4 Waves

Our story so far has involved the bulk motion of fluids, flowing from one place to another, sometimes trying to negotiate obstacles in their way. But fluids are more subtle and interesting than this. They contain mechanisms to transfer energy through space, but *without* the bulk of the fluid travelling very far. This is achieved this through oscillatory behaviour known as *waves*.

Waves are familiar, both from our everyday experience as well as from other areas of physics. Our purpose in this section is to explore some of large variety of waves that can occur in fluids. This includes, in Section 4.4, sound waves which gives us an opportunity to look at some of the novelties that arise with compressible fluids.

4.1 Surface Waves

> "Now, the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light; they have all the complications that waves can have."
>
> *Richard Feynman*

We start with waves travelling on the surface of a fluid. These include waves on the ocean. As Feynman points out, there are a surprisingly large number of subtleties that arise in understanding these waves.

Viscosity will not play a leading role in our story, so we return to the Euler equation of Section 2,

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P + \rho\mathbf{g} \tag{4.1}$$

We've included the effects of gravity on the right-hand side. As we will see, this provides the restoring force needed to create waves.

We will shortly solve the Euler equation using the same techniques that we met in Section 2. All of the novelties come, like so many things in fluid dynamics, from the boundary conditions. So before we get going, we need to think about the kind of boundary condition we should impose on the surface of a fluid.

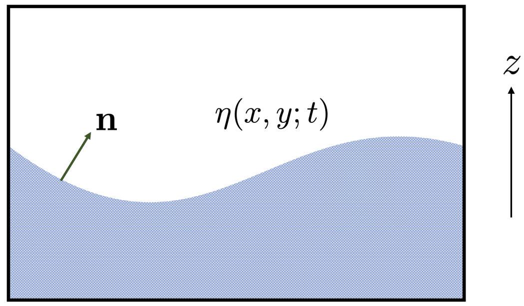


Figure 19. An interface between two fluids.

4.1.1 Free Boundary Conditions

The surface of a fluid is best viewed as the interface between two different fluids. In the case of the ocean, this is the water and the air above. But we could also have a situation where we have two immiscible liquids, like oil and water. The surface is free to move, and so is sometimes referred to as a *free boundary.*

Suppose that the boundary lies close to some $z \approx$ constant surface, as shown in Figure 19. Clearly this is appropriate for the surface of the ocean. The surface can fluctuate and, in general, is described by some function

$$F(\mathbf{x}, t) = z - \eta(x, y; t) = 0 \tag{4.2}$$

The normal to such a surface is parallel to ∇F (as shown, for example, in the lectures on Vector Calculus),

$$\mathbf{n} \sim \nabla F = (-\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1)$$

Meanwhile, the velocity of the interface is, by construction, in the z direction and given by

$$\mathbf{U} = (0, 0, \frac{\partial \eta}{\partial t})$$

The appropriate boundary condition on the fluid velocity $\mathbf{u}$ is the same as we saw in (2.25) for a solid surface moving with some velocity $\mathbf{U}$: one fluid cannot permeate the other. This means that if we write the fluid velocity as $\mathbf{u} = (u_x, u_y, u_z)$, then we have

$$\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \mathbf{U} \quad \Rightarrow \quad -u_x \frac{\partial \eta}{\partial x} - u_y \frac{\partial \eta}{\partial y} + u_z = \frac{\partial \eta}{\partial t}$$

We can alternatively write this as

$$u_z - \mathbf{u} \cdot \nabla \eta = \frac{\partial \eta}{\partial t} \quad \Rightarrow \quad \frac{D\eta}{Dt} = u_z \tag{4.3}$$

Alternatively, if we return to our original definition of the interface as the surface $F(\mathbf{x}, t) = 0$ given in (4.2), the fact that one fluid cannot invade the other can be written in the elegant form,

$$\frac{DF}{Dt} = 0$$

In addition, there is a further dynamical boundary condition that comes from the requirement that the stress tensor is continuous over the surface, ensuring that all forces are balanced. For an inviscid fluid, there is no tangential stress. The component of the stress tensor perpendicular to the interface simply tells us that the pressure must be continuous

$$P(x, y, \eta(x, y)) = P_0 \tag{4.4}$$

where, for the waves on the ocean, P_0 is atmospheric pressure. For example, if the water is stationary, so $\mathbf{u} = 0$, with a flat, free boundary at $z = 0$, then the Euler equation, together with this boundary condition, tells us that

$$P(z) = P_0 - \rho g z$$

We'd like to understand how to generalise this to the case where waves propagate on the boundary.

There are further complications that we could add to this story. In particular, there is an additional force that acts on the boundary known as *surface tension*. We'll postpone a discussion of this to Section 4.1.3.

4.1.2 The Equations for Surface Waves

We will look for irrotational flows which, as in Section 2, allows us to introduce a velocity potential

$$\nabla \times \mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{u} = \nabla \phi$$

The incompressibility of the fluid then tells us that the potential ϕ must obey the Laplace equation

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \nabla^2 \phi = 0$$

All the subtleties lie in the boundary conditions.

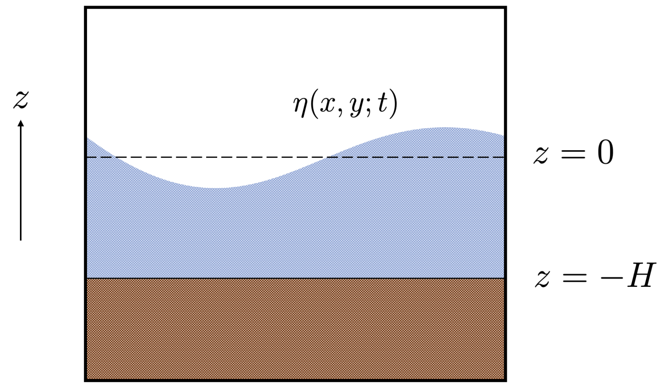


Figure 20. Surface waves.

We'll take the waves the be propagating in an ocean of height H, as shown in Figure 20. The bottom of the ocean lies at $z = -H$ while the surface of the ocean is at $z = \eta(x, y, t)$, some height above or below the equilibrium value of $z = 0$. The boundary condition at the bottom of the ocean is straightforward: the water can't flow into the ocean floor so

$$u_z(z = -H) = \left.\frac{\partial \phi}{\partial z}\right|_{z=-H} = 0 \tag{4.5}$$

Meanwhile, on the surface $z = \eta(x, y, t)$, we impose the free boundary condition (4.3)

$$u_z = \frac{D\eta}{Dt} \quad \Rightarrow \quad \left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = \frac{\partial \eta}{\partial t} + \left.\frac{\partial \phi}{\partial x}\right|_{z=\eta} \frac{\partial \eta}{\partial x} + \left.\frac{\partial \phi}{\partial y}\right|_{z=\eta} \frac{\partial \eta}{\partial y} \tag{4.6}$$

This boundary condition is all well and good if we're given the equation of the surface $z = \eta(x, y, t)$. But here, of course, the surface is something that arises dynamically and our goal is to find it. The equation that relates the velocity potential ϕ and the surface η comes from the continuity of pressure (4.4). To implement this, we write the Euler equation (4.1) as

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla |\mathbf{u}|^2 - \mathbf{u} \times \boldsymbol{\omega} \right) = -\nabla (P + \rho g z)$$

But we're dealing with irrotational flows, so $\boldsymbol{\omega} = 0$, and this becomes

$$\rho \left(\frac{\partial \nabla \phi}{\partial t} + \frac{1}{2} \nabla \left| \nabla \phi \right|^2 \right) = -\nabla (P + \rho g z)$$

But both sides are now total derivatives, so we have

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2} \rho |\nabla \phi|^2 + P + \rho g z = f(t)$$

where the function $f(t)$ on the right-hand-side can depend on time, but not on space. This is the time-dependent version of Bernoulli's principle that we derived earlier in (2.12) for stationary flows. For our final boundary condition, we simply require this condition on the surface $z = \eta(x, y, t)$, but with the pressure replaced by the atmospheric pressure (4.4),

$$\rho \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right)_{z=\eta} + P_0 + \rho g \eta = f(t) \tag{4.7}$$

This completes our setting up of the equations. We must solve the Laplace equation $\nabla^2 \phi = 0$ subject to the boundary conditions (4.5), (4.6) and (4.7). The Laplace equation is easy, but these boundary conditions look hard. As with so many other problems in this course, we need to find an appropriate approximation scheme.

The Linearised Approximation

To make progress, we will assume that the waves are small and flat. The first condition is the statement that the amplitude is small,

$$|\eta| \ll H$$

The second condition is the statement that the derivatives of the amplitude are also small

$$\frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y} \ll 1 \tag{4.8}$$

The boundary condition (4.6) is imposed at $z = \eta(x, y, t)$ but, since η is small, we can view this as "close" to a boundary condition at $z = 0$ by Taylor expanding

$$\left. \frac{\partial \phi}{\partial z} \right|_{z=\eta} = \left. \frac{\partial \phi}{\partial z} \right|_{z=0} + \eta \left. \frac{\partial^2 \phi}{\partial z^2} \right|_{z=0} + \ldots$$

The second order term is smaller than the first and can be dropped. Relatedly, the velocities u_x and u_y will be assumed to be small and we will drop all quadratic terms in the above boundary conditions. This means that we can ignore the $(\partial \phi / \partial x)(\partial \eta / \partial x)$ terms in (4.6) and the $|\nabla \phi|^2$ term in (4.7). The upshot is that our rather complicated set of boundary conditions reduce to the linear equations

$$\left. \frac{\partial \phi}{\partial z} \right|_{z=-H} = 0 \quad \text{and} \quad \left. \frac{\partial \phi}{\partial z} \right|_{z=0} = \frac{\partial \eta}{\partial t} \quad \text{and} \quad \left. \frac{\partial \phi}{\partial t} \right|_{z=0} + g\eta = \tilde{f}(t) \tag{4.9}$$

where, in the last of these, we have absorbed the pressure P_0 and density ρ into the redefined function $\tilde{f}(t)$. It is, it turns out, a significantly easier task to solve the Laplace equation subject to these conditions.

Finally, Some Waves

We will consider wave solutions that move in the x-direction and are independent of the y-direction. We make the ansatz

$$\phi(x,z,t) = \phi_0(z)\, e^{ikx-i\omega t} \quad \text{and} \quad \eta(x,t) = \eta_0\, e^{ikx-i\omega t} \tag{4.10}$$

You may be surprised that the right-hand side is suddenly complex, while both quantities on the left-hand side are clearly real! There's nothing deep going on here, only laziness. Because the equations are linear, if we find a complex solution then the real and imaginary parts are also solutions. But it's often simpler to work with complex numbers $e^{i(\text{something})}$ rather than cos and sin functions. Moreover, this will be particularly useful in Section 5 when we come to study instabilities since these manifest themselves as complex frequencies or wavenumbers for which the solution grows exponentially in time or space. For now, whenever you see equations like those above, you should implicitly think that we are taking the real (or imaginary) part. We'll use the same conventions in other lectures, including those on Electromagnetism.

The ansatz (4.10) depends on two numbers in the exponent, k and ω. Here k is called the *wavenumber*. From the equation, we see the successive peaks – also known as *wavecrests* – are spaced a distance apart given by

$$\lambda = \frac{2\pi}{k}$$

The distance λ is known as the *wavelength*. Meanwhile, ω is the *frequency* of the wave. Part of our goal is to determine the relationship between these. In particular, that will tell us the speed c at which waves travel,

$$c = \frac{\omega}{k} \tag{4.11}$$

The other part of our goal is to fix the function $\phi_0(z)$.

If we substitute the ansatz above into the Laplace equation, we get

$$\frac{d^2\phi_0}{dz^2} = k^2\phi_0$$

This has the solution

$$\phi_0(z) = A\cosh(kz + kH)$$

where we've chosen one integration constant to ensure that the first boundary condition in (4.9) is satisfied, and the overall amplitude A is still to be fixed. The second boundary condition in (4.9) tells us

$$Ak\sinh(kH) = -i\omega\eta_0$$

while the third, Bernoulliesque, condition in (4.9) tells us

$$\left(-iA\omega\cosh(kH) + g\eta_0\right)e^{ikx-i\omega t} = \tilde{f}(t) \quad \Rightarrow \quad A\omega\cosh(kH) = -ig\eta_0$$

where the second equality holds because the function $\tilde{f}(t)$ is a function only of time and so must be independent of x. Dividing these two equations to eliminate A/η_0, we get the relationship between the frequency ω and wavenumber k,

$$\omega^2 = gk\tanh(kH)$$

Equations of this kind are called *dispersion relations*. They are important in many different places in physics. In later courses that take place in the quantum world, we will see similar equations that relate energy (associated to frequency) and momentum (associated to wavenumber).

We find that, for surface waves, the frequency depends on the wavelength. This means that waves of different wavelength travel at different speeds (4.11),

$$c = \sqrt{\frac{g}{k}\tanh(kH)}$$

For many kind of waves, including sound and light, the speed is independent of the wavelength. Not so for surface waves. For fixed H, the speed is a monotonically decreasing function of k. In other words, long wavelength waves travel faster than short wavelength waves.

There are two interesting limits that we can take

- In deep water, $H \gg \lambda = 2\pi/k$ so we have the limit $kH \gg 1$. Here the speed becomes

$$c \approx \sqrt{\frac{g}{k}} \tag{4.12}$$

  We could have anticipated this. When the ocean is very deep, we don't expect the speed of surface waves to depend on H simply because the floor is a long way from the surface. Then, on dimensional grounds, the only thing that we can write down is $c \sim \sqrt{g/k}$.

- For long wavelengths, we have $kH \ll 1$ and the speed becomes

$$c \approx \sqrt{gH} \tag{4.13}$$

  Now the speed is independent of the wavelength of the wave. In this limit, the wave goes faster in deeper water than in shallow. There is a nice consequence of this. When waves come into the beach at an angle, the wave front that is further out travels faster and so the wave rotates until it is parallel to the beach.

Group Velocity and Dispersion

The waves described above extend to infinity in the x-direction, varying over wavelength $\lambda = 2\pi/k$. We can create a localised wavepacket by summing over many different wavenumbers and, because the equations are linear (after linearisation!) this too is a solution. In this way, a bump in the fluid surface can be written as

$$\eta(x,t) = \int \frac{dk}{2\pi}\; a(k) e^{ikx - i\omega t}$$

With some Fourier coefficients $a(k)$. This wavepacket may have some particular shape at time $t = 0$. But because the individual Fourier modes travel at different speeds, this shape will be distorted over time. This is known as *dispersion*.

We can ask: what is the right way to characterise the speed at which the wavepacket moves, rathe than the individual Fourier modes? Suppose that the Fourier modes $a(k)$ are peaked around some particular wavenumber $k = \bar{k}$. Then we can Taylor expand the frequency and write

$$\omega(k) = \omega(\bar{k}) + (k - \bar{k})\, \left.\frac{\partial \omega}{\partial k}\right|_{k=\bar{k}} + \ldots$$

Substituting this into the expression for the wavepacket, we have

$$\eta(x,t) \approx e^{-i\omega(\bar{k})t} \int \frac{dk}{2\pi}\; a(k)\, e^{ik(x - v_g t)} \tag{4.14}$$

where

$$v_g = \left.\frac{\partial \omega}{\partial k}\right|_{k=\bar{k}} \tag{4.15}$$

This is called the *group velocity* of the wave. It's clear from the form (4.14) that v_g is the speed at which the wavepacket moves. If $\omega \sim k$ then the wave doesn't disperse and the group velocity coincides with the speed $c = \omega/k$ that we defined previously,

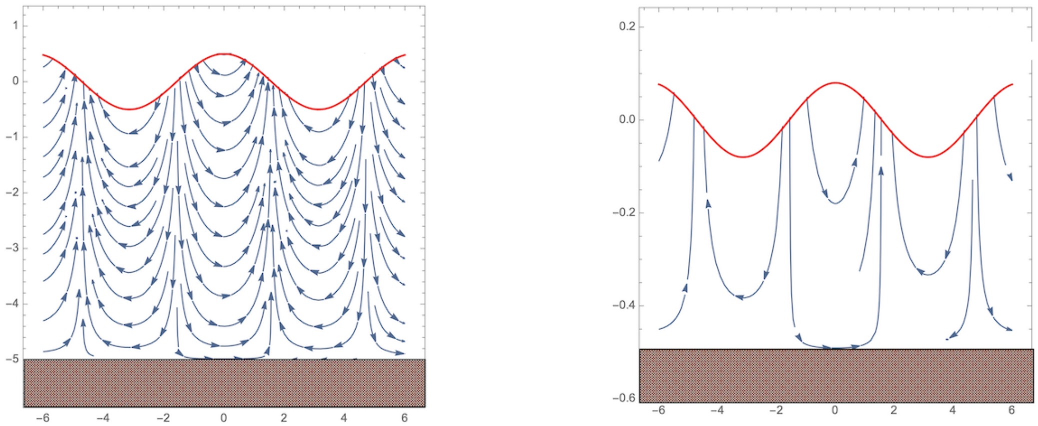


Figure 21. The velocity field for deep water waves on the left, with $kH = 5$, and shallow water on the right with $kH = 0.5$. (Note the different scales on the vertical axes!) The streamplot shows only the direction of the velocity field, not its size. In deep water, the velocity is exponentially smaller at the bottom than the top.

known as the *phase velocity*. But, in general, the two differ. The group velocity is the speed at which energy (and, in other contexts, information) is transported by the wave. For the surface waves considered here, $\omega \sim k^{1/2}$ and so the group velocity and phase velocity are related by $v_g(k) = \frac{1}{2}c(k)$. The wavepackets travel at half the speed of the individual Fourier modes.

The Velocity Field

It is a simple matter to compute the velocity field of the fluid. Substituting for the various integration constants, we have the potential

$$\phi = \text{Re}\left[-i\frac{\omega\eta_0}{k}\frac{\cosh(kz+kH)}{\sinh(kH)}e^{ikx-i\omega t}\right]$$

which now just has a single undetermined integration constant η_0 that fixes the amplitude of the wave. Our approximations above mean that the solution should be trusted only when $\eta_0 k \ll 1$. For once we've explicitly reminded ourselves that we should take the real part of the potential when computing the velocity $\mathbf{u} = \nabla\phi$. We have

$$\begin{pmatrix} u_x \\ u_z \end{pmatrix} = \frac{\omega\eta_0}{\sinh(kH)}\begin{pmatrix} \cosh(kz+kH)\cos(kx-\omega t) \\ \sinh(kz+kH)\sin(kx-\omega t) \end{pmatrix}$$

The velocity profile is plotted in Figure 21 for deep water waves (on the left) and for shallow water waves (on the right). In both cases, the velocity of the water is mostly up/down, despite the fact that the wave travels to the right. In the trough of the wave, the water is moving up on the left and down on the right. In the peak of the wave, this is reversed: the water moves down on the left and up on the right. The net effect is that the wave travels to the right.

There's something misleading about the figure for deep water waves. In this case, $e^{-kH} \approx 0$ and the velocity profile is well approximated by

$$\begin{pmatrix} u_x \\ u_z \end{pmatrix} \approx \omega\eta_0 e^{kz} \begin{pmatrix} \cos(kx - \omega t) \\ \sin(kx - \omega t) \end{pmatrix} \tag{4.16}$$

We see that the magnitude of the velocity $|\mathbf{u}| \approx \omega\eta_0 e^{kz}$ decreases exponentially from its value at the surface $z = 0$. It means that all the action is really taking place within a depth of one wavelength or so from the surface. In contrast, for shallow water waves the speed does not vary greatly with height.

For deep water waves, the ratio of the fluid speed to the wave speed is $|\mathbf{u}|/c \approx k\eta_0 e^{kz}$. The condition (4.8) is tantamount to the requirement that $k\eta_0 \ll 1$. In other words, the wave travels much faster than the fluid from which it's made.

Particle Paths

Suppose that you drop a small ball into the flow that follows an element of fluid on its travels. What path does it take? As we described in Section 1.1, the trajectory $\mathbf{x}(t)$ is called a pathline and is governed by the equation (1.1)

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(t), t) \tag{4.17}$$

which we should solve given some initial starting point $\mathbf{x}(t = 0) = \mathbf{x}_0$.

To solve this, we will assume that the particle doesn't get far from its original starting position and approximate the velocity field $\mathbf{u}(\mathbf{x}, t)$ by its Taylor expansion about $\mathbf{x}_0$,

$$\mathbf{u}(\mathbf{x}, t) \approx \mathbf{u}(\mathbf{x}_0, t) + ((\mathbf{x} - \mathbf{x}_0) \cdot \nabla)\mathbf{u}(\mathbf{x}_0, t) + \ldots \tag{4.18}$$

If we keep just the first term, the equation for the pathline becomes

$$\frac{d\mathbf{x}}{dt} = \frac{\omega\eta_0}{\sinh(kH)} \begin{pmatrix} \cosh(kz_0 + kH)\cos(kx_0 - \omega t) \\ \sinh(kz_0 + kH)\sin(kx_0 - \omega t) \end{pmatrix}$$

$$\Rightarrow \qquad \mathbf{x}(t) = \mathbf{x}_0 + \frac{\eta_0}{\sinh(kH)} \begin{pmatrix} -\cosh(kz_0 + kH)\sin(kx_0 - \omega t) \\ \sinh(kz_0 + kH)\cos(kx_0 - \omega t) \end{pmatrix}$$

This is telling us that the particles travel in ellipses, squashed in the vertical direction. For deep water waves, these ellipses become circles with

$$\mathbf{x}(t) = \mathbf{x}_0 + \eta_0 e^{kz_0} \begin{pmatrix} -\sin(kx_0 - \omega t) \\ \cos(kx_0 - \omega t) \end{pmatrix} \tag{4.19}$$

The ellipses or circles becomes exponentially smaller as the depth increases. The vertical component of the velocity is in phase with the crests of the wave, $\eta \sim \cos(kx - \omega t)$. Meanwhile, the horizontal component ensures that the particle goes clockwise for waves that propagate to the right.

We can also look at the effect of the second term in (4.18). Things are simplest if we restrict attention to deep water waves, with velocity (4.16) and particle position (4.19). If we use our leading order expression (4.19) for $\mathbf{x}(t)$ we find, after a little algebra,

$$((\mathbf{x}(t) - \mathbf{x}) \cdot \nabla)\mathbf{u} = \omega k \eta_0^2 e^{2kz} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

When substituted into (4.17), this has the interpretation of a constant, horizontal drift velocity for the particles, given by

$$v_{\text{drift}} = \omega k \eta_0^2 e^{2kz} = c(k\eta_0 e^{kz})^2$$

This is known as *Stokes' drift*. The ellipses traced by the particles don't quite close, but slowly inch their way in the direction in which the wave propagates. Note that there is a hierarchy of speeds,

$$v_{\text{drift}} \ll |\mathbf{u}| \ll c$$

with $k\eta_0 e^{kz} \ll 1$ the small, dimensionless number that governs successive ratios. The Stokes' drift v_{drift} is the speed at which matter bobbing in the waves moves.

4.1.3 Surface Tension

If you're a molecule, a liquid is a nice, comfortable place to spend your time. You're attracted to all your neighbouring molecules, but are afforded enough freedom to wander off on your own.

Things get more precarious at the surface of the liquid. There are now fewer neighbours to keep you company. As each neighbour offers a welcoming, attractive potential, the fact that you now find yourself a little isolated means that you are sitting in a higher energy state. This, in turn, means that, collectively, the molecules in a liquid can lower their energy by keeping the area of the surface as small as possible. This results in a force called *surface tension*. This force is the reason that droplets of water, or soap bubbles, are round: the sphere has the minimal surface area.

The existence of surface tension means that pressure need no longer be continuous across the surface. Instead, the surface can tolerate a local pressure difference by bending slightly and letting the surface tension push back. Said another way, the surface tension provides another restoring force for the wave motion.

This physics is captured by a change to the boundary condition (4.4). For a surface with embedding $z = \eta(x, y, t)$, the pressure difference should now be

$$P(x, y, \eta(x, y)) - P_0 = -\gamma \nabla^2 \eta \tag{4.20}$$

with γ the surface tension and $\nabla^2 \eta = \partial^2 \eta / \partial x^2 + \partial^2 \eta / \partial y^2$ the 2d Laplacian which is the appropriate characterisation of the curvature of the surface.

We would like to understand how the existence of surface tension affects the dynamics of waves. If we follow through our derivation of the time-dependent Bernoulli principle, equation (4.7) is replaced by

$$\rho \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right)_{z=\eta} + P_0 + \rho g \eta - \gamma \nabla^2 \eta = f(t)$$

After linearisation, the final condition in (4.9) becomes

$$\left. \frac{\partial \phi}{\partial t} \right|_{z=0} + g\eta - \frac{\gamma}{\rho} \nabla^2 \eta = \tilde{f}(t) \tag{4.21}$$

with $\tilde{f}(t)$ a function that can depend on time but, crucially, must be independent of space. We now make our usual ansatz for waves propagating in the x-direction,

$$\phi(x, z, t) = \phi_0(z)\, e^{ikx - i\omega t} \quad \text{and} \quad \eta(x, t) = \eta_0\, e^{ikx - i\omega t}$$

Much proceeds as before. In fact, we can see where how the surface tension affects the story just by staring at (4.21) where we see that it accompanies the gravitational acceleration: we just need to replace g with

$$g \longrightarrow g + \frac{\gamma k^2}{\rho} = g \left(1 + l_c^2 k^2 \right) \tag{4.22}$$

in all our previous formulae. Here we've introduced the length scale

$$l_c = \sqrt{\frac{\gamma}{g\rho}} \tag{4.23}$$

This is known as the *capillary length*. From (4.22), we see that long wavelength modes with $\lambda \gg l_c$, so $l_c k \ll 1$, are pretty much unaffected by surface tension. In contrast, surface tension effects dominate when the wavelength becomes short, $\lambda \ll l_c$, so $l_c k \gg 1$. Waves with $\lambda \lesssim l_c$ are referred to *capillary waves*.

For water at room temperature, $l_c \approx 3$ mm. The capillary waves are little ripples on the water, up to a wavelength of 1 cm or so (with the factor of 2π in the definition of the wavelength raising us above l_c.)

The general dispersion relation is

$$\omega^2 = gk(1 + l_c^2 k^2)\tanh(kH) \tag{4.24}$$

while the phase velocity is

$$c = \sqrt{\frac{g}{k}\,(1 + l_c^2 k^2)\tanh(kH)}$$

For capillary waves, with $l_c k \gg 1$, in deep water, so $kH \gg 1$, we have

$$c \approx \sqrt{\frac{\gamma k}{\rho}}$$

In contrast to surface waves driven by gravity (4.12), the short wavelength modes now travel faster. Furthermore, the group velocity (4.15) is $v_g(k) = \frac{3}{2}c$. The wavepackets now travel faster than the individual Fourier modes.

4.2 Internal Gravity Waves

Gravitational waves are the ripples of the spacetime continuum that emerge from violent events such as the collision of two black holes. That, sadly, is not the topic of discussion here. Instead, "gravity waves" describe the disappointingly mundane phenomenon of fluids bobbing up and down due to gravity. If you want to learn more about gravitational waves, you'll need to open the lectures on General Relativity. Otherwise, read on.

Gravity waves are simply waves in fluids where the restoring force is provided by gravity. The surface waves above are examples (at least those with wavelength longer than the capillary length where surface tension is negligible). In this section we study gravity waves in the bulk of the fluid, as opposed to on the surface.

Stratified Flows and Buoyancy Frequency

A flow is said to be *stratified* if the density ρ varies from place to place. Typically this happens because of gravity and the density is a function of the vertical direction: $\rho = \rho(z)$.

Consider a small ball immersed in a stratified flow. If the ball has density $\rho_0 = \rho(z_0)$ for some height z_0 then, by Archimedes principle, it will naturally sit at height $z = z_0$. This is where the weight of water that it displaces is equal to its own weight.

Suppose now that we displace the ball upwards by some small amount δz. The density of the fluid there is

$$\rho(z_0 + \delta z) \approx \rho(z_0) + \left.\frac{\partial \rho}{\partial z}\right|_{z_0} \delta z$$

Now the weight of the displaced water differs from that of the ball, resulting in a net *upwards* force,

$$\text{Upwards Force} \approx g \left.\frac{\partial \rho}{\partial z}\right|_{z_0} \delta z$$

If $\partial\rho/\partial z > 0$ then the balls original position was unstable, and it flies upwards. But most stratified flows have density larger at the bottom than at the top, so $\partial\rho/\partial z < 0$. In this case, the ball oscillates about its equilibrium position, enacting simple harmonic motion with a frequency

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z} \tag{4.25}$$

This is called the *buoyancy frequency* or, sometimes, the *Brunt-Väisälä frequency*. In what follows, we'll look at similar motion but for the fluid itself.

Note that we haven't specified how $\rho(z)$ depends on the height z. Nor will we do this throughout the rest of this section. This follows only when we introduce an equation of state relating pressure and density. We'll meet this in Section 4.4.

Equations for Gravity Waves

Until now, the incompressibility condition was forced upon by the requirement that the density is constant. For stratified flows, this is no longer the case. Nonetheless, it is still physically sensible to insist on incompressibility (at least for speeds smaller than the sound speed)

$$\nabla \cdot \mathbf{u} = 0$$

With this, mass conservation becomes the requirement,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \Rightarrow \quad \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0$$

In addition, we will ignore viscosity and look at gravity waves in the Euler equation, now in the presence of gravity

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla P-\rho(z)g\hat{\mathbf{z}}$$

We'll consider a boring background, with $\mathbf{u}=0$ and the pressure $P_0(z)$ related to the density $\rho_0(z)$ through the Euler equation, by

$$\frac{dP_0}{dz}=-g\rho_0(z)$$

Now we look at small perturbations around this background. The gravity waves of interest travel in the horizontal x-direction, while bobbing up and down in the vertical z-direction. To this end, we look for solutions of the form

$$\mathbf{u}(\mathbf{x},t)=(u_x,0,u_z)e^{ik_xx+ik_zz-i\omega t}$$

with u_x and u_z constant. Both the density and pressure exhibit the same wavelike behaviour,

$$\rho(x,z,t)=\rho_0(z)+\tilde{\rho}\,e^{ik_xx+ik_zz-i\omega t}\quad\text{and}\quad P(x,z,t)=P_0(z)+\tilde{P}\,e^{ik_xx+ik_zz-i\omega t}$$

The incompressibility condition tells us that

$$k_xu_x+k_zu_z=0 \tag{4.26}$$

Even before we proceed, this equation is telling us that $\mathbf{k}\cdot\mathbf{u}=0$. In other words, the waves are transverse. This is like light waves (which have $\mathbf{E}\cdot\mathbf{k}=\mathbf{B}\cdot\mathbf{k}=0$) but contrasts with the sound waves that we will meet in Section 4.4.

For the other equations, we linearise, throwing away any terms quadratic in perturbations. Mass conservation gives

$$-i\omega\tilde{\rho}+u_z\frac{d\rho_0}{dz}=0$$

and the two components of the Euler equation are

$$-i\rho_0\omega u_x=-ik_x\tilde{P}\quad\text{and}\quad -i\rho_0\omega u_z=-ik_z\tilde{P}-g\tilde{\rho}$$

Solving these simultaneous equations gives us the dispersion relation for the frequency of gravity waves,

$$\omega=\pm N\frac{k_x}{\sqrt{k_x^2+k_z^2}} \tag{4.27}$$

with N the buoyancy frequency (4.25). Note that we necessarily have $\omega \leq N$. Moreover, the frequency is non-vanishing only if $k_x \neq 0$. We can, however, consider the extreme example with $k_z = 0$. In this case $\omega = N$. The incompressibility condition then tells us that we must have $u_x = 0$. This, in turn, means that we have wave in the x-direction since $k_x \neq 0$ but with the motion of the fluid bobbing up and down with buoyancy frequency in the z-direction.

In general, the gravity wave propagates in the direction

$$\mathbf{k} = (k_x, 0, k_z)$$

The slight surprise comes when we compute the group velocity. For a one dimensional wave, this is $v_g = \partial\omega/\partial k$. For a higher dimensional waves, like we have here, the relevant definition is

$$\mathbf{v}_g = \frac{\partial \omega}{\partial k_x}\hat{\mathbf{x}} + \frac{\partial \omega}{\partial k_z}\hat{\mathbf{z}}$$

For the dispersion relation (4.27), this gives

$$\mathbf{v}_g = \frac{N k_z}{(k_x^2 + k_z^2)^{3/2}}(k_z, 0, -k_x)$$

Strangely, group velocity is perpendicular to the direction of the wave, $\mathbf{v}_g \cdot \mathbf{k} = 0$. This means that both wavepackets and energy propagate in the direction $\mathbf{v}_g$, but this is orthogonal to the direction $\mathbf{k}$ of the wave itself! It is somewhat less surprising when you realise that $\mathbf{v}_g$ is parallel to the velocity $\mathbf{u}$ of the fluid.

4.3 Because the Earth Spins

In this section we take something of a diversion. We will explore some novel phenomena that arise when fluids rotate. The main motivation from this comes from the fact that Earth spins and this gives rise to some new types of waves with rather interesting properties.

Recall from the lectures on Dynamics and Relativity that if we sit in a reference frame that rotates with constant angular velocity $\mathbf{\Omega}$ then we experience two fictitious forces. These are the centrifugal force, proportional to $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x})$ and the Coriolis force, proportional to $2\mathbf{\Omega} \times \mathbf{u}$. For fluids, these appear as forces on the right-hand side of the Navier-Stokes equation. Throughout this section, we will neglect viscosity and work with the Euler equation in a rotating frame, so we have

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla P + \mathbf{g} - 2\mathbf{\Omega} \times \mathbf{u} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}) \tag{4.28}$$

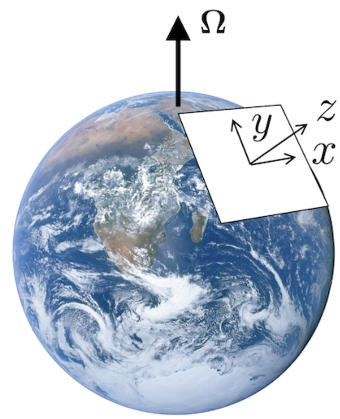
The centrifugal force is not particularly interesting for our purposes. Locally, it simply redefines what we mean by "down" since, like gravity, it can be written as the gradient of a potential energy. We will simply ignore it. As we will see, all the interesting physics arises from the Coriolis force.

4.3.1 The Shallow Water Approximation

In what follows, we will make the so-called *shallow water* approximation. We will assume that the extent of the fluid in the horizontal directions, labelled by x and y, is much greater than the height of the fluid in the vertical z-direction. For our purposes, the Atlantic ocean counts as "shallow" since it is, on average, around 3.5 km deep but several thousand km wide. Similarly, the atmosphere also counts as "shallow" and the phenomena that we describe can be found in both.

Our choice of coordinates is shown in the figure to the right. Locally, "up" is in the z-direction, "North" is in the y-direction, and "East" is the x-direction. We define the *Coriolis parameter*

$$f = 2\mathbf{\Omega} \cdot \hat{\mathbf{z}} \tag{4.29}$$



If we're considering flows where we can neglect the curvature of the Earth, then we restrict attention to a given tangent plane as shown and take f to be constant. In contrast, if we need to take into account the curvature of the Earth, then f will be a function $f = f(y)$, reflecting the fact that as we move along the surface the local "up" direction $\hat{z}$ changes, while the spin $\mathbf{\Omega}$ remains fixed. In what follows, we will consider both situations in which f is taken to be constant and, in Section 4.3.5, situations in which f varies.

Our initial set-up will be similar to that of water waves described in Section 4.1. We'll take the average depth of the water to be H, with a flat, solid base at $z = -H$ and a varying surface at $z = \eta(x, y, t)$ with $|\eta| \ll H$ as shown in Figure 20. (Clearly the flat bottom is more appropriate for the ocean than the atmosphere!)

Next, we assume that the velocities in the horizontal direction are independent of the depth, so

$$\mathbf{u} = (u, v, w) \quad \text{with} \quad u = u(x, y, t), \quad v = v(x, y, t) \quad \text{and} \quad w = w(x, y, z, t)$$

Note that this is where our set-up starts to differ from the water waves of Section 4.1.

The vertical velocity can be eliminated in favour of the height fluctuation $\eta(x,y,t)$ by using the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$$

We integrate over the vertical z-direction, and use the free boundary condition (4.3), which tells us that $w(z=\eta) = D\eta/Dt$ and $w(z=-H)=0$. We then have

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = -(H+\eta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.30}$$

This is the first of our shallow water equations.

Next, we assume that the pressure in the vertical direction adapts to balance the gravitational force. This *hydrostatic approximation* is what led us to Archimedes principle in Section 2.1.3. We also need the boundary condition $P = P_0$ on the surface at $z = \eta$, meaning that we take the pressure to be

$$P = P_0 - \rho g(z - \eta) \tag{4.31}$$

In the Navier-Stokes equation (4.28), we can then replace $-\frac{1}{\rho}\nabla P + \mathbf{g} = -g\nabla\eta$.

With these pieces in place, the remaining two Navier-Stokes equations read

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = fv - g\frac{\partial \eta}{\partial x} \tag{4.32}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -fu - g\frac{\partial \eta}{\partial y} \tag{4.33}$$

As usual, we want an excuse to drop the non-linear terms to make life easy. If a flow has characteristic velocity U, changing over some length scale L then these non-linear terms scale as U^2/L. This should be compared with the Coriolis terms which scale as fU. We introduce a dimensionless number, this time called the *Rossby number* Ro,

$$Ro = \frac{U}{fL}$$

It's appropriate to drop the non-linear terms for flows with $Ro \ll 1$. The rotation of the Earth is $\Omega \approx 10^{-4}\ \mathrm{s}^{-1}$ while typical atmospheric or oceanic speeds are around $U \sim 10\mathrm{ms}^{-1}$. That means that

$$Ro \approx \frac{10^5\ \mathrm{m}}{L}$$

We see that we can think about dropping the non-linear terms only for very long wavelength perturbations. For $L \sim 10^3$ km, we have $Ro \approx 0.1$ which, while admittedly < 1 is barely $\ll 1$. Nonetheless, this is the approximation that we will make. We further linearise the first equation (4.30), leaving us with our three linear shallow water equations

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x} - H\frac{\partial v}{\partial y} \tag{4.34}$$

$$\frac{\partial u}{\partial t} = fv - g\frac{\partial \eta}{\partial x} \tag{4.35}$$

$$\frac{\partial v}{\partial t} = -fu - g\frac{\partial \eta}{\partial y} \tag{4.36}$$

In the rest of this section, we will solve these equations in various scenarios for $u(x,y,t)$, $v(x,y,t)$ and $\eta(x,y,t)$.

4.3.2 Geostrophic Balance and Poincaré Waves

We're going to find a number of different solutions to the linearised shallow water equations (4.34), (4.35) and (4.36). Among these will be wave-like solutions. But, more surprisingly, we will also find some time independent solutions that are more interesting than just an ocean with a flat surface $h =$ constant.

It's simple to see the existence of time independent solutions by setting $\partial/\partial t = 0$ in (4.34), (4.35) and (4.36). Solutions can be built from any divergent free flow, with $\nabla \cdot \mathbf{u} = 0$, that obeys

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y} \quad \text{and} \quad v = +\frac{g}{f}\frac{\partial \eta}{\partial x} \tag{4.37}$$

Here the height η acts like a streamfunction of the kind we met in Section 1.1.4. Steady-state solutions of this form are said to be in *geostrophic balance*.

It's easy to understand the balance of forces underlying geostrophic balance. Suppose that there is some bump in the height of the fluid. Gravity, of course, wants to pull this down but, because the underlying fluid is incompressible, it results in a horizontal force in the direction $\nabla\eta$. The velocity in geostrophic balance is such that it gives rise to Coriolis force that balances gravity.

Flows in geostrophic balance (4.37) obey $\mathbf{u} \cdot \nabla\eta = 0$. In other words, the flow is along lines of constant height η. But, from hydrostatic balance (4.31), we know that the pressure in the fluid is proportional to the height. In other words, the flow is along isobars. This is familiar from weather maps, where wind blows along lines of constant pressure, rather than from high to low pressure as one might naively expect. The large scale flow of both the ocean and atmosphere is largely in geostrophic balance.

Potential Vorticity

Our next task is to understand time-dependent solutions to the shallow water equations. To do this, it's best to first look more closely at the various conserved quantities.

In fact, it's best if we briefly return to the full non-linear equations (4.30), (4.32) and (4.33). These admit two conserved quantities. The first is simply the height, whose conservation follows from the underlying conservation of mass

$$\frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{u} h) = 0 \quad \text{with} \quad h = H + \eta \tag{4.38}$$

The second is conservation of vorticity. It can be checked that

$$\frac{\partial W}{\partial t} + \nabla \cdot (\mathbf{u} W) = 0 \quad \text{with} \quad W = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \tag{4.39}$$

In this equation, both ∇ and $\mathbf{u}$ are now 2d vectors, rather than 3d. Note that the vorticity includes the extra $+f$ contribution from the Coriolis force.

Both (4.38) and (4.39) are continuity equations, which is the the usual conservation law that we know and love. Elsewhere in these lectures, we've been able to use the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ to extract the velocity $\mathbf{u}$ from the clutches of the spatial derivative and write equations of this form as the vanishing of a material derivative. But we're not allowed to do this in the present context because the 2d velocity $\mathbf{u}$ does not necessarily obey $\nabla \cdot \mathbf{u} = 0$. The fluid is still incompressible of course, but the 2d velocity $\mathbf{u}$ can pile up at some point at the expense of increasing the height. Indeed, this is what our first equation (4.38) is telling us. Nonetheless, we can combine (4.38) and (4.39) to construct a quantity that has vanishing material derivative. This is

$$\frac{D\mathcal{Q}}{Dt} = \frac{\partial \mathcal{Q}}{\partial t} + \mathbf{u} \cdot \nabla \mathcal{Q} = 0 \quad \text{with} \quad \mathcal{Q} = \frac{W}{h} = \frac{1}{H + \eta} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right) \tag{4.40}$$

The quantity $\mathcal{Q}$ is called the *potential vorticity*. The equation $D\mathcal{Q}/Dt = 0$ is telling us that the value of the potential vorticity doesn't change as we follow the flow.

The discussion above is for the full non-linear equations. Something rather striking happens when we restrict to the linear equations. We linearise the conservations laws (4.38) and (4.39) about $h = H$ and $W = f$, to find

$$\frac{\partial h}{\partial t} + H\nabla\cdot\mathbf{u} = 0 \quad \text{and} \quad \frac{\partial W}{\partial t} + f\nabla\cdot\mathbf{u} = 0$$

The surprising fact is that these both have the same current: it is simply the velocity $\mathbf{u}$. This means that we can eliminate the current to find the linearised conservation law

$$\frac{\partial Q}{\partial t} = 0 \quad \text{with} \quad Q = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{f\eta}{H} \tag{4.41}$$

The quantity Q is (up to constant term and a scaling by H) the linearised potential vorticity. We see that Q is independent of time. That's a much stronger statement than our usual conservation laws. Usually when something is conserved, its value can change at some point in space by moving to a neighbouring point. That's the physics of the continuity equation. But now we learn that the function Q simply can't change at any point in space! That adds a rigidity to the system that will be responsible for some of the features we'll see below.

Poincaré Waves

With this understanding of potential vorticity in hand, we'll now turn to some wave solutions of the linearised shallow water equations (4.34), (4.35) and (4.36).

If there were no rotation, it's clear what would happen. With $f = 0$, it's simple to check that the equations (4.34), (4.35) and (4.36) become the wave equation $\ddot{\eta} = c^2\nabla^2\eta$ with $c^2 = gH$. This describes surface waves propagating with speed c and reproduces our previous result (4.13) for long wavelength waves.

The Coriolis force changes this. If we assume that $f =$ constant (which means that we are neglecting the effects of the curvature of the Earth), then the wave equation that we derive from (4.34), (4.35) and (4.36) is

$$\frac{\partial^2\eta}{\partial t^2} = c^2\nabla^2\eta - Hf\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad \text{with} \quad c^2 = gH$$

The additional terms can be rewritten in terms of the potential vorticity (4.41) to get

$$\frac{\partial^2\eta}{\partial t^2} - c^2\nabla^2\eta + f^2\eta = -HfQ \tag{4.42}$$

where Q is the potential vorticity which, as we have seen above, is a constant function that doesn't change with time. For a given problem, one might have to solve (4.42) for some fixed Q. But, in addition, one can always add solutions to the complimentary solution which solves the homogeneous equation

$$\frac{\partial^2\eta}{\partial t^2} - c^2\nabla^2\eta + f^2\eta = 0 \tag{4.43}$$

where, as before, $c = \sqrt{gH}$. This is a rather famous equation that, in the world of Quantum Field Theory, it is known as the *Klein-Gordon equation*. It is a simple matter to find solutions by writing

$$\eta(\mathbf{x}, t) = \tilde{\eta} e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$$

with $\mathbf{x} = (x, y)$ and $\mathbf{k} = (k_x, k_y)$. This solves (4.43) provided that the frequency ω and wavevector $\mathbf{k}$ obey the dispersion relation

$$\omega^2 = c^2\mathbf{k}^2 + f^2 \tag{4.44}$$

These are known as *Poincaré waves*. They are a form of gravity wave, since gravity acts as the restoring force, as seen in the speed $c = \sqrt{gH}$. But their properties are affected by the Coriolis force. They are sometimes referred to as *inertia-gravity waves*.

For long wavelengths, $k \to 0$, Poincaré waves have a finite frequency, set by the Coriolis parameter $\omega \to f$. In the language of quantum mechanics, we say that the spectrum is *gapped*, the "gap" being the smallest frequency at which the system oscillates. (In quantum mechanics this translates into a gap in the energy spectrum because $E = \hbar\omega$.)

The cross-over from "short" to "long" wavelengths happens at the length scale

$$R = \frac{c}{f} = \frac{\sqrt{gH}}{f} \tag{4.45}$$

This is known as the *Rossby radius of deformation*. It is the characteristic length scale in the shallow water equations. (For the ocean at mid-latitudes, one has $R \approx 1000$ km. Short wavelength modes, with $k \gg R^{-1}$, act just like usual surface waves, with $\omega \approx ck$. It's the long wavelength modes, with $k \ll R^{-1}$, that feel the effect of the Coriolis force. In this limit, we can neglect the η-terms in (4.34) and (4.35) to find that the velocities obey $\dot{u} = fv$ and $\dot{v} = -fu$. This tells us that the wave velocity in the x and y-directions are $\pi/2$ out of phase.

In preparation for what follows, it's worth redoing the above calculation in a slightly different way. We write our three, linearised shallow water equations (4.34), (4.35) and (4.36) as a combined matrix eigenvalue equation

$$i\frac{\partial \Psi}{\partial t} = \begin{pmatrix} 0 & -ic\partial_x & -ic\partial_y \\ -ic\partial_x & 0 & if \\ -ic\partial_y & -if & 0 \end{pmatrix} \Psi \quad \text{with} \quad \Psi = \begin{pmatrix} \sqrt{g/H}\eta \\ u \\ v \end{pmatrix} \tag{4.46}$$

We've done some cosmetic manipulations to get the equation in this form. In addition to rescaling the η variable, we've also multiplied everything by a factor of i. This makes the resulting equation look very much like a time-dependent Schrödinger equation. In particular, the matrix is Hermitian. With our wave ansatz $\Psi = \tilde{\Psi} e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$, this becomes a standard eigenvalue problem

$$\begin{pmatrix} 0 & ck_x & ck_y \\ ck_x & 0 & -if \\ ck_y & if & 0 \end{pmatrix} \tilde{\Psi} = \omega \tilde{\Psi} \tag{4.47}$$

Because this is a Hermitian matrix, the eigenvalues are guaranteed to be real. They are

$$\omega = \pm\sqrt{c^2\mathbf{k}^2 + f^2} \quad \text{and} \quad \omega = 0 \tag{4.48}$$

We recognise the first of these as the dispersion relation for Poincaré waves (4.44). In addition, there are a collection of solutions with $\omega = 0$. In the context of condensed matter physics, this is known as a *flat band* (because if you plot ω vs k it is a flat plane.) The existence of the flat band follows from the functional conservation of the potential vorticity. It is telling us that there are additional, time independent equilibrium solutions. These are solutions like (4.45) that exhibit geostrophic balance.

4.3.3 We Need to Talk About Kelvin Waves

Everyone likes a trip to the coast. Now it's our turn. For the purposes of this course, the coast is not going to be very exciting. It's simply a boundary of our fluid, which we will take to run North/South. The fluid exists only in the $x \geq 0$ direction. For $x < 0$, there is only land.

Obviously we must put a boundary condition $u = 0$ at $x = 0$, ensuring that no flow passes the boundary. In fact, we'll do something more extreme than this. We will search for solutions that have $u = 0$ everywhere. The linearised shallow water equation (4.35) then becomes

$$v = \frac{g}{f}\frac{\partial \eta}{\partial x} \tag{4.49}$$

This is telling us that the fluid lives in geostrophic balance in the x-direction, with the pressure gradient from $\partial\eta/\partial x$ pushing against the Coriolis force that arises because the fluid has velocity v in the y-direction. Meanwhile, the other two shallow water equations (4.34) and (4.35) become

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y}$$

These are standard wave-like equations. If we make the usual ansatz that $v = v_0(x)\, e^{i\omega t - iky}$ and $\eta = \eta_0(x)\, e^{i\omega t - iky}$, these become

$$\omega\eta_0 = kHv_0 \;\; \text{and} \;\; \omega v_0 = gk\eta_0 \quad \Rightarrow \quad \omega^2 = c^2k^2$$

with the speed given by $c = \sqrt{gH}$ as for our previous examples. So far, things look fairly standard. But there's a slight twist in the tail. This arises when we return to (4.49) which tells us the profile of the water near the boundary. We have

$$\frac{\partial \eta_0}{\partial x} = \frac{f\omega}{kc^2}\eta_0$$

Our dispersion relation $\omega^2 = c^2k^2$ naively suggests that we have two options: $\omega = +ck$ or $\omega = -ck$. But that's not right. Suppose that we take $f > 0$, which is appropriate if we are in the Northern hemisphere. Then if we pick $\omega = +ck$ we're in trouble, because the height of the water will grow exponentially away from the boundary: $\eta_0(x) \sim e^{+fx/c}$. And that's bad. It means that we should throw away this solution. The only physical solution has

$$\omega = -ck$$

with the water profile decaying exponentially away from the boundary, $\eta_0(x) \sim e^{-fx/c}$. This means that the boundary waves propagate only in one direction which, in the current set-up, is the negative y-direction, also known as South. These are known as *Kelvin waves.*

Waves that propagate only in one direction are said to be *chiral.* In the Northern hemisphere, with $f > 0$, Kelvin waves propagate so that the land always sits to their right. (In other words, if these waves are propagating on the boundary of a lake, then they move in an anti-clockwise direction.) In the Southern hemisphere, where $f < 0$, the same argument tells us the we must have the $\omega = +ck$ solution, so Kelvin waves propagate with the land to its left as it moves.

Chiral waves also make an appearance in various condensed matter systems where, as here, they typically live at the edge of some system. In that context, there is often some deep topological reason for the emergence of such chiral waves. The same is also true here and we will elaborate on this further in Section 4.3.6.

4.3.4 Rossby Waves

As we've seen, the linearised shallow water equations admit time independent solutions in geostrophic balance, solving (4.37). But objects that are strictly unmoving are rare in Nature. One can ask: is there something that can coax flows to geostrophic balance to move? The answer, as we shall see, is yes. In this section, we will see that if we look at scales over which the Coriolis parameter f is no longer constant, then flows in geostrophic balance start to evolve in time. This is known as *quasi-geostrophic* balance.

Crucially, the evolution of flows in quasi-geostrophic balance happens much more slowly than the dynamics of Poincaré waves that we saw above. That means that it is this quasi-geostrophic flow governs the long-time dynamcis of the ocean and atmosphere. The purpose of this section is to construct the equations that describe this flow.

At different latitudes θ, the Coriolis parameter is given by $f = 2\Omega \sin\theta$, where $\Omega = 2\pi \text{ day}^{-1}$. To capture the variation of the Coriolis parameter, it will suffice to consider just the leading term in the Taylor expansion

$$f = f_0 + \beta y$$

with y the direction points North. Our strategy will be to turn again to the conservation of potential vorticity (4.40),

$$\frac{D\mathcal{Q}}{Dt} = 0 \quad \text{with} \quad \mathcal{Q} = \frac{1}{H+\eta}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right)$$

You can check that this equations remains valid even when $f = f(\mathbf{x})$. We will consider flows with Rossby number $Ro \ll 1$ that are very close to geostrophic balance (4.37). This means that we can replace the vorticity in the expression with,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{g}{f}\nabla^2\eta \approx \frac{g}{f_0}\nabla^2\eta$$

We further assume that variations in the height are small, so $\eta \ll H$, and the potential vorticity can be written solely in terms of the height fluctuations η. Ignoring an overall constant term, we have

$$\mathcal{Q} \approx \frac{f_0}{H^2}\left(\frac{c^2}{f_0^2}\nabla^2\eta - \eta + \frac{\beta H}{f_0}y\right) \quad . \tag{4.50}$$

As we've seen, potential vorticity is materially conserved and, using the geostrophic balance condition (4.37), this too becomes an equation that can be written solely in terms of the height

$$\frac{D\mathcal{Q}}{Dt} = 0 \quad \Rightarrow \quad \dot{\mathcal{Q}} - \frac{g}{f_0}\frac{\partial \eta}{\partial y}\frac{\partial \mathcal{Q}}{\partial x} + \frac{g}{f_0}\frac{\partial \eta}{\partial x}\frac{\partial \mathcal{Q}}{\partial y} = 0 \tag{4.51}$$

This is now a dynamical equation for the height η. It is known as the shallow water *quasi-geostrophic* equation.

The quasi-geostrophic equation looks a little daunting. But we can easily extract some simple physics. We linearise about a flat surface with $\eta = 0$ and drop any term quadratic in η. The equation then becomes

$$\frac{\partial}{\partial t}\left(c^2\nabla^2\eta - f_0^2\eta\right) + c^2\beta\frac{\partial \eta}{\partial x} = 0$$

We see clearly that the term with β, which captures the variation of the Coriolis parameter, is driving the dynamics. If we look for plane wave solutions with $\eta = \eta_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}$, we find the dispersion relation

$$\omega = -\beta c^2 \frac{k_x}{c^2\mathbf{k}^2 + f_0^2} \tag{4.52}$$

When $\beta = 0$, this gives us the flat band $\omega = 0$ that corresponds to steady-state geostrophically balanced flows. But once we take into account the variation of the Coriolis parameter, these flows start to move. The resulting waves are called *Rossby waves*. The minus sign in (4.52) is important. It is telling us that long wavelength (small k) waves travel in a westward direction. This is indeed the dominant motion of the ocean seen in satellite images. These images clearly reveal Rossby waves that take months, or even years, to cross the Pacific ocean.

It's useful to summarise what we've seen here. The shallow water equations admit two classes of solutions: fast-moving Poincaré waves and slow-moving quasi-geostrophic flows, including Rossby waves. The magic of the quasi-geostrophic equation (4.51) is that it has successfully filtered out the fast-moving Poincaré waves, leaving us just with the slow-moving modes. It is what is referred to in other areas of physics as the "low energy (or frequency) effective field theory". Historically, it the development of the quasi-geostrophic equation was crucial in developing successful weather prediction.

4.3.5 Equatorial Waves

We now ask: what happens when we sit at the equator. Here the Coriolis parameter (4.29) vanishes,

$$f = 2\boldsymbol{\Omega}\cdot\hat{\mathbf{z}} = 0$$

and one might naively think that there can't be any interesting physics due to the Coriolis force. In fact, things are more subtle and more interesting.

To find the more interesting physics, we look a little away from the equator. If we Taylor expand, Coriolis parameter becomes position dependent

$$f(y) = \beta y$$

Here the y-direction is North, and $y = 0$ corresponds to the equator. The parameter β has dimension $[\beta] = L^{-1}T^{-1}$. We can form a distance scale

$$L_{\rm eq} = \sqrt{\frac{c}{\beta}}$$

For the Earth's oceans, this is around $L_{\rm eq} \approx 250$ km. It is somewhat larger for the atmosphere.

We again arrange the height perturbation $\eta(x, y, t)$ and the velocities $u(x, y, t)$ and $v(x, y, t)$ as a vector $\Psi(x, y, t)$ as in (4.46). This time we will look for solutions that are localised near the equator but propagate as waves in the x-direction (i.e. East/West),

$$\Psi(x, y, t) = \tilde{\Psi}(y)e^{i\omega t - ikx} \tag{4.53}$$

The shallow water equations now become

$$\begin{pmatrix} 0 & ck & ic\partial_y \\ ck & 0 & -i\beta y \\ ic\partial_y & i\beta y & 0 \end{pmatrix} \tilde{\Psi} = \omega\tilde{\Psi}$$

Again, we're looking for eigenmodes of this equation. As in the case when f was constant, we expect different branches.

Equatorial Kelvin Waves

To kick us off, there is a special solution to (4.53). This occurs when $v = 0$, so there is no velocity in the y-direction. The equations coming from the first two components of (4.53) are simply algebraic. They relate $\tilde{u} = (\omega/kH)\tilde{\eta}$ and result in the dispersion relation

$$\omega^2 = c^2k^2 \quad \Rightarrow \quad \omega = \pm ck \tag{4.54}$$

We're left just with the third component of (4.53), which governs the profile of $\tilde{\eta}(y)$ and $\tilde{u}(y)$ in the y-direction,

$$\frac{c^2}{H}\frac{\partial\tilde{\eta}}{\partial y} = -\beta y\tilde{u} \quad \Rightarrow \quad \frac{\partial\tilde{\eta}}{\partial y} = -\frac{\omega}{ck}\frac{y}{L_{\rm eq}^2}\tilde{\eta}$$

The key feature of the solution comes from that factor of ω/ck on the right-hand side. From the dispersion relation (4.54), this is either ± 1. However, the resulting solution is only normalisable, and localised around the equator, if we take the positive sign

$$\omega = +ck \quad \Rightarrow \quad \tilde{\eta} = \eta_0 e^{-y^2/2L_{\rm eq}^2}$$

The other choice of sign, with $\omega = -ck$, leads to a divergent solution $\tilde{\eta} \sim e^{+y^2}$ which is not physically permissible. The upshot is rather nice: we have waves at the equator that only travel in the positive x-direction. In other words, they only go east. In analogy with the coastal waves that we met in Section 4.3.3, these are known as *equatorial Kelvin waves.*

Rossby, Poincaré and Yanai Waves

Let's now return to the general problem of equatorial waves, given by the Schrödinger-like equation (4.53). The second component of (4.53) is algebraic and allows us to eliminate $\tilde{u}$ in favour of $\tilde{v}$ and $\tilde{\eta}$. This results in a pair of coupled, first order differential equations

$$\begin{aligned} i\left(\frac{\partial}{\partial y} - \frac{\beta k y}{\omega}\right)\tilde{v} &= \frac{1}{H}\left(\omega - \frac{c^2k^2}{\omega}\right)\tilde{\eta} \\ i\left(\frac{\partial}{\partial y} + \frac{\beta k y}{\omega}\right)\tilde{\eta} &= \frac{H}{c^2}\left(\omega - \frac{\beta^2 y^2}{\omega}\right)\tilde{v} \end{aligned} \qquad (4.55)$$

We can eliminate $\tilde{\eta}$ to manipulate this into a second order differential equation for $\tilde{v}$ alone. After a little bit of algebra, this is

$$\left(-c^2\frac{\partial^2}{\partial y^2} + \beta^2 y^2\right)\tilde{v} = \left(\omega^2 - c^2k^2 - \frac{\beta c^2 k}{\omega}\right)\tilde{v}$$

But this is an equation that we've seen elsewhere: it is the Schrödinger equation for the harmonic oscillator that we met in our first course in Quantum Mechanics. In that context, we write

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\bar{\omega}^2 y^2\right)\tilde{v} = E_n\tilde{v}$$

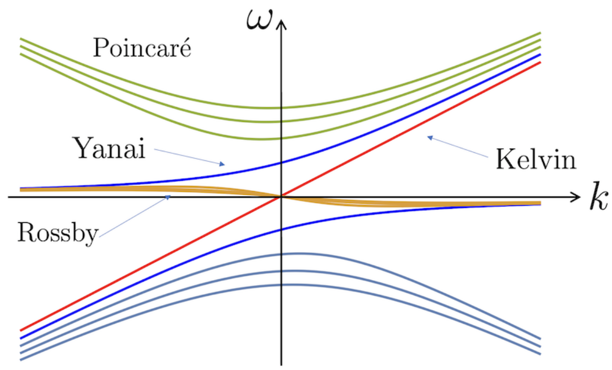


Figure 22. Chiral Kelvin (in red) and Yanai (in dark blue) waves, together with a discretum of Poincaré and Rossby waves.

where m is the mass of the particle and $\bar{\omega}$ is the frequency of the harmonic oscillator (not to be confused with ω, the frequency of our waves the we're trying to determine). Because we are again interested in normalisable solutions, we can simply import our results from quantum mechanics. The velocity $\tilde{v}(y)$ is given by Hermite polynomials. More importantly, the energies of the harmonic oscillator are, famously,

$$E_n = \hbar\bar{\omega}\left(\frac{1}{2}+n\right) \quad \text{with} \quad n = 0, 1, \ldots$$

Translating back into the variables of our equatorial waves, the dispersion relation is given by

$$\omega^3 - \omega\left(c^2k^2 + \beta c(1+2n)\right) - \beta c^2 k = 0 \quad \text{with} \quad n = 0, 1, 2, \ldots \tag{4.56}$$

We'll now look at these for different n. We'll see, the $n = 0$ waves are somewhat different from the $n \geq 1$ waves.

Let's start with the $n = 0$ waves. First note that in this case (4.56) has a root $\omega = -ck$. Naively, this looks like a wave moving in the opposite direction to the Kelvin wave. But it is a spurious solution. This is because although $\tilde{v}(y)$ is normalisable, when we plug this solution into (4.55) we find that $\tilde{\eta}(y)$ is non-normalisable: it has a piece that diverges as $\tilde{\eta} \sim e^{+y^2/2L_{\rm eq}^2}$. So this solution should be thrown out. It turns out that it's the only spurious solution and all others are fine.

If we factor out the spurious $\omega = -ck$ solution, then we find a single $n = 0$ wave, with dispersion relation

$$\omega = \frac{ck}{2} \pm \frac{1}{2}\sqrt{c^2k^2 + 4\beta c}$$

This too is a chiral wave. At large wavenumber, it has the same dispersion relation $\omega \sim +ck$ as the Kelvin wave. However, it differs at small wavenumber, with the dispersion relation affected by the Coriolis force. These are known as *Yanai* waves. (They are also sometimes called mixed Rossby-gravity waves.) The velocity profile is Gaussian around the equator, with $\tilde{v} \sim e^{-y^2/2L_{\rm eq}^2}$.

For $n \geq 1$, the general shape of the dispersion relation takes the same form. There are three branches of modes, which are modified versions of the dispersion relations (4.48) that we saw when f is constant. We again see the dispersion relations corresponding to Poincaré waves, with their characteristic gapped spectrum, asymptoting to $\omega \to \pm ck$.

In addition, we see that our flat band, which previously had $\omega = 0$, is also deformed. Now, it is no longer flat, but asymptotes to $\omega \to -\beta/k$ for large $|k|$. These are equatorial *Rossby waves*. The various modes for $n = 0, 1, 2, 3$, together with the Kelvin wave, are shown in Figure 22. Note that the dispersion relation for the Rossby waves is much flatter than those of the Poincaré waves. Correspondingly, the group velocity of the Rossby waves will be much slower.

4.3.6 Chiral Waves are Topologically Protected

As we mentioned previously, chiral waves appear in various condensed matter systems. The most familiar example is the Quantum Hall Effect, where a sample of electrons in a magnetic field has chiral modes propagating on its edge.

In the context of condensed matter, it turns out that the presence of chiral edge modes can be traced to some interesting topological features of the system, an observation that led to many new developments in the field. The purpose of this section is to point out that, rather wonderfully, the same is true for chiral waves in fluids. I should warn you that this section is something of a departure from the rest of the notes and the motivation is, in part, simply to illustrate the unity of physics.

We will describe the topology associated to equatorial chiral modes. (There is a similar, but more complicated, story for coastal Kelvin waves.) The idea is that the existence of the two chiral modes – Kelvin and Yanai – is a direct consequence of topology in momentum space.

To set the scene, we will return to the case of constant Coriolis parameter f. As we've seen in (4.48), there are three bands with dispersion

$$\omega = \pm\sqrt{c^2\mathbf{k}^2 + f^2} \quad \text{and} \quad \omega = 0 \tag{4.57}$$

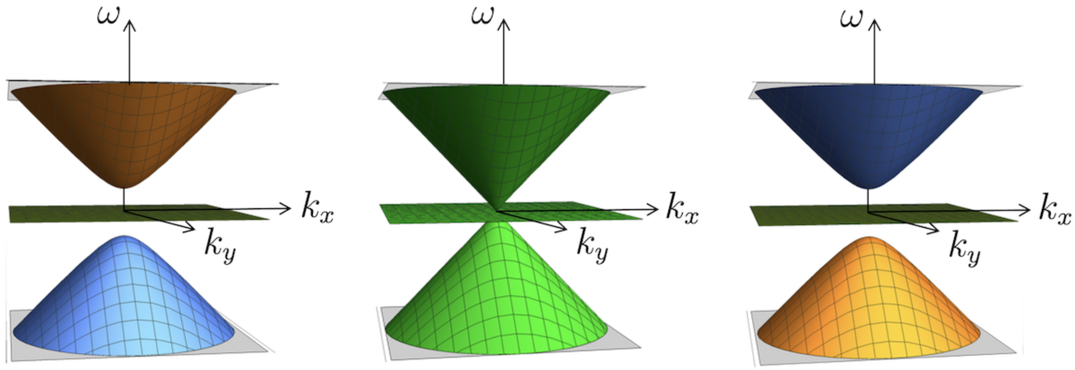


Figure 23. The band structure as a function of constant Coriolis parameter f.

The resulting bands are shown in Figure 23 for three cases: $f > 0$, $f = 0$ and $f < 0$. For $f \neq 0$, there is a gap between the geostrophic flat band and the Poincaré waves. This gap closes when $f = 0$. The fact that the gap closes at $f = 0$ is closely related to the existence of the chiral equatorial waves.

The question that the topological approach addresses is: how robust is this situation? Could we, for example, add some further parameters to the problem so that, as we vary f from positive to negative, the gap never closes? Topology tells us that the answer to this is: no. There must always be some point that looks like the $f = 0$ figure where the gap closes.

The reason for this is that there is a subtle difference between the $f > 0$ and $f < 0$ situations. This difference doesn't show up in dispersion relations (4.57) which are clearly symmetric under $f \rightarrow -f$. Instead, we have to look more closely at what's going on in each band.

Recall from (4.47) that the frequencies arise as the solution to the following eigenvalue problem

$$\begin{pmatrix} 0 & ck_x & ck_y \\ ck_x & 0 & -if \\ ck_y & if & 0 \end{pmatrix} \tilde{\Psi} = \omega \tilde{\Psi}$$

We will focus on the positive frequency band of Poincaré waves, with

$$\omega(\mathbf{k}) = +\sqrt{c^2\mathbf{k}^2 + f^2}$$

As we've already mentioned, the eigenvalues are clearly invariant under $f \to -f$. To see the difference between $+f$ and $-f$ we need to look at the eigenvector. This is given by

$$\tilde{\Psi}_+(\mathbf{k}, f) = \frac{1}{\sqrt{2\omega^2 \mathbf{k}^2}} \begin{pmatrix} c\mathbf{k}^2 \\ k_x\omega - ifk_y \\ k_y\omega + ifk_x \end{pmatrix}$$

Obviously, the eigenvector depends on the wavenumber $\mathbf{k}$. This means that as we move around momentum space, labelled by $\mathbf{k} \in \mathbb{R}^2$, the eigenvector Ψ_0 evolves in $\mathbb{C}^3$. The key idea is that as we explore all of momentum space, the eigenvector may twist within the larger space $\mathbb{C}^3$. This twist is where topology enters the story.

The fact that eigenvectors twist and turn in a larger space is more familiar in the context of quantum mechanics where it goes by the name of *Berry phase*. (You can read about this both in the lectures on Topics in Quantum Mechanics and in the lectures on the Quantum Hall Effect.) We will not review this in detail, but simply state how to characterise the topology of the eigenvector. First, given an eigenvector Ψ_0 we define the *Berry connection*,

$$A_i(\mathbf{k}) = -i\tilde{\Psi}_+^\dagger \frac{\partial \tilde{\Psi}_+}{\partial k^i} \qquad i = 1, 2$$

A short calculation shows that

$$A_i = \frac{f}{\omega \mathbf{k}^2} (k_y, -k_x)$$

The Berry connection has the same mathematical structure as the gauge potential in electromagnetism. In particular, as the next step we compute something akin the magnetic field,

$$B = \partial_1 A_2 - \partial_2 A_1 = \frac{c^2 f}{(f^2 + c^2\mathbf{k}^2)^{3/2}}$$

This is known as the *Berry curvature*. Finally, we integrate this curvature over momentum space to get an object known as the *Chern number*, which we calculate to be

$$C = \frac{1}{2\pi} \int_{\mathbb{R}^2} d^2k \; B = \text{sign}[f] \tag{4.58}$$

Note that, as promised, the Chern number distinguishes between f positive and f negative: we have $C = +1$ for $f > 0$ and $C = -1$ for $f < 0$.

At this stage the argument becomes slightly delicate. When the Chern number is computed by integrating over a compact space (i.e. one which doesn't stretch to infinity), then there is a mathematical result that says

$$C \in \mathbb{Z}$$

(In physics, this is usually referred to as Dirac quantisation.) The fact that C is integer valued is important. It is telling us that we have some discrete way of characterising the system, even though the underlying fluids are continuous. This is the essence of topology.

However, things are not so straightforward for our fluids because the integral (4.58) is not over a compact space but instead over $\mathbb{R}^2$. (This is not a problem in condensed matter systems because the underlying spatial lattice means that momentum lives in a compact Brillouin zone.) And there's no mathematical theorem that says such an integral should be integer valued. Indeed, if you integrate the magnetic flux through a solenoid then you can get anything at all. There are a couple of ways around this and we will take the cheapest. Note that asymptotically, as $|\mathbf{k}| \to \infty$, we have $A_i \to 0$. In fact, more importantly, we have $\oint A_i dk^i \to 0$ as the integration curve is taken out to infinity. This is a property of short wavelength modes and so should hold for regardless of any deformation of the system which doesn't affect arbitrarily short wavelengths. So we insist that A_i is trivial asymptotically and this allows us to effectively compactify the problem, by adding a point at infinity and viewing $\mathbb{R}^2 + \{\infty\} = \mathbf{S}^2$. Correspondingly, we learn that the Chern number C – which is clearly an integer in (4.58) – should remain an integer no matter how we deform the system.

Now we're in business. For $f > 0$, we have $C = 1$. This is the yellowy-brown, upper band on the left-hand side of Figure 23. But because C is restricted to be an integer, it can't change as we vary parameters. The only exception to this is if the gap to some other band closes, because then the eigenvector Ψ_+ becomes degenerate with another eigenvector and the calculation above breaks down. But the Chern number for $f < 0$ is $C = -1$, depicted in blue in Figure 23, so we learn that here is no path in any enlarged parameter space that takes us from $f > 0$ to $f < 0$ without closing the gap.

In fact, there's more to learn from this. The number of chiral edge modes that appear as we vary from $f > 0$ to $f < 0$ is given by the difference in the Chern numbers. In other words, the number of chiral waves is necessarily $C[f > 0] - C[f < 0] = 2$. And this is indeed what we find, with the Kelvin and Yanai waves appearing at the equator.

As we've mentioned previously, the kind of calculation that we've performed above underlies various properties of materials, notably quantum Hall states and topological insulators. (See, for example, Section 2.3 of the lectures on the Quantum Hall Effect for a closely related computation for the Chern insulator.) In that context, the use of topology blossomed, revealing many deep and new ideas about exotic materials. So far, in fluid mechanics, topology seems to be little more than a curiosity. Hopefully, that will change in the future.

4.4 Sound Waves

Throughout these lectures, we've focussed on incompressible fluids, obeying $\nabla \cdot \mathbf{u} = 0$. It is now time to abandon this assumption. Instead, we want to ask: what new physics arises when the density of a fluid changes? The answer, as we shall see, is sound.

Apart from its inherent interest, this question forces us to re-examine the fundamental equations of fluid mechanics. So far, we have mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{4.59}$$

and the Navier-Stokes equation

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P + \mu \nabla^2 \mathbf{u} + \left(\frac{\mu}{3} + \zeta \right) \nabla (\nabla \cdot \mathbf{u})$$

where, because the fluid is now compressible, there's a second viscosity term that can appear on the right-hand side. This comes with a new coefficient ζ, the *bulk viscosity* in addition to μ, the *dynamic shear viscosity*. (In fact, we'll largely ignore the effects of both viscosities in this section, but it's worth keeping it in play while we return to the fundamentals.)

When the density was constant, these equations were all we needed. They are four equations that govern four independent, dynamical fields, $\mathbf{u}(\mathbf{x}, t)$ and $P(\mathbf{x}, t)$. However, when $\rho = \rho(\mathbf{x}, t)$ is also a dynamical field we need a further equation before we can get going. As we now explain, this additional equation is dictated by thermodynamics and forces us to think about the temperature of the fluid.

4.4.1 Compressible Fluids and the Equation of State

The *equation of state* is a relationship between the pressure P, the volume V and the temperature T of a fluid. The simplest such example is very familiar: it is the ideal gas law

$$PV = Nk_B T$$

Here N is the number of particles in the gas and k_B is Boltzmann's constant. The ideal gas equation describes a gas of non-interacting particles. (Note that the "ideal" in "ideal gas" means something different from the "ideal" in "ideal fluid"! The latter just refers to something that obeys the Euler equation with no viscosity.) Other equations of state can be more complicated, capturing some internal interactions between the constituent molecules. For example, a simple generalisation of the idea gas law is the van der Waals equation

$$\frac{PV}{Nk_BT} = 1 - \frac{N}{V}\left(\frac{a}{k_BT} - b\right)$$

where a and b are two constants that characterise the interactions of the gas. You can find derivations of both these equations of state in the lectures on Statistical Physics.

We can make contact with our previous equation if we replace the volume with the density ρ of the fluid,

$$\rho = \frac{Nm}{V}$$

where m is the mass of the each individual particle in the fluid. Then the ideal gas law becomes

$$P = \frac{\rho k_B T}{m} \tag{4.60}$$

When we first meet the equation of state, we think of P, $\rho \sim 1/V$ and T as numbers that describe the global, equilibrium properties of the system. However, the whole point of fluid mechanics is that we can understand what happens as we move away from equilibrium. To achieve this, we assume that locally the system is still described by P, ρ and T but these are now dynamical fields whose values can vary in space and in time. The equation of state now gives a *local* relationship between these quantities, for example

$$P(\mathbf{x},t) = \frac{\rho(\mathbf{x},t)k_BT(\mathbf{x},t)}{m}$$

The existence of the equation of state tells us why we need to start thinking about temperature. If the pressure P and density ρ are changing, then so too is T. Indeed, this is true even when ρ is constant but throughout these lectures we have implicitly assumed that $T(\mathbf{x},t)$ simply tracks the pressure $P(\mathbf{x},t)$. Now, however, we need to think more carefully about how T changes.

4.4.2 Some Thermodynamics

The correct way to proceed is to derive an equation of motion for the temperature $T(\mathbf{x}, t)$. For now, however, we'll take something of a shortcut. For completeness, we will then describe the better approach in Section 4.4.3.

The shortcut that we have in mind is called the *adiabatic approximation*. Heuristically, this means that we assume that the timescale over which the fluid moves is much shorter than the timescale of heat diffusion within the fluid. Mathematically, it means that we assume a quantity called *entropy* is conserved. The purpose of this section is to review some basic facts about thermodynamics, the purpose of which is to lead us to the following, simple result: under the adiabatic approximation

$$\frac{P}{\rho^\gamma} = \text{constant} \tag{4.61}$$

where γ is the ration of heat capacities $\gamma = c_P/c_V$ and will be defined below. For air, $\gamma \approx 1.4$. Starting in Section 4.4.4 we'll make use of this result to study the properties of sound waves.

For now, we'll revert to the older setting where P, V and T as just numbers that characterise the global property of an equilibrium system. We then need to turn to the laws of thermodynamics. (A much fuller discussion of this material can be found in Section 4 of the lectures on Statistical Physics.).

The *first law of thermodynamics* says that the energy E of a system can change in one of two ways: either by adding heat δQ, or by adding work, δW

$$dE = \delta Q + \delta W$$

The energy is a function of the system, but both heat and work are things that you do to the system. There's no sense in which we can talk about the "work" of a gas or the "heat" of a gas; only the heat added to a gas. Roughly speaking, this is the reason that we write the terms on the right-hand side as δQ and δW instead of dQ and dW.

However, it should be possible to describe the effect of both the work done and the heat added in terms of changes to the state of the system. For the work done, this is straightforward. If the fluid has pressure P and we squeeze it by changing its volume, then the infinitesimal work done is

$$\delta W = -PdV$$

To write a similar statement for the heat added to a gas we need to turn to the second law of thermodynamics. This is the statement that the state of the system in equilibrium can be characterised by a function $S(T, P)$ known as *entropy*. Furthermore, for a reversible change we have

$$\delta Q = T dS$$

This definition, relating entropy to heat, is due to Clausius. Subsequently, Boltzmann understood entropy in terms of counting microscopic arrangements of atoms. A large part of the course on Statistical Physics is to understand why these two definitions are actually equivalent. For our purposes, we'll only need the definition above.

Adiabatic processes, of which sound waves are an example, have $\delta Q = 0$. You might think that this means we can simply ignore the heat term. Sadly, that's not quite true! We need to understand a little better what heat actually is before we can discard it.

Next, we need the idea of a heat capacity. This is straightforward: it measures how much the temperature of a system rises if you add some heat. (Actually, it's defined to be the inverse of this.) The subtle point is that you must specify what you are holding fixed when you do this experiment. You could, for example, hold the volume fixed. The corresponding heat capacity C_V is defined by

$$C_V = T \left.\frac{\partial S}{\partial T}\right|_V = \left.\frac{\partial E}{\partial T}\right|_V$$

where, in the second equality, we've used the first law of thermodynamics $dE = TdS - PdV$ where the $-PdV$ term doesn't contribute precisely because we're holding the volume fixed. Alternatively, we can add heat keeping the pressure fixed, rather than the volume. Again, using the first law, we have

$$C_P = T \left.\frac{\partial S}{\partial T}\right|_P = \left.\frac{\partial E}{\partial T}\right|_P + P \left.\frac{\partial V}{\partial T}\right|_P \tag{4.62}$$

In this case, the temperature is expected to rise less because the energy from the heat must now also do work expanding the volume of the gas. Correspondingly, we expect $C_P > C_V$. We often talk about the *specific heats*, which is the heat capacity per unit volume: $c_V = C_V/V$ and $c_P = C_P/V$.

The Ideal Gas

So far, our discussion has been general. To make progress, we now focus on a specific system: the ideal gas, with the familiar equation of state

$$PV = Nk_BT$$

This is a good approximation for dilute gases, like the air in this room. It's not a good approximation for liquids.

The final fact that we need is known as *equipartition*. It is the statement that, at temperature T, the energy of each microscopic degree of freedom is given by $\frac{1}{2}k_BT$. This means that if we have a gas of N "monatomic" particles, meaning that each particle is itself a structureless object, then

$$E = \frac{3}{2}Nk_BT \quad \text{for monatomic gases}$$

where the $\frac{3}{2}$ comes because each particle can move in three dimensions. However, if the particles comprising the gas have additional internal degrees of freedom then equipartition ensures that these too contribute to the energy. For example, a diatomic molecule can be viewed as a dumbbell-like object. It has three translational degrees of freedom, but also two rotational degrees of freedom. (The rotation about the axis doesn't count[8].) This means that the energy is

$$E = \frac{5}{2}Nk_BT \quad \text{for diatomic gases}$$

Air is mostly N_2 and O_2, both of which are diatomic molecules, so this is the energy of air.

We can now compute the heat capacities for the different ideal gases. We have

$$C_V = \frac{3}{2}Nk_B \quad \text{and} \quad C_P = \frac{5}{2}Nk_B \quad \text{for monatomic gases}$$

and

$$C_V = \frac{5}{2}Nk_B \quad \text{and} \quad C_P = \frac{7}{2}Nk_B \quad \text{for diatomic gases}$$

Note that, in both cases, $C_P - C_V = Nk_B$, which follows from (4.62), together with the equation of state. It will be useful to define the ratio of the heat capacities

$$\gamma = \frac{C_P}{C_V} = \begin{cases} 5/3 & \text{for monatomic gases} \\ 7/5 & \text{for diatomic gases} \end{cases}$$

This is where we get the statement that $\gamma \approx 1.4$ for air.

[8]Rather wonderfully, this is a quantum mechanical effect! Both the rotation about the axis and the vibrational mode of the dumbbell have a minimum energy required to excite them due to quantum mechanics, and this energy is higher than k_BT at room temperatures. The same is true of the rotational mode of a monatomic gas.

Finally, we can use the technology above to compute the entropy of an ideal gas. We start from the first law, now written as

$$dS = \frac{1}{T}dE + \frac{P}{T}dV = \frac{C_V}{T}dT + \frac{Nk_B}{V}dV$$

We now replace $Nk_B = C_P - C_V$ and integrate to get

$$\begin{aligned} S &= C_V \log\left(\frac{T}{T_0}\right) + (C_P - C_V)\log\left(\frac{V}{V_0}\right) \\ &= C_V \log\left(\frac{T}{T_0}\left(\frac{V}{V_0}\right)^{\gamma-1}\right) \\ &= C_V \log\left(\frac{P}{P_0}\left(\frac{V}{V_0}\right)^{\gamma}\right) \end{aligned} \tag{4.63}$$

This means that if entropy is to remain constant under some change, the pressure and volume must scale so that PV^γ is constant. Or, written in terms of the density $\rho \sim 1/V$,

$$\frac{P}{\rho^\gamma} = \text{constant} \tag{4.64}$$

This is the result (4.61) that we advertised at the beginning of this section.

4.4.3 Briefly, Heat Transport

There's a more sophisticated way of stating the result above, in which we focus directly on the dynamics of the temperature field $T(\mathbf{x}, t)$. For an ideal fluid, the temperature is governed by the transport equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)T + (\gamma - 1)T\,\nabla\cdot\mathbf{u} = 0 \tag{4.65}$$

We won't derive this here. (You can find the derivation in the lectures on Kinetic Theory.) But we can at least see how it reproduces the result above. To see this, note that the requirement of constant entropy can also be written as $TV^{\gamma-1} \sim T\rho^{1-\gamma}$ is constant, as in (4.63). But with T and ρ both fields, we can see how these evolve within the fluid. The appropriate meaning of "constant" is that the material derivative vanishes. We have

$$\begin{aligned} \frac{D}{Dt}(T\rho^{1-\gamma}) &= \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)(T\rho^{1-\gamma}) \\ &= (1-\gamma)\rho^{1-\gamma}T\nabla\cdot\mathbf{u} + T\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\rho^{1-\gamma} \end{aligned}$$

Here the first term follows from (4.65). We can evaluate the second term using the conservation of mass (4.59). We find

$$\frac{D}{Dt}(T\rho^{1-\gamma}) = 0$$

as expected for adiabatic evolution.

This language has the advantage that it allows us to go beyond ideal fluids. In fact, the heat transport equation (4.65) should be viewed as analogous to the Euler equation for the velocity: both are missing the effect of dissipation. For the velocity field, this is captured by viscosity. For the temperature field, it is captured by heat conductivity. κ. This appears as an additional term in the heat equation

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)T + (\gamma-1)T\,\nabla\cdot\mathbf{u} = \frac{\gamma-1}{\gamma}\frac{\kappa m}{\rho k_B}\nabla^2 T \tag{4.66}$$

where the strange collection of coefficients on the right-hand side means that the coefficient multiplying $\nabla^2 T$ can be identified as κ/c_P where $c_P = \rho k_B\gamma/(\gamma-1)m$ is the specific heat at constant pressure. These terms tell us how heat diffuses in the fluid. Indeed, in the absence of any flow, so $\mathbf{u} = 0$, it reduces to the heat equation

$$\frac{\partial T}{\partial t} = \frac{\kappa}{c_P}\nabla^2 T$$

The adiabatic approximation that we invoked above is essentially the statement that the diffusion of heat can be neglected in the problem of interest. And that problem is, of course, sound waves

4.4.4 The Equations for Sound Waves

Finally, after that long preamble, we can turn to the subject of interest: sound waves. We will initially ignore viscosity (remedying this in Section 4.4.5) and work with the Euler equation

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P$$

Our starting point is the simplest possible solution to the Euler equation: a stationary fluid, with constant density and pressure

$$\mathbf{u} = 0 \quad , \quad \rho = \rho_0 \quad , \quad P = P_0$$

We then study small perturbations about this background. We will take

$$\rho = \rho_0 + \tilde{\rho} \quad \text{and} \quad P = P_0 + \tilde{P}$$

with the perturbations small, meaning $\tilde{\rho} \ll \rho_0$ and $\tilde{P} \ll P_0$. We'll also take $\mathbf{u}$ to be small, in the sense that we keep terms only linear in $\mathbf{u}$, $\tilde{\rho}$ and $\tilde{P}$. The linearised Euler equation then becomes

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla \tilde{P} \tag{4.67}$$

We augment this with the equation of mass conservation,

$$\frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = 0 \tag{4.68}$$

We can combine these by taking the gradient ∇ of the first and the time derivative of the second. This gives

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - \nabla^2 \tilde{P} = 0 \tag{4.69}$$

At this point, we need to invoke the adiabatic approximation (4.64) which, after linearising, becomes

$$\frac{(P_0 + \tilde{P})}{(\rho_0 + \tilde{\rho})^\gamma} = \text{constant} \quad \Rightarrow \quad \tilde{P} - \frac{P_0}{\rho_0}\gamma\tilde{\rho} = 0$$

The equation (4.69) then becomes

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - c_s^2 \nabla^2 \tilde{\rho} = 0 \tag{4.70}$$

This is the wave equation. The speed of sound is given by

$$c_s = \sqrt{\frac{\gamma P_0}{\rho_0}} \tag{4.71}$$

For an ideal gas, the equation of state (4.60) relates this to the temperature T_0 of the background fluid, and the mass m of the constituent particles,

$$c_s = \sqrt{\frac{\gamma k_B T_0}{m}} \tag{4.72}$$

We see that the speed of sound depends on the temperature. For the air at 20°, the speed is $c_s \approx 340 \text{ ms}^{-1}$. This was first measured by Newton by clapping his hands in Nevile's court, Trinity College. (He got a value around 300 ms^{-1}.)

A General Fluid

The equation (4.69) holds for any fluid while, the subsequent derivation of the wave equation, we restricted to the ideal gas. But we get the same wave equation for any equation of state; it's just the speed of sound that changes.

It's useful to think of the pressure as a function of

$$P = P(\rho, S)$$

(It's perhaps more natural to think of $P = P(V,T)$. The density is trivially related to the volume by $\rho \sim 1/V$. But the entropy S is a conjugate variable to the temperature T and it is also possible to think of pressure as a function of entropy. This kind of "what function depends on what variable" is a large part of the game of thermodynamics.) Taylor expanding, the fluctuations in pressure and density are then related by

$$\tilde{P} = \left.\frac{\partial P}{\partial \rho}\right|_S \tilde{\rho}$$

In general, the speed of sound is then given by

$$c_s = \sqrt{\left.\frac{\partial P}{\partial \rho}\right|_S} \tag{4.73}$$

Measurements of this derivative are usually given in terms of the *bulk modulus*, defined to be $K = \rho\, \partial P/\partial \rho$. For water at 20°, this is $K \approx 200\ \mathrm{Nm}^{-2}$. It's much higher than the corresponding value for gases, reflecting the fact that it is more difficult to squeeze water than air. The density of water is $\rho_0 \approx 10^3 \mathrm{kg\,m}^{-3}$. The speed of sound in water is then much higher than in air, $c_s \approx 1500\ \mathrm{ms}^{-1}$.

Sound Waves are Longitudinal

The wave equation (4.70) is solved by any Fourier mode

$$\tilde{\rho}(\mathbf{x}, t) = \hat{\rho}\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \tag{4.74}$$

Here $\hat{\rho}$ is the constant amplitude of the wave. In the exponent, ω is the frequency and $\mathbf{k}$ is the wavevector which points in the direction of propagation. The two are related by the dispersion relation

$$\omega = c_s |\mathbf{k}|$$

This is now a dispersion relation that doesn't disperse, in the sense that all wavelengths propagate with the same speed. As we've seen, this contrasts with the surface waves of

Section 4.1. Because the wave equation is linear, we can combine many Fourier modes to make a wavepacket. If this is made from wavevectors $\mathbf{k}$ that all point in the same direction, then the wavepacket will keep its shape as it moves. We can also see this directly from the wave equation. If the wave is moving in the x-direction, then the wave equation is solved by any function of the form

$$\tilde{\rho} = F(t - x/c_s) + G(t + x/c_s) \tag{4.75}$$

Here F and G are the profiles of two wave packets, moving to the right and left respectively.

We can reconstruct the pressure and velocity oscillations from our original, first order equations. The pressure perturbations are simply given by $\tilde{P} = c_s^2 \tilde{\rho}$. From (4.68) we have

$$\mathbf{u}(\mathbf{x}, t) = \frac{\hat{\mathbf{k}}}{\rho_0 c_s} \tilde{P}(\mathbf{x}, t) = \frac{c_s \hat{\mathbf{k}}}{\rho_0} \tilde{\rho}(\mathbf{x}, t)$$

The oscillations of the fluid velocity and the pressure are all in phase with the density The velocity oscillations are also parallel to the direction $\mathbf{k}$ in which the wave travels. Such waves are called *longitudinal*.

Spherically Symmetric Waves

Although we can construct any solution from the Fourier modes (4.74), that's often not the best way to proceed. For example, if we have some localised source which, for convenience, we will assume is spherically symmetric then it's clear that we are best served by working in spherical polar coordinates. Ignoring the angular directions, the wave equation becomes

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - c_s^2 \nabla^2 \tilde{\rho} = 0 \quad \Rightarrow \quad \frac{\partial^2 (r\tilde{\rho})}{\partial t^2} - c_s^2 \frac{\partial^2 (r\tilde{\rho})}{\partial r^2} = 0$$

This is now a 1d wave equation. It is solved, analogously to (4.75) by any two functions

$$\tilde{\rho}(r, t) = \frac{1}{4\pi r} \left[F(t - r/c_s) + G(t + r/c_s) \right]$$

The factor of 4π is just for convenience. The function F describes the outgoing wave, while G describes the incoming wave. In many situations, there's no wave coming in from infinity so we set $G = 0$. This is the choice we make here.

The associated velocity field is most simply computed from (4.67) using $\tilde{P} = c_s^2 \tilde{\rho}$. To write down the solution, we need to integrate the wave profile. We write

$$F(t - r/c_s) = \dot{Q}(t - r/c_s)$$

In spherical polars, we then have

$$\nabla \tilde{P} = -\frac{c_s^2}{4\pi} \left[\frac{\dot{Q}(t - r/c_s)}{r^2} + \frac{\ddot{Q}(t - r/c_s)}{c_s r} \right] \hat{\mathbf{r}}$$

and, comparing to (4.67), the velocity field is radial, with

$$\mathbf{u}(r, t) = \frac{c_s^2}{4\pi\rho_0} \left[\frac{Q(t - r/c_s)}{r^2} + \frac{\dot{Q}(t - r/c_s)}{c_s r} \right] \hat{\mathbf{r}} \tag{4.76}$$

Close to the source, the first term dominates; far away the second term dominates.

As an example, consider the sound waves generated by a pulsating sphere of radius a. We'll take this sphere to beat in and out, with frequency ω and amplitude ϵ, so the radius changes with time as

$$R(t) = a + \epsilon e^{i\omega t} \quad \Rightarrow \quad \dot{R} = i\omega\epsilon e^{i\omega t}$$

The solution must take the form (4.76) for some $Q(t) = Ae^{i\omega t}$. This means that

$$\mathbf{u}(r, t) = \frac{Ac_s^2}{4\pi\rho_0} \left[\frac{1}{r^2} + \frac{i\omega}{c_s r} \right] e^{i\omega(t - r/c_s)} \hat{\mathbf{r}}$$

This is subject to the requirement that the fluid velocity matches that of the sphere on its surface, i.e.

$$\mathbf{u}(R(t), t) = \dot{R}\, \hat{\mathbf{r}} \quad \Rightarrow \quad \mathbf{u}(a, t) + \frac{\partial \mathbf{u}}{\partial r} \epsilon e^{i\omega t} + \ldots = i\omega\epsilon e^{i\omega t} \hat{\mathbf{r}}$$

Since $\mathbf{u} \sim \mathcal{O}(\omega\epsilon)$, the second term in the above expression is lower order and it will suffice to set

$$\mathbf{u}(a, t) = i\omega\epsilon e^{i\omega t} \hat{\mathbf{r}} \quad \Rightarrow \quad \frac{Ac_s^2}{4\pi\rho_0 a^2} \left[1 + \frac{i\omega a}{c_s} \right] e^{-i\omega a/c_s} = i\omega\epsilon$$

which fixes the overall coefficient A.

4.4.5 Viscosity and Damping

It is natural to ask: how does viscosity affect the propagation of sound? Because viscosity is dissipative, any process will necessarily increase the entropy and so is no longer adiabatic. This means that we can't just use the simple relation $P\rho^{-\gamma}$ and must instead turn to the more sophisticated description in terms of the temperature field.

We met the heat transport equation in (4.66)

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)T + (\gamma-1)T\,\nabla\cdot\mathbf{u} = \frac{\gamma-1}{\gamma}\frac{\kappa m}{\rho k_B}\nabla^2 T$$

This should be augmented with the Navier-Stokes equation

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P + \mu\nabla^2\mathbf{u} + \left(\frac{\mu}{3}+\zeta\right)\nabla(\nabla\cdot\mathbf{u})$$

together with mass conservation and an appropriate equation of state that relates P, ρ and T. We'll stick with the ideal gas equation of state, so

$$P = \frac{k_B T\rho}{m}$$

and we substitute this into the Navier-Stokes equation. For dilute gases, it turns out that $\zeta \approx 0$ so we choose to set it to zero. (It doesn't qualitatively change the physics because, as you can see, the shear viscosity μ already appears in the relevant term.) Our goal is to reproduce our previous results about sound waves in this framework, and then to understand how these results are affected by the viscosity μ and the heat conductivity κ.

As before, we start with a stationary fluid but now also include the fact that it has constant temperature

$$\mathbf{u} = 0 \quad , \quad \rho = \rho_0 \quad , \quad T = T_0$$

We then consider time-dependent perturbations,

$$\begin{aligned}\mathbf{u} &= \hat{u}\hat{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\\ \rho &= \rho_0 + \hat{\rho}\,e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\\ T &= T_0 + \hat{T}\,e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\end{aligned}$$

Note that we're looking for longitudinal waves, with $\mathbf{u}$ parallel to $\mathbf{k}$. Linearising, the mass conservation equation tells us that

$$\omega\hat{\rho} = \rho_0 k\hat{u}$$

The linearised heat transport equation is

$$-i\omega\hat{T} + i(\gamma-1)T_0 k\hat{u} = -\frac{\kappa}{c_P}k^2\hat{T}$$

where $c_P = \rho_0 k_B \gamma/(\gamma-1)m$. Last, the linearised Navier-Stokes equation is

$$-i\rho_0\omega\hat{u} = -\frac{ik_Bk}{m}\left(T_0\hat{\rho} + \rho_0\hat{T}\right) - \frac{4}{3}\mu k^2\hat{u}$$

We can write these simultaneous equations as a matrix,

$$M\begin{pmatrix}\hat{\rho}\\ \hat{u}\\ \hat{T}\end{pmatrix} = \omega\begin{pmatrix}\hat{\rho}\\ \hat{u}\\ \hat{T}\end{pmatrix} \quad \text{with} \quad M = \begin{pmatrix} 0 & \rho_0 k & 0 \\ k_BkT_0/m\rho_0 & -\frac{4i}{3}\mu k^2/\rho_0 & k_Bk/m \\ 0 & (\gamma-1)T_0k & -i\kappa k^2/c_V \end{pmatrix} \tag{4.77}$$

The frequencies of the perturbations ω are given by the eigenvalues of the matrix M. As we will see, this will give the dispersion relation between ω and k. Note, moreover, that the elements of the matrix are real except for those that multiply the dissipative coefficients μ and κ. We'll see what this means for the physics shortly.

First let's look at what happens when $\mu = \kappa = 0$. There are solutions

$$\begin{pmatrix}\hat{\rho}\\ \hat{u}\\ \hat{T}\end{pmatrix} = \epsilon\begin{pmatrix}\rho_0\\ \omega/k\\ (\gamma-1)T_0\end{pmatrix}$$

with ϵ some small, dimensionless parameter needed for the linearised approximation to be valid. This immediately solves the first two and third equations, while the second requires

$$m\omega^2 = \gamma k_BT_0k^2 \quad\Rightarrow\quad \omega = \pm\sqrt{\frac{\gamma k_BT_0}{m}}k = \pm c_s k$$

But this is just our previous result (4.72) for the speed of sound. Moreover, we see that this perturbation has $(\gamma-1)T_0\tilde{\rho} - \rho_0\tilde{T} = 0$ which means that $T/\rho^{\gamma-1}$ is constant. But this is the expected behaviour (4.63) for an adiabatic deformation of the fluid. So, in the limit that the dissipative effects vanish, we do indeed recover the adiabatic sound waves of the previous section.

There is also a novel solution to the equation (4.77) with $\mu = \kappa = 0$ that we haven't seen previously. This has $\hat{u} = 0$ and

$$T_0\hat{\rho} + \rho_0\hat{T} = 0$$

a combination that ensures that $P \sim \rho T$ is constant in this perturbation. It solves the matrix equation above only when $\omega = 0$. Because the pressure is constant, there is no restoring force for this perturbation.

Having made contact with our previous result, we can now see how it's changed when we turn on viscosity μ and heat conductivity κ. Rather than directly finding the eigenvectors, we can take a bit of a shortcut to extract just the eigenvalues ω. First note that the determinant and trace of M are given by

$$\det M = i\frac{\kappa c_s^2}{\gamma c_V}k^4 \quad \text{and} \quad \mathrm{Tr}\, M = -i\left(\frac{\kappa}{c_V} + \frac{4}{3}\frac{\mu}{\rho_0}\right)k^2$$

The product of the three eigenvalues must be equal to $\det M$. When $\mu = \kappa = 0$, we know that one of the eigenvalues vanishes and the other two were $\pm c_s k$. But now we see that the three must multiply to give something proportional to κ. This means that, to leading order in κ, the zero eigenvalue that arose from perturbations of constant pressure must change to

$$-c_s^2 k^2 \omega = \det M \quad \Rightarrow \quad \omega = -i\frac{\kappa}{\gamma c_V}k^2$$

The frequency is imaginary and negative. This is telling us that the modes decay exponentially quickly. To see this, write $\omega = -i\Gamma$. Then the behaviour of all modes goes as $e^{-i\omega t} = e^{-\Gamma t}$. The behaviour that we find above scales as $\omega \sim -ik^2$. This is characteristic of diffusion. It is the kind of behaviour that we get from the heat equation.

The two remaining modes are what becomes of sound waves. These too are expected to get a dissipative contribution. If we anticipate that they take the form

$$\omega = \pm c_s k - i\tilde{\Gamma}$$

possibly with some change to the sound speed, then we can compute $\tilde{\Gamma}$ by noting that the trace must equal the sum of all three eigenvalues, so

$$-2i\tilde{\Gamma} - i\frac{\kappa}{\gamma c_V}k^2 = \mathrm{Tr}\, M \quad \Rightarrow \quad \tilde{\Gamma} = \frac{1}{2}\left(\frac{4}{3}\frac{\mu}{m\rho_0} + \frac{\gamma - 1}{\gamma}\frac{\kappa}{c_V}\right)k^2$$

We see that the effect of viscosity and of heat conduction is similar: the sound waves diffuse and decay over time, with their lifetime set by $1/\tilde{\Gamma}$.

In addition, we can ask about velocity perturbations that are transverse to the wave, so that $\mathbf{k} \cdot \mathbf{u} = 0$. These are known as *shear perturbations*. It's straightforward to see that mass conservation and heat transport require $\tilde{\rho} = \tilde{T} = 0$, while the linearised Navier-Stokes equation gives the dispersion relation

$$\omega = -i\frac{\mu}{\rho_0}k^2$$

We see that these modes also behave diffusively.

4.5 Non-Linear Sound Waves

So far, throughout this section we've only considered linear wave equations. For surface waves we went to some lengths to pick an approximation which made our equations linear and for sound wave we dropped the $\mathbf{u} \cdot \nabla \mathbf{u}$ term in the Navier-Stokes equation. This is a good first step since linear equations are significantly easier to solve than non-linear equations. But it's natural to wonder: under what circumstances are the non-linearities important? And what effect do they have? Here we start to address such questions, albeit in the somewhat restricted context of waves propagating in one dimension.

We'll revisit our analysis of sound waves, but now restricted to 1d. Our defining equations are the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \tag{4.78}$$

and the Euler equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{4.79}$$

Previously we dropped the $u\partial u/\partial x$ term. Our goal now is to understand what role it plays.

So far we have two equations for three variables: u, P and ρ. As we stressed previously in Section 4.4, we must add one further equation. Rather than getting all hot and bothered by introducing temperature, we will instead work directly with an adiabatic equation of state that relates the pressure to the density,

$$P = P(\rho)$$

For example, for the ideal gas undergoing adiabatic deformations, we showed that the relevant equation is $P\rho^{-\gamma} = \text{constant}$ with $\gamma = c_P/c_V$ the ratio of specific heats. (See equation (4.64).) We'll turn to this example later but for now we keep things general. We also saw that in the previous section that, in the linearised approximation, the speed of sound is given by (4.73),

$$c_s^2(\rho) = \frac{dP}{d\rho}$$

(Previously we wrote this as a partial derivative, keeping entropy S fixed. In this section we assume that entropy is fixed and view P only as a function of ρ.) One of the things we would like to learn is the sense in which c_s retains its interpretation as the speed of sound waves beyond the linearised approximation.

4.5.1 The Method of Characteristics

From the definition of c_s^2, together with (4.78), we have

$$\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} = c_s^2\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right) = -\rho c_s^2 \frac{\partial u}{\partial x} \tag{4.80}$$

To make progress, we're going to rewrite the Euler equation (4.79) and our equation for pressure (4.80) in a clever way. Starting from (4.79), we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} + (u - c_s)\frac{\partial}{\partial x}\right) u &= -\frac{1}{\rho}\frac{\partial P}{\partial x} - c_s \frac{\partial u}{\partial x} \\ &= \frac{1}{\rho c_s}\left(\frac{\partial}{\partial t} + (u - c_s)\frac{\partial}{\partial x}\right) P \end{aligned} \tag{4.81}$$

where, to get to the second line, we've used (4.80). There's a nice symmetry between the left- and right-hand side of this equation, with the same differential operator appearing in both. The only difference between them is that extra function $1/\rho c_s$ sitting on the right-hand side. To make things look even more symmetric, we define the new variable,

$$Q(\rho) = \int_{\rho_0}^{\rho} d\rho' \; \frac{c_s(\rho')}{\rho'} \tag{4.82}$$

with ρ_0 some useful fiducial, constant density such as the asymptotic value of the density if such a thing exists. This has the property that

$$\frac{\partial Q}{\partial t} = \frac{c_s}{\rho}\frac{\partial \rho}{\partial t} = \frac{1}{\rho c_s}\frac{\partial P}{\partial t} \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{1}{\rho c_s}\frac{\partial P}{\partial x}$$

This means that we can write (4.81) as

$$\left(\frac{\partial}{\partial t} + (u - c_s)\frac{\partial}{\partial x}\right)(u - Q) = 0$$

The same argument, with some minus signs flipped, also gives

$$\left(\frac{\partial}{\partial t} + (u + c_s)\frac{\partial}{\partial x}\right)(u + Q) = 0$$

We introduce the *Riemann invariants*

$$R_\pm = u \pm Q \tag{4.83}$$

These obey the *Riemann wave equation*

$$\left(\frac{\partial}{\partial t} + (u \pm c_s)\frac{\partial}{\partial x}\right) R_\pm = 0 \tag{4.84}$$

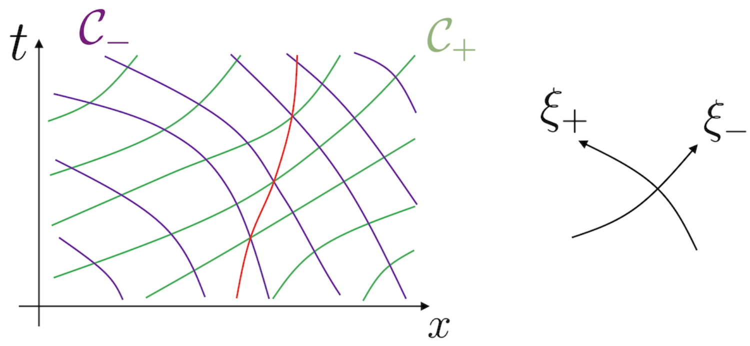


Figure 24. The characteristic curves $\mathcal{C}_+$, in green, run from bottom left to top right, and the curves $\mathcal{C}_-$, in purple, run bottom right to top left. They curves depend locally on both the flow $u(x,t)$ and, through $c_s(\rho)$, the density $\rho(x,t)$. The red line depicts one integral curve of the fluid flow $u(x,t)$. The coordinates $\xi_\pm$ are constant on $\mathcal{C}_\pm$ respectively. This means that, as shown in the axes on the right, ξ_+ increases as we move in the up-left direction, while ξ_- increases as we move in the up-right direction.

We next want to understand what this equation is telling us. To this end, for a given flow $u(x,t)$ with density $\rho(x,t)$, we construct two collections of *characteristic curves*, $\mathcal{C}_\pm$. These are worldlines in the spacetime parameterised by (x,t), defined by

$$\mathcal{C}_\pm : \quad \frac{dx}{dt} = u(x,t) \pm c_s(x,t) \tag{4.85}$$

We introduce two new coordinates in spacetime: ξ_+ and ξ_-. These have the property that $\xi_\pm$ are constant on the characteristic curves $\mathcal{C}_\pm$ respectively. Then the meaning of (4.84) is:

Claim: $R_\pm$ is constant on characteristic curves $\mathcal{C}_\pm$.

Proof: To show this, we just need to think carefully about what depends on what. Suppose that we vary both ξ_+ and ξ_- a tiny tiny bit. Then we move in the t direction an infinitesimal amount

$$dt = \left.\frac{\partial t}{\partial \xi_+}\right|_{\xi_-} d\xi_+ + \left.\frac{\partial t}{\partial \xi_-}\right|_{\xi_+} d\xi_-$$

and we move in the x direction an infinitesimal amount

$$dx = \left.\frac{\partial x}{\partial \xi_+}\right|_{\xi_-} d\xi_+ + \left.\frac{\partial x}{\partial \xi_-}\right|_{\xi_+} d\xi_-$$

On the characteristic curves $\mathcal{C}_+$ we know that ξ_+ is constant, so we have

$$\mathcal{C}_+ : \quad d\xi_+ = 0 \quad \Rightarrow \quad \left.\frac{\partial x}{\partial \xi_-}\right|_{\xi_+} = \frac{dx}{dt}\left.\frac{\partial t}{\partial \xi_-}\right|_{\xi_+} = (u + c_s)\left.\frac{\partial t}{\partial \xi_-}\right|_{\xi_+}$$

Now, if we view $R_+(x,t)$ as a function $R_+(\xi_+, \xi_-)$, then

$$\begin{aligned}\left.\frac{\partial R_+}{\partial \xi_-}\right|_{\xi_+} &= \frac{\partial R_+}{\partial t}\left.\frac{\partial t}{\partial \xi_-}\right|_{\xi_+} + \frac{\partial R_+}{\partial x}\left.\frac{\partial x}{\partial \xi_-}\right|_{\xi_+} \\ &= \left(\frac{\partial R_+}{\partial t} + (u + c_s)\frac{\partial R_+}{\partial x}\right)\left.\frac{\partial t}{\partial \xi_-}\right|_{\xi_+} = 0\end{aligned}$$

In other words, $R_+(\xi_-, \xi_+)$ is really just a function of a single variable, $R_+(\xi_+)$. The same argument also tells us that $R_- = R_-(\xi_-)$. So if we move along a characteristic curve $\mathcal{C}_+$, where ξ_+ is constant, then R_+ doesn't change. Similarly, R_- doesn't change if we move along a characteristic curve $\mathcal{C}_-$. □

It's worth taking stock of what we've achieved. Our goal is to solve for the flow $u(x,t)$ and the density $\rho(x,t)$. We haven't done this yet! However, we have showed that, *if* we can solve it, then we can construct characteristic curves $\mathcal{C}_\pm$ on which the variables $R_\pm$ are constant. And $R_\pm$, in turn, depends on u and ρ that we are trying to figure out. All of which means that the Riemann invariants don't immediately solve our problem, but they should contain some information that we can exploit.

Furthermore, if it's possible to somehow figure out $R_\pm(x,t)$ then it's straightforward to reconstruct the velocity field which, from (4.83), is given by

$$u(x,t) = \frac{1}{2}\left(R_+(\xi_+) + R_-(\xi_-)\right)$$

This is the generalisation of the more familiar solution to the linearised wave equation (4.75),

$$u(x,t) = F(x - c_s t) + G(x + c_s t)$$

which describes wave packets moving left and right at a constant speed c_s.

4.5.2 Soundcones

The equations (4.84) are telling us that the something is propagating in the fluid with speed c_s relative to the flow.

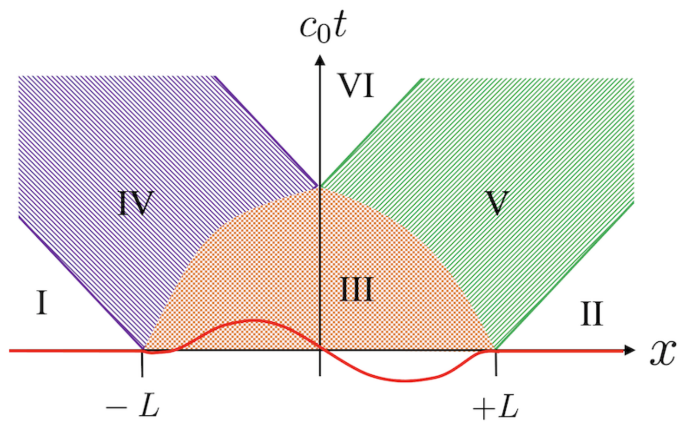


Figure 25. An initial disturbance in a region $|x| < L$ propagates to the left and to the right. Regions I,II and VI are undisturbed; regions IV and V have only left-moving and right-moving waves respectively, and anything can happen in region III.

To see this more clearly, consider some initial disturbance with $u(x,0) \neq 0$ for $|x| < L$ as shown in Figure 25. We'll also assume that the density $\rho(x,t)$ differs from some asymptotic value ρ_0 only within this same region. From (4.82), this ensures that $Q = 0$ outside of this region so so $R_\pm(x,0) = 0$ for $|x| > L$.

We can draw this on a spacetime diagram, with the vertical axis labelled by c_0t where $c_0 = c_s(\rho_0)$ is the asymptotic sound speed. This ensures that linearised sound waves travel at $\pm 45°$ in the diagram, rather like light rays in Minkowski space. In analogy with special relativity, we will say that the pair of characteristic curves $\mathcal{C}_\pm$ emerging from any point form a *soundcones*. (In fact, the analogy works better with general relativity where the lightcones depend on the curvature of spacetime, just like the soundcones depend on the background flow $u(x,t)$.)

Consider the soundcones emerging from the points $x = \pm L$ at time $t = 0$. These are shown in Figure 25 They divide spacetime for $t > 0$ into six distinct regions with the following properties

- Regions I and II have the property that both $\mathcal{C}_+$ and $\mathcal{C}_-$ characteristic curves pass through the x-axis in the region with $R_\pm(x,0) = 0$. Because $R_\pm$ are constant on $\mathcal{C}_\pm$ respectively, this means that we must have $R_+ = R_- = 0$ throughout the regions I and II. This makes sense: the initial disturbance takes time to propagate and, just as signals can't travel faster than the speed of light in special relativity, here they can't travel faster than the (local) speed of sound. Hence the regions

I and II know nothing about the disturbance. Both characteristic curves are straight lines at $\pm 45°$ in these regions.

- Region III has a complicated flow, with both left- and right-travelling waves. Here we would expect both R_+ and R_- to be non-vanishing and there is no reason to think that the characteristic curves $\mathcal{C}_\pm$ will be straight.

- In Region IV, we can trace the $\mathcal{C}_+$ curves back to the region in which $R_+ = 0$. So we know that

$$R_+ = u + Q = 0 \quad \Rightarrow \quad u = -Q$$

Correspondingly, $R_- = u - Q = 2u$. We know that R_- is constant on characteristic curves $\mathcal{C}_-$. So this tells us that both u and Q, and hence ρ and $c_s(\rho)$, must also be constant on $\mathcal{C}_-$. In other words, all of these are functions only of ξ_-,

$$u(x,t) = u(\xi_-) \quad , \quad \rho(x,t) = \rho(\xi_-) \quad , \quad c_s(x,t) = c_s(\xi_-)$$

These are purely left-moving waves. Now, the defining equation for the characteristic curve $\mathcal{C}_-$ is

$$\frac{dx}{dt} = u(\xi_-) - c_s(\xi_-)$$

But for a given $\mathcal{C}_-$ curve, defined by some fixed value of ξ_-, the right-hand side is obviously constant. This means that the $\mathcal{C}_-$ characteristic curves are straight lines in region IV. Although these curves are all straight lines in region IV, they need not necessarily be parallel: the gradient $u(\xi_-) - c_s(\xi_-)$ can depend, as advertised, on ξ_- and typically will. Of course, if they're not parallel then there is the possibility that they will converge and cross at some point. That's somewhat confusing because $R_-(\xi_-)$ is expected to have different values on different curves and if two curves collide then it looks like, say, the velocity $u(x,t)$ will have two different values at that point! We will learn how to think about this shortly.

Region V has similar properties to Region IV, now with the characteristic curves $\mathcal{C}_+$ straight lines. This is a purely right-moving wave. In general, situations where one of the Riemann invariants vanish (or, indeed, is constant) over a region of space are referred to as *simple waves* and are associated to waves that are either purely left-moving or purely right-moving.

- In Region VI, the same arguments apply as for Region I and II: you can trace back both $\mathcal{C}_+$ (to the left) and $\mathcal{C}_-$ (to the right) to regions where $R_+ = 0$ and

$R_- = 0$ respectively. This means that this is once again a region of calm, with $u = 0$ and $\rho = \rho_0$. The disturbance has passed. This is because there is no option to dawdle in this system: all modes must travel at the speed of sound. The only question is how that speed of sound changes.

4.5.3 Wave Steepening and a Hint of Shock

We can illustrate these ideas further. We'll first give a heuristic discussion of the physics and then fill in some of the details. To do this, it's useful to pick a concrete example and we'll choose to look at ideal gas obeying $P\rho^{-\gamma} = \text{constant}$.

The speed of sound for the ideal gas is (4.71)

$$c_s^2(\rho) = \frac{\gamma P}{\rho} \tag{4.86}$$

The function $Q(\rho)$ defined in (4.82) is

$$Q(\rho) = \int_{\rho_0}^{\rho} d\rho' \; \frac{c_s(\rho')}{\rho'} = \frac{2}{\gamma - 1}\left(c_s(\rho) - c_0\right)$$

where $c_0 = c_s(\rho_0)$. The Riemann invariants are then

$$R_\pm = u \pm \frac{2}{\gamma - 1}(c_s - c_0)$$

These are normalised so that in a boring, static flow with $u = 0$ and $\rho = \rho_0$, both Riemann invariants vanish: $R_\pm = 0$.

Now consider a simple, right-moving flow in which $R_- = 0$ everywhere. This means that

$$R_- = u - \frac{2}{\gamma - 1}(c_s - c_0) = 0 \quad \Rightarrow \quad c_s = c_0 + \frac{1}{2}(\gamma - 1)u \tag{4.87}$$

This gives us a relation between the velocity field and the speed of sound. In fact, this contains the key bit of physics. Any region of fluid with $u > 0$ has a speed of sound $c_s > c_0$. And any region of fluid with $u < 0$ has a speed of sound $c_s < c_0$. (Assuming $\gamma > 1$.)

The wave propagates along the characteristic curves $\mathcal{C}_+$, on which u and c_s are both constant. These curves are given by (4.85)

$$\mathcal{C}_+ : \quad \frac{dx}{dt} = u(\xi_+) + c_s(\xi_+) = c_0 + \frac{1}{2}(\gamma + 1)u(\xi_+) \tag{4.88}$$

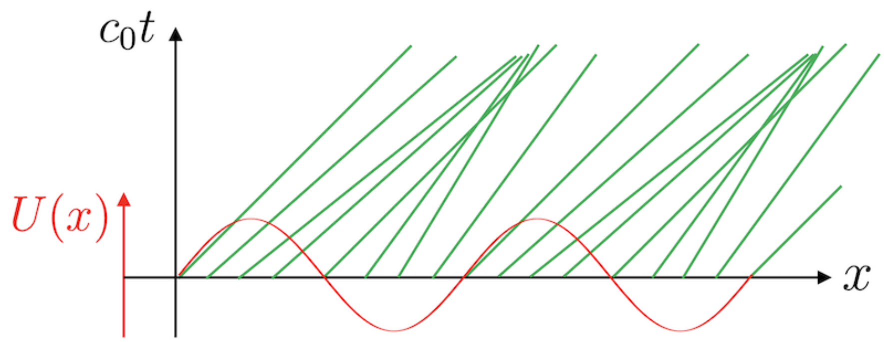


Figure 26. An initial, purely right-moving, sine wave has characteristic curves $\mathcal{C}_+$ that intersect.

Suppose that we're given some initial data at time $t = 0$. We're told that

$$u(x, 0) = U(x) \tag{4.89}$$

The requirement that we've got a purely right-moving wave then fixes the density (or, equivalently, the speed of sound) along this slice using (4.87). We know that ξ_+ labels the different $\mathcal{C}_+$ curves, but we haven't yet got a natural way to fix the normalisation of this coordinate. We'll resolve this by choosing ξ_+ to be the value of x where a given curve $\mathcal{C}_+$ intersects the $t = 0$ axis. Then the characteristic curves (4.88) are simply the straight lines

$$\mathcal{C}_+ : \quad x(t; \xi_+) = \xi_+ + \left[c_0 + \frac{1}{2}(\gamma + 1)U(\xi_+)\right] t \tag{4.90}$$

We can see the slope of the characteristic curves in the square bracket. Those parts of the fluid that had an initial velocity $U > 0$ travel along characteristic curves that have an angle greater than 45° (as measured from the vertical axis). And those parts of the fluid with $U < 0$ travel along lines that sit at less than 45°.

The resulting characteristic curves are shown in green in Figure 26 for the simple initial data where $U(x)$ is a sine wave. Importantly, we see that, as we anticipated previously, the characteristic curves meet. But this is very confusing! On a given characteristic curve, the velocity of the fluid is fixed as $U(x)$. Wherever two curves intersect, the velocity must be multi-valued. In other words, the non-linearities have pushed our nice, simple initial sine wave into a solution that is discontinuous in the velocity field!

In fact, there's a straightforward interpretation of this. As the nonlinearities cause the peak of the wave, where $U > 0$, to move faster than the trough of the wave, where $U < 0$. This means that the wave will become skewed, and increasingly sawtooth-like. This is known as *wave steepening* and is shown in Figure 27 for an initial Gaussian wavepacket rather than the sin wave considered above (because it's easier to draw). Eventually, the peak will catch up with the trough, at which point the velocity field is no longer single-valued. This is the reason that the characteristic curves cross. It turns out that this is telling us that a *shock* forms. We'll understand more about what this means in Section 4.6. For now, we will simply adopt the terminology and say that a shock is tantamount to two characteristic curves intersecting.

We can ask: how long does it take for the shock to form? Two curves intersect whenever a shift to an adjacent curve doesn't require a shift in space. Or, in equations,

$$\left.\frac{\partial x}{\partial \xi_+}\right|_t = 0$$

From (4.90), this gives the requirement

$$t = -\frac{2}{\gamma+1}\frac{1}{U'(\xi_+)} \tag{4.91}$$

We want the value of ξ_+ that minimises $t > 0$ since this is when two of the characteristic curves first cross. We take, as our initial conditions, a sine wave as shown in Figure 26

$$u(x, 0) = U(x) = U \sin kx$$

with U the overall amplitude of the wave. The first shock then forms when

$$t_{\rm shock} = \frac{2}{\gamma+1}\frac{1}{kU}$$

It's useful to compare this time scale with the period T of the wave itself. Recall from Section 4.4 that sound waves have the simple dispersion relation $\omega = c_s|k|$ and the period is $T = 2\pi/\omega \approx 2\pi/c_0|k|$. This then gives

$$\frac{t_{\rm shock}}{T} = \frac{1}{\pi(\gamma+1)}\frac{c_0}{U} \tag{4.92}$$

This is our first sign of an important dimensionless quantity called the *Mach number*, the ratio of the velocity of the fluid flow to the speed of sound

$$M = \frac{U}{c_0} \tag{4.93}$$

The time to shock formation is roughly $t_{\rm shock} \sim T/M$.

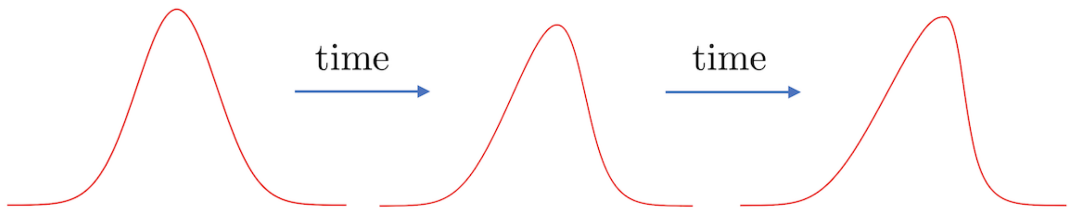


Figure 27. The steepening of an initial Gaussian wavepacket over time.

We can put some numbers into this. The decibel scale is the familiar scale used to measure how loud a sound is. It's a $\log_{10}$ scale such that if the amplitude of the wave U changes by a factor of 10 then the decibels increase additively by 10. A quiet chat down the pub with a friend will involve sound waves with frequency around $\omega \sim 1000s^{-1}$ and a corresponding period of $T \sim 3 \times 10^{-3}$ s. The volume is around 60 dB and this corresponds to a Mach number $M \sim 10^{-13}$. Apparently you need to wait about 1000 years for a shock wave to form! We can instead crank up the volume. If you stand next to a rocket at take off, you will suffer around 180 dB. (And get permanent ear damage.) Now $M \sim 10^{-1}$. Perhaps unsurprisingly, you can expect a shock wave to form in a very short time.

4.5.4 Burgers' Equation

We can elucidate the analysis above a little further with a simple change of variables. We continue to study a purely right-moving wave for which $R_+ = 2u$ and, from (4.87), the local speed of sound c_s is related to the fluid velocity by

$$c_s(u) = c_0 + \frac{1}{2}(\gamma - 1)u$$

The Riemann wave equation (4.84) is now a non-linear equation for u

$$\left(\frac{\partial}{\partial t} + \left[c_0 + \frac{1}{2}(\gamma + 1)u\right]\frac{\partial}{\partial x}\right) u = 0$$

We introduce the co-moving coordinate

$$X = x - c_0 t$$

This would travel along with the wave if the wave were travelling at a constant speed c_0. Of course, it's not, which is where much of the fun lies. We also introduce the rescaled velocity field

$$v = \frac{1}{2}(\gamma + 1)u$$

and we think of $v = v(X, t)$. Then the Riemann wave equation becomes

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial X} = 0 \tag{4.94}$$

where the partial time derivative is now taken with X held fixed, rather than x held fixed as before. This is the *inviscid Burgers' equation*. (Inviscid because throughout this section we've neglected viscosity.)

The Burgers' equation (4.94) takes a particularly simple form. We'll again take the initial data to be

$$v(X, 0) = V(X)$$

and identify X with ξ_+ when $t = 0$. (The function $V(X)$ differs from our previous $U(x)$ defined in (4.89) only by the constant factor $2/(\gamma + 1)$.) The characteristic curves $\mathcal{C}_+$ given in (4.90) are then just

$$\mathcal{C}_+ : \quad X(t; \xi_+) = \xi_+ + V(\xi_+)t \tag{4.95}$$

along which v is constant. This means that $v(X, t) = v(\xi_+, 0)$. Or, substituting the expression for X above, we have the solution

$$v(X, t) = v(X - V(\xi_+)t, 0) \tag{4.96}$$

It's worth pausing to parse what this solution means. First, for a given X and t, we need to use (4.95) to figure out what ξ_+ is. In other words, what characteristic curve $\mathcal{C}_+$ the point (X, t) lies on. Clearly this depends on the initial data $v(\xi_+, 0)$ that we're given. It's an algebraic computation and, for a given initial condition, the answer may not be available in closed form. Nonetheless, it's doable numerically. With this in hand, the solution (4.96) then tells us how the initial data evolves. Indeed, it's just a rewriting of what we saw previously: the points with higher initial velocity propagate at a faster speed.

Steepening Again

The coordinate X has the advantage that it keeps up with the propagating wave, at least on average. The slope of the wave is

$$\left.\frac{\partial v}{\partial X}\right|_t = \frac{\partial v(\xi_+, 0)}{\partial \xi_+} \left.\frac{\partial \xi_+}{\partial X}\right|_t = \frac{V'(\xi_+)}{1 + V'(\xi_+)t}$$

where, in the second equality, we've used (4.95). This now gives us a better handle on the phenomenon of wave steepening. Those parts of the wave with $V'(\xi_+) < 0$ get steeper over time; those parts of the wave with $V'(\xi_+) > 0$ become flatter. The shock occurs when the wave becomes infinitely steep, so

$$\left.\frac{\partial v}{\partial X}\right|_t = \infty \quad \Rightarrow \quad t = -\frac{1}{V'(\xi_+)}$$

This agrees with our earlier condition (4.91).

Very Briefly, the Effect of Viscosity

So far our discussion has neglected viscosity. But we can see at the level of equations what it changes. If we go back to the 1d Euler equation (4.79) and trace through various change of variables, we find that (4.94) is replaced by

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial X} = \nu\frac{\partial^2 v}{\partial X^2}$$

where ν is, as always, the kinematic viscosity and we've inadvertently stumbled upon the rather unfortunate situation of having both ν and v in the same equation. (It won't be for long and we can live with it.) This is the full *Burgers equation*.

We know that viscosity causes the velocity to diffuse, and this mitigates large velocity gradients. This can be shown to remove the formation of the discontinuous shock.

4.6 Shocks

> "This is a manifest absurdity. No step, however, of the reasoning by which this result has been obtained can be controverted. What then is the meaning of it?"

The Rev. James Challis, in 1848, expressing shock on first discovering that initial data in the Euler equations gives rise to discontinuities in the velocity.

We got our first hints of shock waves in the last section where a discontinuity in the velocity field arises as the flow evolves over time. Although this discontinuity is expected to be smoothed out by the effects of viscosity, if we zoom out and look at suitably coarse-grained scales then the discontinuous flow is a good approximation to what actually happens. In this section we'll explore some of the properties of these shocks. We'll see that the discontinuities have a remarkably simple and constrained structure, all of which follows from conservation laws (together with a little bit of thermodynamics).

We're going to upgrade from one dimension to two. We'll consider flows of a compressible fluid in the (x, y)-plane, with a shock that sits at some specific point in the x-direction and extends along the y-direction. The flows themselves will be invariant under translations in the y-direction. This means that we will restrict our attention to flows of the form

$$\mathbf{u}(\mathbf{x}, t) = (u(x, t), v(x, t))$$

We have mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{4.97}$$

and the non-linear Euler equations

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial P}{\partial x} = 0 \tag{4.98}$$

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} = 0 \tag{4.99}$$

where to get the second set of equations we use the continuity equation (4.97) associated to mass conservation. We've rewritten the Euler equations in this way to stress that they too are continuity equations, describing the conservation of momentum.

We've got three equations for four variables, ρ, u, v and P. We need a fourth. This is a road that we've been down before: for adiabatic variations, the fourth condition is a relation

$$P = P(\rho)$$

and our standard example is the ideal gas with $P/\rho^\gamma = \text{constant}$.

As we will see, the physics of the shock is all about understanding conserved quantities. The final conservation law that we need is for energy. But here there's a subtlety because there are additional contributions to the energy for a compressible fluid. As we will now show, these additional contributions are fully determined by the relation $P = P(\rho)$.

We start by introducing the energy density of the fluid

$$\text{Energy} = \frac{1}{2}\rho(u^2 + v^2) + \rho e(\rho)$$

where $e(\rho)$ is some internal energy (per unit mass) of the fluid that we will determine below. In addition, we introduce the energy current

$$\text{Energy Current} = \frac{1}{2}\rho u(u^2 + v^2) + \rho u h(\rho)$$

where the addition quantity $h(\rho)$ is known as the *enthalpy*. Again, we'll figure out what this is shortly. Then using the same kind of manipulations that we saw when first deriving Bernoulli's principle in Section 2.1.4, we can write the conservation of energy as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho(u^2 + v^2) + \rho e(\rho)\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho u(u^2 + v^2) + \rho u h(\rho)\right) = 0 \qquad (4.100)$$

A little algebra shows that this equation holds provided that the internal energy $e(\rho)$ and enthalpy $h(\rho)$ obey the equations

$$\frac{d(\rho e)}{d\rho} = h \quad \text{and} \quad \frac{d(\rho h)}{d\rho} = h + \frac{dP}{d\rho}$$

These equations can then be solved given the relation $P = P(\rho)$. For example, for the ideal gas with $P = A\rho^\gamma$ for some constant A, we can solve these to get

$$e = \frac{A\rho^\gamma}{\gamma - 1} = \frac{1}{\gamma - 1}\frac{P}{\rho} \quad \text{and} \quad h = \frac{A\gamma}{\gamma - 1}\rho^{\gamma - 1} = \frac{\gamma}{\gamma - 1}\frac{P}{\rho} \qquad (4.101)$$

Note that $h = e + P/\rho$.

Equations (4.97), (4.98), (4.99) and (4.100) are our starting point. Our goal now is to search for discontinuous solutions to these equations describing shock waves.

4.6.1 Jump Conditions

A shock is a discontinuous flow. But we don't allow any old discontinuity. Instead, the discontinuity itself has certain properties. And these are derived from the conservation laws described above.

The discontinuity splits the flow into two, as shown in Figure 28. On the left, the flow has values $\mathbf{u}_1$, ρ_1 and P_1; on the right values $\mathbf{u}_2$, ρ_2 and P_2. To make life particularly simple, we'll assume that each of these flows is constant in space and time. All of the physics arises from the discontinuity.

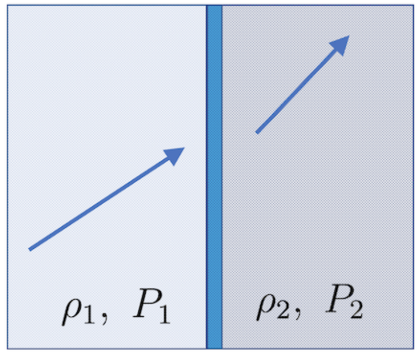


Figure 28. Two flows separated by a shock wave.

We'll assume that the shock itself is not propagating, but is fixed at some position, say $x = 0$. We model the discontinuity as in infinitely thin surface and, as such, it can't carry any conserved charge density. Any mass that enters from one side must exit through the other. The same holds for momentum and energy. This means that each of the currents in (4.97) through (4.100) must coincide on the left and right. Mass conservation (4.97) tells us

$$\rho_1 u_1 = \rho_2 u_2 \tag{4.102}$$

Momentum conservation tells us

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \tag{4.103}$$

and

$$\rho_1 u_1 v_1 = \rho_2 u_2 v_2 \quad \Rightarrow \quad v_1 = v_2 \tag{4.104}$$

where the second equation follows from (4.102). This tells us that the velocity tangent to the shock remains constant. Finally, energy conservation gives

$$\begin{aligned} \rho_1 u_1 \left(\frac{1}{2}(u_1^2 + v_1^2) + h_1\right) &= \rho_2 u_2 \left(\frac{1}{2}(u_2^2 + v_2^2) + h_2\right) \\ \Rightarrow \qquad \frac{1}{2}u_1^2 + h_1 &= \frac{1}{2}u_2^2 + h_2 \end{aligned} \tag{4.105}$$

where now the second equation follows from (4.102) and (4.104). This is Bernoulli's theorem applied to the shock. Equations (4.102), (4.103), (4.104) and (4.105) are called the *Rankine-Hugoniot jump conditions*.

It's simplest to transform to a frame in which $v_1 = v_2 = 0$, so that all the action is taking place transverse to to shock wave. We're then left with three conditions which fix u_2, ρ_2 and P_2 in terms of the initial flow data. To see this in more detail, we first use (4.102) and (4.103) to derive the relation

$$\rho_1 u_1^2 = \left(1 - \frac{\rho_1}{\rho_2}\right)^{-1} (P_2 - P_1) \tag{4.106}$$

Since the left-hand-side is positive, the right-hand side must also be positive. That gives us two possibilities: either pressure and density both increase across the shock

$$P_2 > P_1 \quad \text{and} \quad \rho_2 > \rho_1$$

or the opposite happens. Clearly these two options are related by a parity flip, so we'll assume that the above occurs and the pressure is greater on the right of the shock. Then, from (4.102), we have

$$u_2 = \frac{\rho_1}{\rho_2} u_1 \tag{4.107}$$

This tells us that $|u_2| < |u_1|$, so the *speed* of the flow is smaller on the right of the shock. Note, however, that we haven't yet said anything about the sign of u_1 and u_2, i.e. is the flow left-to-right or right-to-left? We'll come to this shortly.

The Size of the Shock

The *shock compression ratio* is defined to be

$$r = \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \tag{4.108}$$

Obviously it's a measure of how big the discontinuity is. We can also get an expression for r in terms of the pressure difference on each side but, for this, we need our final matching condition associated to consevation of energy (4.105). For an ideal gas, this reads

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma - 1}\frac{P_2}{\rho_2}$$

We can use (4.106) and (4.107) to write this as

$$\frac{1}{2}\frac{\rho_1 + \rho_2}{\rho_1 \rho_2}(P_2 - P_1) = \frac{\gamma}{\gamma - 1}\left(\frac{P_2}{\rho_2} - \frac{P_1}{\rho_1}\right)$$

Now substituting $\rho_2 = r\rho_1$, we find

$$r = \frac{(\gamma - 1)P_1 + (\gamma + 1)P_2}{(\gamma + 1)P_1 + (\gamma - 1)P_2} \tag{4.109}$$

This form of r puts some bounds on the strength of the shock. For a *strong shock*, $P_2 \gg P_1$. In the limit $P_2/P_1 \to \infty$, we have

$$r \to r_{\rm max} = \frac{\gamma + 1}{\gamma - 1}$$

This is the largest compression factor that we can have. For a monatomic gas, with $\gamma = 5/3$, we have $r_{\rm max} = 4$. Note that the discontinuity is very different for pressure and speed: if the pressure changes by an infinite amount, the speed changes only by a factor of 4.

There's also something familiar hiding in this unfamiliar setting. Suppose that we have a *weak shock*, meaning $P_2 = P_1 + \Delta P$ with $\Delta P \ll P_1$. Then we have $r \approx 1 + \Delta P/\gamma P_1$. We can also write, $\rho_2 = \rho_1 + \Delta\rho$ and this gives $r \approx 1 + \Delta\rho/\rho_1$. Equating these, we have

$$\frac{\Delta P}{\Delta \rho} = \frac{\gamma P_1}{\rho_1}$$

But this is the equation for the speed of sound in an ideal gas (see, for example, (4.71) and (4.73))

$$\frac{dP}{d\rho} = c_s^2 = \frac{\gamma P}{\rho} \tag{4.110}$$

We previously derived this result for linearised (i.e. small) sound waves. Here we make contact with the shock waves. A very weak shock wave can be viewed as the limit of a very strong sound wave.

The Entropy Jump

There's a very basic question that we haven't yet addressed. Which way is the flow is going? Is the fluid moving left-to-right, so $u_1, u_2 > 0$ as shown in Figure 28? Or is it moving right-to-left, with $u_1, u_2 < 0$. In other words, does the pressure increase in the direction of the flow or decrease in the direction of the flow? It turns out that the answer to this question lies in the second law of thermodynamics.

The entropy density $s = S/V$ for an ideal gas computed in (4.63): it is given by

$$s = c_V \log\left(\frac{P}{P_0}\left(\frac{\rho_0}{\rho}\right)^\gamma\right)$$

with P_0 and ρ_0 some fiducial values. Indeed, this is where we got the now-familiar equation of state $P\rho^{-\gamma}$ = constant for adiabatic processes which have constant entropy.

We'd like to understand how the entropy changes across the shock,

$$\Delta s = s_2 - s_1 = c_V \log\left(\frac{P_2}{P_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right)$$

The second law of thermodynamics means that entropy must increase. But is the change of entropy Δs or is it $-\Delta s$? In other words, does the flow from region 1 to region 2, in which case Δs is the change in entropy. Or does it go from region 2 to region 1 in which case it's $-\Delta s$?

This is straightforward to answer using the expression for the compression ratio (4.108) and (4.109). The entropy jump Δs is then

$$\Delta s = c_V \log\left(\frac{r(\gamma+1)-(\gamma-1)}{(\gamma+1)-r(\gamma-1)}r^\gamma\right)$$

For $r < r_{\rm max}$, we have $\Delta s > 0$. But the second law of thermodynamics then gives us a direction for the shock: the flow must propagate from left to right, as anticipated in Figure 28, so that Δs is the change of entropy. Correspondingly, the speed of the flow decreases and the density increases. We say that the shock is *compressive*.

Although we've studied the shock only for an ideal gas, it turns out that the result above is general: shocks are always compressive for any equation of state $P(\rho)$, with the velocity decreasing after the shock.

The fact that the entropy is not constant across the discontinuity means that shocks are necessarily dissipative. There's something a little surprising about this. We've worked with the Euler equation which, as mentioned previously, enjoys the symmetry of time reversal. Moreover, we've also used the adiabatic condition $P\rho^{-\gamma}$ = constant for an ideal gas. Nonetheless, the discontinuity is a violent event and allows dissipative behaviour to be hidden in the singularity, even though the underlying equations did not themselves have dissipation.

Physically, we've captured the dissipation by allowing the internal, heat energy $e(\rho)$ to increase downstream. A fuller understanding of the dissipation mechanism would need us to look more closely at the shock wave by understanding the role that viscosity plays in thickening the discontinuity. But the results above tell us that, ultimately, fact these microscopic details don't affect the amount of dissipation: that's fully determined by the properties of the initial flow and some basic conservation laws.

4.6.2 Shocks Start Supersonic

There is more physics to extract from our expressions for the compression ratio. We define the *normal Mach number*

$$\mathcal{M} = \frac{u}{c_s}$$

where the speed of sound is (4.110)

$$c_s^2 = \frac{\gamma P}{\rho}$$

This is not quite the same thing as the mach number (4.93) because we've ignored the tangential velocity v. Each side of the flow has a normal Mach number, $\mathcal{M}_1$ and $\mathcal{M}_2$. Note, in particular, the speed of sound also differs on either side of the shock. We'll now show that we can express the normal Mach numbers $\mathcal{M}_1$ and $\mathcal{M}_2$ on either side of the flow directly in terms of the compression factor r.

Lemma: The algebra is a little fiddly so we'll tread slowly. To begin, we'll need:

$$\begin{aligned}\rho_1 u_1^2 &= \frac{1}{2}\left[(\gamma-1)P_1 + (\gamma+1)P_2\right] \\ \rho_2 u_2^2 &= \frac{1}{2}\left[(\gamma+1)P_1 + (\gamma-1)P_2\right]\end{aligned} \tag{4.111}$$

Proof: We start with two expressions for the compression factor r, the first following from (4.108) and the second (4.109),

$$r = \frac{\rho_1 u_1^2}{\rho_2 u_2^2} = \frac{(\gamma-1)P_1 + (\gamma+1)P_2}{(\gamma+1)P_1 + (\gamma-1)P_2} \tag{4.112}$$

Note that if the first equation in (4.111) is true, then (4.112) immediately implies that the second is also true. So we just need to prove the first. This follows from (4.103) which reads $\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$. If we divide through by $\rho_1 u_1^2$ then, after a little rearranging, we find

$$\rho_1 u_1^2 = \frac{r}{r-1}(P_2 - P_1)$$

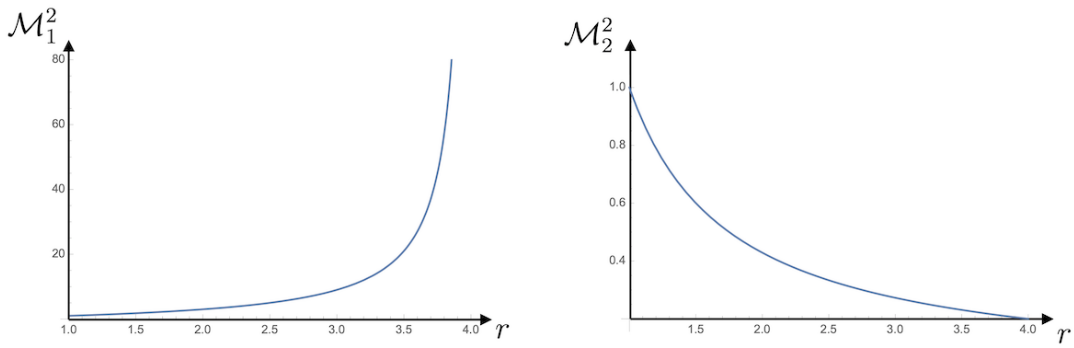


Figure 29. The initial and final Mach numbers as a function of the compression ratio r, plotted for $\gamma = 5/3$. We have $\mathcal{M}_1 \geq 1$ and $\mathcal{M}_2 \leq 1$ for all values of r.

Now compute $r/(r-1)$ using the expression in (4.112) involving pressure. This will give the result (4.111) that we want. □

Now we've done the hard work. We use the expression for the speed of sound $c_s^2 = \gamma P/\rho$ to write the two equations in (4.111) as

$$\gamma \mathcal{M}_1^2 = \frac{1}{2}\left[(\gamma-1) + (\gamma+1)\frac{P_2}{P_1}\right]$$
$$\gamma \mathcal{M}_2^2 = \frac{1}{2}\left[(\gamma+1)\frac{P_1}{P_2} + (\gamma-1)\right]$$

To finish, we just need an expression for the pressure ratios. We can easily get this from (4.112). It is

$$\frac{P_2}{P_1} = \frac{r(\gamma+1)-(\gamma-1)}{(\gamma+1)-r(\gamma-1)} \tag{4.113}$$

Finally we get the results that we wanted: the normal Mach numbers before and after the shock are

$$\mathcal{M}_1^2 = \frac{2r}{(\gamma+1)-r(\gamma-1)}$$
$$\mathcal{M}_2^2 = \frac{2}{r(\gamma+1)-(\gamma-1)} \tag{4.114}$$

The key takeaway from these equations is that, for $1 < r \leq r_{\rm max}$, we always have $\mathcal{M}_1 > 1$ and $\mathcal{M}_2 < 1$, as shown in Figure 29. This means that shocks only form in supersonic flows, where the speed of the fluid exceeds the speed of sound. After the shock, the speed of the fluid is reduced below the sound speed. (Although, as the reduction of the fluid speed is limited by a factor of $r < r_{\rm max}$, for very fast flows this is achieved by increasing the pressure, and hence increasing the sound speed, rather than by reducing the flow speed.)

From a physical perspective, the equations (4.114) are kind of backwards: the compression factor r doesn't determine the initial speed $\mathcal{M}_1$. It's the other way round! We can easily invert these equations to get the compression factor in terms of the initial Mach number,

$$r = \frac{(\gamma+1)\mathcal{M}_1^2}{2+(\gamma-1)\mathcal{M}_1^2}$$

Similarly, the jump in pressure, given in (4.113), is also determined by the initial speed

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(\mathcal{M}_1^2 - 1)$$

These equations are known as the *Rankine-Hugoniot relations*

4.6.3 On Singularities and Physics

There is a general expectation that non-linear, partial differential equations will develop singularities in a finite time. When those non-linear equations describe something physical, these singularities are particularly interesting. A singularity is telling us the equations are no longer sufficient to capture the underlying physics and must be replaced by something more fundamental. This suggests that singularities may offer a window into the microscopic realm.

Within classical physics, there are two pre-eminent sets of non-linear equations. These are the Navier-Stokes equation (or its baby brother, the Euler equation) for fluids, and the Einstein equations for gravity. As we now explain, both these equations are rather special and the way in which singularities form, or fail to form, is surprising and poorly understood.

For fluids, it's useful to distinguish between the compressible and non-compressible cases. As we've seen above, the compressible Euler equation readily develops singularities in finite time. These are the shock waves that we've explored in this section, characterised by a discontinuity in the density ρ and other dynamical variables. As we anticipated above, the presence of the shock does mean that we have to introduce new physics. But the surprise of this section is that this new physics is the most minimal imaginable: it is just the second law of thermodynamics. Once we accept that entropy must increase when the shock develops, we have all we need to tell us what happens to the subsequent evolution. We certainly don't need to resort to any detailed microscopic description involving atoms and quantum world. This is rather remarkable. Shocks may be singular but, from a physical perspective, the singularity is very mild.

It is natural to ask: is this same property shared by all singularities of the compressible Euler equation? Or, indeed singularities of the compressible Navier-Stokes equation? The answer is: we don't know. For example, what happens when many shocks collide and start to interact with each other? Is it still the case that we can track the singular evolution of the Euler equations using only the second law as our guide? This situation is complicated and we don't know the answer. Moreover, one may worry that there are singularities worse than shocks that can arise in the compressible Euler equation. For example, it may be possible that $\rho(\mathbf{x}, t) \to \infty$ in some finite time. This kind of singularity would surely need some detailed understanding of the underlying atoms to resolve. But does such a singularity actually arise? Again, the answer is: we don't know. It can be shown that such singularities occur for very special initial data, but to be physically relevant it should happen for generic initial conditions, meaning initial conditions that lie within some open ball rather than at specific points. And it remains an open problem to show whether or not this occurs.

The situation for the incompressible Euler and Navier-Stokes equations is somewhat simpler to state, but still not well understood. Here there is a conjecture that no singularities occur in a finite time. No counter example is known, but a mathematical proof appears challenging to say the least. Indeed, proving the existence and smoothness of solutions to the Navier-Stokes equation is one of the Millennium Prize problems with a \$1 million dollar prize attached. (If you're genuinely motivated by the money then I would suggest that mathematics may not be your true calling. There are easier ways to be both happy and rich.)

Finally, that leaves us with the Einstein equations of General Relativity. Here the situation is most intriguing of all. It is straightforward to show that singularities do develop in finite time (at least with a suitable definition of "time"!). This arises when matter collapses to form a black hole, with a singularity forming in the centre where the curvature of spacetime becomes infinite. The presence of such a singularity is telling us that the laws of classical gravity are breaking down and must be replaced by something quantum. This means that singularities provide a wonderful opportunity to teach us something new about the "atoms of spacetime", whatever that means. Sadly, however, nature has made these singularities very difficult to access experimentally. It appears that they are generically shielded by an event horizon, so that they can't be seen by anyone sensible who chooses not to jump into the black hole. The idea that singularities necessarily sit behind an event horizon goes by the name of the *cosmic censorship conjecture*. From a mathematical perspective, it appears utterly miraculous and a proof is generally thought to be even more challenging than the Navier-Stokes existence and smoothness conjecture.

The upshot is that the laws of physics appear to be surprisingly robust against the formation of singularities. Even when singularities do arise – as in the compressible Euler equation and the Einstein equations – some poorly understood feature of the equations means that they are more innocuous than we would have naively thought. They are either hidden behind horizons, or neatly resolved by the second law. In both cases, we can largely carry on with our lives without worrying too much about what microscopic physics lurks inside the singularity.

It feels like there is an important lesson hiding within this story. The refusal of both the Navier-Stokes and the Einstein equations to develop readily accessible singularities, that require something atomic or quantum to fully understand, is a striking mathematical fact. It should have a striking physical reason behind it. But I don't know what it is.