-hand side vanishes because it's a total derivative and $\mathbf{v} = 0$ on the boundary ∂V. Moreover, the second term is the integral of a total square so this can be zero only if the integrand vanishes: $\partial_i v_j = 0$. Hence $v_j = 0$ everywhere and our original solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ were the same.

Stokes Flow Dissipates Less Than Any Other Flow

Here's a cute mathematical result. Among all the incompressible flows with the same boundary condition, the Stokes flow dissipates the least energy.

To prove this, suppose that we have a solution $\mathbf{u}$ and P to the Stokes equations with no external force (3.20), and a second flow $\tilde{\mathbf{u}}$ that satisfies the same boundary conditions but is otherwise arbitrary. Recall from (3.9) that the energy dissipated by an arbitrary flow $\tilde{\mathbf{u}}$ is (3.9)

$$\begin{aligned}
\text{Dissipation} &= 2\mu \int_V \tilde{E}_{ij}\tilde{E}_{ij}\ dV \\
&= 2\mu \int_V \left[E_{ij}E_{ij} + (E_{ij} - \tilde{E}_{ij})^2 + 2E_{ij}(\tilde{E}_{ij} - E_{ij}) \right]\ dV \\
&\geq 2\mu \int_V \left[E_{ij}E_{ij} + 2E_{ij}(\tilde{E}_{ij} - E_{ij}) \right]\ dV \\
&= \text{Stokes Dissipation} + 4\mu \int_V E_{ij}(\tilde{E}_{ij} - E_{ij})\, dV
\end{aligned}$$

We'll now show that this second integral actually vanishes. To see this, recall that the stress tensor for the Stokes flow is

$$\sigma_{ij} = -P\delta_{ij} + 2\mu E_{ij}$$

Importantly, the stress tensor is divergence free for the Stokes flow,

$$\partial_i \sigma_{ij} = -\partial_j P + \mu \nabla^2 u_j = 0 \tag{3.28}$$

where the other term in E_{ij} vanishes because it involves $\partial_i u_i = 0$ and the whole thing is equal to zero by virtue of the Stokes equations. This is the special property of the Stokes flow that we need. If we now contract the Stokes stress with the strain tensor $\tilde{E}_{ij}$ for any other flow, we have

$$\sigma_{ij}\tilde{E}_{ij} = 2\mu E_{ij}\tilde{E}_{ij}$$

where the other term $-P\tilde{E}_{ii}$ vanishes because the flow is incompressible and $\tilde{E}_{ii} = \nabla\cdot\tilde{\mathbf{u}}$. We now have

$$\begin{aligned} 4\mu \int_V E_{ij}(\tilde{E}_{ij} - E_{ij})\, dV &= 2\int_V \sigma_{ij}(\tilde{E}_{ij} - E_{ij})\, dV \\ &= 2\int_V \sigma_{ij}\left(\partial_i \tilde{u}_j - \partial_i u_j\right)\, dV \\ &= 2\int_V \partial_i\left[\sigma_{ij}(\tilde{u}_j - u_j)\right]\, dV = 0 \end{aligned}$$

where, in the second line, we've used the fact that σ_{ij} is symmetric and, in the final line, we've used the special property of the Stokes flow (3.28), together with the divergence theorem which means that the integral only cares about the boundary where, by assumption, $\tilde{\mathbf{u}} = \mathbf{u}$. The upshot is that for any flow $\tilde{\mathbf{u}}$ that is *not* a Stokes flow, we necessarily have

$$\int_V \tilde{E}_{ij}\tilde{E}_{ij}\, dV > \int_V E_{ij}E_{ij}\, dV$$

The dissipation from other flows is always greater than the corresponding Stokes flow. This is the Helmholtz minimum dissipation theorem.

There is, it turns out, a deep relationship between drag and dissipation, known as the *fluctuation dissipation theorem*. (We describe this in the lectures on Kinetic Theory.) The fact that the Stokes flow has the smallest dissipation translates into the statement that it also results in the smallest drag. This means that, as we increase the Reynolds number, the drag on the sphere will only increase beyond that given by Stokes law (3.27). Indeed, one can set up a perturbation expansion to understand the effects of the terms in the Navier-Stokes equation that we neglected. This is an expansion in the Reynolds number $Re \ll 1$ and the leading order term turns out to be

$$\text{Drag Force} = 6\pi\mu R\mathbf{U}\left(1 + \frac{3}{8}Re + \ldots\right)$$

3.4.3 Eddies in the Corner

As you might imagine, there are many different flows that exhibit interesting properties. Here is another one. We simply look at fluid passing around a corner. This corner has an opening angle that we denote as 2α. We want to know what happens.

This problem is effectively two-dimensional and can be solved quite straightforwardly by working in cylindrical polar coordinates and introducing a stream function $\Psi(r, \theta)$. Recall from Section 1.1.4 that the stream function allows us to construct a vector field $\mathbf{A} = \Psi\hat{\mathbf{z}}$ and, from that, an incompressible flow $\mathbf{u} = \nabla \times \mathbf{A}$. In cylindrical polar coordinates, the resulting flow is

$$\mathbf{u} = \frac{1}{r}\frac{\partial \Psi}{\partial \theta}\hat{\mathbf{r}} - \frac{\partial \Psi}{\partial r}\hat{\boldsymbol{\theta}} \tag{3.29}$$

The associated vorticity is

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = -(\nabla^2\Psi)\hat{\mathbf{z}} \quad \text{with} \quad \nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial \theta^2}$$

But we've seen in (3.22) that the vorticity $\boldsymbol{\omega}$ is harmonic for Stokes flows, which means that the stream function must be biharmonic

$$\nabla^4\Psi = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \Psi = 0$$

The form of the equation suggests that it might be fruitful to look for scale-invariant, separable solutions of the form

$$\Psi(r, \theta) = r^\lambda f(\theta)$$

for some exponent λ and some function $f(\theta)$. The biharmonic condition then becomes a differential equation for f,

$$\nabla^4\Psi = r^{\lambda-4}\left(\frac{\partial^2}{\partial \theta^2} + \lambda^2\right)\left(\frac{\partial^2}{\partial \theta^2} + (\lambda-2)^2\right) f(\theta) = 0$$

The solution is simply

$$f(\theta) = A\sin\lambda\theta + B\cos\lambda\theta + C\sin(\lambda-2)\theta + D\cos(\lambda-2)\theta$$

with four integrations constants as well as the exponent λ still to be determined. At this point we bring out some boundary conditions. We'll arrange the geometry so that the boundaries lie at $\theta = \pm\alpha$. The fluid comes in close to one boundary, and out close to the other, meaning that the radial component of the flow should be an odd function of θ. The expression (3.29) then tells us that the stream function should be an even function of θ, so $A = C = 0$.

We now have two further boundary conditions since both components of $\mathbf{u}$ must vanish along the boundary. The requirement that no fluid moves into the boundary is

$$\left.\frac{\partial\Psi}{\partial r}\right|_{\theta=\pm\alpha} = 0 \quad \Rightarrow \quad B\cos\lambda\alpha + D\cos(\lambda-2)\alpha = 0$$

Meanwhile, the no-slip condition tells us that

$$\left.\frac{\partial\Psi}{\partial\theta}\right|_{\theta=\pm\alpha} = 0 \quad \Rightarrow \quad B\lambda\sin\lambda\alpha + D(\lambda-2)\sin(\lambda-2)\alpha = 0$$

Or, combined,

$$\lambda\,\sin\lambda\alpha\,\cos(\lambda-2)\alpha = (\lambda-2)\cos\lambda\alpha\,\sin(\lambda-2)\alpha$$

This equation always has the solution $\lambda = 1$, but the conditions above tell us that if $\lambda = 1$ then $B = -D$ and, correspondingly, $\Psi = 0$. This is not what we want. So we'll look for solutions with $\lambda \neq 1$. Expand each sin and cos above in terms of $e^{i(\text{whatever})}$ and rearrange to get

$$\frac{\sin 2(\lambda-1)\alpha}{\lambda-1} = -\sin 2\alpha$$

This equation determines the exponent λ in terms of the opening angle of the corner 2α, admittedly in a slightly opaque form. To understand what it's telling us, write $x = 2(\lambda-1)\alpha$, so the equation becomes

$$\frac{\sin x}{x} = -\frac{\sin 2\alpha}{2\alpha} \tag{3.30}$$

Suppose that the opening angle α is small. Then, as you can see from Figure 11, the value of $\sin 2\alpha/2\alpha$ is large. But there is no value of x for which $\sin x/x$ has the equal negative value. So for small opening angles, we can't solve (3.30), at least not for real x.

As the opening angle gets bigger, we do get solutions. The smallest value of $\sin x/x$ occurs at the first minimum as shown in Figure 11, which sits at

$$x \approx 1.43\pi \quad \Rightarrow \quad \frac{\sin x}{x} \approx -0.217$$

This corresponds to a value of 2α given by

$$2\alpha \approx 0.813\pi \approx 146^\circ \quad \Rightarrow \quad \frac{\sin 2\alpha}{2\alpha} \approx +0.217$$

We learn that there is a critical value of the opening angle, given by

$$2\alpha_{\text{crit}} \approx 146^\circ$$

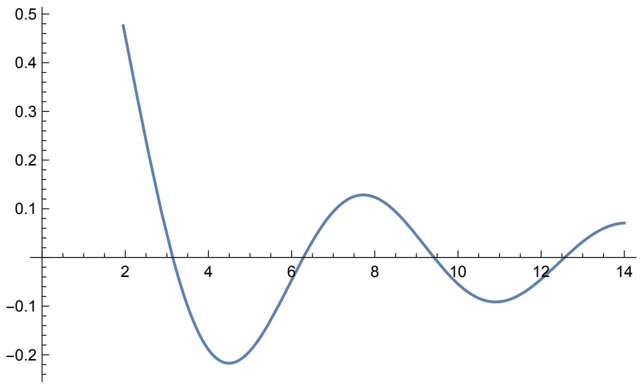
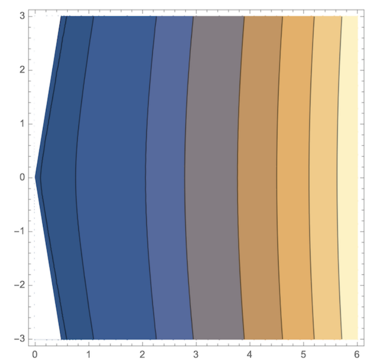


Figure 11. The graph of $\sin x/x$ with the value at the minimum around -0.217.

For opening angles larger than this, we can find solutions to (3.30). A contour plot of the stream function for $2\alpha = 160°$ is shown in the figure to the right. The lines of constant value are the streamlines and they simply flow around the corner undisturbed.



What happens when the opening angle is smaller than 146°? Now, no solutions to (3.30) exist. Or, said more precisely, no *real* solutions exist! There are, however, always complex solutions. For example, suppose that we have a right angle corner, with $2\alpha = \pi/2 < 2\alpha_{\text{crit}}$. Then there is an infinite sequence of complex solutions to (3.30), starting with

$$2\alpha = \frac{\pi}{2} \quad \Rightarrow \quad \begin{aligned} &\lambda \approx 3.74 + 1.12i \\ &\lambda \approx 7.84 + 1.66i \ \ldots \end{aligned}$$

What is the interpretation of these solutions? If we have a solution with

$$\lambda = \lambda_1 + i\lambda_2$$

then, because the velocity (3.29) is a linear function of Ψ, we can take the real part of the stream function to get

$$\Psi(r,\theta) = \text{Re}\left[r^\lambda f(\theta)\right] = r^{\lambda_1}\left[\cos(\lambda_2 \log r)\,\text{Re}f(\theta) - \sin(\lambda_2 \log r)\,\text{Im}f(\theta)\right]$$

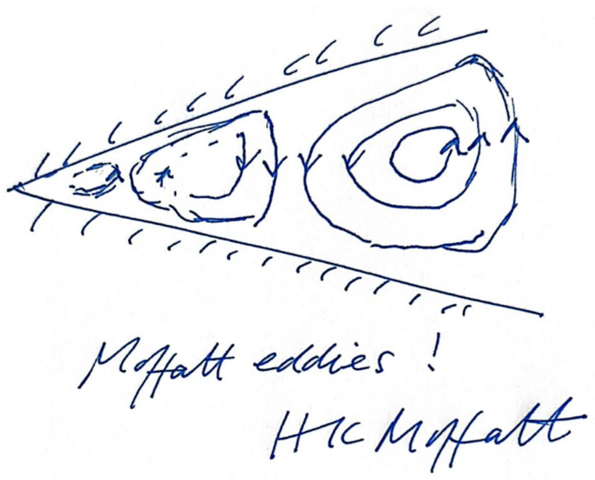
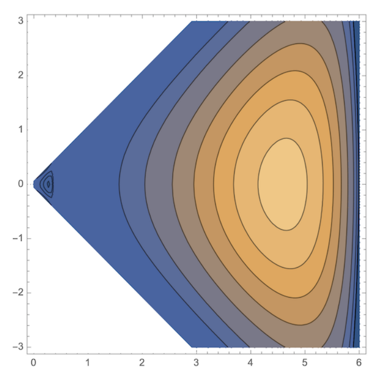


Figure 12. Not just any old Moffatt eddies, but Moffat's Moffatt eddies. My thanks to Keith Moffatt for graciously humouring my fanboy request to sketch these.



That $\cos\log r$ behaviour is striking! For a fixed angle θ, it gives rise to increasingly wild oscillations as $r \to 0$, albeit with decreasing amplitude because of the overall r^{λ_1} scaling. You can check that this means that the angular velocity $\mathbf{u} \cdot \hat{\boldsymbol{\theta}}$ is also oscillating in sign as $r \to 0$. This is telling us that the flow no longer takes the simple form, as shown in the figure for large opening angle, but instead develops eddies. In fact, there are an infinite number of these eddies, becoming increasingly small as $r \to 0$. These are known as *Moffatt eddies*.

The stream function for a right-angle corner is shown in the figure, clearly exhibiting one such eddy. The logarithm means that both the size of the eddies, and the amplitude of the stream function, vary exponentially. The centres of consecutive eddies lie at

$$\lambda_2 \log r_{n+1} = \lambda_2 \log r_n - \pi \quad \Rightarrow \quad \frac{r_{n+1}}{r_n} = e^{-\pi/\lambda_2}$$

and this also characterises the size of the eddies. (If you squint, you can just see a second eddie in the figure centred around $x \approx 0, 2$.) Meanwhile, the size of the stream function scales as

$$\frac{|\Psi(r_{n+1})|}{|\Psi(r_n)|} \sim \left(\frac{r_{n+1}}{r_n}\right)^{\lambda_1} = e^{-\lambda_1\pi/\lambda_2}$$

The magnitude of velocities involves a derivative of stream function, $u_\theta \sim \partial\Psi/\partial r$, and so scale as $(r_{n+1}/r_n)^{\lambda_1-1} = e^{-(\lambda_1-1)\pi/\lambda_2}$. For the right-angle corner shown in this figure, this ratio is around 2000. This exponential scaling doesn't just make it difficult to plot the eddies; it also makes it difficult to experimentally observe more than two or three.

Although the eddies get smaller as you approach the vertex, the flow also becomes slower so it takes significantly longer for a particle to orbit the smaller eddies than the larger ones.

3.4.4 Hele-Shaw Flow

In this short section, we look at a particular way of restricting Stokes flow to two dimensions. However, rather than simply solving the 2d version of Stokes equations, we instead do something more physical. We trap the fluid between two parallel, stationary plates, separated by a distance h. This scale will be much smaller than any other scale, such as the size of any object that the fluid moves around

We separate the plates in the z-direction and consider situations in which the fluid flows only in the (x, y)-plane

$$\mathbf{u} = (u(x, y, z), v(z, y, z), 0)$$

and we now solve the Stokes equation

$$\nabla P = \mu \nabla^2 \mathbf{u}$$

The first thing to realise is that gradients in the z direction are of order $\partial/\partial z \sim 1/h$ and so are much bigger than anything else. (These gradients can't vanish because the no-slip condition means that $\mathbf{u}$ vanishes at $z = 0$ and $z = a$ but we want it to be non-vanishing in the middle.) We work in the approximation that these z-gradients are entirely accounted for by the pressure

$$\mu \frac{\partial^2 \mathbf{u}}{\partial z^2} = \nabla P \quad \Rightarrow \quad u = \frac{1}{2\mu}\frac{\partial P}{\partial x} z(z-a) \quad \text{and} \quad v = \frac{1}{2\mu}\frac{\partial P}{\partial y} z(z-a)$$

where the boundary conditions have been chosen so that the no-slip condition is satisfied. This is the same kind of velocity profile that we saw for Poiseuille flow (3.12), but now in 2d rather than 1d. In the present context, it is known as *Hele-Shaw* flow. (One person, not two! He chose, I think rather unusually, to adopt both his father's and his mother's name.)

But Hele-Shaw flow is something very familiar: we have a situation where the 2d velocity field $\mathbf{u}_{2d} = (u, v)$ is given by

$$\mathbf{u}_{2d} = \nabla_{2d}\phi \quad \text{with} \quad \phi(x, y; z) = \frac{1}{2\mu} P(x, y) z(z - a)$$

and with $\nabla_{2d} = (\partial_x, \partial_y)$. In other words, we're back in the realm of 2d potential flow that we solved in Section 2.4. This means, for example, that if you place a cylinder between the plates, with its axis pointing in the z-direction, then the velocity flow around it coincides with the velocity (2.34) that we previously calculated.

There is an irony here. We originally introduced potential flow as a description of completely inviscid fluids. Yet the same solutions also describe extremely viscous fluids when sandwiched between plates! In fact the irony runs deeper. If you attempt to go to a regime where viscosity can be neglected – which means high Reynolds number – then another effect, known as the boundary layer, kicks in and the flows don't look at all like potential flows near objects. (We describe this in Section 3.5.) So, in fact, the only way to genuinely manufacture the inviscid potential flows of Section 2.4 is to work with very viscous fluids.

There is, however, a difference between Hele-Shaw flows and the general 2d potential flow. Hele-Shaw flows can have no circulation in the (x, y)-plane,

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{x} = 0$$

This is because, as we showed in Section 2.4, circulation arises only from potentials that are not single-valued. In contrast, the potential for Hele-Shaw flows is effectively the pressure $P(x, y)$ and this is certainly single-valued. The upshot is that Hele-Shaw flows don't include those flows shown in Figure 8 which induce a lift force on the obstacle.

3.4.5 Swimming at Low Reynolds Number

Given the obvious constraints of their biology, scallops are remarkably elegant swimmers[5]. They open their shells, then quickly close them, forcing water out through the hinges to propel themselves forward.

This strategy works in the ocean. But it would be hopeless at low Reynolds number. This is because, as we mentioned at the beginning of this section, the lack of time derivatives in the Stokes equations means that motion at low Reynolds number is reversible. When friction dominates, the speed at which a scallop opens or closes its

[5]as this video shows.

shell is irrelevant. It moves in one direction when the shell opens, and comes back the same amount when it closes. A scallop dropped in honey can no longer swim. Although it surely tastes nice.

To swim at low Reynolds number, you need a different strategy. You can't just flap your arms (or your legs or your fins) back and forth, because what is done by the forward flap is undone by the backward one. Instead, you need to change your shape in some way that is not, itself, time reversible. In other words, you need something like breaststroke.

Such non-reversible strategies have been developed by micro-organisms living in water, for whom life is lived at low Reynolds number. For example, the bacterium E. coli has a helical flagellum which rotates to make it swim[6].

In this section, we describe a very simple model that captures the essence of swimming at low Reynolds number. It also captures the tension between finding a mathematical model that is easy to solve, and finding one that looks vaguely like a living creature.

An Infinite, Wavey Plate

As our proxy micro-organism, we take an infinite thin plate, lying in the (x, z)-plane. (Admittedly, this object is unlikely to make a good pet.) The plate "swims" by wriggling so that a wave passes down in the x-direction, meaning that the position of the plate in the y-direction, which is perpendicular to the flat plate, is

$$y(x) = A \sin(kx - \omega t)$$

Here A is the amplitude of the wave, which has wavenumber k and frequency ω. Said another way, the wave has wavelength $\lambda = 2\pi/k$ and travels with speed

$$c = \frac{\omega}{k}$$

We want to understand the flow that results from this wriggling. Our goal is to show that the wriggling induces a constant asymptotic velocity in the fluid. This isn't quite swimming of course: it's staying still and making all the universe move around you. But, by Galilean relativity, this is equivalent to the fluid staying still and the plate moving. And that's what we mean by swimming.

[6]A number of videos of micro-organisms swimming can be found on the webpage of Howard Berg, one of the pioneers of low Reynolds number physics. The story is described further in a famous and charming paper by E. Purcell called "Life at Low Reynolds Number". The underlying theory is closely related to the rotation of deformable (as opposed to rigid) bodies that is covered in the lectures on Classical Dynamics.

To proceed, we introduce a stream function $\Psi(x,y)$, so that $\mathbf{u} = (u,v,0)$ with

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \Psi}{\partial x}$$

Repeating the argument that we saw for corner flows, we know that $\boldsymbol{\omega} = \nabla \times \mathbf{u} = -\nabla^2 \Psi\, \hat{\mathbf{z}}$ and $\nabla^2 \boldsymbol{\omega} = 0$, which, combined, tell us that

$$\nabla^4 \Psi = 0$$

We must solve this, subject to the no-slip requirement that the velocity of the flow matches that of the plate,

$$u = 0 \quad \text{and} \quad v = -A\omega \cos(kx - \omega t) \quad \text{on} \;\; y = A\sin(kx - \omega t) \tag{3.31}$$

We'll also impose suitable boundary conditions asympotically. We'll flag these up as we go along.

Simple as the equations above are, it's not straightforward to solve them because the boundary condition (3.31) is evaluated on the waving plate. To proceed, we need an approximation. Roughly speaking, we want the amplitude of the wave A to be small in the hope that the boundary condition is easier to implement. But A is dimensionful, so it has to be small relative to something else and the only other length scale we have is the wavelength. So the relevant dimensionless expansion parameter is

$$\epsilon = Ak \ll 1$$

To understand how to write our equations in terms of a Taylor expansion, it's useful to introduce dimensionless distances and a dimensionless stream function

$$\tilde{x} = kx \quad , \quad \tilde{y} = ky \quad , \quad \tilde{\Psi} = \frac{k\Psi}{A\omega}$$

The boundary conditions (3.31) then become

$$\frac{\partial \tilde{\Psi}}{\partial \tilde{y}} = 0 \quad \text{and} \quad \frac{\partial \tilde{\Psi}}{\partial \tilde{x}} = \cos(\tilde{x} - \omega t) \quad \text{on} \;\; \tilde{y} = \epsilon \sin(\tilde{x} - \omega t)$$

Now we can see that, for $\epsilon \ll 1$, we do indeed impose the boundary condition on a value of $\tilde{y}$ that is small. It means that we can Taylor expand the function $\tilde{\Psi}$ around $\tilde{y} = 0$, so these boundary conditions read

$$\left.\frac{\partial \tilde{\Psi}}{\partial \tilde{y}}\right|_{\tilde{y}=\epsilon\sin(\tilde{x}-\omega t)} = \left.\frac{\partial \tilde{\Psi}}{\partial \tilde{y}}\right|_{\tilde{y}=0} + \epsilon \sin(\tilde{x} - \omega t) \left.\frac{\partial^2 \tilde{\Psi}}{\partial \tilde{y}^2}\right|_{\tilde{y}=0} + \ldots = 0 \tag{3.32}$$

and

$$\left.\frac{\partial \tilde{\Psi}}{\partial \tilde{x}}\right|_{\tilde{y}=\epsilon \sin(\tilde{x}-\omega t)} = \left.\frac{\partial \tilde{\Psi}}{\partial \tilde{x}}\right|_{\tilde{y}=0} + \epsilon \sin(\tilde{x}-\omega t)\left.\frac{\partial^2 \tilde{\Psi}}{\partial \tilde{x}\partial \tilde{y}}\right|_{\tilde{y}=0} + \ldots = \cos(\tilde{x}-\omega t) \tag{3.33}$$

We now expand the stream function itself in powers of ϵ,

$$\tilde{\Psi} = \tilde{\Psi}_0 + \epsilon \tilde{\Psi}_1 + \ldots$$

Each $\tilde{\Psi}_n$ is biharmonic, meaning that it satisfies $\nabla^4 \tilde{\Psi}_n = 0$ for $n = 0, 1, 2, \ldots$. But each $\tilde{\Psi}_n$ obeys different boundary conditions at $\tilde{y} = 0$. We start with $\tilde{\Psi}_0$ which has boundary conditions

$$\frac{\partial \tilde{\Psi}_0}{\partial \tilde{y}} = 0 \quad \text{and} \quad \frac{\partial \tilde{\Psi}_0}{\partial \tilde{x}} = \cos(\tilde{x}-\omega t) \quad \text{on} \quad \tilde{y} = 0$$

The biharmonic function obeying these boundary conditions in the region above the plate, $\tilde{y} > 0$, is

$$\tilde{\Psi}_0 = (1+\tilde{y})e^{-\tilde{y}} \sin(\tilde{x}-\omega t) \tag{3.34}$$

which obeys $\tilde{\nabla}^2 \tilde{\Psi}_0 = -2e^{-\tilde{y}} \sin(\tilde{x}-\omega t)$ and so $\tilde{\nabla}^4 \tilde{\Psi}_0 = 0$. Note that we've thrown away a similar solution that scales as $e^{+\tilde{y}}$ on the grounds that it gives an unbounded velocity field as $\tilde{y} \to +\infty$. A solution of this kind is relevant below the plate for $\tilde{y} < 0$.

The first correction to this solution is $\tilde{\Psi}_1$ which, from (3.32) and (3.33), obeys

$$\frac{\partial \tilde{\Psi}_1}{\partial \tilde{y}} + \sin(\tilde{x}-\omega t)\frac{\partial^2 \tilde{\Psi}_0}{\partial \tilde{y}^2} = 0 \quad \text{and} \quad \frac{\partial \tilde{\Psi}_1}{\partial \tilde{x}} + \sin(\tilde{x}-\omega t)\frac{\partial^2 \tilde{\Psi}_0}{\partial \tilde{x}\partial \tilde{y}} = 0$$

Both boundary conditions should again be imposed at $\tilde{y} = 0$. Using our solution (3.34) for $\tilde{\Psi}_0$, these become

$$\frac{\partial \tilde{\Psi}_1}{\partial \tilde{y}} = \sin^2(\tilde{x}-\omega t) \quad \text{and} \quad \frac{\partial \tilde{\Psi}_1}{\partial \tilde{x}} = 0 \quad \text{on} \quad \tilde{y} = 0$$

The $\sin^2$ term is where our interest lies. We decompose this into Fourier modes by using the double angle formula $\sin^2 \tilde{x} = \frac{1}{2}(1-\cos 2\tilde{x})$. The $\cos 2\tilde{x}$ term is just telling us that the second harmonic is excited. That's little surprise. The constant term is more interesting as it tells us that there must be a constant component to the fluid motion. Indeed, you can check that the biharmonic function obeying these boundary conditions is

$$\tilde{\Psi}_1 = \frac{1}{2}\tilde{y} - \frac{1}{2}\tilde{y}e^{-2\tilde{y}} \cos(2(\tilde{x}-\omega t))$$

Again, this solution holds above the plate for $\tilde{y} > 0$. There is again an analogous solution with a $e^{+\tilde{y}}$ below the plate but with the same constant $\frac{1}{2}\tilde{y}$ term. That linear term is what we're after. Putting the various constants back in, it gives a contribution to the stream function that looks like $\Psi = \frac{1}{2}A^2k^2cy + \ldots$ where the $\ldots$ are the oscillatory terms that drop off as e^{-ky} or e^{-2ky} as we move away from the plate. The constant term is telling us that, far away from the plate, there is necessarily a constant fluid velocity

$$\mathbf{u} \to \frac{1}{2}A^2k^2c\,\hat{\mathbf{x}} \quad \text{as } y \to +\infty$$

Alternatively, if we boost to another frame so that the fluid is asymptotically stationary, then the plate must be moving to the left with speed $U = \frac{1}{2}A^2k^2c$. In other words, the plate is swimming. The speed is proportional to the speed c with which waves propagate down the plate but, at least in this approximation, suppressed by $\epsilon^2 = A^2k^2 \ll 1$.

3.5 The Boundary Layer

In the previous section, we focussed on very viscous flow at low Reynolds number. Now we turn to the opposite regime of high Reynolds number. We're going to revisit the question of flows around some fixed object, like a sphere or the wing of an aircraft.

When the Reynolds number is large, the inertia term in the Navier-Stokes equation should dominate over the viscosity term,

$$Re = \frac{\text{intertial term}}{\text{viscosity term}} = \frac{|\mathbf{u}\cdot\nabla\mathbf{u}|}{|\nu\nabla^2\mathbf{u}|} \gg 1$$

For example, for a plane flying we have $Re \sim 10^7$. Given this, it's tempting to think that we can drop the viscosity term completely. But this brings us back to the Euler equation and, as we have seen in Section 2, inviscid flows do not give rise to any drag on an object. Something is amiss! In fact it turns out that, no matter how small the viscosity, it still plays an important role.

Mathematically this is because the character of the Navier-Stokes equation changes if we set $\nu = 0$. With $\nu \neq 0$, we have an equation that is second order in spatial derivatives. When $\nu = 0$, it changes to an equation that is first order. As we have commented previously, this means that we must impose two boundary conditions when solving the $\nu \neq 0$ Navier-Stokes equation, but only a single boundary condition when solving the Euler equation. The boundary condition that is expendable is the no-slip condition and, in its absence, solutions exhibit no drag force. However, as soon as we have ν, no matter how small, we're back in business and we can impose the no-slip condition to our heart's content.

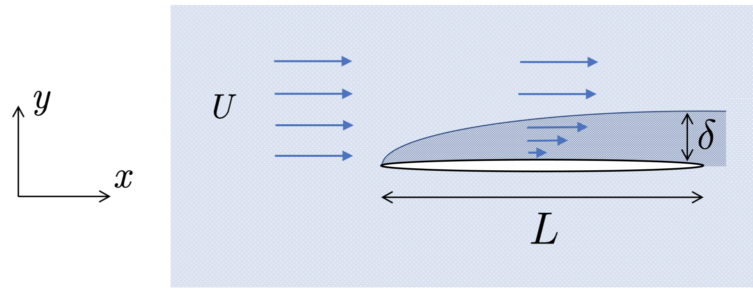


Figure 13. An exaggerated picture of the boundary layer forming over a thin plate. A second boundary layer will, of course, also form below.

Physically, a continuous flow with a no-slip boundary condition must have a layer of almost-stationary fluid sitting next to the object. This is the *boundary layer*. The purpose of this section is to understand some of its properties.

We can make progress with some simple dimensional analysis, coupled with a little intuition built on what we've learned so far. For example, one of the most basic questions that we can ask is: what is the width of the boundary layer? It seems plausible that when the fluid first hits the leading edge of the object, only those molecules immediately in contact know about its existence. But, as we look further down the flow, more and more of the fluid should be affected. How much?

Suppose that our object has length L, and travels relative to the fluid with speed U. The Reynolds number is then

$$Re = \frac{UL}{\nu}$$

By assumption, $Re \gg 1$.

Any fluid element takes a time $T = L/U$ to move past the object. Close to the object, the fluid will be affected by the no-slip condition and it is reasonable to think that the near-boundary behaviour mimics that of Couette or Poiseuille flow. One of the simple, yet important facts about these flows is that they have vorticity, as the fluid near the boundary travels at different speeds. And we know from the vorticity equation (3.13) that viscosity causes vorticity to diffuse, with diffusion constant ν. Importantly, diffusion spreads as $\sqrt{\text{time}}$ rather than linearly in time. This means that in the time scale T, the vorticity will diffuse a distance

$$\delta \sim \sqrt{\nu T} \sim \sqrt{\frac{\nu L}{U}} \sim \frac{L}{\sqrt{Re}} \tag{3.35}$$

This is the result for the width of the boundary layer that we wanted. It suggests that, at high Reynolds number, there are actually two length scales in the game. The first is the size L of the object. The second, $\delta \ll L$, is the width of a boundary layer that surrounds the object where the effects of both viscosity and vorticity are important. The existence of this thin boundary layer is the 1905 insight of Prandtl.

Outside of the boundary layer, we may neglect viscosity and the fluid is well described by the Eulerian flows of Section 2.3. But much of the physics is dictated by what happens inside the boundary layer where there are large velocity gradients. We want to better understand the properties of this boundary layer.

3.5.1 Prandtl's Boundary Layer Equation

As usual, we don't want to attack the full Navier-Stokes equations. Instead, we will extract the relevant equations that will suffice to model the boundary layer.

We'll set things up as follows. We consider a two-dimensional flow in the (x, y)-plane. As shown in Figure 13, we'll take a thin plate that extends in the x-direction sitting at $y = 0$. The flow is two-dimensional and we write

$$\mathbf{u} = (u, v)$$

Asymptotically, $\mathbf{u} \to (U, 0)$. We impose the no-slip boundary condition $u = v = 0$ on the plate at $y = 0$. Incompressibility tells us that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.36}$$

We'll also look only at steady flows, so there are no time derivatives. The full Navier-Stokes equations then read

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.37}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3.38}$$

We want to ask: which of these terms can we safely ignore? And which should we keep in the boundary layer?

We look at how the flow changes over a horizontal scale L. We start with the assumption that velocities vary in the x-direction only over the scale L, but may vary in the y-direction on the much smaller scale $\delta \ll L$. Our goal is to construct a consistent truncation of (3.37) and (3.38) such that the terms we're omitting are systematically smaller by a factor of the dimensionless parameter δ/L.

Our first piece of information comes from the incompressibility condition (3.36), with the terms scaling as

$$\frac{\partial u}{\partial x} \sim \frac{U}{L} \quad \text{and} \quad \frac{\partial v}{\partial y} \sim \frac{v}{\delta} \quad \Rightarrow \quad v \sim \frac{\delta}{L} U \tag{3.39}$$

So the vertical velocity v is much smaller than the horizontal velocity U. This equation is telling us that the fluid flow is deflected only through a small angle $\sim \delta/L$.

Now let's look to the Navier-Stokes equations (3.37) and (3.38). Both terms on the left-hand side of (3.37) scale as U^2/L, while both terms on the left-hand side of (3.38) scale as $U^2\delta/L^2$. This means that the equation (3.38) is significantly less important than (3.37). In particular, if we assume that the pressure terms have the same order of magnitude then this tells us that

$$\left|\frac{\partial P}{\partial y}\right| \sim \frac{\delta}{L}\left|\frac{\partial P}{\partial x}\right|$$

So, to leading order, pressure becomes a function only of the horizontal distance: $P = P(x)$.

Now we turn to the second order terms on the right-hand-side of (3.37). We have

$$\frac{\partial^2 u}{\partial x^2} \sim \frac{U}{L^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} \sim \frac{U}{\delta^2}$$

The second of these is clearly the most important, and we may ignore the $\partial^2 u/\partial x^2$ term. Moreover, assuming that the $\partial^2 u/\partial y^2$ term has the same order of magnitude as those on the left-hand side tells us that

$$\frac{U^2}{L} \sim \frac{\nu U}{\delta^2} \quad \Rightarrow \quad \delta \sim \sqrt{\frac{\nu L}{U}} \sim \frac{L}{\sqrt{Re}}$$

which confirms our earlier estimate (3.35) and reassures us that the whole approximation scheme is valid at large Reynolds number.

The upshot is that, when solving for the fluid in the boundary layer, we may ignore the y-component of the Navier-Stokes equation (3.38) and the x-component (3.37) simplifies to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{3.40}$$

This is the *Prandtl boundary layer equation*. It should be solved in conjunction with the incompressibility condition (3.36).

There is one final finesse. We know that the pressure is approximately a function only of x. This means that we are at liberty to evaluate the pressure $P(x)$ far from the boundary layer, $y \gg \delta$. But here the viscosity terms may be neglected completely, and the flow is governed by the Euler equation. The velocity field takes some profile

$$\mathbf{u} \to \ (U(x), 0) \quad \text{as} \quad y/\delta \to \infty$$

where $U(x) \to U$ as $x \to -\infty$. The Euler equation then tells us that, for a steady flow,

$$-\frac{1}{\rho}\frac{dP}{dx} = U\frac{\partial U}{\partial x} \tag{3.41}$$

which can be substituted into (3.40).

Our next task is to solve (3.40). Far from the plate, the term proportional to ν is unimportant. There is a mathematical framework to solve equations of this kind equation, whose characteristic form differs in some limit such as $\nu \to 0$. This is the theory of "matched asymptotic expansion". We won't need this in what follows. Instead, we'll look just at some simple examples.

3.5.2 An Infinite Flat Plate

Our simple example is a semi-infinite flat plate. The plate starts at $x = 0$, which we refer to as the leading edge. It then continues indefinitely.

Asymptotically, the flow is constant, $\mathbf{u} \to (U, 0)$ as $y/\delta \to \infty$ so, from (3.41), we have $dP/dx = 0$ and the Prandtl equation becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{3.42}$$

The flow is two-dimensional so we can again use a stream function $\Psi(x, y)$, such that $\mathbf{u} = (u, v)$ with

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \Psi}{\partial x}$$

If we take the stream function to scale as $\Psi \sim U\delta$ then, with the scalings described above, we expect to get $u \sim U$ and $v \sim (\delta/L)U$ which is what we want. In looking for a solution, we'll be guided by Figure 13. We know that as we move further in the x-direction, the width δ of the boundary layer grows. We will search for "self-similar" solutions in which the velocity profile within the boundary layer remains the same, but gets stretched in the y direction as the the layer grows. Mathematically, this means that we'll search for solutions of the form

$$\Psi(x,y) = U\,\delta(x)\,f(\eta)$$

where η is the rescaled y coordinate,

$$\eta = \frac{y}{\delta(x)}$$

where $\delta(x)$ is the size of the boundary layer (3.35)

$$\delta(x) = \sqrt{\frac{\nu x}{U}} \tag{3.43}$$

(Note: for once $\delta(x)$ has nothing to do with the Dirac delta function!) For our whole approximation to be valid, we required $\delta \ll L$ which, in the present context means

$$\delta(x) \ll x \quad \Rightarrow \quad x \gg \frac{\nu}{U}$$

In other words, we can only trust what follows a distance ν/U from the leading edge of the plate. It only gives a good description beyond that point.

The velocity in the x-direction is

$$u = Uf'$$

Meanwhile, the y-direction, we have

$$v = -U\delta' f - U\delta f'\frac{\partial\eta}{\partial x} = -U(f - \eta f')\delta' \tag{3.44}$$

Now we can start building the various terms in the Prandtl equation (3.42). We have

$$\frac{\partial u}{\partial x} = -Uf''\frac{\eta}{\delta}\delta' \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{U}{\delta}f'' \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \frac{U}{\delta^2}f'''$$

So putting it all together, the Prandtl equation (3.42) becomes

$$-U^2\eta\frac{\delta'}{\delta}f''f' - U^2\frac{\delta'}{\delta}(f - \eta f')f'' = \nu\frac{U}{\delta^2}f'''$$

Two of the terms happily cancel, and we're left with

$$U\delta'\delta ff'' + \nu f''' = 0$$

But, from (3.43), we have $\delta'\delta = \nu/2U$ so our problem reduces to an ordinary, third order differential equation for $f(\eta)$,

$$f''' + \frac{1}{2} f f'' = 0 \tag{3.45}$$

We need to solve this subject to the no-slip boundary condition

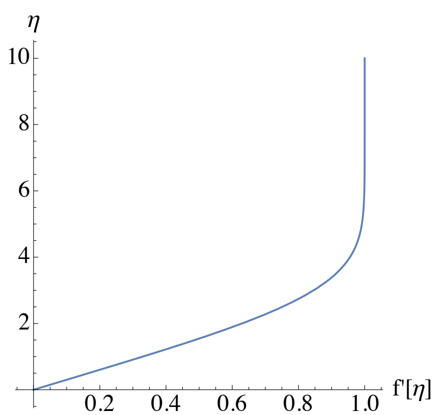
$$f = f' = 0 \quad \text{at} \quad \eta = 0$$

and the asymptotic requirement

$$f' \to 1 \quad \text{as} \quad \eta \to \infty$$

which ensures that, far from the plate, $\mathbf{u} \to (U, 0)$.

There's no analytic solution to this equation. But it's straightforward to solve the equation numerically. The resulting velocity profile is shown in the figure on the right and is known as the *Blasius boundary layer*. The distance from the plate $y \sim \eta$ is plotted vertically and the velocity $u \sim f'[\eta]$ plotted horizontally. You can see that the velocity interpolates from its zero value on the plate, to the asymptotic value. The graph also gives a more accurate estimate of the thickness of boundary layer as something like $\sim 4 - 5$ times δ, by which point the velocity is pretty much at its asymptotic value.



The numerical solution tells us something else. Asymptotically, as $\eta \to \infty$, we find that

$$f(\eta) \approx \eta - 1.72 + \mathcal{O}(1/\eta)$$

This means that, far from the plate, there is vertical component to the velocity (3.44),

$$v \approx 1.72\sqrt{\frac{\nu U}{4x}} \quad \text{as } y/\delta \to \infty$$

This is capturing what we saw previously in (3.39): the fluid is deflected by an angle $\sim \delta/L$. This angle gets smaller as we get further from the leading edge. This is because the boundary layer increases, and so the velocity gradient – which is always such that the velocity changes from zero to U – decreases as x gets larger, and this fact is reflected by the velocity component in the y-direction infinitely far from the plate. The would-be divergence at $x = 0$ is mitigated by the fact that, as we have seen, our solution only makes sense for distances $x \gg \nu/U$ from the leading edge.

The Drag Force on a Finite Plate

Strictly, the calculation above holds for an infinite plate. We've also seen that it fails within a distance ν/U of the leading edge, and one may expect that it similarly fails near the trailing edge. But we may hope that, for large L, it gives a suitable approximation of the boundary layer over much of a finite plate. With this assumption, we can compute the drag force.

The force on the plate comes from the appropriate component of the stress tensor (3.5). For a single boundary layer, we have

$$\sigma_{ij} = \rho\nu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_{y=0} = \rho\nu \frac{U}{\delta} f''(0) \tag{3.46}$$

where only the $\partial u/\partial y$ term contributes because $\partial v/\partial x$ vanishes at $y = 0$. We use the numerical solution to evaluate $f''(0) \approx 0.33$. We also need to remember that there are two boundary layers, one on each side. So the total drag force is

$$F_{\text{drag}} = 2 \times 0.33 \times \rho\nu^{1/2} U^{3/2} \int_0^L dx\, \frac{1}{\sqrt{x}} = 1.33\, \rho\nu^{1/2} U^{3/2} L^{1/2}$$

Note that the drag force increases as $\sqrt{L}$ rather than proportional to L as one might naively expect. This is because, as the boundary layer thickens, the velocity gradients decrease and, hence, so too does the stress on the plate.

This is our first honest resolution of d'Alembert's paradox: the drag force for an object at high Reynold's number, where one might think that the Euler equation is sufficient, is non-zero. We see explicitly that the drag does vanish if we set $\nu = 0$. If we embed the viscosity in the dimensionless Reynolds number $Re = UL/\nu$, we have

$$F_{\text{drag}} = 1.33\rho \frac{U^2 L}{\sqrt{Re}}$$

Taken at face value, this says that the drag force is, in fact, vanishing in the limit $Re \to \infty$. But, sadly, there's another catch awaiting us. The calculation above breaks down at large Reynolds numbers due to the effects of turbulence. Experimentally, this is found to happen at $Re \sim 10^5$ or 10^6.

3.5.3 Boundary Layers with Pressure Gradients

There is a generalisation of the ideas above that exhibits some novel behaviour within the boundary layer. This will be important in the next section when we look at the fate of the boundary layer when it leaves an object.

The generalisation involves looking at boundary layers in flows that are accelerating or decelerating asymptotically. We will again take a semi-infinite flat plate. Far from the boundary layer, the fluid flow takes the form $\mathbf{u} \to (U(x), 0)$, now with

$$U(x) = U\left(\frac{x}{l}\right)^m \tag{3.47}$$

with l some length scale and m a parameter that determines the acceleration. Note that when $m < 0$ our velocity profile (3.47) diverges at $x = 0$. We deal with this by ignoring it: our interest is only in the behaviour of the boundary layer downstream at $x > 0$.

From (3.41), we must have a pressure gradient driving this flow

$$\frac{1}{\rho}\frac{dP}{dx} = -U\frac{dU}{dx} = -\frac{mU^2}{l}\left(\frac{x}{l}\right)^{2m-1}$$

There are two distinct cases that will interest us:

- $m > 0$: Accelerating flow with $dP/dx < 0$.
- $m < 0$: Decelerating flow with $dP/dx > 0$.

This is the asymptotic pressure gradient. But, by the arguments of Section 3.5.1, there is no change in the pressure in the y-direction, perpendicular to the plate. This means that the boundary layer also experiences the pressure gradient dP/dx. Our goal is to understand how the boundary layer reacts to this gradient.

The Prandtl equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{3.48}$$

We again seek a self-similar solution, now of the form

$$\Psi(x, y) = U(x)\,\delta(x)\,f(\eta)$$

Here $U(x)$ is given by (3.47), while $\delta(x)$ is a generalisation of our previous expression for the boundary layer thickness,

$$\delta(x) = \sqrt{\frac{\nu x}{U(x)}}$$

which takes into account the x-dependence of the asymptotic velocity. Note that, for accelerating flows, the boundary layer becomes thinner, relative to the $m = 0$ case, as the flow proceeds. It becomes thicker for decelerating flows. Finally, $\eta = y/\delta(x)$ is the rescaled y-coordinate, as before.

The velocity in the x-direction and y-directions are now

$$u = Uf' \quad \text{and} \quad v = -(U\delta)'f + U\eta f'\delta'$$

After a small amount of algebra, the Prandtl equation (3.48) becomes

$$UU'f'^2 - \frac{U}{\delta}(U\delta)'ff'' = UU' + \frac{\nu U}{\delta^2}f'''$$

Now we use the explicit expression for the asymptotic velocity (3.47), which tells us that $U \sim x^m$ and $U\delta \sim x^{(m+1)/2}$. Substituting these into the equation above, we see that all terms scale as U^2/x and we may divide by this. Happily, the partial differential equation reduces once again to an ordinary differential equation,

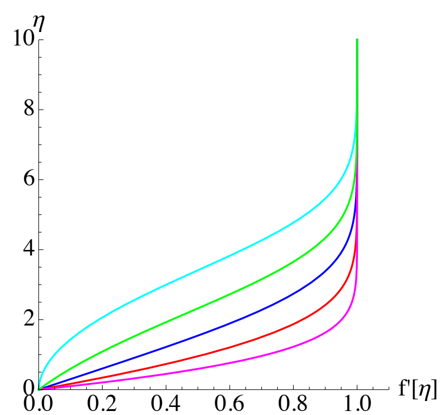
$$mf'^2 - \frac{1}{2}(m+1)ff'' = m + f'''$$

This reduces to our previous equation (3.45) when $m = 0$.

Again, we solve this subject to the boundary conditions

$$f = f' = 0 \quad \text{at } \eta = 0 \quad \text{and} \quad f' \to 1 \quad \text{as} \quad \eta \to \infty$$

The solutions are known as the *Falkner-Scan family of boundary layers*. The velocity profiles $u \sim f'(\eta)$ for a number of different flows are shown in the figure. The colours correspond, from top to bottom, to $m = -0.09$ (in cyan), $m = -0.07$ (in green), $m = 0$ (in blue), $m = 0.2$ (in red) and $m = 0.7$ (in magenta).



For accelerating flows, with $m > 0$, there isn't a great deal of difference from our previous results. One can show that the solution to the equations is unique and, as you can see from the graph, the velocity profiles all live underneath the $m = 0$ curve, coming in at ever more acute angles at the origin. This can be understood because there is a greater transfer of momentum from the accelerating fluid above. It also has consequence: the angle at which the graph intersects the origin is related to (the inverse of) $f''(0)$. As the acceleration increases, so too does $f''(0)$. But, from (3.46), means that the force imparted on the plate due to the boundary layer also increases.

At first glance, things don't look too different for decelerating flows with $m < 0$ either. Two are shown in the figure: $m = -0.07$ (in green) and $m = -0.09$ (in cyan). Now the graphs come in more steeply at the origin, corresponding to a smaller value of $f''(0)$ and, correspondingly, a smaller stress on the plate. But when we look more closely, there is a surprise waiting us: numerically, we find that for some critical value $m_{\rm crit}$, the solution actually comes into the origin vertically,

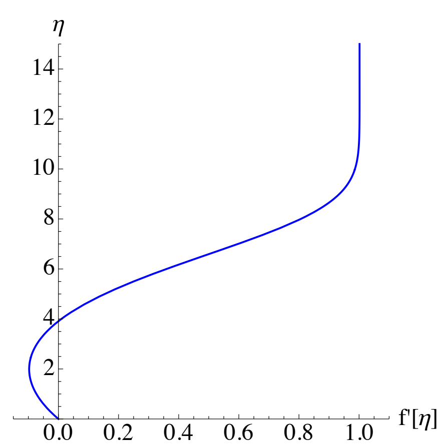
$$m = m_{\rm crit} \approx -0.0904 \quad \Rightarrow \quad f''(0) = 0$$

In other words, for a critical deceleration, there is no friction force between the plate and fluid!

What's going on here? Consider an element of fluid near the boundary. It has a force to the right due to the fluid moving above it. But there are also forces to the left, both from the pressure gradient $dP/dx > 0$ and from the viscous force of the boundary. At $m = m_{\rm crit}$, these precisely cancel. The result is that not only is $u = 0$ on the boundary, but also $du/dy = 0$.

What happens if we decrease m below the value $m_{\rm crit}$? Naively, one might have thought that one would find solutions with $du/dy < 0$, which would mean the the fluid closest to the boundary actually flows in the opposite direction. It turns out that this *doesn't* happen. There are no solutions for $m < m_{\rm crit}$.

However, there are further solutions that do exhibit reverse flows. It turns out that these solutions exist for any $m_{\rm crit} < m < 0$ where there are two branches of solutions. The first, given above, has $u > 0$ everywhere. The second has a region with $u < 0$ close to the plate. An example is shown in the figure for $m = -0.05$. It has $f''(0) \approx -0.1$. In this case, a fluid element in the region closest to the boundary has a velocity in the opposite direction to the rest of the flow. This reverse flow can be understood as the pressure gradient pushing to the left, while the force from both the fluid above it, and also from the plate, pushes to the right.



It seems that these boundary solutions with reverse flow cannot be set-up in experiment because they are thought to be unstable in this particular context. Nonetheless, the existence of such reversed boundary layers is crucial to understand the next topic that we turn to. This is the fate of the boundary layer when the boundary ends.

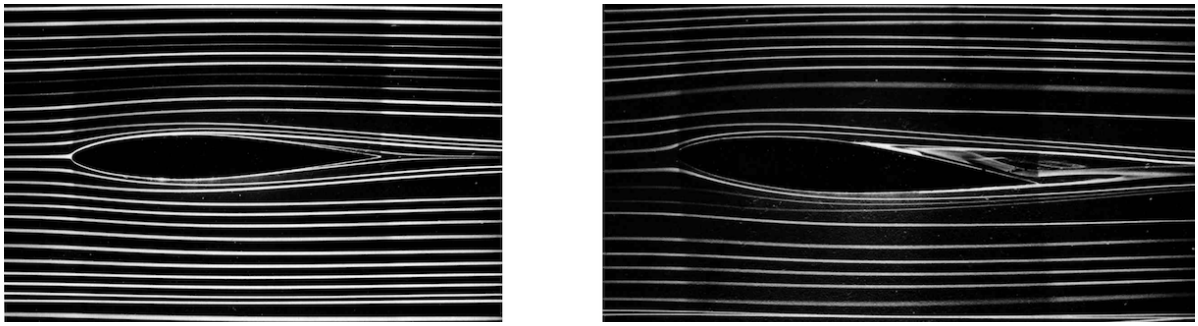


Figure 14. The flow, from left to right, around a streamlined object at $Re \approx 7000$. On the left, the object is aligned with the streamlines and the boundary layer merges smoothly into the flow at the trailing edge. On the right, the object is inclined by 5°. The boundary layer separates from the object on the upper edge.

3.5.4 Separation

So far we've understood how the boundary layer develops, but only by restricting to a flat, semi-infinite plate. Needless to say, that's not particularly realistic. Most objects are neither flat, nor semi-infinite. Clearly, we need to understand the physics of the boundary layer for objects that are curved and finite.

This, it turns out, is not so easy. Until now, we've made progress by finding clever ways to reduce the Navier-Stokes equations to an ordinary differential equation which can then easily be solved. But the problem that we're now interested in offers no such simplification. That means that to get a complete handle on the problem we must resort to solving partial differential equations numerically. Which is possible, but challenging, and beyond the scope of these lectures. Instead we will make do with some rather qualitative arguments, piecing together various bits of physics that we've learned so far.

First, we can gain some intuition for what's going on by turning to experiment[7]. Figure 14 shows the stream lines for a high Reynolds number flow ($Re \approx 7000$) around an elegantly pointy object. In the figure on the left, the object is aligned with the streamlines, which glide around much like the flows that we've discussed so far in these lectures. Such flows, where there is little mixing between adjacent layers of the fluid, are called *laminar*. A boundary layer forms around the object but, at least as far as

[7]The photos in Figures 14, 15, 16 and 18 are taken from the beautiful book "An Album of Fluid Motion" by Milton Van Dyke.

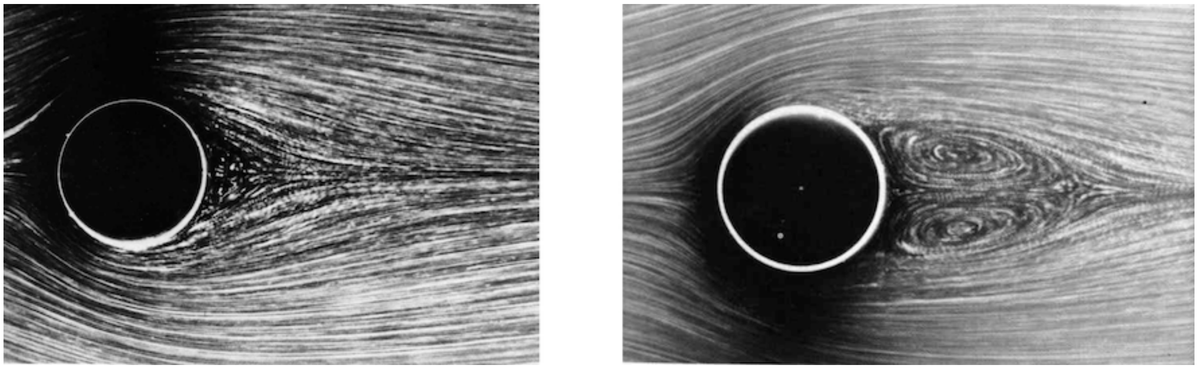


Figure 15. The flow around a circular cylinder with $Re \approx 10$ on the left, and $Re \approx 26$ on the right. In both cases, the flow separates from the cylinder at some point, leaving two trailing eddies in the wake (more visible in the second picture).

the photograph shows, appears to merge seamlessly back into the bulk fluid at the tail end.

On the right of Figure 14 is the same object, again at $Re \sim 7000$, but now tilted at an angle of $5°$. The flow is again laminar at the front and below the object. But you can see that something screwy is happening on the upper trailing edge. There is clearly a streamline that moves away from the object, leaving a swirling indeterminate flow beneath it.

The same phenomenon occurs for less aerodynamic objects. Figures 15 and 16 show flows moving past a circular cylinder. The first flow, at $Re \approx 10$, clearly shows an anti-symmetry between the front and back of the cylinder as the streamlines separate from the body. This is unsurprising, but sits in stark contrast to the potential flows and Stokes flows that we've seen previously, where it's difficult to see by eye the difference between the front and back of the flow. (See, for comparison, Figure 7 or Figure 10.) In the second picture in Figure 15, the Reynolds number has increased to $Re \approx 26$ and we again see the flow separating from the body, this time clearly leaving two counter-circulating eddies in its wake.

The Reynolds numbers in Figure 15 are fairly low and it's not at all obvious that we can use the theory of boundary layers, which relies on the approximation $Re \gg 1$. But this is surely valid for the picture in Figure 16, now at $Re \approx 2000$. Now we clearly see that the laminar flow at the front of the cylinder separates somewhere near the top of the cylinder, leaving a turbulent flow in its wake.

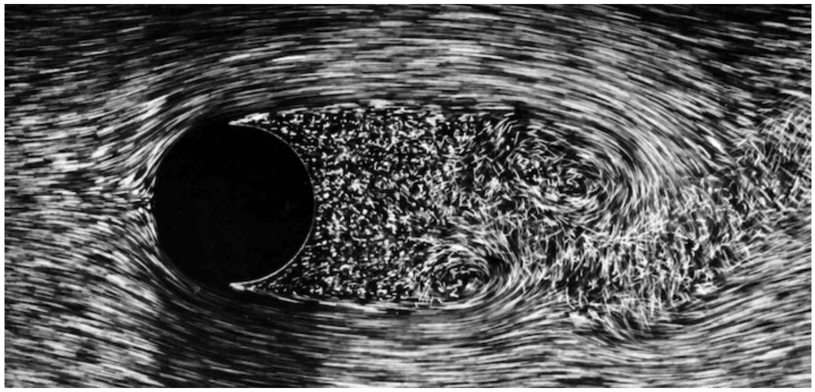


Figure 16. A flow around the circular cylinder, now with $Re \approx 2000$. We're now at values where the boundary layer theory should work. The picture clearly shows laminar flow at the front of the cylinder, where the boundary layer remains attached. It separates somewhere near the top and bottom of the cylinder, leaving a turbulent wake.

There are a bunch of things to unpack here. First, how do we extend the theory of a boundary layer to a curved object like those shown in the figures? Second, why does the flow separate from the object at some point? And, finally, how can we understand the physics of the wake left behind? We'll deal with each of these in turn.

Here is a cartoon of the physics. First, extending the theory of the boundary layer to a curved object turns out to be fairly straightforward. We use the same equations as before, but with x and y now curvilinear coordinates: x is the coordinate along the boundary and y the coordinate perpendicular. The boundary layer is so thin that, locally, it barely notices the curvature. All we must do is ensure that the pressure in the boundary layer is given by (3.41),

$$-\frac{1}{\rho}\frac{dP}{dx} = U\frac{\partial U}{\partial x}$$

Here, as a first approximation to $U(x)$, we should take the *near*-boundary limit of the flow that surrounds the boundary layer. Provided that this flow isn't turbulent, we can use the near-boundary limit of the inviscid potential flows that we described in Section 2.3. But we know how the pressure changes over the sphere or cylinder due to a potential flow. (The answer for the sphere was given in (2.31) and the result for the cylinder is similar.) There we saw that the pressure directly at the front and back is the same as the asymptotic pressure, but the pressure reduces as you move up or down over the sphere and takes its minimum value at the top and bottom. Crucially,

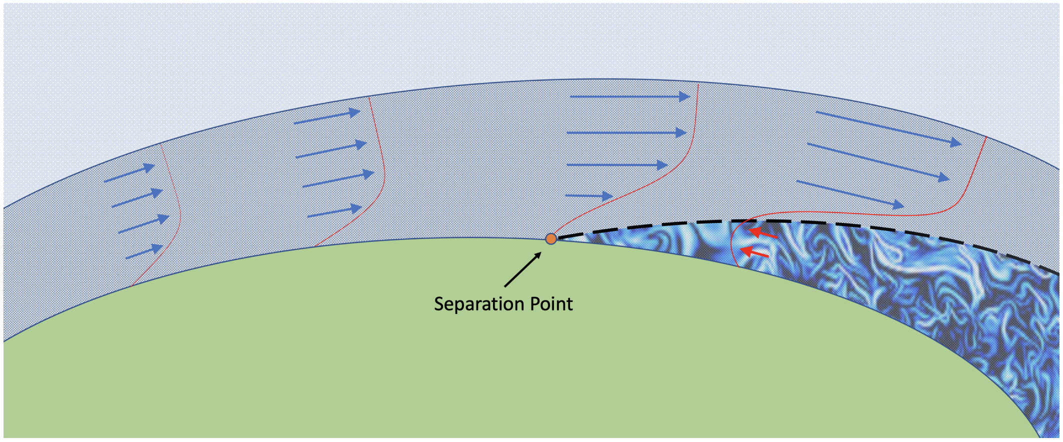


Figure 17. A cartoon of the evolution of the boundary layer and its ultimate separation from the boundary as the flow becomes reversed.

the pressure for an inviscid potential flow is symmetric on the front and back: this, of course, was what lead to d'Alembert's paradox.

Now we can see what this means for the boundary layer. On the front edge of the cylinder, the pressure is decreasing, $P' < 0$. This corresponds to an accelerating flow. But on the back edge, the pressure is increasing, $P' > 0$, and the flow is decelerating. This suggests that we might get the kind of behaviour that we observed for decelerating flows in the Falkner-Scan family of boundary layers. In particular, at some point the velocity u tangential to the boundary will obey

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$$

where y is the direction perpendicular to the boundary. This is the *separation point*, with the streamline bifurcating and leaving the boundary. Beyond this point, one expects reverse flow close to the boundary. Beyond the separation point, the boundary layer moves off into the bulk of the fluid, leaving behind the wake. A sketch of the scenario is shown in Figure 17.

The boundary layer itself cannot just dissolve once it has separated from the boundary. One might reasonably wonder what distinguishes it from the bulk of the fluid. After all, they're made from the same stuff. The answer is that the boundary layer has vorticity, generated by the no-slip condition

$$\boldsymbol{\omega} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{z}} \approx -\frac{\partial u}{\partial y}\hat{\mathbf{z}}$$

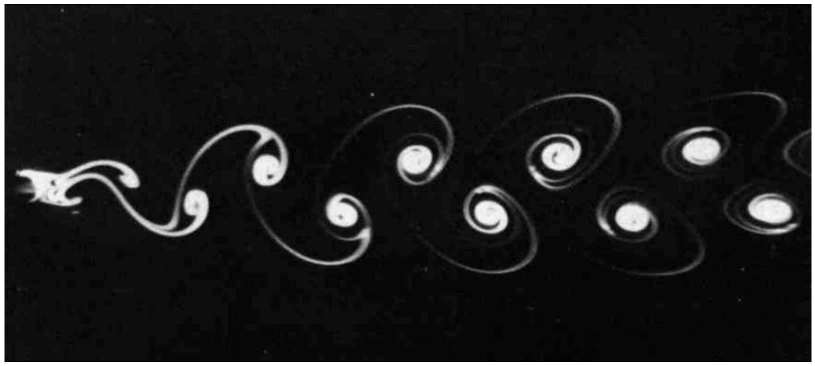


Figure 18. The von Kármán vortex street from air passing over a circular cylinder at Reynolds number $Re \approx 105$.

where the first term dominates in the boundary layer approximation. For the boundary layers described above, we have $|\boldsymbol{\omega}| = Uf''(0)/\delta$. Meanwhile, as we saw previously, the outer laminar flow is irrotational. The vorticity persists in the wake that trails the objects.

For low Reynolds number, the stream flow is low and this vorticity has time to diffuse due to the effects of viscosity. The result is the two large eddies trailing the object seen in Figure 15. The flow is steady. These are steady eddies.

But as the Reynolds number is increased to around $Re \sim 100$, something more interesting happens. One of the eddies grows until it peels off from the boundary in a process known as *vortex shedding*. The flow then curls back around the boundary and a new eddy forms. Meanwhile, the eddie on the other side then undergoes the same process. The result is a gorgeous flow pattern of alternating eddies known as the *von Kármán vortex street*. An example is shown in Figure 18. At these Reynolds numbers, there is no steady flow of the kind that we've searched for in these lectures. Instead, the flow is time dependent, but periodic.

There is much that we have swept under the carpet in the discussion above. The elephant in the room is turbulence. As the pictures clearly show, for large Reynolds number the flow is far from laminar. Indeed, the flow is no longer even two dimensional, but twists and turns in a noisy fashion in three dimensions. This occurs for $Re \gtrsim 10^4$ when the wake becomes turbulent as shown in Figure 16. A process known as *turbulent mixing* causes the pressure to be uniform across the turbulent wake, and equal to its value at the point of separation. This means that there is a much lower pressure behind the object and, correspondingly, a much larger drag force.

As the Reynolds number is increased yet further to $Re \gtrsim 10^5$ something novel happens: now the boundary layer itself becomes turbulent. The same turbulent mixing means that vorticity can be transferred vertically much more efficiently, and the result is that the boundary layer gets thicker. This has two, competing effects. The first is that the drag due to the turbulent boundary layer increases compared to the laminar boundary layer. The second is that the separation of the boundary layer is delayed, with the reversed flow happening further downstream. This results in a narrower wake which reduces the drag. It turns out that this reduced drag from the narrower wake is more than sufficient to compensate for the increased drag due to the turbulent boundary layer, and the result is that, surprisingly, the drag force actually drops suddenly at this Reynolds number. This goes by the name of the *drag crisis*.