4. Lattice Gauge Theory

Quantum field theory is hard. Part of the reason for our difficulties can be traced to the fact that quantum field theory has an infinite number of degrees of freedom. You may wonder whether things get simpler if we can replace quantum field theory with a different theory which has a finite, albeit very large, number of degrees of freedom. We will achieve this by discretizing space (and, as we will see, also time). The result goes by the name of *lattice gauge theory*.

There is one, very practical reason for studying lattice gauge theory: with a discrete version of the theory at hand, we can put it on a computer and study it numerically. This has been a very successful programme, especially in studying the mass spectrum of Yang-Mills and QCD, but it is not our main concern here. Instead, we will use lattice gauge theory to build better intuition for some of the phenomena that we have met in these lectures, including confinement and some subtle issues regarding anomalies.

There are different ways that we could envisage trying to write down a discrete theory:

- Discretize space, but not time. We could, for example, replace space with a cubic, three dimensional lattice. This is known as *Hamiltonian lattice gauge theory*.

  This has the advantage that it preserves the structure of quantum mechanics, so we can discuss states in a Hilbert space and the way they evolve in (continuous) time. The resulting quantum lattice models are conceptually similar to the kinds of things we meet in condensed matter physics. The flip side is that we have butchered Lorentz invariance and must hope that it emerges at low energies.

  This is the approach that we will use when we first introduce fermions in Section 4.3. But, for other fields, we will be even more discrete...

- Discretize spacetime. We might hope to do this in such a way that preserves some remnant of Lorentz invariance, and so provide a natural discrete approximation to the path integral.

  There are two ways we could go about doing this. First, we could try to construct a lattice version of Minkowski space. This, it turns out, is a bad. Any lattice clearly breaks the Lorentz group. However, while a regular lattice will preserve some discrete remnant of the rotation group $SO(3)$, it preserves no such remnant of the Lorentz boosts. The difference arises because $SO(3)$ is compact, while $SO(3,1)$ is non-compact. This means that if you act on a lattice with $SO(3)$, you will come back to your starting point after, say, a π rotation. In contrast, acting with a Lorentz boost in $SO(3,1)$ will take you further and further away from your starting point. The upshot is that lattices in Minkowski space are not a good idea.

The other option is to work with Euclidean spacetime. Here there is no problem in writing down a four dimensional lattice that preserves some discrete subgroup of $SO(4)$. The flip side is that we have lost the essence of quantum mechanics; there is no Hilbert space, and no concept of entanglement. Instead, we have what is essentially a statistical mechanics system, with the Euclidean action playing the role of the free energy. Nonetheless, we can still compute correlation functions and, from this, extract the spectrum of the theory and we may hope that this is sufficient for our purposes.

Throughout this section, we will work with a cubic, four-dimensional Euclidean lattice, with lattice spacing a. We introduce four basis vectors, each of unit length. It is useful, albeit initially slightly unfamiliar, to denote these as $\hat{\mu}$, with $\mu = 1,2,3,4$. A point x in our discrete Euclidean spacetime is then restricted to lie on the lattice Γ, defined by

$$\Gamma = \left\{ x : x = \sum_{\mu=1}^{4} a n_\mu \hat{\mu} \ , \ n_\mu \in \mathbf{Z} \right\} \tag{4.1}$$

The lattice spacing plays the role of the ultra-violet cut-off in our theory

$$a = \frac{1}{\Lambda_{UV}}$$

For the lattice to be a good approximation, we will need a to be much smaller than any other physical length scale in our system.

Because our system no longer has continuous translational symmetry, we can't invoke Noether's theorem to guarantee conservation of energy and momentum. Instead we must resort to Bloch's theorem which guarantees the conservation of "crystal momentum", lying in the Brillouin zone, $|k| \leq \pi/a$. (See, for example, the lectures on Applications of Quantum Mechanics.) Umklapp processes are allowed in which the lattice absorbs momentum, but only in units of $2\pi/a$. This means that provided we focus on low-momentum processes, $k \ll \pi/a$, we effectively have conservation of momentum and energy.

(An aside: the discussion above was a little quick. Bloch's theorem is really a statement in quantum mechanics in which we have continuous time. It applies directly only in the framework of Hamiltonian lattice gauge theory. In the present context, we really mean that the implications of momentum conservation on correlation functions will continue to hold in our discrete spacetime lattice, provided that we look at suitably small momentum.)

4.1 Scalar Fields on the Lattice

To ease our way into the discrete world, we start by considering a real scalar field $\phi(x)$. A typical continuum action in Euclidean space takes the form

$$S = \int d^4x \; \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \tag{4.2}$$

Our first task is to construct a discrete version of this, in which the degrees of freedom are

$$\phi(x) \text{ with } x \in \Gamma$$

This is straightforward. The kinetic terms are replaced by the finite difference

$$\partial_\mu \phi(x) \;\longrightarrow\; \frac{\phi(x + a\hat{\mu}) - \phi(x)}{a} \tag{4.3}$$

while the integral over spacetime is replaced by the sum

$$\int d^4x \;\longrightarrow\; a^4 \sum_{x \in \Gamma}$$

Our action (4.2) then becomes

$$S = a^4 \sum_{x\in\Gamma} \; \frac{1}{2} \sum_\mu \left(\frac{\phi(x + a\hat{\mu}) - \phi(x)}{a} \right)^2 + V(\phi(x))$$

As always, this action sits in the path integral, whose measure is now simply a whole bunch of ordinary integrals, one for each lattice point:

$$Z = \int \prod_{x \in \Gamma} d\phi(x) \; e^{-S}$$

With this machinery, computing correlation functions of any operators reduces to performing a large but (at least for a lattice of finite size) finite number of integrals.

It's useful to think about the renormalisation group (RG) in this framework. Suppose that we start with a potential that takes the form

$$V(\phi) = \frac{m_0^2}{2}\phi^2 + \frac{\lambda_0}{4}\phi^4 \tag{4.4}$$

As usual in quantum field theory, m_0^2 and λ_0 are the "bare" parameters, appropriate for physics at the lattice scale. We can follow their fate under RG by performing the kind of blocking transformation that was introduced in statistical mechanics by Kadanoff. This is a real space RG procedure in which one integrates out the degrees of freedom on alternate lattice sites, say all the sites in (4.1) in which one or more n_μ is odd. This then leaves us with a new theory defined on a lattice with spacing $2a$. This will renormalise the parameters in the action. In particular, the mass term will typically shift to

$$m^2 \sim m_0^2 + \frac{\lambda_0}{a^2}$$

This is the naturalness issue for scalar fields. If we want to end up with a scalar field with physical mass $m_{\text{phys}}^2 \ll 1/a^2$, then the bare mass must be delicately tuned to be of order the cut-of, $m_0^2 \sim -\lambda_0/a^2$, so that it cancels the contribution that arises when performing RG. This makes it rather difficult in practice to put scalar fields on the lattice. As we will see below, life is somewhat easier for gauge fields and, after jumping through some hoops, for fermions.

As usual, RG does not leave the potential in the simple, comfortable form (4.4). Instead it will generate all possible terms consistent with the symmetries of the theory. These include higher terms such as ϕ^6 and ϕ^8 in the potential, as well as higher derivative terms such as $(\partial_\mu\phi\partial^\mu\phi)^2$. (Here, and below, we use the derivative notation as shorthand for the lattice finite difference (4.3).) This doesn't bother us because all of these terms are irrelevant (in the technical sense) and so don't affect the low-energy physics.

However, this raises a concern. The discrete rotational symmetry of the lattice is less restrictive than the continuous rotational symmetry of $\mathbf{R}^4$. This means that RG on the lattice will generate some terms involving derivatives $\partial\phi$ that would not arise in the continuum theory. If these terms are irrelevant then they will not affect the infra-red physics and we can sleep soundly, safe in the knowledge that the discrete theory will indeed give a good approximation to the continuum theory at low energies. However, if any of these new terms are relevant then we're in trouble: now the low-energy physics will not coincide with the continuum theory.

So what are the extra terms that arise from RG on the lattice? They must respect the $\mathbf{Z}_2$ symmetry $\phi \to -\phi$ of the original action, which means that they have an even number of ϕ fields. They must also respect the discrete rotation group that includes, for example, $x_1 \to x_2$. This rules out lone terms like $(\partial_1\phi)^2$. The lowest dimension term involving derivatives that respects these symmetries is

$$\sum_{\mu=1}^{4}(\partial_\mu\phi)^2$$

But this is, of course, the usual derivative term in the action. The first operator that is allowed on the lattice but prohibited in the continuum is

$$\sum_{\mu=1}^{4}\phi\,\partial_\mu^4\,\phi \tag{4.5}$$

This has dimension 6, and so is irrelevant. Happily, we learn that the lattice scalar field theory differs from the continuum only by irrelevant operators. Provided that we fine tune the mass, we expect the long wavelength physics to well approximate a continuum theory of a light scalar field.

4.2 Gauge Fields on the Lattice

We now come to Yang-Mills. Our task is write down a discrete theory on the lattice that reproduces the Yang-Mills action. For concreteness, we will restrict ourselves to $SU(N)$ gauge theory, with matter in the fundamental representation.

As a first guess, it's tempting to follow the prescription for the scalar field described above and introduce four, Lie algebra valued gauge fields $A_\mu(x)$, with $\mu = 1, 2, 3, 4$ at each point $x \in \Gamma$. This, it turns out, is not the right way to proceed. At an operational level, it is difficult to implement gauge invariance in such a formalism. But, more importantly, this approach completely ignores the essence of the gauge field. It misses the idea of holonomy.

4.2.1 The Wilson Action

Mathematicians refer to the gauge field as a connection. This hints at the fact that the gauge field is a guide, telling the internal, colour degrees of freedom or a particle or field how to evolve through parallel transport. The gauge field "connects" these internal degrees of freedom at one point in space to those in another.

We saw this idea earlier in Section 2 after introducing the Yang-Mills field. (See Section 2.1.3.) Consider a test particle which carries an internal vector degree of freedom w_i, with $i = 1, \ldots, N$. As the particle moves along a path C, from x_i to x_f, this vector will evolve through parallel transport

$$w(\tau_f) = U[x_i, x_f]w(\tau_i)$$

where the holonomy, or Wilson line, is given by the path ordered exponential

$$U[x_i, x_f] = \mathcal{P} \exp\left(i \int_{x_i}^{x_f} A\right) \tag{4.6}$$

Note that $U[x_i, x_f]$ depends both on the end points, and on the choice of path C.

This is the key idea that we will implement on the lattice. We will not treat the Lie-algebra valued gauge fields A_μ as the fundamental objects. Instead we will work with the group-valued Wilson lines U. These Wilson lines are as much about the journey as the destination: their role is to tell other fields how to evolve. The matter fields live on the sites of the lattice. In contrast, the Wilson lines live on the links.

Specifically, on the link from lattice site x to $x + \hat{\mu}$, we will introduce a dynamical variable

$$\text{link } x \to x + \hat{\mu} : \qquad U_\mu(x) \in G$$

The fact that the fundamental degrees of freedom are group valued, rather than Lie algebra valued, plays an important role in lattice gauge theory. It means, for example that there is an immediate difference between, say, $SU(N)$ and $SU(N)/\mathbf{Z}_N$, a distinction that was rather harder to see in the continuum. We will see other benefits of this below.

At times we will wish to compare our lattice gauge theory with the more familiar continuum action. To do this, we need to re-introduce the A_μ gauge fields. These are related to the lattice degrees of freedom by

$$U_\mu(x) = e^{iaA_\mu(x)} \tag{4.7}$$

The placing of the μ subscripts on the left and right hand side of this equation should make you feel queasy. It looks bad because if one side transforms covariantly under $SO(4)$ rotations, then the other does not. But we don't want these variables to transform under continuous symmetries; only discrete ones. This is the source of your discomfort.

We will wish to identify configurations related by gauge transformations. In the continuum, under a gauge transformation $\Omega(x)$, the Wilson line (4.6) transforms as

$$U[x_i, x_f; C] \to \Omega(x_i)\, U[x_i, x_f; C]\, \Omega^\dagger(x_f)$$

We can directly translate this into our lattice. The link variable transforms as

$$U_\mu(x) \to \Omega(x)\, U_\mu(x)\, \Omega(x + \hat{\mu}) \tag{4.8}$$

The next step is to write down an action that is invariant under gauge transformations. We can achieve this by multiplying together a string of neighbouring Wilson lines, and then taking the trace. With no dangling ends, this is guaranteed to be gauge invariant. This is the lattice version of the Wilson loop (2.15) that we met in Section 2.

We can construct a Wilson loop for any closed path C in the lattice. When the path goes from the site x to $x + \hat{\mu}$, we include a factor of $U_\mu(x)$; when the path goes from site x to site $x - \hat{\mu}$, we include a factor of $U_\mu^\dagger(x + \hat{\mu})$. The simplest such path is a square which traverses a single plaquette of the lattice as shown in the figure. The corresponding Wilson loop is

$$W_\Box = \operatorname{tr} U_\mu(x)\, U_\nu(x + \hat{\mu})\, U_\mu^\dagger(x + \hat{\nu})\, U_\nu^\dagger(x)$$

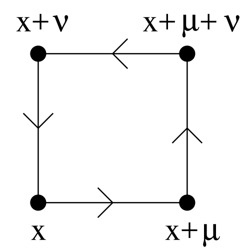

Figure 34:

To get some intuition for this object, we can write it in terms of the gauge field (4.7). We will assume that we can Taylor expand the gauge field so that, for example, $A_\nu(x+\hat{\mu}) \approx A_\nu(x) + a\partial_\mu A_\nu(x) + \ldots$. Then we have

$$\begin{aligned} W_\Box &\approx \operatorname{tr} e^{iaA_\mu(x)}\, e^{ia(A_\nu(x) + a\partial_\mu A_\nu(x))}\, e^{-ia(A_\mu(x) + a\partial_\nu A_\mu(x))}\, e^{-iaA_\nu(x)} \\ &\approx \operatorname{tr} e^{ia(A_\mu(x) + A_\nu(x) + a\partial_\mu A_\nu(x) + \frac{ia}{2}[A_\mu(x), A_\nu(x)])}\, e^{-ia(A_\nu(x) + A_\mu(x) + a\partial_\nu A_\mu(x)) - \frac{ia}{2}[A_\mu(x), A_\nu(x)])} \end{aligned}$$

where, to go to the second line, we've used the BCH formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots}$. On both lines we've thrown away terms of order a^3 in the exponent. Using BCH just once more, we have

$$\begin{aligned} W_\Box &= \operatorname{tr} e^{ia^2 F_{\mu\nu}(x) + \ldots} = \operatorname{tr}\left(1 + ia^2 F_{\mu\nu} - \frac{a^4}{2} F_{\mu\nu} F_{\mu\nu} + \ldots\right) \\ &= -\frac{a^4}{2} \operatorname{tr} F_{\mu\nu} F_{\mu\nu} + \ldots \end{aligned}$$

where, as usual, $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i[A_\mu(x), A_\nu(x)]$ and the $\ldots$ include both a constant term and terms higher in order in a^2. Note that there is no sum over μ, ν in this expression; instead these μ, ν indices tell us of the orientation of the plaquette.

By summing over all possible plaquettes, we get something that reproduces the Yang-Mills action at leading order. The Wilson loop $W_\Box$ itself is not real so we need to add the conjugate $W_\Box^\dagger$, which is the loop with the opposite orientation. This then gives us the *Wilson action*

$$S_{\text{Wilson}} = -\frac{\beta}{2N}\sum_{\Box}\left(W_\Box + W_\Box^\dagger\right) = \frac{a^4\beta}{4N}\int d^4x \ \text{tr}\, F_{\mu\nu}F_{\mu\nu} + \ldots \tag{4.9}$$

where now we are again using summation convention for μ,ν indices. An extra factor of $1/2$ has appeared because the sum over plaquettes differs by a factor of 2 from the sum over μ,ν. It is convention to put a factor of $1/N$ in front of the action. The coupling β is related to the continuum Yang-Mills coupling (2.8) by

$$\frac{\beta}{2N} = \frac{1}{g^2}$$

The Wilson action only coincides with the Yang-Mills action at leading order. Expanding to higher orders in a will give corrections. The next lowest dimension operator to appear is $F_{\mu\nu}\mathcal{D}_\mu^2 F_{\mu\nu}$. It has dimension 6 and does not correspond to an operator that respects continuous $O(4)$ rotational symmetry. In this way, it is analogous to the operator (4.5) that we saw for the scalar field. Happily, it is irrelevant.

The Wilson action is far from unique. For example, we could have chosen to sum over double plaquettes $\Box\!\Box$ as opposed to single plaquettes. Expanding these, or any such Wilson loop, will result in a $F_{\mu\nu}F_{\mu\nu}$ term simply because this is the lowest dimension, gauge invariant operator. These Wilson loops differ in the relative coefficients of the expansion.

For numerical purposes, this lack of uniqueness can be exploited. We could augment the Wilson action with additional terms corresponding to double, or larger, plaquettes. This can be done in such a way that the Yang-Mills action survives, but the higher dimension operators, such as $F_{\mu\nu}\mathcal{D}_\mu^2 F_{\mu\nu}$ cancel. This means that the leading higher derivative terms are even more irrelevant, and helps with numerical convergence. We won't pursue this (or, indeed, any numerics) here.

Adding Dynamical Matter

As we mentioned before, matter fields live on the sites of the lattice. Consider a scalar field $\phi(x)$ transforming in the fundamental representation of the gauge group. (Fermions will come with their own issues, which we discuss in Section 4.3.) Under a gauge transformation we have

$$\phi(x) \to \Omega(x)\phi(x)$$

We can now construct gauge invariant objects by topping and tailing the Wilson line with particle and anti-particle matter insertions. The simplest example has the particle and anti-particle separated by just one lattice spacing, $\phi^\dagger(x)U_\mu(x)\phi(x+\hat{\mu})$. More generally, we can separate the two as much as we like, as the long as the Wilson line forges a continuous path between them.

To write down a kinetic term for this scalar, we need the covariant version of the finite difference (4.3). This is given by

$$\int d^4x\ |\mathcal{D}_\mu\phi(x)|^2 \ \longrightarrow\ a^2\sum_{(x,\mu)}\left[2\phi^\dagger(x)\phi(x)-\phi^\dagger(x)U_\mu(x)\phi(x+\hat{\mu})-\phi^\dagger(x+\hat{\mu})U^\dagger_\mu(x)\phi(x)\right]$$

In this way, it is straightforward to coupled scalar matter to gauge fields. We won't have anything more to say about dynamical matter here, but we'll return to the question in Section 4.3 when we discuss fermions on the lattice.

4.2.2 The Haar Measure

To define a quantum field theory, it's not enough to give the action. We also need to specify the measure of the path integral.

Of course, usually in quantum field theory we're fairly lax about this, and the measure certainly isn't defined at the level of rigour that would satisfy a mathematician. The lattice provides us an opportunity to do better, since we have reduced the path integral to a large number of ordinary integrals. For lattice gauge theory, the appropriate measure is something like

$$\prod_{(x,\hat{\mu})} dU_\mu(x) \tag{4.10}$$

so that we integrate over the $U\in G$ degree of freedom on each link. The question is: what does this mean?

Thankfully this is a question that is well understood. We want to define an integration measure over the group manifold G. We will ask that the measure obeys the following requirements:

- Left and right invariance. This means that for any function $f(U)$, with $U\in G$, and for any $\Omega\in G$,

$$\int dU\ f(U)=\int dU\ f(\Omega U)=\int dU\ f(U\Omega) \tag{4.11}$$

This will ensure that our path integral respects the gauge symmetry (4.8). By a change of variables, this is equivalent to the requirement that $d(U\Omega) = d(\Omega U) = dU$ for all $\Omega \in G$.

- Linearity:

$$\int dU\ \left(\alpha f(U) + \beta g(U)\right) = \alpha \int dU\ f(U) + \beta \int dU\ g(U)$$

This is something that we take for granted in integration, and we would very much like to retain it here.

- Normalisation condition:

$$\int dU\ 1 = 1 \tag{4.12}$$

A difference between gauge theory on the lattice and in the continuum is that the dynamical degrees of freedom live in the group G, rather than its Lie algebra. The group manifold is compact, so that $\int dU\ 1$ just gives the volume of G. There's no real meaning to this volume, so we choose to normalise it it to unity.

It turns out that there is a unique measure with these properties. It is known as the *Haar measure*.

We won't need to explicitly construct the Haar measure in what follows, because the properties above are sufficient to calculate what we'll need. Nonetheless, it may be useful to give a sense of where it comes from. We start in a neighbourhood of the identity. Here we can write any $SU(N)$ group element as

$$U = e^{i\alpha^a T^a}$$

with T^a the generators of the $su(N)$ algebra. In this neighbourhood, the Haar measure becomes (up to normalisation)

$$\int dU = \int d^{N^2-1}\alpha\ \sqrt{\det\gamma} \tag{4.13}$$

where γ is the canonical metric on the group manifold,

$$\gamma_{ab} = \mathrm{tr}\left(U^{-1}\frac{\partial U}{\partial\alpha^a}U^{-1}\frac{\partial U}{\partial\alpha^b}\right)$$

This measure is both left and right invariant in the sense of (4.11), since the group action corresponds to shifting $\alpha^a \to \alpha^a + \text{constant}$.

Now suppose that we want to construct the measure in the neighbourhood of any other point, say U_0. We can do this by using the group multiplication to transport the neighbourhood around the identity to a corresponding neighbourhood around U_0. In this way, we can construct the measure over various patches of the group manifold.

One way to transport the measure from one neighbourhood to another is by right multiplication. We write

$$U = e^{i\alpha^a T^a}\, U_0 \tag{4.14}$$

We then again use the definition (4.13) to define the measure. This measure is left invariant, satisfying $dU = d(\Omega U)$ since multiplying U on the left by Ω corresponds to shifting $\alpha^a \to \alpha^a + \text{constant}$. In fact, this is the unique left invariant measure.

But is the measure right invariant? If we multiply U on the right then the group element Ω must make its way past U_0 before we can conclude that it shifts α^a by a constant. But Ω and U_0 do not necessarily commute. Nonetheless, the measure is right invariant. This follows from the fact that we have constructed the unique left invariant measure which means that, if we consider the measure $d(U\Omega)$, which is also left invariant then, by uniqueness, it must be the same as the original. So $d(U\Omega) = dU$.

Integrating over the Group

In what follows, we will need results for some of the simpler integrations.

We start by computing the integral $\int dU\; U$. Because the measure is both left and right invariant, we must have

$$\int dU\; U = \int dU\; \Omega_1 U \Omega_2$$

for any Ω_1 and $\Omega_2 \in G$. But there's only one way to achieve this, which is

$$\int dU\; U = 0 \tag{4.15}$$

More generally, we will only get a non-vanishing answer if we integrate objects which are invariant under G. This will prove to be a powerful constraint, and we'll discuss it further below.

The simplest, non-trivial integral is therefore $\int dU\; U^\dagger_{ij} U_{kl}$, where we've included the gauge group indices $i,j = 1,\ldots,N$. This must be proportional to an invariant tensor, and the only option is

$$\int dU\; U^\dagger_{ij} U_{kl} = \frac{1}{N}\delta_{jk}\delta_{il} \tag{4.16}$$

To see that the $1/N$ factor is correct, we can contract the jk indices and reproduce the normalisation condition (4.12). One further useful integral comes from the baryon vertex, which gives

$$\int dU\; U_{i_1 j_1}\ldots U_{i_N j_N} = \frac{1}{N!}\,\epsilon_{i_1\ldots i_N}\,\epsilon_{j_1\ldots j_N}$$

Elitzur's Theorem

Let's now return to our lattice gauge theory. We wish to compute expectation values of operators $\mathcal{O}$ by computing

$$\langle \mathcal{O}\rangle = \frac{1}{Z}\int \prod_{(x,\mu)} dU_\mu(x)\;\mathcal{O}\;e^{-S_{\rm Wilson}}$$

This is simply lots of copies of the group integration defined above. The fact that any object which transforms under G necessarily vanishes when integrated over the group manifold has an important consequence for our gauge theory: it ensures that we have

$$\langle \mathcal{O}\rangle = 0$$

for any operator $\mathcal{O}$ that is not gauge invariant. This is known as *Elitzur's theorem*. Note that this statement has nothing to do with confinement. It is just as valid for electromagnetism as for Yang-Mills, and is a statement about the operators we should be considering in a gauge theory.

Elitzur's theorem follows in a straightforward manner from (4.15). To illustrate the basic idea, we will show how it works for a link variable, $\mathcal{O} = U_\nu(y)$. We want to compute

$$\langle U_\nu(y)\rangle = \frac{1}{Z}\int \prod_{(x,\mu)} dU_\mu(x)\;U_\nu(y)\;e^{-S_{\rm Wilson}}$$

The specific link variable $U_\nu(y)$ will appear in a bunch of different plaquettes that arise in the Wilson action. For example, we could focus on the plaquette Wilson loop

$$W_\Box = {\rm tr}\,U_\nu(y)\,U_\rho(y+\hat\nu)\,U^\dagger_\nu(y+\hat\rho)\,U^\dagger_\rho(y) \tag{4.17}$$

But we know that the measure is invariant under group multiplication of any link variable. We can therefore make the change of variable

$$U_\rho(y) \to U_\nu^\dagger(y) U_\rho(y) \tag{4.18}$$

in which case the particular plaquette Wilson loop (4.17) becomes

$$W_\Box \;\to\; \mathrm{tr}\, U_\rho(y+\hat{\nu})\, U_\nu^\dagger(y+\hat{\rho})\, U_\rho^\dagger(y)$$

and is independent of $U_\nu(y)$. You might think that this will screw up some other plaquette action, where $U_\nu(y)$ will reappear. There are 8 links emanating from the site y, as shown in the disappointingly 3d figure on the right. You can convince yourself that if you make the same change of variables (4.18) for each of them then S_{Wilson} no longer depends on the specific link variable $U_\nu(y)$. We can then isolate the integral over the link variable $U_\nu(y)$, to get

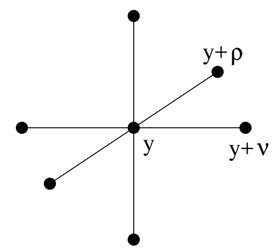

Figure 35:

$$\langle U_\nu(y) \rangle = \text{other stuff} \times \int dU_\nu(y)\; U_\nu(y) = 0$$

which, as shown, vanishes courtesy of (4.15). This tells us that a single link variable cannot play the role of an order parameter in lattice gauge theory. But this is something we expected from our discussion in the continuum.

We see that the Wilson action is rather clever. It's constructed from link variables $U_\nu(y)$, but doesn't actually depend on them individually. Instead, it depends only on gauge invariant quantities that we can construct from the link variables. These are the Wilson loops.

A Comment on Gauge Fixing

The integration measure (4.10) will greatly overcount physical degrees of freedom: it will integrate over many configurations all of which are identified by gauge transformations. What do we do about this? The rather wonderful answer is: nothing at all.

In the continuum, we bend over backwards worrying about gauge fixing. This is because we are integrating over the Lie algebra and will get a divergence unless we fix a gauge. But there is no such divergence in the lattice formulation because we are integrating over the compact group G. Instead, the result of failing to fix the gauge will simply be a harmless normalisation constant.

4.2.3 The Strong Coupling Expansion

We now have all the machinery to define the partition function of lattice gauge theory,

$$Z = \int \prod_{(x,\mu)} dU_\mu(x)\ e^{-S_{\rm Wilson}} \tag{4.19}$$

with $S_{\rm Wilson}$ the sum over plaquette Wilson loops,

$$S_{\rm Wilson} = -\frac{\beta}{2N}\sum_{\Box}\left(W_\Box + W_\Box^\dagger\right) \tag{4.20}$$

Because we're in Euclidean spacetime, the parameter β plays the same role as the inverse temperature in statistical mechanics. It is related to the bare Yang-Mills coupling as $\beta = 2N/g^2$.

We expect this theory to give a good approximation to continuum Yang-Mills when the lattice spacing a is suitably small. Here "small" is relative to the dynamically generated scale Λ_{QCD}. Thinking of $1/a$ as the UV cut-off of the theory, the physical scale is defined by

$$\Lambda_{QCD} = \frac{1}{a}e^{1/2\beta_0 g^2} \tag{4.21}$$

where β_0 is the one-loop beta-function which, despite the unfortunate similarity in their names, has nothing to do with the lattice coupling β that we introduced in the Wilson action. We calculated the one-loop beta function in Section 2.4 and, importantly, $\beta_0 < 0$.

In the expression (4.21), g^2 is the bare gauge coupling. We see that we have a separation of scales between Λ_{QCD} and the cut-off provided our theory is weakly coupled in the UV,

$$g^2 \ll 1 \quad \Leftrightarrow \quad \beta \gg 1$$

In this case, we expect the lattice gauge theory to closely match the continuum. We only have to do some integrals. Lots of integrals. I can't do them. You probably can't either. But a computer can.

We could also ask: what happens in the opposite regime, namely

$$g^2 \gg 1 \quad \Leftrightarrow \quad \beta \ll 1$$

It's not obvious that this regime is of interest. From (4.21), we see that there is no separation between the physical scale, Λ_{QCD}, and the cut-off scale $1/a$, so this is unlikely to give us quantitative insight into continuum Yang-Mills. Nonetheless, it does have one thing going for it: we can actually calculate in this regime! We do this by expanding the partition function (4.19) in powers of β. This is usually referred to as the *strong coupling expansion*; it is analogous to the high temperature expansion in statistical lattice models. (See the lectures on Statistical Physics for more details of how this works in the Ising model.)

Confinement

We'll use the strong coupling expansion to compute the expectation value of a large rectangular Wilson loop, $W[C]$,

$$W[C] = \frac{1}{N}\,\mathrm{tr}\left(\mathcal{P}\prod_{(x,\mu)\in C} U_\mu(x)\right) \tag{4.22}$$

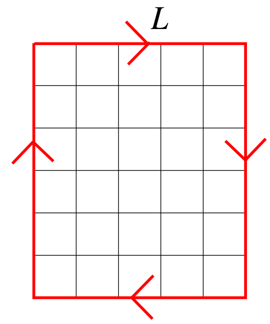

Figure 36:

Here the factor of $1/N$ is chosen so that if all the links are $U = 1$ then $W[C] = 1$. We'll place this loop in a plane of the lattice as shown in the figure, and give the sides length L and T. (Each of these must be an integer multiple of a.)

We would like to calculate

$$\langle W[C]\rangle = \frac{1}{Z}\int \prod_{(x,\mu)} dU_\mu(x)\; W[C]\, e^{-S_{\text{Wilson}}}$$

In the strong coupling expansion, we achieve this by expanding $e^{-S_{\text{Wilson}}}$ in powers of $\beta \ll 1$. What is the first power of β that will give a non-zero answer? If a given link variable U appears in the integrand just once then, as we've seen in (4.15), it will integrate to zero. This means, for example, that the β^0 term in the expansion of $e^{-S_{\text{Wilson}}}$ will not contribute, since it leaves the each of the links in $W[C]$ unaccompanied.

The first term in the expansion of $e^{-S_{\text{Wilson}}}$ that will give a non-vanishing answer must contribute a $U^\dagger$ for each link in C. But any $U^\dagger$ that appears in the expansion of S_{Wilson} must be part of a plaquette of links. The further links in these plaquettes must also have companions, and these come from further plaquettes. It is best to think graphically. The links U of the Wilson loop are shown in red. They must be compensated by a corresponding $U^\dagger$ from S_{Wilson} plaquettes; these are shown in blue in the next figure. The simplest way to make sure that no link is left behind is to tile a surface bounded by C by plaquettes. We have shown some of these tiles in the figure. Note that each of the plaquettes $W_\Box$ must have a particular orientation to cancel the Wilson loop on the boundary; this orientation then dictates the way further tiles are laid.

There are many different surfaces S that we could use to tile the interior of C. The simplest is the one that lies in the same plane as C and covers each lattice plaquette exactly once. However, there are other surfaces, including those that do not lie in the plane. We can compute the contribution to $\langle W[C]\rangle$ from any given surface S. Only the plaquettes of a specific orientation in the Wilson action (4.20) will contribute (e.g. $W_\Box$, but not $W_\Box^\dagger$). The beta dependence is therefore

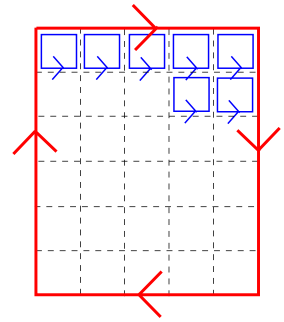

Figure 37:

$$\left(\frac{\beta}{2N}\right)^{\#\text{ of plaquettes}}$$

Each link in the surface (including those in the original C) will give rise to an integral of the form (4.16). This then gives a term of the form

$$\left(\frac{1}{N}\right)^{\#\text{ of links}}$$

Finally, for every site on the surface (including those on the original C), we'll be left with a summation $\delta_{ij}\delta_{ji} = N$. This gives a factor of

$$N^{\#\text{ of sites}}$$

Including the overall factor of $1/N$ in the normalisation of the Wilson loop (4.22), we have the contribution to the Wilson loop

$$\langle W[C]\rangle = \frac{1}{N}\left(\frac{\beta}{2N}\right)^{\#\text{ of plaquettes}}\left(\frac{1}{N}\right)^{\#\text{ of links}} N^{\#\text{ of sites}}$$

where we've used the fact that $Z = 1$ at leading order in β. This is the answer for a general surface. The leading order contribution comes from the minimal, flat surface which bounds C which has

$$\#\text{ of plaquettes} = \frac{LT}{a^2}$$

and

$$\#\text{ vertical links} = \frac{(L+a)T}{a^2} \quad \text{and} \quad \#\text{ horizontal links} = \frac{L(T+a)}{a^2}$$

and

$$\#\text{ sites} = \frac{(L+a)(T+a)}{a^2}$$

The upshot is that the leading order contribution to the Wilson loop is

$$\langle W[C]\rangle = \left(\frac{\beta}{2N^2}\right)^{LT/a^2}$$

But this is exactly what we expect from a confining theory: it is the long sought area law (2.75) for the Wilson loop,

$$\langle W[C]\rangle = e^{-\sigma A}$$

where $A = LT$ is the area of the minimal surface bounded by C and the string tension σ is given by

$$\sigma = -\frac{1}{a^2}\log\left(\frac{\beta}{2N^2}\right)$$

At the next order, this will get corrections of $\mathcal{O}(\beta)$. Note that the string tension is of order the UV cut-off $1/a$, which reminds us that we are not working in a physically interesting regime. Nonetheless we have demonstrated, for the first time, the promised area law of Yang-Mills, the diagnostic for confinement.

A particularly jarring way to illustrate that we're not computing in the continuum limit is to note that the computation above makes no use of the non-Abelian nature of the gauge group. We could repeat everything for Maxwell theory, in which the link variables are $U \in U(1)$. Nothing changes. We again find an area law in the strong coupling regime, indicating the existence of a confining phase.

What are we to make of this? For $U(1)$ gauge theory, there clearly must be a phase transition as we vary the coupling from $\beta \ll 1$ to $\beta \gg 1$ where we have the free, continuum Maxwell theory that we know and love. But what about Yang-Mills? We may hope that there is no phase transition for non-Abelian gauge groups G, so that the confining phase persists for all values of β. It seems that this hope is likely to be dashed. At least as far as the string tension is concerned, it appears that there is a finite radius of convergence around $\beta = 0$, and the string tension exhibits an essential singularity at a finite value of β. It is not known if there is a different path – say by choosing a different lattice action – which avoids this phase transition.

The Mass Gap

We can also look for the existence of a mass gap in the strong coupling expansion. Since we're in Euclidean space, we have neither Hilbert space nor Hamiltonian so we can't talk directly about the spectrum. However, we can look at correlation functions between two far separated objects.

The objects that we have to hand are the Wilson loops. We take two, parallel plaquette Wilson loops $W_\Box$ and $W_{\Box'}$, separated along a lattice axis by distance R. We expect the correlation function of these Wilson loops to scale as

$$\langle W_\Box W_{\Box'} \rangle \sim e^{-mR} \tag{4.23}$$

with m the mass of the lightest excitation. If the theory turns out to be gapless, we will instead find power-law decay.

We can compute this correlation function in the strong coupling expansion. The argument is the same as that above: to get a non-zero answer, we must form a tube of plaquettes. The minimum such tube is depicted in the figure, with the source Wilson loops shown in red, and the tiling from the action shown in blue. (This time we have not shown the orientation of the Wilson loops to keep the figure uncluttered.) It has

$$\#\text{ of plaquettes} = \frac{4R}{a}$$

$$\#\text{ links} = \frac{4(2R+a)}{a}$$

$$\#\text{ sites} = \frac{4(R+a)}{a}$$

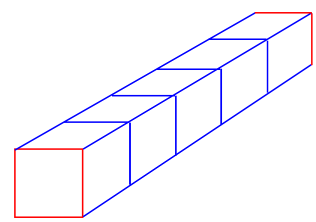

Figure 38:

The leading order contribution to the connected part of the correlation function is therefore

$$\langle W_\Box W_{\Box'} \rangle = \frac{1}{N^2}\left(\frac{\beta}{2N^2}\right)^{4R/a}$$

Comparing to the expected form (4.23), we see that we have a mass gap

$$m = -\frac{4}{a}\log\left(\frac{\beta}{2N^2}\right)$$

Once again, it's comforting to see the expected behaviour of Yang-Mills. Once again, we see the lack of physical realism highlighted in the fact that the mass scale is the same order of magnitude as the UV cut-off $1/a$.

4.3 Fermions on the Lattice

Finally we turn to fermions. Here things are not so straightforward. The reason is simple: anomalies.

Even before we attempt any calculations, we can anticipate that things might be tricky. Lattice gauge theory is a regulated version of quantum field theory. If we work on a finite, but arbitrarily large lattice, we have a finite number of degrees of freedom. This means that we are back in the realm of quantum mechanics. There is no room for the subtleties associated to the chiral anomaly. There is no infinite availability at the Hilbert hotel.

This means that we're likely to run into trouble if we try to implement chiral symmetry on the lattice or, at the very least, if we attempt to couple gapless fermions to gauge fields. We might expect even more trouble if we attempt to put chiral gauge theories on the lattice. In this section, we will see the form that this trouble takes.

4.3.1 Fermions in Two Dimensions

We can build some intuition for the problems ahead by looking at fermions in $d = 1+1$ dimensions. Here, Dirac spinors are two-component objects. We work with the gamma matrices

$$\gamma^0 = \sigma^1 \quad , \quad \gamma^1 = i\sigma^2 \quad , \quad \gamma^3 = -\gamma^0\gamma^1 = \sigma^3$$

The Dirac fermion then decomposes into chiral fermions $\chi_\pm$ as

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

In the continuum, the action for a massless fermion is

$$S = \int d^2x \; i\bar{\psi}\,\partial\!\!\!/\psi = \int d^2x \; i\psi_+^\dagger \partial_- \psi_+ + i\psi_-^\dagger \partial_+ \psi_- \tag{4.24}$$

with $\partial_\pm = \partial_t \pm \partial_x$. The equations of motion tell us $\partial_-\psi_+ = \partial_+\psi_- = 0$. This means that ψ_+ is a left-moving fermion, while ψ_- is a right-moving fermion.

As in Section 3.1, it is useful to think in the language of the Dirac sea. The dispersion relation $E(k)$ for fermions in the continuum is drawn in the left hand figure. All states with $E < 0$ are to be thought of as filled; all states with $E > 0$ are empty.

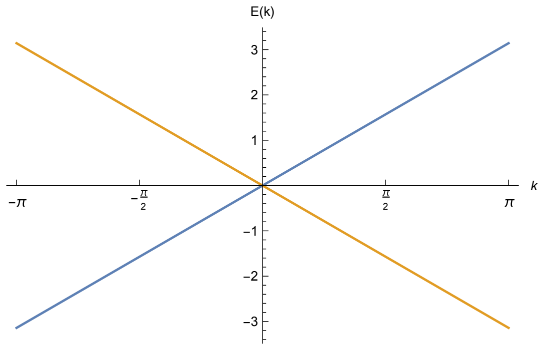

Figure 39: The dispersion relation for a Dirac fermion in the continuum

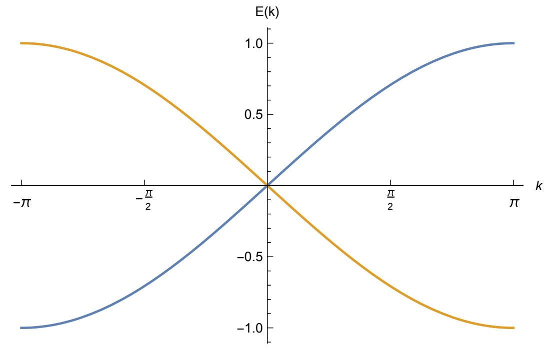

Figure 40: A possible deformation to keep the dispersion periodic in the Brillouin zone (with $a = 1$).

The (blue) line with positive gradient describes the excitations of the right-moving fermion ψ_-: the particles have momentum $k > 0$ while the filled states have momentum $k < 0$ which means that the anti-particles (a.k.a holes) again have momentum $k > 0$. Similarly, the (orange) line with negative gradient describes the excitations of the left-moving fermion ψ_+.

The chiral symmetry of the action (4.24) means that the left- and right-handed fermions are individually conserved. As we have seen Section 3.1, this is no longer the case in the presence of gauge fields. But, for now, we will consider only free fermions so the chiral symmetry remains a good symmetry, albeit one that has a 't Hooft anomaly.

So much for the continuum. What happens if we introduce a lattice? We will start by keeping time continuous, but making space discrete with lattice spacing a. This is familiar from condensed matter physics, and we know what happens: the momentum takes values in the Brillouin zone

$$k \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right)$$

Importantly, the Brillouin zone is periodic. The momentum $k = +\pi/a$ is identified with the momentum $k = -\pi/a$.

What does this mean for the dispersion relation? We'll look at some concrete models shortly, but first let's entertain a few possibilities. We require that the dispersion relation $E(k)$ remains a continuous, smooth function, but now with $k \in \mathbf{S}^1$ rather than $k \in \mathbf{R}$. This means that the dispersion relation must be deformed in some way.

One obvious possibility is shown in the right hand figure above: we deform the shape of the dispersion relation so that it is horizontal at the boundary of the Brillouin zone

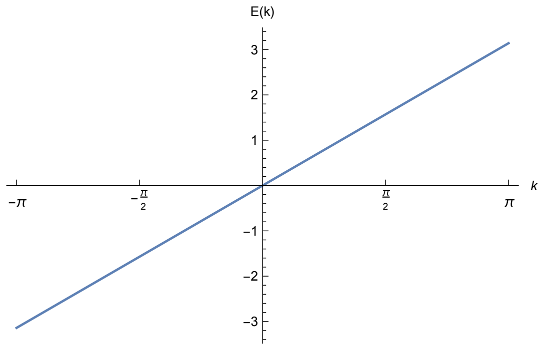

Figure 41: The dispersion relation for a right-handed fermion in the continuum

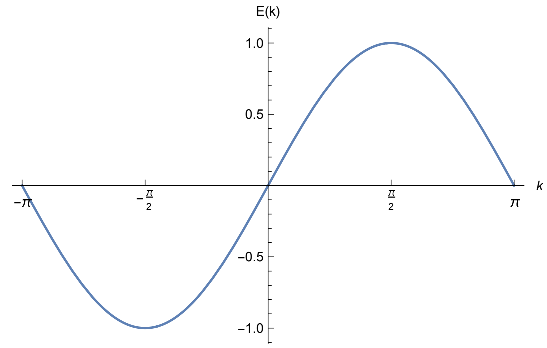

Figure 42: A possible deformation to keep the dispersion periodic in the Brillouin zone (with $a = 1$).

$|k| = \pi/a$. We then identify the states at $k = \pm\pi/a$. Although this seems rather mild, it's done something drastic to the chiral symmetry. If we take, say, a right-moving excitation with $k > 0$ and accelerate it, it will eventually circle the Brillouin zone and come back as a left-moving excitation. This is shown graphically by the fact that the blue line connects to the orange line at the edge of the Brillouin zone. (This is similar to the phenomenon of Bloch oscillations observed in cold atom systems; see the lectures on Applications of Quantum Mechanics.) Said another way, to get such a dispersion relation we must include an interaction term between ψ_+ and ψ_-. This means that, even without introducing gauge fields, there is no separate conservation of left and right-moving particles: we have destroyed the chiral symmetry. Note, however, that we have to excite particles to the maximum energy to see violation of chiral symmetry, so it presumably survives at low energies.

Suppose that we insist that we wish to preserve chiral symmetry. In fact, suppose that we try to be bolder and put just a single right-moving fermion ψ_+ on a lattice. We know that the dispersion relation $E(k)$ crosses the $E = 0$ axis at $k = 0$, with $dE/dk > 0$. But now there's no other line that it can join. The only option is that the dispersion relation also crosses the $E = 0$ at some other point $k \neq 0$, now with $dE/dk < 0$. An example is shown in right hand figure above. Now the lattice has an even more dramatic effect: it generates another low energy excitation, this time a left-mover. We learn that we don't have a theory of a chiral fermion at all: instead we have a theory of two Weyl fermions of opposite chirality. Moreover, once again a right-moving excitation can evolve continuously into a left-moving excitation. This phenomenon is known as *fermion doubling*.

You might think that you can simply ignore the high momentum fermion. And, of course, in a free theory you essentially can. But as soon as we turn on interactions — for example, by adding gauge fields — these new fermions can be pair produced just as easily as the original fermions. This is how the lattice avoids the gauge anomaly: it creates new fermion species!

More generally, it is clear that the Brillouin zone must house as many gapless left-moving fermions as right-moving fermions. This is for a simple reason: what goes up, must come down. This is a precursor to the Nielsen-Ninomiya theorem that we will discuss in Section 4.3.3

Quantising a Chiral Fermion

Let's now see how things play out if we proceed in the obvious fashion. The Hamiltonian for a chiral fermion on a line is

$$H = \pm \int dx \; i\psi_\pm^\dagger \partial_x \psi_\pm$$

The form of the Hamiltonian is the same for both chiralities; only the $\pm$ sign out front determines whether the particle is left- or right-moving. As we will see below, the requirement that the Hamiltonian is positive definite will ultimately translate this sign into a choice of vacuum state above which all excitations move in a particular direction.

For concreteness, we'll work with right-moving fermions ψ_-. We discretise this system in the obvious way: we consider a one-dimensional lattice with sites at $x = na$, where $n \in \mathbf{Z}$, and take the Hamiltonian to be

$$H = -a \sum_{x \in a\mathbf{Z}} i\psi_-^\dagger(x) \left[\frac{\psi_-(x+a) - \psi_-(x-a)}{2a} \right]$$

The Hamiltonian is Hermitian as required. We introduce the usual momentum expansion

$$\psi_-(x) = \int_{-\pi/a}^{+\pi/a} \frac{dk}{2\pi} \, e^{ikx} \, c_k$$

Note that we have momentum modes for both $k > 0$ and $k < 0$, even though this is a purely right-moving fermion. Inserting the mode expansion into the Hamiltonian gives

$$H = \frac{1}{2a} \int_{-\pi/a}^{+\pi/a} \frac{dk}{2\pi} \; 2\sin(ka) \, c_k^\dagger c_k$$

From this we can extract the one-particle dispersion relation by constructing the state $|k\rangle = c_k^\dagger|0\rangle$, to find the energy $H|k\rangle = E(k)|k\rangle$, with

$$E(k) = \frac{1}{a}\sin(ka)$$

This gives a dispersion relation of the kind we anticipated above: it has zeros at both $k = 0$ and at the edge of the Brillouin zone $k = \pi/a$. As promised, we started with a right-moving fermion but the lattice has birthed a left-moving partner.

Finally, a quick comment on the existence of states with $k < 0$. The true vacuum is not $|0\rangle$, but rather $|\Omega\rangle$ which has all states with $E < 0$ filled. This is the Dirac sea or, Fermi sea since the number of such states are finite. This vacuum obeys $c_k|\Omega\rangle = 0$ for $k > 0$ and $c_k^\dagger|\Omega\rangle = 0$ for $k < 0$. In this way, $c_k^\dagger$ creates a right-moving particle when $k > 0$, and c_k creates a right-moving anti-particle with momentum $|k|$ when $k < 0$.

4.3.2 Fermions in Four Dimensions

A very similar story plays out in $d = 3 + 1$ dimensions. A Weyl fermion $\psi_\pm$ is a 2-component complex spinor and obeys the equation of motion

$$\partial_0\psi_\pm = \pm\sigma^i\partial_i\psi_\pm$$

The Hamiltonian for a single Weyl fermion takes the form

$$H = \pm\int d^3x\; i\psi_\pm^\dagger\sigma^i\partial_i\psi_\pm$$

Once again, we wish to write down a discrete version of this Hamiltonian on a cubic spatial lattice Γ. For concreteness, we'll work with ψ_-. We take the Hamiltonian to be

$$H = -a^3\sum_{\mathbf{x}\in\Gamma} i\psi_-(\mathbf{x})\sum_{i=1,2,3}\sigma^i\left[\frac{\psi_-(\mathbf{x}+a\hat{\mathbf{i}}) - \psi_-(\mathbf{x}-a\hat{\mathbf{i}})}{2a}\right]$$

where $i = 1, 2, 3$ labels the spatial directions. In momentum space, the spinor is

$$\psi_-(\mathbf{x}) = \int_{BZ}\frac{d^3k}{(2\pi)^3}\; e^{i\mathbf{k}\cdot\mathbf{x}}\, c_{\mathbf{k}}$$

where $c_{\mathbf{k}}$ is again a two-component spinor. Here the momentum is integrated over the Brillouin zone

$$k_i \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right) \qquad i = 1, 2, 3$$

The Hamiltonian now takes the form

$$H = \frac{1}{2a}\int_{BZ}\frac{d^3k}{(2\pi)^3}\sum_{i=1,2,3} 2\sin(k_i a)\, c^\dagger_{\mathbf{k}}\,\sigma^i\, c_{\mathbf{k}} \tag{4.25}$$

If we focus on single particle excitations, the spectrum now has two bands, corresponding to a particle and anti-particle, and is given by

$$E(\mathbf{k}) = \frac{1}{a}\sum_{i=1,2,3}\sin(k_i a)\,\sigma^i$$

Close to the origin, $k \ll 1/a$, the Hamiltonian looks like that of the continuum fermion, with dispersion

$$E(\mathbf{k}) \approx \mathbf{k}\cdot\boldsymbol{\sigma} \tag{4.26}$$

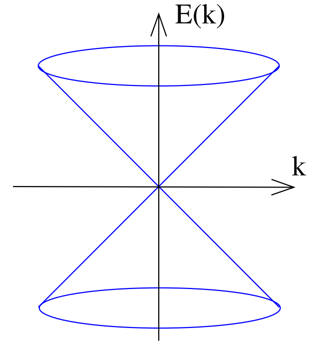

Figure 43:

This is referred to as the *Dirac cone*; it is sketched in the figure. Note that the bands cross precisely at $E = 0$ which, in a relativistic theory, plays the role of the Fermi energy. If the dispersion relation were to cross anywhere else, we would have a Fermi surface.

The fact that the Dirac cone corresponds to a right-handed fermion ψ_- shows up only in the overall + sign of the Hamiltonian. A left-handed fermion would have a minus sign in front. In fact, our full lattice Hamiltonian (4.25) has both right- and left-handed fermions since, like the $d = 1+1$ example above, it exhibits fermion doubling. There are gapless modes at momentum

$$k_i = 0 \text{ or } \frac{\pi}{a}$$

This gives $2^3 = 8$ gapless fermions in total. If we expand the dispersion relation around, say $\mathbf{k}_1 = (\pi/a, 0, 0)$, it looks like

$$E(\mathbf{k}') \approx -\mathbf{k}'\cdot\boldsymbol{\sigma} \quad \text{ where } \mathbf{k}' = \mathbf{k} - \mathbf{k}_1$$

which is left-handed. Of the 8 gapless modes, you can check that 4 are right-handed and 4 are left-handed. We see that, once again, the lattice has generated new gapless modes. Anything to avoid that anomaly.

4.3.3 The Nielsen-Ninomiya Theorem

We saw above that a naive attempt to quantise a $d = 3+1$ chiral fermion gives equal numbers of left and right-handed fermions in the Brillouin zone. The Nielsen-Ninomiya theorem is the statement that, given certain assumptions, this is always going to be the case. It is the higher dimensional version of "what goes up must come down".

The Nielsen-Ninomiya theorem applies to free fermions. We will work in terms of the one-particle dispersion relation, rather than the many-body Hamiltonian. To begin with, we consider a dispersion relation for a single Weyl fermion (we will generalise shortly). In momentum space, the most general Hamiltonian is given by

$$H = v_i(\mathbf{k})\sigma^i + \epsilon(\mathbf{k})\,\mathbf{1}_2 \tag{4.27}$$

where $\mathbf{k}$ takes values in the Brillouin zone.

In the language of condensed matter physics, this Hamiltonian has two bands, corresponding to the fact that each term is a 2×2 matrix. The first question that we will ask is: when do the two bands touch? This occurs when each $v_i(\mathbf{k}) = 0$ for $i = 1,2,3$. This is three conditions, and so we expect to generically find solutions at points, rather than lines, in the Brillouin zone BZ $\subset \mathbf{R}^3$. Let us suppose that there are D such points, which we call $\mathbf{k}_\alpha$,

$$v_i(\mathbf{k}_\alpha) = 0 \quad , \quad \alpha = 1,\ldots,D$$

Expanding about any such point, the dispersion relation becomes

$$H \approx v_{ij}(\mathbf{k}_\alpha)\,(\mathbf{k}-\mathbf{k}_\alpha)^j\sigma^i \quad \text{ with } \; v_{ij} = \frac{\partial v_i}{\partial k^j}$$

This now takes a similar form to (4.26), but with an anisotropic dispersion relation. The chirality of the fermion is dictated by

$$\text{chirality} = \text{sign}\det v_{ij}(\mathbf{k}_\alpha) \tag{4.28}$$

The assumption that the band crossing occurs only at points means that $\det v_{ij}(\mathbf{k}_\alpha) \neq 0$. The Nielsen-Ninomiya theorem is the statement that, for any dispersion (4.27) in a Brillouin zone, there are equal numbers of left- and right-handed fermions.

We offer two proofs of this statement. The first follows from some simple topological considerations. For $\mathbf{k} \neq \mathbf{k}_\alpha$, we can define a unit vector

$$\hat{\mathbf{v}}(\mathbf{k}) = \frac{\mathbf{v}}{|\mathbf{v}|}$$

The key idea is that this unit vector can wind around each of the degenerate points $\mathbf{k}_\alpha$. To see this, surround each such point with a sphere $\mathbf{S}^2_\alpha$. Evaluated on these spheres, $\hat{\mathbf{v}}$ provides a map

$$\hat{\mathbf{v}} : \mathbf{S}^2_\alpha \mapsto \mathbf{S}^2$$

But we know that such maps are characterised by $\Pi_2(\mathbf{S}^2) = \mathbf{Z}$. Generically, this winding will take values ± 1 only. In non-generic cases, where we have, say, winding $+2$, we can perturb the $\mathbf{v}$ slightly and the offending degenerate point will split into two points each with winding $+1$. This is the situation we will deal with.

This winding $\{+1, -1\} \subset \Pi_2(\mathbf{S}^2)$ is precisely the chirality (4.28). One, quick argument for this is the a spatial inversion will flip both the winding and the sign of the determinant.

To finish the argument, we need to show that the total winding must vanish. This follows from the compactness of the Brillouin zone. Here are some words. We could consider a sphere $\mathbf{S}^2_{\text{bigger}}$ which encompasses more and more degenerate points. The winding of around this sphere is equal to the sum of the windings of the $\mathbf{S}^2_\alpha$ which sit inside it. By the time we get to a sphere $\mathbf{S}^2_{\text{biggest}}$ which encompasses all the points, we can use the compactness of the Brillouin zone to contract the sphere back onto itself on the other side. The winding around this sphere must, therefore, vanish.

Here are some corresponding equations. The winding number ν_α is given by

$$\nu_\alpha = \frac{1}{8\pi} \int_{\mathbf{S}^2_\alpha} d^2 S_i \; \epsilon^{ijk} \epsilon^{abc} \hat{v}^a \frac{\partial \hat{v}^b}{\partial k^j} \frac{\partial \hat{v}^c}{\partial k^k} = \pm 1$$

We saw this expression previously in (2.89) when discussing 't Hooft-Polyakov monopoles. Let us define BZ′ as the Brillouin zone with the balls inside $\mathbf{S}^2_\alpha$ excised. This means that the boundary of BZ′ is

$$\partial(\text{BZ}') = \sum_{\alpha=1}^{D} \mathbf{S}^2_\alpha$$

Note that this is where we've used the compactness of the Brillouin zone: there is no contribution to the boundary from infinity. We can then use Stokes' theorem to write

$$\sum_{\alpha=1}^{D} \nu_\alpha = \frac{1}{8\pi} \int_{\text{BZ}'} d^3 k \; \frac{\partial}{\partial k^i} \left(\epsilon^{ijk} \epsilon^{abc} \hat{v}^a \frac{\partial \hat{v}^b}{\partial k^j} \frac{\partial \hat{v}^c}{\partial k^k} \right)$$

But the bulk integrand is strictly zero,

$$\frac{\partial}{\partial k^i}\left(\epsilon^{ijk}\epsilon^{abc}\hat{v}^a\frac{\partial\hat{v}^b}{\partial k^j}\frac{\partial\hat{v}^c}{\partial k^k}\right)=\epsilon^{ijk}\epsilon^{abc}\frac{\partial\hat{v}^a}{\partial k^i}\frac{\partial\hat{v}^b}{\partial k^j}\frac{\partial\hat{v}^c}{\partial k^k}=0$$

because each of the three vectors $\partial\hat{v}^a/\partial k^i$, $i=1,2,3$ is orthogonal to $\hat{v}^a$ and so all three lie must in the same plane. This tells us that

$$\sum_{\alpha=1}^{D}\nu_\alpha=0$$

as promised.

Note that the Nielsen-Ninomiya theorem counts only the points of degeneracy in the dispersion relation (4.27): it makes no comment about the energy $\epsilon(\mathbf{k}_\alpha)$ of these points. To get relativistic physics in the continuum, we require that $\epsilon(\mathbf{k}_\alpha)=0$. This ensures that the bands cross precisely at the top of the Dirac sea, and there is no Fermi surface. This isn't as finely tuned as it appears and arises naturally if there is one electron per unit cell; we saw an example of this phenomenon in the lectures on Applications of Quantum Field Theory when we discussed graphene.

Another Proof of Nielsen-Ninomiya: Berry Phase

There is another viewpoint on the Nielsen-Ninomiya theorem that is useful. This places the focus on the Hilbert space of states, rather than the dispersion relation itself[9].

For each $\mathbf{k}\in$ BZ, there are two states. As long as $\mathbf{k}\neq\mathbf{k}_\alpha$, these have different energies. In the language of the Dirac sea, the one with lower energy is filled and the one with higher energy is empty. We focus on the lower energy, filled states which we refer to as $|\psi(\mathbf{k})\rangle$, $\mathbf{k}\neq\mathbf{k}_\alpha$. The Berry connection is a natural $U(1)$ connection on these filled states, which tells us how to relate their phases for different values of $\mathbf{k}$,

$$\mathcal{A}_i(\mathbf{k})=-i\langle\psi(\mathbf{k})|\,\frac{\partial}{\partial k^i}\,|\psi(\mathbf{k})\rangle$$

You can find a detailed discussion of Berry phase in both the lectures on Applications of Quantum Field Theory and the lectures on Quantum Hall Effect. From the Berry phase, we can define the Berry curvature

$$\mathcal{F}_{ij}=\frac{\partial\mathcal{A}_j}{\partial k^i}-\frac{\partial\mathcal{A}_i}{\partial k^j}$$

[9]This is closely related to the Nobel winning TKKN formula that we discussed the lectures on the Quantum Hall Effect.

The Berry curvature for the dispersion relation (4.27) is the simplest example that we met when we first came across the Berry phase and is discussed in detail in both previous lectures. The chirality of the gapless fermion can now be expressed in terms of the curvature $\mathcal{F}$, which has the property that, when integrated around any degenerate point $\mathbf{k}_\alpha$,

$$\nu_\alpha = \frac{1}{2\pi}\int_{\mathbf{S}^2_\alpha} \mathcal{F} = \pm 1$$

Now we complete the argument in the same way as before. We have

$$\frac{1}{2\pi}\int_{\rm BZ'} d\mathcal{F} = \frac{1}{2\pi}\sum_{\alpha=1}^{D}\int_{\mathbf{S}^2_\alpha}\mathcal{F} = \sum_{\alpha=1}^{D}\nu_\alpha = 0$$

Again, we learn that there are equal numbers of left- and right-handed fermions.

We can extend this proof to systems with multiple bands. Suppose that we have a system with q bands, of which p are filled. This state of affairs persists apart from at points $\mathbf{k}_\alpha$ where the $p^{\rm th}$ band intersects the $(p+1)^{\rm th}$. Away from these points, we denote the filled states as $|\psi_a(\mathbf{k})\rangle$ with $a = 1,\ldots p$. These states then define a $U(p)$ Berry connection

$$(\mathcal{A}_i)_{ba} = -i\langle\psi_a|\,\frac{\partial}{\partial k^i}\,|\psi_b\rangle$$

and the associated $U(p)$ field strength

$$(\mathcal{F}_{ij})_{ab} = \frac{\partial(\mathcal{A}_j)_{ab}}{\partial k^i} - \frac{\partial(\mathcal{A}_i)_{ab}}{\partial k^j} - i[\mathcal{A}_i,\mathcal{A}_j]_{ab}$$

This time the winding is

$$\nu_\alpha = \frac{1}{2\pi}\int_{\mathbf{S}^2_\alpha}{\rm tr}\,\mathcal{F}$$

The same argument as above tells us that, again, $\sum_\alpha \nu_\alpha = 0$.

4.3.4 Approaches to Lattice QCD

So far our discussion of fermions has been in the Hamiltonian formulation, where time remains continuous. The issues that we met above do not disappear when we consider discrete, Euclidean spacetime. For example, the action for a single massless Dirac fermion is

$$S = \int d^4x\; i\bar{\psi}\gamma^\mu\partial_\mu\psi$$

The obvious discrete generalisation is

$$S = a^4 \sum_{x\in\Gamma} i\bar{\psi}(x) \sum_\mu \gamma^\mu \left[\frac{\psi(x+a\hat{\mu}) - \psi(x-a\hat{\mu})}{2a}\right] \tag{4.29}$$

Working in momentum space, this becomes

$$S = \frac{1}{a}\int_{BZ} \frac{d^4k}{(2\pi)^4}\; \bar{\psi}_{-k}\, D(k)\, \psi_k \tag{4.30}$$

with the inverse propagator

$$D(k) = \sum_\mu \gamma^\mu \sin(k_\mu a) \tag{4.31}$$

We again see the fermion doubling problem, now in the guise of poles in the propagator $D^{-1}(k)$ at $k_\mu = 0$ and $k_\mu = \pi/a$. Since we have also discretised time, the problem has become twice as bad: there are now $2^4 = 16$ poles.

The Nielsen-Ninomiya theorem that we met earlier has a direct translation in this context. It states that it is not possible to write down a $D(k)$ in (4.30) that obeys the following four conditions,

- $D(k)$ is continuous within the Brillouin zone. This means, in particular, that it is periodic in k.
- $D(k) \approx \gamma^\mu k_\mu$ when $k \ll 1/a$, so that the theory looks like a massless Dirac fermion when the momentum is small.
- $D(k)$ has poles only at $k = 0$. This is the requirement that there are no fermion doublers. As we've seen, this requirement doesn't hold if we follow the naive discretization (4.31).
- $\{\gamma^5, D(k)\} = 0$. This is the statement that the theory preserves chiral symmetry. It is true for our naive approach (4.31), but this suffered from fermionic doublers. As we will see below, if we try to remove these we necessarily screw with chiral symmetry. Indeed, we saw a very similar story in Section 4.3.1 when we discussed fermions in $d = 1+1$ dimensions.

What to make of this? Clearly, we're not going to be able to simulate chiral gauge theories using these methods. But what about QCD? This is a non-chiral theory that involves only Dirac fermions. Even here, we have some difficulty because if we try to remove the doublers to get the right number of degrees of freedom, then we are going to break chiral symmetry explicitly. Of course, ultimately chiral symmetry will be broken by the anomaly anyway, but there's interesting physics in that anomaly and that's going to be hard to see if we've killed chiral symmetry from the outset.

What to do? Here are some possible approaches. We will discuss a more innovative approach in the following section.

SLAC Fermions

We're going to have to violate one of the requirements of the Nielsen-Ninomiya theorem. One possibility is to give up on periodicity in the Brillouin zone. Now what goes up need not necessarily come down. We make the dispersion relation discontinuous at some high momentum. For example, you could just set $D(\mathbf{k}) = \gamma^\mu k_\mu$ everywhere, and suffer the discontinuity at the edge of the Brillouin zone. This, it turns out, is bad. A discontinuity in momentum space corresponds to a breakdown of locality in real space. The resulting theories are not local quantum field theories. They do not behave in a nice manner.

Wilson Fermions

As we mentioned above, another possibility is to kill the doublers, at the expense of breaking chiral symmetry. One way to implement this, first suggested by Wilson, is to add to the original action (4.30) the term

$$S = ar \int d^4x \; \bar{\psi}\, \partial^2 \psi = a^3 r \sum_{x\in\Gamma} \bar{\psi}(x) \sum_\mu \left[\frac{\psi(x + a\hat{\mu}) - 2\psi(x) + \psi(x - a\hat{\mu})}{a^2} \right]$$

In momentum space, this becomes

$$S = \frac{4r}{a^2} \int_{BZ} \frac{d^4k}{(2\pi)^4} \; \bar{\psi}_{-k} \sin^2\left(\frac{k_\mu a}{2}\right) \psi_k$$

and we're left with the inverse propagator

$$D(k) = \gamma^\mu \sin(k_\mu a) + \frac{4r}{a} \sin^2\left(\frac{k_\mu a}{2}\right) \tag{4.32}$$

This now satisfies the first three of the four requirements above, with all the spurious fermions at $k_\mu = \pi/a$ lifted. The resulting dispersion relation is analogous to what we saw in $d = 1 + 1$ dimensions. The down side is that we have explicitly broken chiral symmetry, which can be seen by the lack of gamma matrices in the second term above. This becomes problematic when we consider interacting fermions, in particular when we introduce gauge fields. Under RG, we no longer enjoy the protection of chiral symmetry and expect to generate any terms which were previously prohibited, such as mass terms $\bar{\psi}\psi$ and dimension 5 operators $\bar{\psi}\gamma^\mu\gamma^\nu F_{\mu\nu}\psi$. Each of these must be fine tuned away, just like the mass of the scalar in Section 4.1.

Staggered Fermions

The final approach is to embrace the fermion doublers. In fact, as we will see, we don't need to embrace all 16 of them; only 4.

To see this, we need to return to the real space formalism. At each lattice site, we have a 4-component Dirac spinor $\psi(x)$. We denote the position of the lattice site as $x = a(n_1, n_2, n_3, n_4)$, with $n_\mu \in \mathbf{Z}$. We then introduce a new Dirac spinor $\chi(x)$, defined by

$$\psi(x) = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}\,\chi(x) \tag{4.33}$$

In the action (4.29), we have $\bar{\psi}(x)\,\gamma^\mu\,\psi(x \pm a\hat{\mu})$. Written in the χ variable, the term $\gamma^\mu\,\psi(x \pm a\hat{\mu})$ will have two extra powers of γ^μ compared to $\bar{\psi}(x)$; one from the explicit γ^μ out front, and the other coming from the definition (4.33). Since we have $(\gamma^\mu)^2 = +1$ in Euclidean space, we will find

$$\gamma^\mu\,\psi(x \pm a\hat{\mu}) = (-1)^{\text{some integer}}\gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}\chi(x \pm a\hat{\mu})$$

where the integer is determined by commuting various gamma matrices past each other. But this means that the integrand of the action has terms of the form

$$\bar{\psi}(x)\,\gamma^\mu\psi(x + a\hat{\mu}) = \eta_{x,\mu}\bar{\chi}(x)\chi(x + a\hat{\mu})$$

where there's been some more commuting and annihilating of gamma matrices going on, resulting in the signs

$$\eta_{x,1} = 1 \quad , \quad \eta_{x,2} = (-1)^{n_1} \quad , \quad \eta_{x,3} = (-1)^{n_1+n_2} \quad , \quad \eta_{x,4} = (-1)^{n_1+n_2+n_3}$$

The upshot is that the transformation (4.33) has diagonalised the action in spinor space. One can check that this same transformation goes through unscathed if we couple the fermion to gauge fields. This means that, on the lattice, we have

$$\det(i\not{D}) = \det{}^4(\hat{D})$$

for some operator $\hat{D}$. The operator $\hat{D}$ still includes contributions from the 16 fermions dotted around the Brillouin zone, but only one spinor index contribution from each. We may then take the fourth power and consider $\det(\hat{D})$ by itself. Perhaps surprisingly, one still finds a relativistic theory in the infra-red, with 4 of the 16 doublers providing the necessary spinor degrees of freedom.

Roughly speaking, you can think of the staggered fermions as arising from placing just a single degree of freedom on each lattice site. After doubling, we have 16 degrees of freedom living at the origin of momentum space and the corners of the Brillouin zone. The staggering trick is to recombine these 16 degrees of freedom back into 4 Dirac spinors. The idea that some subset of the fermion doublers may play the role of spin sounds strange at first glance, but is realised in $d = 2 + 1$ dimensions in graphene.

This staggered approach still leaves us with $16/4 = 4$ Dirac fermions. At high energy, these are coupled in a way which is distinct from four flavours in QCD. Nonetheless, it is thought that, when coupled to gauge fields, the continuum limit coincides with QCD with four flavours which, in this context, are referred to as *tastes*. The lattice theory has a $U(1) \times U(1)$ chiral symmetry, less than the $U(4) \times U(4)$ chiral symmetry of the (classical) continuum but still sufficient to prevent the generation of masses. This is a practical advantage of staggered fermions.

In fact, there are further reasons to be nervous about staggered fermions. As we've seen, the continuum limit results in 4 Dirac fermions. Let's call them $\psi_{\alpha i}$, where $\alpha = 1, 2, 3, 4$ is the spinor index and $i = 1, 2, 3, 4$ is the taste (flavour) index. However, these spinor and tase indices appear on the same footing in the lattice: both come from doubling. This suggests that they will sit on the same footing in the continuum limit. But that's rather odd. It means that, upon a Lorentz transformation Λ, the resulting Dirac spinors will transform as

$$\psi_{\alpha i} \to S[\Lambda]_\alpha{}^\beta S[\Lambda]_i{}^j \psi_{\beta j}$$

with $S[\Lambda]$ the spinor representation of the Lorentz transformation. (Since we're in Euclidean space, it is strictly speaking just the rotation group $SO(4)$.) The first term $S[\Lambda]_\alpha{}^\beta$ is the transformation property that we would expect of a spinor, but the second term $S[\Lambda]_i{}^j$ is very odd, since these are flavour indices. In particular, it means that if we rotate by 2π, we never see the famous minus sign acting on the staggered fermions. Instead we get two minus signs, one acting on the two indices, and the resulting object actually has integer spin!

What's going on here is that the object $\psi_{\alpha i}$ is really a bi-spinor, in the sense that both α and i are spinor indices. In representation theory language, a Dirac spinor transforms as $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The staggered fermions then transform in

$$[(\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2})] \otimes [(\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2})] = 2(0, 0) \oplus 2(\tfrac{1}{2}, \tfrac{1}{2}) \oplus (1, 0) \oplus (0, 1)$$

Here $(0, 0)$ are scalars, $(\frac{1}{2}, \frac{1}{2})$ is the vector representation, while $(1, 0)$ and $(0, 1)$ are the self-dual and anti-self-dual representations of 2-forms. In fact, formally, the collection of objects on the right can be written as a sum of forms of different degrees,

$$\Phi = \phi^{(0)} + \phi^{(1)}_\mu dx^\mu + \phi^{(2)}_{\mu\nu} dx^\mu \wedge dx^\nu + \phi^{(3)}_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho + \phi^{(4)}_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

where Poincaré duality means that the 4-form has the same degrees of freedom as a scalar and the 3-form the same as a vector. The 16 degrees of freedom that sit in the staggered fermions $\psi_{\alpha i}$ can then be rearranged to sit in Φ. Moreover, the Dirac equation on ψ has a nice description in terms of these forms; it becomes

$$(d - {*d*} + m)\Phi = 0$$

This is sometimes called a Dirac-Kähler field.

The upshot is that staggered fermions don't quite give rise to Dirac fermions, but a slightly more exotic object constructed in terms of forms. Nonetheless, this doesn't stop people using them in an attempt to simulate QCD, largely because of the numerical advantage that they bring. Given the discussion above, one might be concerned that this is not quite a legal thing to do and it is, in fact, simulating a different theory.

This is not the only difficulty with staggered fermions. The four tastes necessarily have the same mass meaning that, the problems above notwithstanding, staggered fermions do not allow us to get close to a realistic QCD theory, where the masses of the four lightest quarks are very different. To evade this issue, one sometimes attempts to simulate a single quark by taking yet another fourth-root, $\det^{1/4}(\hat{D})$. It seems clear that this does not result in a local quantum field theory. Arguments have raged about how evil this procedure really is.

4.4 Towards Chiral Fermions on the Lattice

A wise man once said that, when deciding what to work on, you should first evaluate the importance of the problem and then divide by the number of people who are already working on it. By this criterion, the problem of putting chiral fermions on the lattice ranks highly. There is currently no fully satisfactory way of evading the Nielsen-Ninomiya theorem. This means that there is no way to put the Standard Model on a lattice.

On a practical level, this is not a particularly pressing problem. It is the weak sector of the Standard Model which is chiral, and here perturbative methods work perfectly well. In contrast, the strong coupling sector of QCD is a vector-like theory and this is where most effort on the lattice has gone. However, on a philosophical level, the lack of lattice regularisation is rather disturbing. People will bang on endlessly about whether or not we live “the matrix’”, seemingly unaware that there are serious obstacles to writing down a discrete version of the known laws of physics, obstacles which, to date, no one has overcome.

In this section, I will sketch some of the most promising ideas for how to put chiral fermions on a lattice. None of them quite works out in full – yet – but may well do in the future.

4.4.1 Domain Wall Fermions

Our first approach has its roots in the continuum, which allows us to explain much of the basic idea without invoking the lattice. We start by working in $d = 4{+}1$ dimensions. The fifth dimension will be singled out in what follows, and we refer to it as $x^5 = y$.

In $d = 4 + 1$, the Dirac fermion has four components. The novelty is that we endow the fermion with a spatially dependent mass, $m(y)$

$$i\partial\!\!\!/\psi + i\gamma^5\partial_y\psi - m(y)\psi = 0 \tag{4.34}$$

where we pick the boundary conditions

$$m(y) \to \pm M \quad \text{as } y \to \pm\infty$$

with $M > 0$. We will take the profile $m(y)$ to be monotonic, with $m(y) = 0$ only at $y = 0$. A typical form of the mass profile is shown in the figure. Profiles of this kind often arise when we solve equations which interpolate between two degenerate vacua. In that context, they are referred to as *domain walls* and we’ll keep the same terminology, even though we have chosen $m(y)$ by hand.

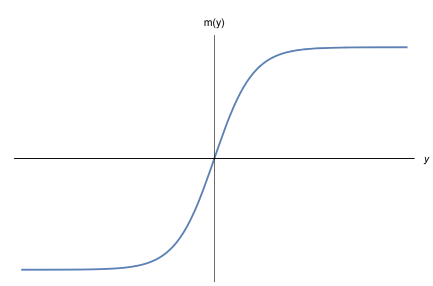

Figure 44:

The fermion excitation spectrum includes a continuum of scattering states with energies $E \geq M$ which can exist asymptotically in the y direction. At these energies, physics is very much five dimensional. But there are also states with $E < M$ which are bound to the wall. If we restrict to these energies then physics is essentially four dimensional. In this sense, the mass M can be thought of as an unconventional cut-off for the four dimensional theory on the wall.

In the chiral basis of gamma matrices,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \ i = 1,2,3 \quad , \quad \gamma^5 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

where the factors of i in γ^5 reflect the fact that we're working in signature $(+,-,-,-,-)$. The Dirac equation becomes

$$\begin{aligned} i\partial_0\psi_- + i\sigma^i\partial_i\psi_- - \partial_5\psi_+ &= m(y)\psi_+ \\ i\partial_0\psi_+ - i\sigma^i\partial_i\psi_+ + \partial_5\psi_- &= m(y)\psi_- \end{aligned}$$

where $\psi = (\psi_+, \psi_-)^T$. There is one rather special solution to these equations,

$$\psi_+(x,y) = \exp\left(-\int^y dy'\ m(y')\right)\chi_+(x) \quad \text{and} \quad \psi_-(x,y) = 0$$

The profile is supported only in the vicinity of the domain wall; it dies off exponentially $\sim e^{-M|y|}$ as $y \to \pm\infty$. Importantly, there is no corresponding solution for ψ_-, since the profile must be of the form $\exp\left(+\int dy' m(y')\right)$ which now diverges exponentially in both directions.

The two-component spinor $\chi_+(x)$ obeys the equation for a right-handed Weyl fermion,

$$\partial_0\chi_+ - \sigma^i\chi_+ = 0$$

We see that we can naturally localise chiral fermions on domain walls. The existence of this mode, known as a *fermion zero mode*, does not depend on any of the detailed properties of $m(y)$. We met a similar object in Section 3.3.4 when discussing the topological insulator.

This is interesting. Our original 5d theory had no hint of any chiral symmetry. But, at low-energies, we find an emergent chiral fermion and an emergent chiral symmetry.

Implications for the Lattice

So far, our discussion in this section has taken place in the continuum. How does it help us in our quest to put chiral fermions on the lattice?

The idea to apply domain wall fermions to lattice gauge theory is due to Kaplan. At first sight, this doesn't seem to buy us very much: a straightforward discretisation of the Dirac equation (4.34) shows that the domain wall does nothing to get rid of the doublers: in Euclidean space there are now 2^4 right-handed fermions χ_+, with the new modes sitting at the corners of the Brillouin zone as usual. Moreover, on the lattice one also finds a further 2^4 left-moving fermions χ_-. This brings us right back to a vector-like theory, with 2^4 Dirac fermions.

However, the outlook is brighter when we add a 5d Wilson term (4.32) to the problem. By a tuning the coefficient to lie within a certain range, we can not only remove all of the 16 left-handed fermions χ_-, but we can remove 15 of the 16 right-handed fermions. This leaves us with just a single right-handed Dirac fermion localised on the domain wall.

It is surprising that the Wilson term (4.32) can remove an odd number of gapless fermions from the spectrum since everything we learned up until now suggests that gapless modes can only be removed in pairs. But we have something new here, which is the existence of the infinite fifth dimension. This gives a novel mechanism by which zero modes can disappear: they can become non-normalisable.

There is an alternative way to view this. Suppose that we make the fifth direction compact. Then the domain wall must be accompanied by an anti-domain wall that sits at some distance L. While the domain wall houses a right-handed zero mode, the anti-domain wall has a left-handed zero mode. Now Nielsen-Ninomiya is obeyed, but the two fermions are sequestered on their respective walls, with any chiral symmetry breaking interaction suppressed by $e^{-L/a}$.

I will not present that analysis that leads to the conclusions above. But we will address a number of questions that this raises. First, what happens if we couple the chiral mode on the domain wall to a gauge field? Second, how has the single chiral mode evaded the Nielsen-Ninomiya theorem?

4.4.2 Anomaly Inflow

We have seen that a domain wall in $d = 4 + 1$ dimension naturally localises a chiral $d = 3 + 1$ fermion. This may make us nervous: what happens if we now couple the system to gauge fields?

At low energies, the only degree of freedom is the zero mode on the domain wall, so we might think it makes sense to restrict our attention to this. (We'll see shortly that things are actually a little more subtle.) Let us introduce a $U(1)$ gauge field everywhere in $d = 4+1$ dimensional spacetime, under which the original Dirac fermion ψ has charge $+1$.

We haven't yet discussed gauge theories in $d = 4 = 1$ dimensions, although we'll learn a few things below. The first statement we'll need is that there are no chiral anomalies in odd spacetime dimensions. This is because there is no analog of γ^5. We might, therefore, expect that a $U(1)$ gauge theory coupled to a single Dirac fermion is consistent in $d = 4 + 1$ dimensions. We will revisit this expectation shortly.

However, from a low energy perspective we seem to be in trouble, because there is a single massless chiral fermion χ_+ on the domain wall which has charge $+1$ under the gauge field. The fact that the gauge field extends in one extra dimension does not stop the anomaly which is now restricted to the region of the domain wall. Under the assumption that the zero mode is restricted to the $y = 0$ slice, the anomaly (3.34) for the gauge current

$$\partial_\mu j^\mu = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,\delta(y) \tag{4.35}$$

It is a factor of 1/2 smaller than the chiral anomaly for a Dirac fermion because we have just a single Weyl fermion. This is bad: if the $U(1)$ gauge field is dynamical then this is precisely the form of gauge anomaly that we cannot tolerate. Indeed, as we saw in (3.33), under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\omega(x,y)$, the measure for the 4d chiral fermion will transform as

$$\int \mathcal{D}\chi\mathcal{D}\bar{\chi} \;\longrightarrow \int \mathcal{D}\chi\mathcal{D}\bar{\chi}\;\exp\left(-\frac{i}{32\pi^2}\int d^4x\;\omega(x;0)\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right) \tag{4.36}$$

Fortunately, there is another phenomenon which will save us. Let's return to $d = 4+1$ dimensions. Far from the domain wall, the fermion is massive and we can happily integrate it out. You might think that as $m \to \infty$, the fermion simply decouples from the dynamics. But that doesn't happen in odd spacetime dimensions. Instead, integrating out a massive fermions generates a term that is proportional to sign(m),

$$S_{CS} = -\frac{k}{24\pi^2}\int d^5x\;\epsilon^{\mu\nu\rho\sigma\lambda}A_\mu\,\partial_\nu A_\rho\,\partial_\sigma A_\lambda \tag{4.37}$$

with

$$k = \frac{1}{2}\frac{m}{|m|}$$

This is a *Chern-Simons term* and k is referred to as the *level*. We will discuss the corresponding term in $d = 2+1$ dimensions in some detail in Section 8.4. We will also perform the analogous one-loop calculation in Section 8.5 and show how the Chern-Simons term, proportional to the sign of the mass, is generated when a Dirac fermion is integrated out. The calculation necessary to generate (4.37) is entirely analogous.

Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\omega$, the Chern-Simons action (4.37) transforms as

$$\delta S_{CS} = -\frac{k}{24\pi^2}\int d^5x\;\partial_\mu\left(\epsilon^{\mu\nu\rho\sigma\lambda}\omega\partial_\nu A_\rho\,\partial_\sigma A_\lambda\right)$$

This is a total derivative. Under most circumstances, we can simply throw this away. But there are some circumstances when we cannot, and the presence of a domain wall is one such an example. We take the thin wall limit, in which we approximate

$$\frac{m}{|m|} = \begin{cases} -1 & y < 0 \\ +1 & y > 0 \end{cases}$$

Since the level is now spatially dependent, we should put it inside the integral. After some integration by parts, we then find that the change of the Chern-Simons term is then

$$\begin{aligned} \delta S_{CS} &= -\frac{3}{24\pi^2}\int d^5x \; \frac{m}{2|m|}\partial_y\left(\omega\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu\,\partial_\rho A_\sigma\right) \\ &= +\frac{1}{8\pi^2}\int d^5x\; \delta(y)\,\omega\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu\,\partial_\rho A_\sigma \end{aligned}$$

We see that this precisely cancels the gauge transformation that comes from the chiral fermion (4.36). A very similar situation occurs in the integer quantum Hall effect, where a 2d chiral fermion on the boundary compensates the lack of gauge invariance of a $d = 2+1$ dimensional Chern-Simons theory in the bulk. This was described in the lectures on the Quantum Hall Effect.

We learn that the total theory is gauge invariant, but only after we combine two subtle effects. In particular, the anomalous current (4.35) on the domain wall is real. A low energy observer, living on the wall, would see that the number of fermions is not conserved in the presence of an electric and magnetic field. But, for a higher dimensional observer there is no mystery. The current is generated in the bulk (strictly speaking, at infinity) by the Chern-Simons term,

$$J^\mu = \frac{\delta S_{CS}[A]}{\delta A_\mu} = -\frac{1}{32\pi^2}\frac{m}{|m|}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\nu\rho}F_{\sigma\lambda}$$

The current is conserved in the bulk, but has a non-vanishing divergence on the domain wall where it is cancelled by the anomaly. This mechanism is referred to as *anomaly inflow*.

There is one final subtlety. I mentioned above that the five-dimensional Maxwell theory coupled to a single Dirac fermion is consistent. This is not quite true. Even in the absence of a domain wall, one can show that the 5d Chern-Simons (4.37) theory is invariant under large gauge transformations only if we take $k \in \mathbf{Z}$. (We'll explain why this is for 3d Chern-Simons theories in Section 8.4.) But integrating out a massive fermion gives rise to a half-integer k rather than integer. In other words, even in the absence of a domain wall the 5d theory is not quite gauge invariant. This doesn't invalidate our discussion above; we can simply need to add a bare Chern-Simons term with level $k = 1/2$ so that, after integrating out the massive fermion, the effective level is $k = 1$ when $y > 0$ and $k = 0$ when $y < 0$. (This discussion is slightly inaccurate: we'll have more to say on these issues in Section 8.5.)

4.4.3 The Ginsparg-Wilson Relation

We have not yet addressed exactly how the domain wall fermion evades the Nielsen-Ninomiya theorem. Here we explain the loophole. The idea that follows is more general than the domain wall, and goes by the name of *overlap fermions*.

Rather than jump straight to the case of a Weyl fermion, let's first go back and think about a Dirac fermion. We take the action in momentum space to be

$$S = \frac{1}{a}\int_{BZ} \frac{d^4k}{(2\pi)^4}\; \bar{\psi}_{-k}\, D(k)\, \psi_k$$

for some choice of inverse propagator $D(k)$. As explained in Section 4.3.4, the Nielsen-Ninomiya theorem can be cast as four criterion which cannot all be simultaneously satisfied by $D(k)$. One of these is the requirement that the theory has a chiral symmetry, in the guise of

$$\{\gamma^5, D(k)\} = 0$$

The key idea is to relax this constraint, but relax it in a very particular way. We will instead require

$$\{\gamma^5, D(k)\} = aD\,\gamma^5\, D \tag{4.38}$$

This is the *Ginsparg-Wilson relation*. Note the presence of the lattice spacing a on the right-hand-side. This means that in the continuum limit, which is naively $a \to 0$, we expect to restore chiral symmetry.

In fact, the Ginsparg-Wilson relation ensures that a chiral symmetry exists at all scales. However, it's rather different from the chiral symmetry that we're used to. It's simple to check that the action is invariant under

$$\delta\psi = i\gamma^5\left(1 - \frac{a}{2}D\right)\psi \quad , \quad \delta\bar{\psi} = i\bar{\psi}\left(1 - \frac{a}{2}D\right)\gamma^5 \tag{4.39}$$

These transformation rules have the strange property that the amount a fermion is rotated depends on its momentum. In real space, this means that the symmetry does not act in the same way on all points of the lattice. In the language of condensed matter physics, it is not an *onsite* symmetry. This will cause us a headache shortly.

So the Ginsparg-Wilson relation (4.38) is sufficient to guarantee a chiral symmetry, albeit an unconventional one. The next, obvious question is: what form of D obeys this relation? It's perhaps simplest to give a solution in the continuum, where $a = 1/M$ is simply interpreted as some high mass scale. You can check that, in real (Euclidean) space, the following operator obeys the Ginsparg-Wilson relation,

$$D = \frac{1}{a}\left(1 - \frac{1 - a\,\partial\!\!\!/}{\sqrt{1 - a^2\partial^2}}\right) \quad \Rightarrow \quad D(k) = \frac{1}{a}\left(1 - \frac{1 - i k\!\!\!/}{\sqrt{1 + a^2 k^2}}\right) \tag{4.40}$$

This is the overlap operator. It obeys the Hermiticity property $D^\dagger = \gamma^5 D \gamma^5$. At low momenta, $a\partial \ll 1$, we reproduce the usual Dirac operator,

$$D = \partial\!\!\!/ + \ldots$$

At high momentum, things look stranger. In particular, the derivatives in the denominator mean that this operator is non-local. However, it's not very non-local, and can be shown to fall off exponentially at large distances.

The Ginsparg-Wilson relation relies only on the gamma matrix structure of the operator (4.40). This means that we can also write down operators on the lattice, simply by replacing $\partial\!\!\!/$ by the operator appropriate for, say, Wilson fermions (4.32). Moreover, we can couple our fermions to gauge fields simply by replacing $\partial\!\!\!/$ with $D\!\!\!\!/$, or its lattice equivalent.

Next, we can try to use this chiral symmetry to restrict the Dirac fermion to an analog a Weyl fermion. Usually this is achieved by using the projection operators

$$P_\pm = \frac{1}{2}\left(1 \pm \gamma^5\right)$$

For overlap fermions, we need a different projection operator. This is

$$\hat{P}_\pm = \frac{1}{2}\left(1 \pm \gamma^5(1 - aD)\right)$$

You can check that this obey $P_\pm^2 = P_\pm$ and $P_\pm P_\mp = 0$, using the Ginsparg-Wilson relation (4.38). To write down the theory in terms of chiral fermions, we actually need both projection operators: the action can be expressed as

$$\begin{aligned} S &= \frac{1}{a}\int_{BZ} \frac{d^4k}{(2\pi)^4}\; \bar{\psi}_{-\mathbf{k}}(P_+ + P_-)\, D(\mathbf{k})\, (\hat{P}_+ + \hat{P}_-)\, \psi_{\mathbf{k}} \\ &= \frac{1}{a}\int_{BZ} \frac{d^4k}{(2\pi)^4}\; \left[\bar{\psi}_{-\mathbf{k}} P_+\, D(\mathbf{k})\, \hat{P}_+\, \psi_{\mathbf{k}} + \bar{\psi}_{-\mathbf{k}} P_-\, D(\mathbf{k})\, \hat{P}_-\, \psi_{\mathbf{k}}\right] \end{aligned}$$

Throwing away one of these terms can then be thought of as a chiral fermion. It can be shown that if one writes down a strict 4d action for the domain wall fermion, it takes a form similar to that above.

It seems like we have orchestrated a way to put a chiral fermion on the lattice, albeit with a number of concessions forced upon us by the strange Ginsparg-Wilson relation. So what's the catch? The problem comes because, although the action is invariant under (4.39), the measure is not. The measure for a Dirac fermion transforms as

$$\begin{aligned}\delta\left[\mathcal{D}\bar{\psi}\mathcal{D}\psi\right] &= \mathcal{D}\bar{\psi}\mathcal{D}\psi\ \text{Tr}\left[i\gamma^5\left(1-\frac{a}{2}D\right)+i\left(1-\frac{a}{2}D\right)\gamma^5\right]\\ &= \mathcal{D}\bar{\psi}\mathcal{D}\psi\ \text{Tr}\left[-ia\gamma^5 D\right]\end{aligned}$$

This now smells like the way the anomaly shows up in the continuum. Except here, the lack of invariance shows up even before we couple to gauge fields. If we also include gauge fields, and project onto a chiral fermions, then we run into trouble. In general, the measure will not be gauge invariant. This, of course, is the usual story of anomalies. However, now life has become more complicated, in large part because of the non-onsite nature of the chiral transformation. What we would like to show is that the measure remains gauge invariant if and only if the matter coupling does not suffer a gauge anomaly. This was studied in some detail by Lüscher. The current state of the art is that this technique can be shown to be consistent for Abelian, chiral gauge theories, but open questions remain in the more interesting non-Abelian case.

4.4.4 Other Approaches

There is one final assumption of the Nielsen-Ninomiya theorem that we could try to leverage in an attempt to put chiral fermions on the lattice: this is the assumption that the fermions are free, so that we can talk in terms of a one-particle dispersion relation. One might wonder if it's possible to turn on some interactions to lift collections of gapless fermions in a manner consistent with 't Hooft anomalies, while preserving symmetries which you might naively have thought should be broken. There has been a large body of work on this topic, which now goes by the name of *symmetric mass generation*, starting with Eichten and Preskill. It's interesting.

4.5 Further Reading

Kenneth Wilson is one of the more important figures in the development of quantum field theory. His work in the early 1970s on the renormalisation group, largely driven by the need to understand second order phase transitions in statistical physics, had an immediate impact on particle physics. The older ideas of renormalisation, due to Schwinger, Tomonaga, Feynman and Dyson, appeared to be little better than sweeping infinites under the carpet. Viewed through Wilson's new lens, it was realised that these infinities are telling us something deep about the way Nature appears on different length scales.

Wilson's pioneering 1974 paper on lattice gauge theory showed how to discretize a gauge theory, and demonstrated the existence of confinement in the strong coupling regime [214]. He worked only with $U(1)$ gauge group, although this was quickly generalised to a large number of non-Abelian gauge theories [8]. The Hamiltonian approach to lattice gauge theory was developed soon after by Kogut and Susskind [124].

In fact, Wilson was not the first to construct a lattice gauge theory. A few years earlier, Wegner described a lattice construction of what we now appreciate as $\mathbf{Z}_2$ gauge theory [201]. (Arguably, the first lattice gauge theory paper was earlier still: in 1966 an electrical engineer called Kane Yee wrote down a discrete version of Maxwell theory, with the gauge fields on the links, with the goal of solving the classical equations numerically rather than simulating the quantum theory [233].) The lattice continued to play a prominent role in many subsequent conceptual developments of quantum field theory, not least because such a (Hamiltonian) lattice really exists in condensed matter physics. Elitzur's theorem was proven in [52].

Wilson's original lattice gauge theory paper does not mention that a discrete version of the theory lends itself to numerical simulation, but this was surely on his mind. He later used numerical renormalisation group techniques [215] to solve the Kondo problem – a sea of electrons interacting with a spin impurity — which also exhibits asymptotic freedom [125]. It wasn't until the late 1970s that people thought seriously about simulating Yang-Mills on the lattice. The first Monte Carlo simulation of four dimensional Yang-Mills was performed by Creutz in 1980 [33].

More details on the basics of lattice gauge theory can be found in the book by Creutz [34] or the review by Guy Moore [138].

Fermions on the Lattice

Wilson introduced his approach to fermions, giving mass to the doublers at the corners of the Brillouin zone, in Erice lectures in 1975. To my knowledge, this has never been published. Other approaches soon followed: the discontinuous SLAC derivative in [48], and the staggered approach in [124]. The general problem of putting fermions on the lattice was later elaborated upon by Susskind [188]. The "rooting" trick, to reduce the number of staggered fermions, is prominently used in lattice simulations, but its validity remains controversial: see [180, 35] for arguments.

The idea that placing fermions on the lattice is a deep, rather than irritating, problem is brought into sharp focus by the theorem of Nielsen and Ninomiya [146, 147].

The story of domain wall fermions has its origins firmly in the continuum. Jackiw and Rebbi were the first to realise that domain walls house chiral fermions [111], a result which now underlies the classification of certain topological insulators. The interaction of these fermions with gauge fields was studied by Callan and Harvey who introduced the idea of anomaly inflow [25]. The fact that this continuum story can be realised in the lattice setting was emphasised by David Kaplan [117].

In a parallel development, the Ginsparg-Wilson relation was introduced in [73] in a paper that sat unnoticed for many years. Maybe it would have helped if the authors were more famous. The first (and, to date, only) solution to this relation was discovered by Neuberger [143, 144], and the resulting exact chiral symmetry on the lattice was shown by Lüscher [126]. The relationship between domain wall fermions and the Ginsparg-Wilson relation was shown in [121].

The idea that strong coupling effects could lift the fermion doublers, in a way consistent with ('t Hooft) anomalies, was first suggested by Eichten and Preskill [51]. This subject has had a renaissance of late, starting with the pioneering work of Fidkowski and Kitaev on interacting 1d topological insulators [58, 59]. They show that Majorana zero modes can be lifted, preserving a particular time reversal symmetry, only in groups of 8. There are more conjectural extensions to higher dimensions where, again, it is thought that only specific numbers of fermions can be gapped together. In $d = 3 + 1$, the conjecture is that Weyl fermions can become gapped in groups of 16; the fact that the Standard Model (with a right-handed neutrino) has $16n$ Weyl fermions has not escaped attention [202, 234].

The lectures by Witten on topological phases of matter include a clear discussion of the Nielsen-Ninomiya theorem [229]. Excellent reviews on the issues surrounding chiral fermions on the lattice have been written by Lüscher [127] and Kaplan [118].