8. Quantum Field Theory on the Plane

In this section, we step up a dimension. We will discuss quantum field theories in $d = 2 + 1$ dimensions. Like their $d = 1 + 1$ dimensional counterparts, these theories have application in various condensed matter systems. However, they also give us further insight into the kinds of phases that can arise in quantum field theory.

8.1 Electromagnetism in Three Dimensions

We start with Maxwell theory in $d = 2 = 1$. The gauge field is A_μ, with $\mu = 0, 1, 2$. The corresponding field strength describes a single magnetic field $B = F_{12}$, and two electric fields $E_i = F_{0i}$. We work with the usual action,

$$S_{\text{Maxwell}} = \int d^3x \;\; -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + A_\mu j^\mu \tag{8.1}$$

The gauge coupling has dimension $[e^2] = 1$. This is important. It means that $U(1)$ gauge theories in $d = 2 + 1$ dimensions coupled to matter are strongly coupled in the infra-red. In this regard, these theories differ from electromagnetism in $d = 3 + 1$.

We can start by thinking classically. The Maxwell equations are

$$\frac{1}{e^2}\partial_\mu F^{\mu\nu} = j^\nu$$

Suppose that we put a test charge Q at the origin. The Maxwell equations reduce to

$$\nabla^2 A_0 = Q\,\delta^2(\mathbf{x})$$

which has the solution

$$A_0 = \frac{Q}{2\pi}\log\left(\frac{r}{r_0}\right) + \text{constant}$$

for some arbitrary r_0. We learn that the potential energy $V(r)$ between two charges, Q and $-Q$, separated by a distance r, increases logarithmically

$$V(r) = \frac{Q^2}{2\pi}\log\left(\frac{r}{r_0}\right) + \text{constant} \tag{8.2}$$

This is a form of confinement, but it's an extremely mild form of confinement as the log function grows very slowly. For obvious reasons, it's usually referred to as *log confinement*.

In the absence of matter, we can look for propagating degrees of freedom of the gauge field itself. As explained in the previous section, we expect the gauge field to have a single, propagating polarisation state in $d = 2 + 1$ dimensions.

8.1.1 Monopole Operators

Something special happens for $U(1)$ gauge theories in $d = 2 + 1$ dimensions: they automatically come an associated global $U(1)$ symmetry that we will call $U(1)_{\rm top}$, the "top" for "topological". The associated current is

$$J^\mu_{\rm top} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} \tag{8.3}$$

which obeys the conservation condition $\partial_\mu J^\mu_{\rm top} = 0$ by the Bianchi identity on $F_{\mu\nu}$. The associated conserved quantity is simply the magnetic flux

$$Q_{\rm top} = \int d^2x\ J^0_{\rm top} = \frac{1}{2\pi}\int d^2x\ B$$

In quantum field theory, symmetries act on local operators. The operators that transform under $U(1)_{\rm top}$ are not the usual fields of the theory. Rather, they are *disorder operators*, entirely analogous to the 't Hooft lines that we introduced in Section 2.6. In the present context, they are referred to as *monopole operators*.

We work in Euclidean space. A monopole operator $\mathcal{M}(x)$ inserted at a point $x \in \mathbf{R}^3$ is defined in the path integral by requiring that we integrate over field configurations in which there is a Dirac monopole inserted inserted at the point x. This means that, for an $\mathbf{S}^2$ surrounding x, we have

$$\frac{1}{4\pi}\int_{\mathbf{S}^2} d^2S_\mu\ \epsilon^{\mu\nu\rho}F_{\nu\rho} = 1 \tag{8.4}$$

This operator creates a single unit of magnetic flux so that, in the presence of $\mathcal{M}(x)$, the topological current is no longer conserved; instead it has a source

$$\partial_\mu J^\mu_{\rm top} = \delta^3(x) \tag{8.5}$$

Equivalently, the monopole operator is charged under $U(1)_{\rm top}$ so that

$$U(1)_{\rm top} : \mathcal{M}(x) \mapsto e^{i\alpha}\mathcal{M}(x) \tag{8.6}$$

The definition of monopole operators given above is somewhat abstract. As we will now see, in certain phases of the theory it is possible to give a more concrete definition.

Consider free Maxwell theory. Alternatively, consider $U(1)$ gauge theory coupled to charged fields with masses $m \gg e^2$. In both cases, the theory lies in the *Coulomb phase*, meaning that low energy spectrum contains just a single, free massless photon. The partition function is particularly straightforward; ignoring gauge fixing terms, we have

$$Z = \int \mathcal{D}A_\mu \ \exp\left(-\int d^3x \ -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}\right)$$

Because the action depends only on $F_{\mu\nu}$, and not explicitly on A_μ, we can choose instead to integrate over the field strength. However, we shouldn't integrate over all field strengths; in the absence of monopole operators, we should integrate only over those that satisfy the Bianchi identity $\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} = 0$. We can do this by introducing a Lagrange multiplier field $\sigma(x)$,

$$Z = \int \mathcal{D}F_{\mu\nu}\mathcal{D}\sigma \ \exp\left(-\int d^3x \ -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{4\pi}\sigma\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}\right) \tag{8.7}$$

If the field strength obeys the Dirac quantisation condition, then σ has periodicity 2π. But in this formulation, it is particularly straightforward to implement a monopole operator. We simply add to the path integral

$$\mathcal{M}(x) \sim e^{i\sigma(x)} \tag{8.8}$$

This ensures that the topological current has a source (8.5) or, equivalently, inserts a monopole at x.

We can now go one step further, and integrate out the field strength $F_{\mu\nu}$. We're left with an effective action for the Lagrange multiplier field $\sigma(x)$ which, in this context, is usually referred to as the *dual photon*. We're left with the effective action,

$$Z = \exp\left(-\int d^3x \ \frac{e^2}{8\pi^2}\,\partial_\mu\sigma\partial^\mu\sigma\right) \tag{8.9}$$

Clearly this describes a single, propagating degree of freedom. But this is what we expect for a photon in $d = 2+1$ which has just a single polarisation state.

In this formulation, the global symmetry $U(1)_{\rm top}$ is manifest, and is given by

$$U(1)_{\rm top} : \sigma \mapsto \sigma + \alpha \tag{8.10}$$

This agrees with our expected symmetry transformation (8.6) given the identification (8.8). The associated current can be read off from (8.9); it is

$$J^\mu_{\rm top} = \frac{e^2}{(2\pi)^2}\,\partial^\mu\sigma \tag{8.11}$$

There's one, nice twist to this story. The theory (8.9) has a degeneracy of ground states, given by constant $\sigma \in [0, 2\pi)$. These degenerate ground states reflects the fact that if we place some magnetic flux in the Coulomb phase then it spreads out. In any of these ground states, the global symmetry $U(1)_{\rm top}$ acts like (8.10) and so is spontaneously broken. The associated Goldstone boson is simply σ itself. But this is equivalent to the original photon. We have the chain of ideas

$$\begin{aligned} \text{Coulomb Phase}: \quad \text{Unbroken } U(1)_{\rm gauge} \quad &\Leftrightarrow \quad \text{Spontaneously Broken } U(1)_{\rm top} \\ &\Leftrightarrow \quad \text{Goldstone Mode} = \text{Photon} \end{aligned}$$

A related set of ideas also holds in higher dimensions, but now with the $U(1)_{\rm top}$ a generalised symmetry, which acts on higher dimensional objects, as we discussed in Section 3.6.2. $d = 2 + 1$ dimensions is special because the disorder operator $\mathcal{M}(x)$ is a local operator, ensuring that $U(1)_{\rm top}$ is a standard global symmetry, rather than the less familiar generalised symmetry.

8.2 The Abelian-Higgs Model

We can get some more intuition for the role of monopole operators, and 3d gauge theories in general, by looking at the Abelian-Higgs model. This is a $U(1)$ gauge theory coupled to a scalar field ϕ which we take to have charge 1. The action is

$$S_{AH} = \int d^3x \;\; -\frac{1}{4e^2} F_{\mu\nu}^2 + |\mathcal{D}_\mu \phi|^2 - m^2 |\phi|^2 - \frac{\lambda}{2} |\phi|^4 \tag{8.12}$$

We will look at what happens to this theory as we vary the mass m^2 from positive to negative. This is a game that we've already played in both $d = 3 + 1$ dimensions (in Section 2.5.2) and in $d = 1 + 1$ dimensions (in Section 7.2). In both cases, the interesting physics came from vortices in the $m^2 < 0$ phase, and the same will be true here.

When the mass is small, $|m| \ll e^2$, the theory is strongly coupled in the infra-red. It is difficult to get a handle on the physics here, although we will ultimately be able understand what happens. In contrast, when $|m| \gg e^2$, we can first understand the dynamics of the scalar in a regime where the gauge field is weakly coupled, and then figure out what's left. We first look at these two phases.

$\underline{m^2 \gg e^4}$: When $m^2 > 0$ we can simply integrate out the scalar, to leave ourselves with free Maxwell theory below the scale of m^2. This is the gapless Coulomb phase, in which we have an unbroken $U(1)$ gauge symmetry. As we explained above, this means that the global symmetry $U(1)_{\rm top}$ is spontaneously broken. The Goldstone mode is the photon.

There are also massive, charged excitations in this phase that come from the ϕ field. They interact through the Coulomb force which means that charges of opposite sign experience a logarithmically confining potential (8.2).

$\underline{m^2 \ll -e^4}$: This is the *Higgs phase*. The scalar condenses,

$$|\phi|^2 = -\frac{m^2}{\lambda}$$

giving the photon a mass. This phase is gapped. The $U(1)_{\text{gauge}}$ symmetry is spontaneously broken. But now the global topological symmetry $U(1)_{\text{top}}$ is unbroken.

The finite energy states of the theory which carry non-vanishing Q_{top} charge are the vortices. We discussed these in detail in both $d = 3 + 1$ dimensions where the vortices are strings (see section 2.5.2) and in $d = 1 + 1$ dimensions where the vortices are instantons (see Section 7.2). In $d = 2 + 1$, vortices are particle-like excitations. They are classical configurations in which the phase of ϕ winds asymptotically in the spatial plane $\mathbf{R}^2$. They have finite energy, and finite quantised magnetic flux

$$\oint dx^i\ \partial_i \phi = \frac{1}{2\pi} \int d^2x\ B = Q_{\text{top}} \in \mathbf{Z}$$

This is what monopole operators do in the Higgs phase: they create vortices. The upshot is that we can characterise the Higgs phase of the theory as

$$\begin{aligned} \text{Higgs Phase}: \quad \text{Spontaneously broken } U(1)_{\text{gauge}} \quad &\Leftrightarrow \quad \text{Unbroken } U(1)_{\text{top}} \\ &\Leftrightarrow \quad \text{Charged Excitation} = \text{Vortex} \end{aligned}$$

$\underline{m^2 = 0}$: In $d = 2 + 1$, the two phases at $m^2 > 0$ and $m^2 < 0$ are clearly different since they have a different global symmetry $U(1)_{\text{top}}$. (This is in contrast to the story in $d = 1 + 1$ where vortices are instantons and blur the distinction between the two phases.)

We can ask: what happens as we dial m^2 from positive to negative. We expect a phase transition to occur at some point, we which we heuristically refer to as $m^2 = 0$. (In practice, this point can be shifted away from zero). Is this a first order phase transition, or second order? If second order, what universality class does the theory lie in? Because the theory is strongly coupled in the regime $|m| \lesssim e^2$ it is difficult to perform any quantitative calculations to answer this question. Instead, we will guess.

To guide our guess, we use the symmetries of the problem. Since we have identified the global $U(1)_{\text{top}}$ symmetry as distinguishing phases, it seems reasonable to postulate that the phase transition lies in the same universality class as other theories governed by a $U(1)$ global symmetry. This turns out to be true, and underlies a rather beautiful feature of 3d gauge theories known as particle-vortex duality.

8.2.1 Particle-Vortex Duality

In quantum field theories, there are very often two kinds of particle excitations that can appear. The first kind is the familiar excitation that we get when we quantise a local field. This is that kind that we learned about in our Quantum Field Theory course. The second kind we've seen a number of times in these lectures: they are solitons.

Despite the fact that these two kinds of particles arise in different ways, there is really little difference between them in the quantum theory. In particular, both are described as states in the Fock space. Typically at weak coupling, the solitons are much heavier than the "elementary particles", but that's more a limitation of our need to work at weak coupling. It may be – and often is – that as we move into strongly coupled regimes, the solitons become light.

This opens up an intriguing possibility. Is it possible to write down a different quantum field theory in which the roles of solitons and elementary particles are reversed. These two quantum field theories would describe the same physics, but what appears as a soliton in one would appear as an elementary particle in the other, and vice versa. This is referred to a *duality*.

In fact, we've already met a simple example of a duality in these lectures. In Section 7.5, we used bosonization to demonstrate the equivalence between a massive fermion and the Sine-Gordon model. The elementary fermion arises as a kink in the Sine-Gordon model.

Typically, dualities get harder to construct with any conviction as the number of dimensions increases. There wonderful examples of dualities in $d = 3 + 1$, which exchange electric and magnetic excitations, but they need supersymmetry to keep control over the dynamics and so are beyond the scope of these lectures. However, things are somewhat easier in $d = 2 + 1$. Here we do have examples of dualities. In contrast to the bosonization story of Section 7.5, we are unable to prove the $d = 2 + 1$ dualities from first principles, but nonetheless have convincing evidence that they are true. We will see a number of these dualities as we proceed.

As we've seen above, in $d = 2+1$ dimensions the appropriate solitons are vortices. We will now propose a second theory, whose classical dynamics is different from the Abelian-Higgs model (8.12), but whose quantum dynamics is argued to be identical. The vortices in one theory are identified with the elementary particles of the other. For this reason, the claimed equivalence of the two theories is referred to as *particle-vortex duality*.

The XY-Model

The theory which is claimed to be dual to the 3d Abelian-Higgs model is simply a theory of a complex scalar field $\tilde{\phi}$, without any gauge field,

$$S_{XY} = \int d^3x \; |\partial_\mu \tilde{\phi}|^2 - \tilde{m}^2 |\tilde{\phi}|^2 - \frac{\tilde{\lambda}}{2} |\tilde{\phi}|^4 \tag{8.13}$$

This is known as the *XY-Model*. At first glance, the physics of this model is rather different from the XY-model. Indeed, at first glance it appears to have fewer degrees of freedom because it is missing the gauge field. Nonetheless, as we now explain, they describe the same physics, albeit in a non-obvious and interesting way.

Let's first address the issue of degrees of freedom. The XY-model clearly has two degrees of freedom in the UV where it is weakly coupled. But the Abelian-Higgs model has the same number: the gauge redundancy removes one degree of freedom from ϕ, but this is replenished by the single polarization state of the photon. We learn an interesting lesson: gauging a $U(1)$ symmetry in $d = 2+1$ changes the dynamics, but does not change the overall number of degrees of freedom. This will be important in later developments.

We can also match the symmetries between the XY-model and the Abelian-Higgs model. The XY-model clearly has a $U(1)$ global symmetry which rotates the phase of $\tilde{\phi}$. The associated current is

$$J^\mu_{XY} = i \left(\tilde{\phi}^\dagger \partial^\mu \tilde{\phi} - (\partial^\mu \tilde{\phi}^\dagger) \tilde{\phi} \right)$$

The Abelian-Higgs model also has a single global symmetry that we called $U(1)_{\rm top}$. You might worry that the Abelian-Higgs model also has a gauge symmetry, which is clearly not shared by the XY-model. But, as we have stressed many times, gauge symmetries are not symmetries at all, but redundancies. This gives another important lesson: there is no need for gauge symmetries to match on both sides of a duality.

We can now look at how the physics of the XY-model changes as we vary the mass:

$\underline{\tilde{m}^2 > 0}$: This is a gapped phase. The $\tilde{\phi}$ excitations are massive and carry charge under the unbroken $U(1)$ global symmetry. We see that, at least with broad brush, this looks similar to the the Higgs phase of the Abelian-Higgs model, in which the $U(1)_{\rm top}$ symmetry was unbroken. In that case, the vortices carried charge under $U(1)_{\rm top}$.

$\underline{\tilde{m}^2 < 0}$: In this phase, $\tilde{\phi}$ gets a vacuum expectation value and the $U(1)$ global symmetry is broken. We can write $\tilde{\phi} = \rho e^{i\sigma}$. The fluctuations of ρ are massive, while the σ field is massless: it is the Goldstone mode for the broken $U(1)$. Notice that we've given this field the same name as the dual photon in the Abelian-Higgs model. This is not a coincidence.

Again, with broad brush this looks similar to the gapless Coulomb phase of the Abelian-Higgs model. However, the Coulomb phase was also characterised by the existence of massive, charged ϕ excitations that were logarithmically confined. Can we see similar excitations in the XY-model? The answer is yes.

The ordered phase of the XY-model also has vortices. As before, these arise from the phase of $\tilde{\phi}$ winding asymptotically, but now there is no gauge field to cancel the log divergence in their energy,

$$\int d^2x\ |\partial_i \tilde{\phi}|^2 = \int d\theta dr\ r\ \frac{1}{r^2}|\partial_\theta \tilde{\phi}|^2 + \ldots = 2\pi \int_0^\infty dr\ \frac{n^2}{r}|\tilde{\phi}|^2 + \ldots$$

The energy of a single vortex is logarithmically divergent. But this divergence can be cancelled by placing an anti-vortex at some distance r. It's not hard to convince yourself that the logarithm reappears in the potential energy between the vortex and anti-vortex, which scales as

$$V = \frac{1}{2\pi} \log\left(\frac{r}{r_0}\right)$$

for some cut-off r_0. In other words, the vortices are logarithmically confined. This, of course, is the same behaviour exhibited by charged particles in 3d electromagnetism.

$\underline{\tilde{m}^2 = 0}$: Lying between the two phases above is a critical point. Once again, we are being a little careless in describing this as sitting at $\tilde{m} = 0$; strictly, you should tune both $\tilde{m}$ and the other parameters to hit the critical point.

This time, the physics of the critical point is well understood: this is the XY Wilson-Fischer fixed point. We studied this in some detail in the lectures on Statistical Field Theory using the epsilon expansion.

The essence of particle-vortex duality is the claim that the Abelian-Higgs model also flows to the XY Wilson-Fisher fixed point at $m = 0$. This claim can be traced back to work of Peskin in the 1970s, but was brought to prominence by Dasgupta and Halperin in the early 1980s. Given the similarity in their phase structure, this would seem to be a reasonable claim. There is currently no proof of the duality, but there is now convincing numerical evidence that it is true.

The Duality Dictionary

The key to particle-vortex duality is really the idea of universality: the two theories (8.12) and (8.13) share the same critical point. We can then attempt to map the operators of the two theories at the critical point. We have only an incomplete dictionary at the moment, but our discussion above allows us to start to fill in some entries. For example, we have seen how the currents match on both sides

$$J^\mu_{\rm top} \quad \longleftrightarrow \quad J^\mu_{XY}$$

With two theories flowing to the same critical point, we can now turn on relevant operators in each. As long as we turn on the same relevant operator, we are guaranteed that the theories coincide in the neighbourhood of the fixed points. We have seen above how this plays out: when the scalar condenses in one theory, it matches the phase in which the scalar is not condensed in the other. Roughly speaking, we have

$$m^2 \approx -\tilde{m}^2$$

Alternatively, we can write this in terms of the relevant operators at the critical point as

$$|\phi|^2 \quad \longleftrightarrow \quad -|\tilde{\phi}|^2 \tag{8.14}$$

although since the critical points are strongly coupled, this relation is likely to have corrections, with operators on both sides mixing with others.

Far from the critical point, we have seen that the theories have the same qualitative features. In particular, the duality inherits its name from the map between massive excitations,

$$\begin{array}{rcl} \text{gauge vortex} & \longleftrightarrow & \tilde{\phi}\ \text{excitation} \\ \phi\ \text{excitation} & \longleftrightarrow & \text{global vortex} \end{array}$$

Only the first of these describes a map between finite energy excitations. In this case, it is better to phrase the map in terms of local operators, rather than solitons: the essence of particle-vortex duality is that the monopole operator on one side is a traditional field in the Lagrangian on the other,

$$\mathcal{M}(x) \quad \longleftrightarrow \quad \tilde{\phi}(x) \tag{8.15}$$

We could ask: do the interactions between these massive excitations agree in detail? The answer is most likely no. One could add irrelevant operators to both the Abelian-Higgs model and the XY model which will affect the interactions between these massive particles. We would have to work much harder to get quantitative agreement away from the critical point. For what it's worth, it is possible to do this matching in certain supersymmetric versions of the duality. Here, particle-vortex duality is referred to as *3d mirror symmetry*.

The View from Statistical Physics

The claim of particle-vortex duality offers a very clear experimental prediction. Although we have phrased our discussion in the context of physics in $d = 2+1$ dimensions, everything goes through in the the Euclidean $d = 3+0$ world. Here, the theories (8.12) and (8.13) can be viewed as statistical field theories, with the path integral describing thermal rather than quantum fluctuations. More details can be found in the lecture notes on Statistical Field Theory.

In this context, the 3d XY-model (8.13) governs the phase transition of a number of systems, including the superfluid transition of liquid helium. Similarly, the 3d Abelian-Higgs model (8.12) governs the superconducting phase transition, with the field strength F_{ij}, $i, j = 1, 2, 3$ describing the fluctuating magnetic field.

In both cases, the mass^2 term determines the deviation from the critical temperature T_c at which the phase transition occurs. But that makes the map (8.14) between the masses rather surprising. It means that the duality maps the high temperature phase of the superfluid to the low temperature phase of the superconductor, and vice versa.

The claim that both theories share a critical point then becomes the claim that the two phase transitions have the same critical exponents. Experimentally, however, this claim is incorrect: the two phase transitions are *not* the same. While the superfluid transition exhibits the XY Wilson-Fisher exponents, the superconducting transition has mean field exponents. It would seem that particle-vortex duality has been ruled out experimentally!

In fact this is too quick. Recall that the XY-model has two critical points. The mean field critical point is unstable, with $|\phi|^4$ a relevant operator that drives the theory to the Wilson-Fisher point. The same should be true of the Abelian-Higgs model. It is thought that the mean field exponents seen in the superconducting transition reflect the fact that the experiments haven't got close enough to the true critical point, and are instead probing the unstable mean field point. Calculations suggest that one would start to see Wilson-Fisher critical exponents in the superconducting transition only at $T - T_c \sim 10^{-9}$ K. Such a level of precision is not technologically feasible.

But this brings its own issues. It appears that we have a system in Nature which is fine-tuned. The natural scale of the superconducting phase transition is $T_c \sim 10$ K or so. In the experiments, we tune the coefficient of $|\phi|^2$ by hand to hit the critical temperature. But why is the coefficient of the $|\phi|^4$ relevant operator so small that it only shows up when $T - T_c \sim 10^{-9}$ K? This is similar to the famous hierarchy problem in the Standard Model, where again the coefficient of a relevant operator appears to be fine-tuned.

Particle physicists have sleepless nights over fine tuning, and desperately search for an explanation. In large part, this is because of experience with RG in statistical physics, where any fine-tuning seen in Nature must also have an explanation. In the case of superconductors, the apparent fine tuning is understood: it arises because the underlying scalar field ϕ is not fundamental, but instead comprises of a Cooper pair of electrons. (The analogous possibility for the Higgs fine tuning goes by the name of technicolour.) A full explanation would take us too far from the purpose of these lectures, but this suffices to ensure that the smallness of the $|\phi|^4$ relevant operator seen in the superconducting transition is technically natural.

8.3 Confinement in $d = 2 + 1$ Electromagnetism

We've seen that classical electromagnetism in $d = 2 + 1$ dimensions confines particles, but only weakly with a log potential

$$V(r) = \frac{Q^2}{2\pi} \log\left(\frac{r}{r_0}\right)$$

There is, however, an important effect in the quantum theory that turns the logarithmic confining potential into a more powerful linearly confining potential. This effect, first discovered by Polyakov, is due to instantons.

We've met instantons in $d = 3 + 1$ Yang-Mills theory in Section 2.3, and again in the $d = 1 + 1$ Abelian-Higgs model in Section 7.2. In the latter case, vortices that play the role of instantons. Now that we are living in $d = 2+1$ dimensions, the instantons should be objects localised in three Euclidean dimensions. But these are very familiar: they are magnetic monopoles.

We've already introduced the idea of monopole operators in Section 8.1. These can be thought of Dirac monopoles at a point. They are not quite what we want for the present purposes. As a starting point for a semi-classical calculation, we would like the monopoles to be smooth configurations with finite action. But we've seen such objects before: we can use the 't Hooft Polyakov monopole described in Section 2.8.

Recall that the 't Hooft Polyakov monopoles arise in an $SU(2)$ gauge theory (or, more generally, any non-Abelian gauge theory) broken down to its Cartan subalgebra. To achieve this, we couple the $SU(2)$ gauge theory to a real, adjoint scalar ϕ and work with the action

$$S = \int d^3x \; -\frac{1}{2g^2}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{1}{g^2}\mathrm{tr}\,(\mathcal{D}_\mu\phi)^2 - \frac{\lambda}{4}\left(\mathrm{tr}\,\phi^2 - \frac{v^2}{2}\right)^2 \tag{8.16}$$

The ground state of the system has, up to a gauge transformation, $\phi = v\sigma^3$, and breaks the gauge symmetry

$$SU(2) \to U(1)$$

At low energies, the spectrum contains just a single massless photon and looks like pure electromagnetism. In addition, there is a neutral scalar with mass $\sim \sqrt{\lambda} g v$ and a charged W-boson of mass $\sim v$.

In this way, we can view the model as $U(1)$ gauge theory, with a UV cut-off at the scale v. The dimensionless gauge coupling constant is g^2/v and to trust any semi-classical calculation, we must take $g^2/v \ll 1$.

8.3.1 Monopoles as Instantons

Our main reason for introducing the action (8.16) is that, in Euclidean spacetime, it admits smooth monopole solutions. These are the 't Hooft Polyakov monopoles that we introduced in Section 2.8, but now localised in Euclidean spacetime meaning that they play the role of instantons, rather than particles. Here we recount the basics.

The existence of the monopoles can be traced to topology. Any finite action configuration must obey $\mathrm{tr}\,\phi^2 \to v^2$ as $x \to \infty$. This defines a sphere $\mathbf{S}^2$ in field space, so all finite action configurations are classified by a winding number $\Pi_2(\mathbf{S}^2) = \mathbf{Z}$, defined as

$$\nu = \frac{1}{8\pi v^3}\int_{\mathbf{S}^2_\infty} d^2S_i \; \epsilon^{ijk}\epsilon_{abc}\phi^a\partial_j\phi^b\partial_k\phi^c \in \mathbf{Z} \tag{8.17}$$

However, winding comes at a cost. Any purely scalar configuration that winds has linearly divergent action. This can be compensated by turning on a gauge field and this, in turn, endows the soliton with magnetic charge in the unbroken $U(1) \subset SU(2)$, (2.91),

$$m = -\frac{1}{v}\int d^2S_i\ \frac{1}{2}\epsilon^{ijk}\,\mathrm{tr}\,(F_{jk}\phi) = 4\pi\nu$$

The solution for a single monopole, with winding $\nu = 1$, has asymptotic form

$$\phi^a \to v\frac{x^a}{r} \quad \text{and} \quad A_i^a \to -\epsilon_{aij}\frac{x^j}{r^2} \quad \text{as } x \to \infty$$

The action of this configuration is finite, and given by

$$S_{\text{mono}} = \frac{8\pi v}{g^2}\, f(\lambda g^2)$$

with $f(\lambda g^2)$ a monotonically increasing function. It has the property that $f(0) = 1$, so that the action above coincides with that of a BPS monopole (2.93) when $\lambda = 0$.

We're used to the idea that finite action configurations in Euclidean space tunnel between different vacua of the theory. But what vacua does the monopole tunnel between? Clearly, it changes the magnetic flux $\Phi = \int d^2x\ B$ on a spatial slice. If we were living on a compact space, this would change the energy of a state, which is given by

$$\Delta E = \int d^2x\ \frac{1}{2}B^2 \sim \frac{1}{2}\,\text{Area}\left(\frac{\Phi}{\text{Area}}\right)^2$$

with "Area" the area of a spatial slice. However, as the area tends to infinity, the flux is suitably diluted and the cost in energy is vanishingly small. These are the different vacua that the monopoles tunnel between.

A Dilute Gas of Monopoles and Anti-Monopoles

With our monopole solution in hand, we can use it as the starting point for a semi-classical evaluation of the path integral. We should be getting used to this by now, and we follow the structure of the calculation laid out in Section 2.3, and again in Section 7.2.

One key step in the calculation is to invoke the use of a dilute gas of instantons. In the present case, this means we treat configurations of widely separated monopoles and anti-monopoles, with magnetic charges $m_i = \pm 4\pi$, as saddle points in the path integral. In the previous situations, we argued that the action of a dilute gas of N (anti)-instantons was roughly $S \approx NS_{\text{inst}}$, reflecting the fact that these are approximate solutions when the objects are far separated.

For monopoles, however, we should treat this step more carefully. Viewed as particles in $d = 3 + 1$ dimensions, we know that the energy will pick up contributions from the long range Coulomb forces between the monopoles. This translates into a contribution to the action in our context. If a monopole of charge $m_i = \pm 4\pi$ sits at position X_i, the total action will be

$$S = S_{\rm mono} \sum_i \left(\frac{m_i}{4\pi}\right)^2 + \frac{1}{4\pi g^2} \sum_{i\neq j} \frac{m_i m_j}{|X_i - X_j|}$$

where the second term reflects the long range Coulomb interaction.

We evaluate the path integral by summing over these dilute gas configurations, containing N constituents of either type. This results in the expression,

$$Z = \sum_{N=0}^{\infty} \sum_{m_i = \pm 4\pi} \frac{1}{N!} \left(K e^{-S_{\rm mono}}\right)^N \int \prod_{i=1}^{N} d^3 X_i \ \exp\left(-\frac{1}{8\pi g^2} \sum_{i\neq j} \frac{m_i m_j}{|X^i - X^j|}\right) \quad (8.18)$$

Here K is the usual contribution from one-loop determinants and Jacobian factors. We could compute it, but it does not give any qualitatively new insights into the physics so we will not. The second factor in the expression above is the novelty. When the instantons are non-interacting, this just gives a power of V^N to the path integral, with V the spacetime volume. Now that we have long range interactions between the instantons, we must work a little harder.

There is a useful way to rewrite the final expression. We use the fact that the $1/r$ factor also arises in the Green's function of the Laplacian in three dimensions. In general, for a scalar field $\sigma(x)$, and any fixed function $f(x)$, we have

$$\int \mathcal{D}\sigma \ \exp\left(-\int d^3x \ \frac{1}{2}(\partial_\mu \sigma)^2 + f(x)\sigma(x)\right) \sim \exp\left(\frac{1}{8\pi}\int d^3x d^3y \ \frac{f(x)f(y)}{|x-y|}\right)$$

Using this, we rewrite the sum over the Coulomb gas in (8.18) as a path integral

$$\exp\left(-\frac{1}{8\pi g^2}\sum_{i\neq j}\frac{m_i m_j}{|X^i - X^j|}\right) = \int \mathcal{D}\sigma \exp\left(-\int d^3x \ \frac{g^2}{8\pi^2}(\partial_\mu\sigma)^2 + \frac{i}{2\pi}\sum_i m_i \sigma \, \delta^3(x - X_i)\right)$$

(We used a very similar trick in the lectures on Statistical Field Theory when treating the 2d Coulomb gas in the XY model.)

In fact, we've met this field $\sigma(x)$ before: it is precisely the dual photon that we introduced in Section 8.1. To see this, note that the coupling to the magnetic charge above coincides with the coupling in (8.7)

Continuing with our calculation, the partition function becomes

$$
\begin{aligned}
Z &= \int \mathcal{D}\sigma \;\exp\left(-\int d^3x\;\frac{g^2}{8\pi^2}(\partial_\mu\sigma)^2\right)\;\sum_{N=0}^{\infty}\frac{(Ke^{-S_{\rm mono}})^N}{N!} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times\int\prod_{i=1}^{N}d^3X_i\sum_{m_i=\pm 4\pi}e^{-\frac{i}{2\pi}\sum_i m_i\sigma(X_i)} \\
&= \int \mathcal{D}\sigma \;\exp\left(-\int d^3x\;\frac{g^2}{8\pi^2}(\partial_\mu\sigma)^2\right)\;\sum_{N=0}^{\infty}\frac{1}{N!}\left(Ke^{-S_{\rm mono}}\int d^3x\cos(2\sigma(x))\right)^N \\
&= \int \mathcal{D}\sigma \;\exp\left(-\int d^3x\;\frac{g^2}{8\pi^2}(\partial_\mu\sigma)^2 - Ke^{-S_{\rm mono}}\cos(2\sigma)\right)
\end{aligned} \tag{8.19}
$$

We can now see the net effect of the instantons: they have generated a potential for the dual photon σ. Expanding about the minimum at $\sigma = 0$, we find that the dual photon has acquired a mass,

$$
m^2_{\rm photon} = \frac{4\pi^2 Ke^{-S_{\rm mono}}}{g^2}
$$

On dimensional grounds, the one-loop determinants and Jacobian factors that we lumped into the constant K must have dimension $[K] = 3$. For small λ, it turns out to scale as $K \sim v^{7/2}/g$. At weak coupling $g^2/v \ll 1$ and $S_{\rm mono} \gg 1$, where our semi-classical analysis is valid, we find that the mass of the dual photon is exponentially smaller than all other scales in the game. This means that we can read off the effective action from (8.19)

$$
S_{\rm eff} = \int d^3x\;\frac{g^2}{8\pi^2}(\partial_\mu\sigma)^2 + Ke^{-S_{\rm mono}}\cos(2\sigma) \tag{8.20}
$$

We recognise this as the Sine-Gordon model that we met in $d = 1+1$ dimensions in Section 7.5.5. Now it arises as the effective, low-energy description of a gauge theory in $d = 2+1$ dimensions.

8.3.2 Confinement

What does it mean for the dual photon to get a mass? To answer this, we can see how the ground state responds to various provocations.

First, let's try to turn on an electric field in the ground state, say $F_{01} \neq 0$. To understand what this means in terms of the dual photon, we need to relate $F_{\mu\nu}$ with

σ. We can do this by comparing our expressions for the topological current (8.3) and (8.11),

$$J^\mu_{\rm top} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}F_{\mu\nu} = \frac{e^2}{(2\pi)^2}\partial^\mu\sigma$$

We find that an electric field corresponds to

$$F_{01} = \frac{e^2}{2\pi}\,\partial_2\sigma$$

However, the configuration $\partial_2\sigma =$ constant does not obey the equations of motion of our effective action (8.20). This means that the vacuum does not support a constant, background electric field. Instead, solutions to the equations of motion with $\partial_2\sigma \neq 0$ are kinks, or domain walls, in which σ interpolates from, say, $\sigma = 0$ as $x_2 \to -\infty$, to $\sigma = 2\pi$ as $x_2 \to +\infty$. We already met these kinks in Section 7.5.5 when discussing the Sine-Gordon model in $d = 1 + 1$ dimensions. In the present context, the domain walls are string-like configurations stretched in the x^1 direction, with width $\sim 1/m_{\rm photon}$ in the x^2 direction, and tension,

$$\gamma = \frac{4}{\pi}\sqrt{2Kg^2e^{-S_{\rm mono}}}$$

a result which follows from translating our earlier result (7.69). (Up until now, we've always referred to the string tension as σ. Obviously that's a bad choice for our current discussion.)

The domain wall, or string, is a collimated flux tube of electric field $F_{01} \neq 0$. This is the expected behaviour of a gauge theory that is linearly confining. In other words, the classical log potential (8.2) of 3d gauge theories has been replaced with a more severe,

$$V(r) = \gamma r$$

We could explicitly compute the Wilson loop in this framework and confirm that it does indeed exhibit an area law.

We have seen that 3d electromagnetism exhibits linear confinement due to instantons which, in this context, are monopoles. It is crucial that these monopoles have a finite action, which we achieved by embedding the theory in a non-Abelian gauge group. If we introduce other UV completions of the theory, with a finite cut-off, Λ_{UV}, these too will have monopoles, typically with action $S_{\rm mono} \sim \Lambda_{UV}/g^2$. (Lattice gauge theory provides a good example of this). These too will then exhibit linear confinement.

8.4 Chern-Simons Theory

Gauge theories in $d = 2+1$ dimensions admit a rather special interaction that does not have a counterpart in even spacetime dimensions. This is the famous *Chern-Simons* interaction. It plays a key role in many areas of theoretical and mathematical physics, from the physics of the quantum Hall effect, to the mathematics of the knot invariants. Many details on the former application can be found in the lecture notes on the Quantum Hall Effect.

For $U(1)$ gauge theory, the Chern-Simons term takes the form

$$S_{CS} = \frac{k}{4\pi}\int d^3x \; \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{8.21}$$

We could consider this term on its own, or in conjunction with the Maxwell action (8.1). In either case, the dimensionless coefficient k is known as the *level*. We can write down similar terms in any odd spacetime dimension; we briefly met the $d = 4+1$ dimensional version in Section 4.4.1.

Let's start by studying the symmetries of the Chern-Simons action. It is Lorentz invariant, courtesy of the $\epsilon^{\mu\nu\rho}$ invariant tensor. At an operational level, the existence of this tensor means that the term is exclusive to $d = 2+1$ dimensions. However, this same $\epsilon^{\mu\nu\rho}$ tensor means that the Chern-Simons interaction breaks both parity and time-reversal invariance. Here we focus on parity. In even dimensions we can always take parity to act as $\mathbf{x} \mapsto -\mathbf{x}$ (see, for example, (1.25)). But, in odd dimensions, this coincides with a rotation. We should instead take parity to flip the sign of just a single spatial coordinate,

$$x^0 \rightarrow x^0 \quad , \quad x^1 \rightarrow -x^1 \quad , \quad x^2 \rightarrow x^2 \tag{8.22}$$

and, correspondingly, $A_0 \rightarrow A_0$, $A_1 \rightarrow -A_1$ and $A_2 \rightarrow A_2$. This means that, as advertised, the Chern-Simons action is odd under parity.

8.4.1 Quantisation of the Chern-Simons level

At first glance, it's not obvious that the Chern-Simons term is gauge invariant since it depends explicitly on A_μ. However, under a gauge transformation, $A_\mu \rightarrow A_\mu + \partial_\mu \omega$, we have

$$S_{CS} \rightarrow S_{CS} + \frac{k}{4\pi}\int d^3x \; \partial_\mu \left(\omega \epsilon^{\mu\nu\rho} \partial_\nu A_\rho\right)$$

The change is a total derivative. In many situations we can simply throw this total derivative away and the Chern-Simons term is gauge invariant. However, there are some situations where the total derivative does not vanish. As we will now show, in these cases the Chern-Simons partition function is gauge invariant provided that

$$k \in \mathbf{Z} \tag{8.23}$$

For Abelian Chern-Simons theories, it's a little subtle to see the requirement (8.23) since it only shows up in the presence of magnetic flux. (This is to be contrasted with the situation for non-Abelian Chern-Simons theories described in Section 8.4.3 where one can see the analogous quantisation condition around the vacuum state.)

Perhaps the simplest way is to consider the theory on Euclidean spacetime $\mathbf{S}^1 \times \mathbf{S}^2$. We then add a single unit of magnetic flux through the $\mathbf{S}^2$. As we've seen many times in these lectures, if we take the gauge group to compact $U(1)$, the flux is quantised, in the minimal unit

$$\frac{1}{2\pi}\int_{\mathbf{S}^2} F_{12} = 1 \tag{8.24}$$

We then consider large gauge transformations of this background that wind around the $\mathbf{S}^1$. We denote the radius of this $\mathbf{S}^1$ as R, and parameterise it by the coordinate $x^0 \in [0, 2\pi R)$. Consider a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \omega$ which winds around the $\mathbf{S}^1$, with

$$\omega = \frac{x^0}{R} \tag{8.25}$$

Under such a transformation, any matter field ϕ with charge $q \in \mathbf{Z}$ remains single valued, since $\phi \rightarrow e^{iq\tau/R}\phi$. Even in the absence of charged matter, the statement that we're working with a compact $U(1)$ gauge group, rather than a non-compact $\mathbf{R}$ gauge group, means that the theory admits fluxes (8.24) and gauge transformations (8.25).

Under the gauge transformation (8.25), we have

$$A_0 \rightarrow A_0 + \frac{1}{R} \tag{8.26}$$

This means that the zero mode of A_0 is a periodic variable, with periodicity $1/R$. (We came to the same conclusion in Section 7.1 where we discussed two dimensional electromagnetism on a spatial circle.)

We can now see what becomes of our Chern-Simons action under such a gauge transformation? Evaluated on a configuration with constant A_0, we have

$$S_{CS} = \frac{k}{4\pi}\int d^3x\ A_0F_{12} + A_1F_{20} + A_2F_{01}$$

Now it's tempting to throw away the last two terms when evaluating this on our background. But we should be careful as it's topologically non-trivial configuration. We can safely set all terms with ∂_0 to zero, but integrating by parts on the spatial derivatives we get an extra factor of 2,

$$S_{CS} = \frac{k}{2\pi}\int d^3x\ A_0 F_{12}$$

Evaluated on the flux (8.24), with constant $A_0 = a$, we have

$$S_{CS} = 2\pi k R a$$

And under the gauge transformation (8.26), we have

$$S_{CS} \to S_{CS} + 2\pi k$$

The Chern-Simons action is not gauge invariant. But all is not lost. The partition function depends only on $e^{iS_{CS}}$ and this remains gauge invariant provided $k \in \mathbf{Z}$, which is our claimed result. This last part of the argument is exactly the same as the one we met in Section 2.1.3 when we discussed Chern-Simons terms in quantum mechanics, and in a number of other places when we've discussed WZW terms.

8.4.2 A Topological Phase of Matter

So what is the physics of Chern-Simons theory? Despite the simplicity of the action, the physics is remarkably subtle. Let's start with the basics. We'll take the $d = 2+1$ dimensional gauge field to be governed by

$$S = S_{\text{Maxwell}} + S_{CS} = \int d^3x\ -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$$

We can start by gaining some intuition from the classical equation of motion,

$$\partial_\mu F^{\mu\nu} + \frac{ke^2}{4\pi}\epsilon^{\nu\rho\sigma}F_{\rho\sigma} = 0 \tag{8.27}$$

In terms of the electric field $E_i = F_{0i}$ and the magnetic field $B = F_{12}$, Gauss' law becomes

$$\partial_i E_i = \frac{ke^2}{2\pi}B \tag{8.28}$$

which tells us that a magnetic field acts as a source for the electric field. This simple observation will underlie much of the physics of Section 8.6 where we discuss bosonization in 3d.

What are the propagating excitations of the equations of motion (8.27)? Taking one further derivative of the equations of motion, we can decouple electric and magnetic fields to show that each component obeys the massive wave equation,

$$\partial^2 E_i - \left(\frac{ke^2}{2\pi}\right)^2 E_i = \partial^2 B - \left(\frac{ke^2}{2\pi}\right)^2 B = 0$$

(To do this, it's perhaps simplest to first define the field $G^\mu = \epsilon^{\mu\nu\rho} F_{\nu\rho}$ and show that G^μ obeys the massive wave equation.) We see that, at least classically, the excitations do not propagate at the speed of light. Instead, they are exponentially damped. In the quantum theory, which means that we have a theory of massive excitations. The mass of the photon is

$$m_{CS} = \frac{ke^2}{2\pi}$$

Yet again, we find ourselves in a situation with a massive gauge boson. How should we think of this phase?

We've already met other situations in $d = 2+1$ dimensions where the photon gets a mass. There is the confining phase, driven by instantons, that we saw in Section 8.3, in which the Wilson loop has an area law. And there is, of course, the Higgs phase in which a charged scalar field condenses and the Wilson line has a perimeter law. It turns out that the Chern-Simons phase differs from both of these. Instead, it is a novel phase of matter, referred to as a *topological phase*.

Topological phases of matter are subtle. They typically have interesting things going on at energies $E \ll m_{CS}$ way below the gap, even though there are no physical excitations beyond the vacuum. We'll explain below what these interesting things are.

Chern-Simons Terms are Topological

Before we address the novel physics of Chern-Simons theory, we first point out an important property of the Chern-Simons action (8.21): it doesn't depend on the metric of the background spacetime manifold. It depends only on the topology of the manifold. To see this, let's first look at the Maxwell action for comparison. If we were to couple this to a background metric $g_{\mu\nu}$, the action becomes

$$S_{\text{Maxwell}} = \int d^3x \sqrt{-g}\; -\frac{1}{4e^2}\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$$

We see that the metric plays two roles: first, it is needed to raise the indices when contracting $f_{\mu\nu} f^{\mu\nu}$; second it provides a measure $\sqrt{-g}$ (the volume form) which allows us to integrate in a diffeomorphism invariant way. Recall from our first lectures on *Quantum Field Theory* that this allows us to quickly construct the stress-tensor of the theory by differentiating with respect to the metric,

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}$$

In contrast, we have no need to introduce a metric when generalising (8.21) to curved spacetime. This is best stated in the language of differential geometry: $A \wedge dA$ is a 3-form, and we can quite happily integrate this over any three-dimensional manifold

$$S_{CS} = \frac{k}{4\pi} \int A \wedge dA$$

This means that pure Chern-Simons theory knows nothing length scales. In particular, the Wilson loop can exhibit neither area nor perimeter law, since both of these are statements about lengths. Moreover, pure Chern-Simons theory has vanishing stress tensor.

Chern-Simons Theory on a Torus

If Chern-Simons theory has vanishing stress tensor, and no physical excitations, then what can it possibly do? The answer is that the theory responds to low-energy probes in interesting ways.

Here is a simple, yet dramatic way to probe the theory. We will place it on a spatial 2-dimensional manifold Σ. As we have seen, Chern-Simons theory knows nothing about the metric on Σ. However, as we now show, it does know about the topology and responds accordingly.

For pure Chern-Simons theory (or, equivalently, the $e^2 \to \infty$ limit of Maxwell-Chern-Simons theory), Gauss' law (8.28) becomes

$$F_{12} = 0$$

Although this equation is very simple, it can still have interesting solutions if the background has some non-trivial topology. These are called, for obvious reason, *flat connections*. It's simple to see that such solutions exist on the torus $\Sigma = \mathbf{T}^2$, where one example is to simply set each A_i to be constant. Our first task is to find a gauge-invariant way to parameterise this space of solutions.

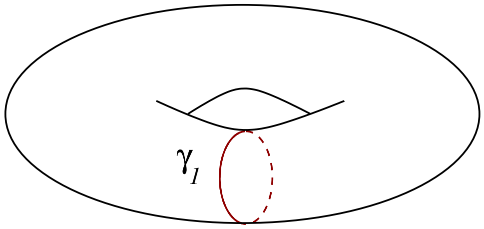


Figure 54:

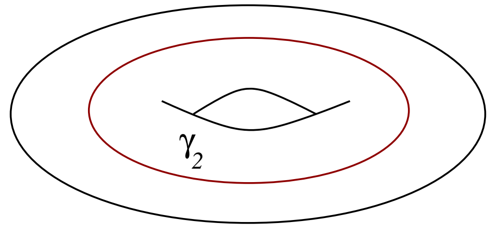


Figure 55:

We'll denote the radii of the two circles of the torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ as R_1 and R_2. We'll denote two corresponding non-contractible curves shown in the figure as γ_1 and γ_2. The simplest way to build a gauge invariant object from a gauge connection is to integrate

$$w_i = \oint_{\gamma_i} dx^j \, A_j$$

This is invariant under most gauge transformations, but not those that wind around the circle. By the same kind of arguments that led us to (8.26), we can always construct gauge transformations which shift $A_j \to A_j + 1/R_j$, and hence $w_i \to w_i + 2\pi$. The correct gauge invariant objects to parameterise the solutions are therefore the Wilson loops

$$W_i = \exp\left(i \oint_{\gamma_i} A_j dx^j\right) = e^{iw_i}$$

Because the Chern-Simons theory is first order in time derivatives, these Wilson loops are really parameterising the phase space of solutions, rather than the configuration space. Moreover, because the Wilson loops are complex numbers of unit modulus, the phase space is compact. On general grounds, we expect that when we quantise a compact phase space, we get a finite-dimensional Hilbert space. (We met an example of this in Section 2.1.3 when first describing Wilson lines.) Our next task is to understand how to quantise the space of flat connections.

The canonical commutation relations can be read off from the Chern-Simons action (8.21)

$$[A_1(x), A_2(x')] = \frac{i}{k}\,\delta^2(x - x') \quad \Rightarrow \quad [w_1, w_2] = \frac{2\pi i}{k}$$

The algebraic relation obeyed by the Wilson loops then follows from the usual Baker-Campbell-Hausdorff formula,

$$e^{iw_1} e^{iw_2} = e^{[w_1, w_2]/2} e^{i(w_1 + w_2)}$$

which tells us that

$$W_1 W_2 = e^{2\pi i/k}\, W_2 W_1 \tag{8.29}$$

But such an algebra of operators can't be realised on a single vacuum state. This immediately tells us that the ground state must be degenerate. The smallest representation of (8.29) has dimension k, with the action

$$W_1|n\rangle = e^{2\pi n i/k}|n\rangle \quad \text{and} \quad W_2|n\rangle = |n+1\rangle$$

We have seen that on a torus $\Sigma = \mathbf{T}^2$, an Abelian Chern-Simons theory has k degenerate ground states. The generalisation of this argument to a genus-g Riemann surface tells us that the ground state must have degeneracy k^g. Notice that we don't have to say anything about the shape or sizes of these manifolds. The number of ground states depends only on the topology. This is an example of *topological order*.

8.4.3 Non-Abelian Chern-Simons Theories

We've not had much to say about non-Abelian gauge theories in low dimensions. This is not because they're boring, but simply because there is enough to keep us busy elsewhere. Here we make an exception and give a brief description of non-Abelian Chern-Simons theory.

Like Yang-Mills, Chern-Simons is based on a Lie algebra valued gauge connection A_μ. The non-Abelian Chern-Simons action is

$$S_{CS} = \frac{k}{4\pi}\int d^3x\; \epsilon^{\mu\nu\rho}\,\mathrm{tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) \tag{8.30}$$

We've met this term before: the theta term in $d=3+1$ dimensions can be written as a derivative of the Chern-Simons term (2.24). (It also arose in the same context when discussing canonical quantisation of Yang-Mills (2.35).) Chern-Simons theories with gauge group G and level k are sometimes denoted as G_k.

Once again, we will find that the level must be integer, $k \in \mathbf{Z}$. This time, however, the computation is more direct than in the Abelian case. Under a gauge transformation, we have

$$A_\mu \rightarrow \Omega^{-1}A_\mu\Omega + i\Omega^{-1}\partial_\mu\Omega$$

with $\Omega \in G$. The field strength transforms as $F_{\mu\nu} \rightarrow \Omega^{-1}F_{\mu\nu}\Omega$. A simple calculation shows that the Chern-Simons action changes as

$$S_{CS} \rightarrow S_{CS} + \frac{k}{4\pi}\int d^3x\;\left\{\epsilon^{\mu\nu\rho}\partial_\nu\mathrm{tr}\,(\partial_\mu\Omega\,\Omega^{-1}a_\rho) + \frac{1}{3}\epsilon^{\mu\nu\rho}\mathrm{tr}\,\left((\Omega^{-1}\partial_\mu\Omega)(\Omega^{-1}\partial_\nu\Omega)(\Omega^{-1}\partial_\rho\Omega)\right)\right\}$$

The first term is a total derivative. The same kind of term arose in Abelian Chern-Simons theories. However, the second term is novel to non-Abelian gauge theories, and this is where the quantisation requirement now comes from. In fact, we have seen this calculation before in Section 2.2.2 when discussing the theta angle in $d = 3 + 1$ Yang-Mills. On a spacetime manifold $\mathbf{S}^3$ (or on $\mathbf{R}^3$ with the requirement that gauge transformations asymptote to the same value at infinity), gauge transformations are characterised by the homotopy group $\Pi_3(SU(N)) \cong \mathbf{Z}$. The winding is counted by the function

$$n(\Omega) = \frac{1}{24\pi^2} \int_{\mathbf{S}^3} d^3S \; \epsilon^{\mu\nu\rho} \mathrm{tr}\,(\Omega^{-1}\partial_\mu\Omega\,\Omega^{-1}\partial_\nu\Omega\,\Omega^{-1}\partial_\rho\Omega) \quad \in \mathbf{Z} \tag{8.31}$$

We recognise this as the final term that appears in the variation of the Chern-Simons action. This means that the Chern-Simons action is not invariant under these large gauge transformations; it changes as

$$S_{CS} \to S_{CS} + \frac{k}{12\pi}\,24\pi^2\,n(\Omega) = S_{CS} + 2\pi k\,n(\Omega)$$

Insisting that the path integral, with its weighing $e^{iS_{CS}}$ is gauge invariant then gives us immediately our quantisation condition $k \in \mathbf{Z}$.

Wilson Loops

We have so far avoided talking about Wilson lines in Chern-Simons theories. There is rather a lot to say. We will not describe this in detail here, but just sketch the key idea.

In $d = 3$ Euclidean spacetime dimensions, a Wilson loop can get tangled. Mathematicians call closed curves in three dimensions *knots*, and there has been a great deal of effort in trying to classify the ways in which they can get tangled. It turns out that Chern-Simons theories provide one of the most powerful tools. For a given knot C, we can compute the Wilson loop $\langle W[C]\rangle$. In Chern-Simons theory the Wilson loop exhibits neither an area law, nor a perimeter law. Instead, it depends on the details of the topology of the knot C. For each gauge group G, the Wilson loop gives a topological invariant which is a polynomial (roughly in $q = e^{2\pi i/k}$.) In simple cases, these topological invariants coincide with ones already understood by mathematicians (such as the Jones polynomial), but they also offer a large number of generalisations. Edward Witten was awarded the Fields medal, in large part for understanding this connection.

8.5 Fermions and Chern-Simons Terms

There is an intricate interplay between fermions in $d = 2+1$ dimensions and Chern-Simons terms.

In signature $\eta^{\mu\nu} = \text{diag}(+1, -1, -1)$, the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ is satisfied by the 2×2 gamma matrices,

$$\gamma^0 = \sigma^2 \quad , \quad \gamma^1 = i\sigma^1 \quad , \quad \gamma^3 = i\sigma^3$$

The Dirac spinor is then a two-component complex object. In odd spacetime dimensions, there is no "γ^5" matrix and, correspondingly, no Weyl fermions. In $d = 2+1$, we can take the gamma matrices as above to be purely imaginary, which means that we can have Majorana fermions. However, we won't have a need for this real representation in what follows.

It will prove useful to understand the action of parity on fermions. As we saw in (8.22), in three dimensions parity acts as

$$x^0 \to x^0 \quad , \quad x^1 \to -x^1 \quad , \quad x^2 \to x^2$$

The Dirac action is then invariant if we take parity to act as

$$\mathcal{P} : \ \psi \mapsto \gamma^1 \psi \tag{8.32}$$

But this means that the fermion mass term necessarily breaks parity,

$$\mathcal{P} : \bar{\psi}\psi = \psi^\dagger \gamma^0 \psi \mapsto -\bar{\psi}\psi$$

where, to see this, you need to remember that $(\gamma^1)^\dagger = -\gamma^1$ and $(\gamma^1)^2 = -1$.

This is different from what happens in $d = 3+1$ dimensions or, indeed, in any even spacetime dimension. There parity flips the sign of all spatial dimensions and, correspondingly, the Dirac action is invariant if we take $\mathcal{P} : \psi \mapsto \gamma^0 \psi$. This means that in even spacetime dimensions, $\bar{\psi}\psi$ is even under parity; in odd spacetime dimensions $\bar{\psi}\psi$ is odd.

We can understand why this is by counting degrees of freedom. In $d = 3+1$ dimensions, the Dirac spinor has 4 components. When we quantise a massive fermion, we get two particle states – spin up and spin down – and the same anti-particle states. But a Dirac fermion in $d = 2+1$ dimensions has only two components, and so we must have half the number of particle states of the $d = 3+1$ theory. The pair that we keep is dictated by the sign of the mass, and by $\mathcal{CPT}$ invariance: if we have a particle with spin, or angular momentum, $+\frac{1}{2}$, the theory must also include an anti-particle of spin $-\frac{1}{2}$. But this necessarily breaks parity: the theory has a particle of spin $+\frac{1}{2}$ but no particle of spin $-\frac{1}{2}$.

8.5.1 Integrating out Massive Fermions

Let us take a single Dirac fermion, of mass m, coupled to a $U(1)$ gauge field A_μ. The action is

$$S = \int d^3x \; i\bar{\psi}\not{D}\psi + m\bar{\psi}\psi$$

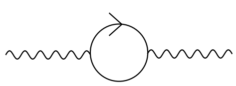
If we care about physics at energies below the fermion mass m, we can integrate out the fermion. We work in Euclidean space. The fermion then gives a contribution to the low-energy effective action for the gauge field,

$$S_{\text{eff}} = \log\det\big(i\not{D} + m\big) = \operatorname{Tr}\log\big(i\not{\partial} + \gamma^\mu A_\mu + m\big)$$

We expand this as,

$$S_{\text{eff}} = \operatorname{Tr}\log\big(i\not{\partial} + m\big) + \operatorname{Tr}\log\left(\frac{1}{i\not{\partial}+m}\gamma^\mu A_\mu\right) + \frac{1}{2}\operatorname{Tr}\log\left(\frac{1}{i\not{\partial}+m}\gamma^\mu A_\mu \frac{1}{i\not{\partial}+m}\gamma^\nu A_\nu\right) + \ldots$$

The first term is an overall constant, and the second term cannot lead to anything gauge invariant. But the third term holds something interesting. If we give the background field A_μ momentum p, then the trace over momenta corresponds to the diagram,



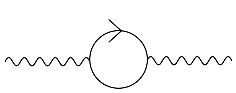
$$\begin{aligned} &= \frac{1}{2}A_\mu(-p)A_\nu(p)\int \frac{d^3k}{(2\pi)^3}\;\operatorname{tr}\left(\frac{1}{(\not{p}+\not{k})+m}\gamma^\mu \frac{1}{\not{k}+m}\gamma^\nu\right) \\ &= \frac{1}{2}A_\mu(-p)A_\nu(p)\int \frac{d^3k}{(2\pi)^3}\;\operatorname{tr}\left(\frac{\not{p}+\not{k}-m}{(p+k)^2+m^2}\gamma^\mu \frac{\not{k}-m}{k^2+m^2}\gamma^\nu\right)\end{aligned}$$

where we've used the fact that, after the Wick rotation, each gamma matrix squares to -1.

The trace picks out the non-vanishing gamma matrix structure. There will be a contribution to the Maxwell term; that doesn't interest us here. Instead, we care about the term we get when three gamma matrices are multiplied together. The trace structure gives

$$\operatorname{tr}\gamma^\rho\gamma^\mu\gamma^\nu = -2\epsilon^{\mu\nu\rho}$$

The resulting term is

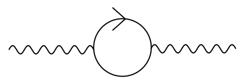


$$= \epsilon^{\mu\nu\rho}A_\mu(-p)A_\nu(p)\,p_\rho\int \frac{d^3k}{(2\pi)^3}\;\frac{m}{((p+k)^2+m^2)(k^2+m^2)}$$

We're interested in this integral in the infra-red limit, $p \to 0$, where it is given by

$$\int \frac{d^3k}{(2\pi)^3}\, \frac{m}{(k^2+m^2)^2} = \frac{1}{2\pi^2}\int_0^\infty dk\, \frac{mk^2}{(k^2+m^2)^2} = \frac{1}{8\pi|m|}$$

Putting this together, the 1-loop diagram gives

$$\lim_{p\to 0}$$  $$= \frac{1}{8\pi}\frac{m}{|m|}\epsilon^{\mu\nu\rho}A_\mu(-p)A_\nu(p)\,p_\rho$$

Back in real space, this gives us the leading term to the low energy effective action

$$S_{\text{eff}} = \frac{i}{4\pi}\,\frac{\text{sign}(m)}{2}\int d^3x\; \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{8.33}$$

There are a number of interesting things to point out about this result. First, the effective action comes with a power of i; this is expected for the Chern-Simons term in Euclidean space, and follows from Wick rotating terms with an ϵ symbol.

Second, and more surprisingly, the fermion does not decouple in the limit $m \to \infty$. After integrating out a massive field, one typically generates terms in the effective action that scale as a power of $1/m$. Not so for the Chern-Simons term: it is proportional to the sign of the mass. This behaviour holds for fermions in any odd spacetime dimensions; we met a similar example in $d = 4+1$ when discussing anomaly inflow in Section 4.4.2.

Finally, and most importantly, the effective action (8.33) is not gauge invariant! It is a Chern-Simons term (8.21) with level $k = \pm\frac{1}{2}$. Yet, we saw in the previous section, that the Chern-Simons term is only gauge invariant for $k \in \mathbf{Z}$. With $k = \pm\frac{1}{2}$, the sign of the partition function can flip under gauge transformations.

What are we to make of this? It appears that a single massive Dirac fermion, coupled to a $U(1)$ gauge field, is inconsistent. This is very much reminiscent of the gauge anomalies that we met in $d = 3+1$ dimensions in Section 3. However, we shouldn't be too hasty. After all, anomalies in $d = 3+1$ dimensions were strictly related to massless fermions, and here we're dealing with a massive fermion. What's going on?

Indeed, we were sloppy in how we deal with UV divergences in the calculation above. They do not arise in the calculation of the Chern-Simons term, but they will surely be important if we compute other quantities and, as in any quantum field theory, we need a way to regulate them. To achieve this, we introduce a Pauli-Villars regulator field, together with suitable counterterms. We take the Pauli-Villars field to have real mass $\Lambda_{UV} > 0$. The regulated Dirac determinant is then

$$\frac{\det(i\not{D} + m)}{\det(i\not{D} + \Lambda_{UV})}$$

This gives two contributions to the Chern-Simons term; one from our fermion, and one from the regulator. The effective action for the gauge field then becomes

$$\frac{\det(i\not{D} + m)}{\det(i\not{D} + \Lambda_{UV})} = \frac{1}{2\pi}\left(\text{sign}(m) - \frac{1}{2}\right)\int d^3x\ \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$$

which vanishes when $m > 0$ but gives a Chern-Simons term of level $k = -1$ when $m < 0$. In other words, when the regulated fermion determinant is defined more carefully, there is no problem with gauge invariance.

The resulting situation is notationally inconvenient. Usually we would like to write down an action as shorthand for a quantum field theory, even though we know that to fully define the theory really requires a statement about how we regulate. The issue above means that the sign of the mass of the Pauli-Villars regulator matters in a crucial fashion. To avoid this, we are often sloppy and pretend that we've already integrated out the Pauli-Villars field to generate a bare Chern-Simons term with level $k = -\frac{1}{2}$ in the action.

More generally, we can couple N_f Dirac fermions to a $U(1)$ gauge field with the leading terms in the action given by

$$S = \int d^3x\ \ -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \sum_{i=1}^{N_f} i\bar{\psi}_i\not{D}\psi_i + m_i\bar{\psi}_i\psi_i$$

Using the convention that the Chern-Simons term already includes the contributions from Pauli-Villars fields, gauge invariance requires

$$k + \frac{N_f}{2} \in \mathbf{Z}$$

This interplay between the level k and the number of fermions is sometimes referred to as the *parity anomaly*. It's not a great name since the theory with fermion masses is not parity invariant to begin with.

8.5.2 Massless Fermions and the Parity Anomaly

We can gain a slightly different perspective on the ideas above by considering a massless fermion coupled to a $U(1)$ gauge field, A_μ. The action is now

$$S = \int d^3x \; i\bar{\psi}\not{D}\psi$$

The transformation (8.32) ensures that the classical action is invariant under parity, provided that we also act with $A_1 \to -A_1$.

The classical action is invariant under parity. But what about the partition function. To answer this, we must make sense of the determinant of the Dirac operator,

$$Z[A] = \det\left(i\not{D}\right)$$

As above, we work in Euclidean space. The Dirac operator is Hermitian, which means that it has real eigenvalues,

$$i\not{D}\phi_n = \lambda_n\phi_n \quad \lambda_n \in \mathbf{R}$$

So formally we can write

$$Z = \prod_n \lambda_n$$

Of course, this formula is divergent and so we must work to make sense of it. For now, we would like to ask the following question: what is the sign of $\det(i\not{D})$. Roughly speaking, this must be the difference between the number of negative eigenvalues and the number of positive eigenvalues. But, as there are an infinite number of each, it is not clear how to count them.

Why do we care so much about the sign? The problem comes if we try to reconcile a given sign with the requirements of gauge invariance. Suppose that we start with some gauge configuration $A_\mu^\star$ and decide that $\det(i\not{D})$ has a specific sign. Then it better be the case that, for any gauge configuration A_μ^ω, related to $A_\mu^\star$ by a gauge transformation, the sign of $\det(i\not{D})$ remain the same.

At this point, the discussion may be ringing bells. It is entirely analogous to the $SU(2)$ anomaly that we described in Section 3.4.3. We proceed in a very similar way. Consider the 1-parameter family of gauge configurations,

$$A_\mu(s;x) = (1-s)A_\mu^\star(x) + sA_\mu^\omega(x) \tag{8.34}$$

This has the property that it interpolates from $A_\mu^\star$ when $s = 0$ to A_μ^ω when $s = 1$. The question that we would like to answer is: how many eigenvalues pass through zero and change sign as we vary $s \in [0, 1]$. To answer this, we can consider the gauge configuration $A_\mu(s; x)$ in (8.34) to live on the four manifold $I \times \mathbf{R}^3$, where I is the interval parameterised by $0 \leq s \leq 1$.

The number of times that the an eigenvalue crosses zero is given by the index of the Dirac operator. This is the object that we introduced in Section 3.3.1 where, on a closed four manifold, the Atiyah-Singer index theorem allowed us to write

$$\text{Index}(i\not{D}_{4d}) = \frac{1}{32\pi^2} \int_{I \times \mathbf{R}^3} d^4x \; \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

In 4d, the index counts the difference between the number of left-handed and right-handed zero modes. For our purposes, it tells us the difference between the number of eigenvalues that switch from positive to negative, and those which switch from negative to positive. In other words, under the gauge transformation $A_\mu^0 \to A_\mu^\omega$, the partition function of the massless fermion changes as

$$Z \to Z\,(-1)^{\text{Index}(i\not{D}_{4d})}$$

There is no reason for this index to be even. We see, once again, that without regularisation the sign of the partition function can change under a suitable gauge transformation.

What happens if we now include a regulator? In mathematics, a suitably regulated sum of the signs of the eigenvalues of $i\not{D}$ is known as the Atiyah-Patodi-Singer eta-invariant. It is defined by

$$\eta(A) = \lim_{\epsilon \to 0^+} \sum_n e^{-\epsilon\,\lambda_n^2}\,\text{sign}(\lambda_n)$$

We then define a regulated version of the fermion partition function as

$$Z = |\det(i\not{D})|\, e^{-i\pi\eta(A)/2}$$

The η invariant depends on the background gauge field A. The Atiyah-Patodi-Singer index theorem provides an expression for η in terms of the gauge field. If we restrict to the generic situation where the gauge field has no zero modes, then one can show that

$$\pi\,\eta(A) = \frac{1}{4\pi} \int d^3x \; \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

This reproduces the expression that we found previously from the Pauli-Villars regularisation. In general, the eta-invariant is the more mathematically rigorous way to describe what's happening as it allows one to track what happens as eigenvalues pass through zero.

8.6 3d Bosonization

In two spacetime dimensions, there is not much of a distinction between bosons and fermions. The map between them is known as bosonization and was described in Section 7.5.

In three spacetime dimensions, bosons are not the same as fermions. We can tell which one we have in the same way as we would in four dimensions. Given a pair of particles we can rotate them by 180°, keeping them well separated. The wavefunction for a pair of bosons will come back to itself, while the wavefunction for a pair of fermions comes back with a minus sign.

Nonetheless, it is possible to use Chern-Simons terms to change statistics of an excitation from a boson to a fermion. This process is referred to as *3d bosonization*.

8.6.1 Flux Attachment

To get a feel for what's going on, it's useful to first revert to some non-relativistic physics. Consider Chern-Simons theory coupled to a current J^μ

$$S = \int d^3x \; \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + A_\mu J^\mu \tag{8.35}$$

We can insert a test particle of unit charge by taking $J^\mu = \delta^2(\mathbf{x})$. How does the gauge field respond? Gauss' law tells us that the charged particle is accompanied by a fractional magnetic flux,

$$\frac{1}{2\pi}B = \frac{1}{k}\,\delta^2(\mathbf{x}) \tag{8.36}$$

This is referred to as *flux attachment*.

Now consider two such particles. We will exchange them to determine their quantum statistics. The wavefunction will pick up a factor of ± 1 depending on whether the original particles were fermions. However, there is a second contribution to the phase of the wavefunction that comes from the Aharonov-Bohm effect.

Recall that a particle of charge q moving around a flux Φ picks up a phase $e^{iq\Phi}$. But because of flux attachment (8.36), the particles carry both charge $q = 1$ and flux $\Phi = 2\pi/k$. If we move one particle all the way around another, we will get a phase $e^{iq\Phi}$. But the statistical phase is defined by exchanging particles, which consists of only half an orbit (followed by a translation which contributes no phase). So, after exchange, the expected statistical phase is

$$\pm\, e^{iq\Phi/2} = \pm e^{i\pi/k}$$

where we take the $+$ sign if our original particles are bosons and the $-$ sign if they were fermions. We see that the effect of the Chern-Simons term is to transmute the quantum statistics of the particles. In particular, if we take a Chern-Simons term at level $k = \pm 1$, what were bosons become fermions and vice versa. Once again, we see that the topological nature of the Chern-Simons term endows it with seemingly magic infra-red properties: it can change the behaviour of far separated particles, even though it has no propagating degrees of freedom.

For $|k| > 1$, the particles are neither bosons nor fermions. Instead they carry fractional quantum statistics. Such particles are called *anyons* and are allowed only in $d = 2+1$ dimensions. You can read more about them in the lecture notes on the Quantum Hall Effect where they play a prominent role.

A Famously Fiddly Factor of 2

The calculation above contains an annoying factor of 2 that we've swept under the rug. Here's the issue. As the charge q in the first particle moved around the flux Φ in the second, we picked up a phase $e^{iq\Phi}$. But you might think that the flux Φ of the first particle also moved around the charge q of the second. So surely this should give another factor of $e^{iq\Phi}$. Right? Well, no. To see why, it's best to just do the calculation.

For generality, let's take N particles sitting at positions $\mathbf{x}_a(t)$ which, as the notation shows, we allow to change with time. The charge density and currents are

$$J^0(\mathbf{x},t) = \sum_{a=1}^{N} \delta^2(\mathbf{x} - \mathbf{x}_a(t)) \quad \text{and} \quad \mathbf{J}(\mathbf{x},t) = \sum_{a=1}^{N} \dot{\mathbf{x}}_a\, \delta^2(\mathbf{x} - \mathbf{x}_a(t))$$

The equation of motion from (8.35) is

$$\frac{1}{2\pi} F_{\mu\nu} = \frac{1}{k} \epsilon_{\mu\nu\rho} J^\rho$$

and can be easily solved even in this general case. We work in Coulomb gauge with $A_0 = 0$ and $\nabla \cdot \mathbf{A} = 0$. The solution is then

$$A_i(\mathbf{x},t) = \frac{1}{k} \sum_{a=1}^{N} \epsilon^{ij}\, \frac{x^j - x^j_a(t)}{|\mathbf{x} - \mathbf{x}_a(t)|^2} \tag{8.37}$$

This follows from the standard methods that we know from our Electromagnetism lectures, but this time using the Green's function for the Laplacian in two dimensions: $\nabla^2 \log|\mathbf{x}-\mathbf{y}| = 2\pi\delta^2(\mathbf{x}-\mathbf{y})$. This solution is again the statement that each particle carries flux $1/k$. However, we can also use this solution directly to compute the phase change when one particle – say, the first one – is transported along a curve C. It is simply

$$\exp\left(i\oint_C \mathbf{A}\cdot d\mathbf{x}_1\right)$$

If the curve C encloses one other particle, the resulting phase change can be computed to be $e^{2\pi i/m}$. As before, if we exchange two particles, we get half this phase, or $e^{i\pi/k}$. This, of course, is the same result we got above.

8.6.2 A Bosonization Duality

The discussion above shows that Chern-Simons terms can turn bosons into fermions and vice-versa. However, it holds only for massive particles, and cannot be easily generalised to massless particles, let alone to relativistic quantum field theories. Nonetheless, it is suggestive that it may be possible to write down a quantum field theory of bosons coupled to Chern-Simons terms that has a dual interpretation in terms of fermions. As we now explain, it is thought that this is indeed the case.

Before we proceed, we're going to make a small change in notation. In what follows, there will be lots of $U(1)$ gauge fields floating around. Some of them will be dynamical, while others will be background gauge fields that we couple to currents. To distinguish between these, we use the following convention: dynamical gauge fields will be written in lower case, e.g. a_μ. Meanwhile, background gauge fields will be written in upper case, e.g. A_μ.

This convention differs from what we've used throughout these lectures, where we typically refer to all gauge fields, dynamical or background, as A_μ. It is, however, a standard convention in condensed matter physics where the true electromagnetic gauge field A_μ is typically a background field, describing electric or magnetic fields that the experimenter has chosen to turn on. In contrast, 3d dynamical gauge fields a_μ are always emergent excitations, arising from some collective behaviour of strongly coupled electrons.

Consider the following theory, that we refer to as Theory A: a complex scalar field coupled to a $U(1)$ gauge field, with Chern-Simons term at level $k=1$,

$$S_A[\phi,a] = \int d^3x \ -\frac{1}{4e^2}f_{\mu\nu}f^{\mu\nu} + \frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + |\mathcal{D}_\mu\phi|^2 - m^2|\phi|^2 - \frac{\lambda}{2}|\phi|^4 \quad (8.38)$$

This is the Abelian Higgs model (8.12), but with the addition of a Chern-Simons term. Just as before, it is straightforward to analyse in the limits $m^2 \gg e^2$ and $m^2 \ll -e^2$ where it is a theory of weakly interacting massive particles. But we'd like to understand what happens in the strongly coupled regime. We will argue below that as we vary the m^2 from positive to negative, there is a unique second order phase transition, roughly at $m = 0$. You can think of this gapless theory as the XY critical point, coupled to a Chern-Simons gauge field $U(1)_1$. Below, we will conjecture an alternative, and somewhat simpler, description.

In the infra-red limit $e^2 \to \infty$, the Gauss' law constraint gives rise to the local flux attachment condition,

$$\frac{f_{12}}{2\pi} + \rho_{\text{scalar}} = 0 \tag{8.39}$$

where ρ_{scalar} is the charge density of the scalar field ϕ. In the non-relativistic setting – which can be invoked when $m^2 \gg e^2$ – we viewed this as attaching flux to every scalar excitation and saw that, for $k = 1$, this turns a boson into a fermion. In the relativistic setting, it turns out to be more appropriate to think of attaching a scalar to every flux.

To see this, first note that the theory has a conserved global symmetry, with the topological current (8.3)

$$j^\mu_{\text{top}} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho \tag{8.40}$$

We know from our earlier discussion in Section 8.1 that the local operators which carry charge under this current are monopole operators $\mathcal{M}(x)$, which insert magnetic flux at a point. The flux attachment (8.39) is telling us that, in the presence of a Chern-Simons term, these monopole operators are not gauge invariant. We can make them gauge invariant only by dressing them with some scalar charge ρ_{scalar}. Schematically, we refer to the gauge invariant composite operator as $\mathcal{M}\phi$.

How do we do this less schematically? The right way to proceed is to solve the equation of motion for the scalar in the presence of a Dirac monopole. We then treat each mode quantum mechanically: the flux attachment condition (8.39) tells us that we should excite a single mode. The monopole operator with the lowest dimension will correspond to exciting the lowest energy scalar mode.

We won't go through this full calculation. However, the key physics can be seen from a simple calculation that we did back in Section 1.1: a charged particle moving in a minimal Dirac monopole receives a shift of $\hbar/2$ to its angular momentum. (See, in particular, equation (1.9).) This means that exciting any bosonic mode will shift the angular momentum of the monopole to become 1/2-integer. But, in a relativistic theory, the spin-statistics relation must hold. If our gauge invariant monopole operator $\mathcal{M}\phi$ has spin 1/2, then it must also be a fermion.

We see that this argument leads to the same result as before: a bosonic theory coupled to a $U(1)$ Chern-Simons gauge field at level $k = 1$ is really a theory of fermions. The obvious question is: what theory of fermions?

It is conjectured that, close to the critical point, the bosonic theory (8.38) is really just a free Dirac fermion! In other words, it can be equivalently described as

$$S_B[\psi] = \int d^3x \; i\bar{\psi}\,\partial\!\!\!/\psi - m'\bar{\psi}\psi \tag{8.41}$$

The map is very similar to that of particle-vortex duality that we saw in Section 8.2.1. In particular, the fermion is described by the dressed monopole operator in Theory A,

$$\mathcal{M}\phi \quad \longleftrightarrow \quad \psi$$

while the $U(1)$ currents map between themselves

$$j^\mu_{\text{top}} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho \quad \longleftrightarrow \quad j^\mu = \bar{\psi}\gamma^\mu\psi \tag{8.42}$$

Checking the Topological Phases

Let's now look for some evidence that this claimed duality is correct. In the case of particle-vortex duality, we checked that the theories looked similar in the weakly coupled regimes $|m^2| \gg e^2$. We can try to do something similar here.

This is simplest for Theory B. To study the relevant physics, we couple the current (8.42) to a background gauge field A_μ. The partition function for each theory then depends on this background field. For Theory B it is

$$Z_B[A] = \int \mathcal{D}\psi \; \exp\left(iS_B[\psi] + i\int d^3x \; j_\mu A_\mu - \frac{1}{2}\frac{1}{4\pi}\epsilon^{\mu\nu\rho}A_\nu\partial_\nu A_\rho\right)$$

Note that we are using the convention described in Section 8.5, in which the half-integer Chern-Simons term arising from the Pauli-Villars regulator field is shown explicitly in the action. We have chosen to add this term with level $k = -1/2$.

When the fermions are massive, $m' \neq 0$, we can integrate them out and generate an effective theory for the background fields A_μ. The lowest dimension term is a Chern-Simons interaction for A_μ,

$$Z[A] = \exp\left(i\frac{\tilde{k}}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \ldots\right) \tag{8.43}$$

From our discussion in Section 8.5, we know that after integrating out the massive fermion ψ the Chern-Simons level for the background gauge field will be

$$\tilde{k} = \frac{1}{2}\left(-1 + \text{sign}(m')\right) = \begin{cases} 0 & m' > 0 \\ -1 & m' < 0 \end{cases}$$

It may seem odd to write down an action for background fields which don't fluctuate, but there's important information in the coefficient $\tilde{k}$; it is the Hall conductivity of the topological gapped phase. This follows by using the partition function $Z[A]$ to compute the response of the current j^μ to a background electric field

$$\langle j^\mu(x)\rangle = -i\frac{\delta \log Z[A]}{\delta A_\mu(x)} \quad \Rightarrow \quad \langle j_i\rangle = -\frac{\tilde{k}}{2\pi}\epsilon_{ij}E_i$$

You can read (a lot) more about the Hall conductivity in the lectures on the Quantum Hall Effect.

We would like to see how this effect is encoded in the bosonic Theory A. We couple the background gauge field A_μ to the topological current (8.40) to get the partition function

$$Z_A[A] = \int \mathcal{D}\phi\mathcal{D}a\ \exp\left(iS_A[\phi,a] + i\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho\right)$$

where we're neglecting gauge fixing terms. This time we only have a scalar field, which does not shift the level of the Chern-Simons term when integrated out. Nonetheless, we can still reproduce the result (8.44) for the Hall conductivity. To see how this works, let's start with the mass $m^2 \gg e^2$ where, at low energies, the scalar field simply decouples, leaving us with the effective action

$$S_{\text{eff}}[a,A] = \int d^3x\ \frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho$$

The equation of motion for the dynamical gauge field a is simply $a = -A$. Substituting this back in, given the effective action (8.43) with $\tilde{k} = -1$.

What happens when $m^2 \ll -e^2$? In this case the scalar field condenses and the dynamical gauge field a becomes gapped. This extra term kills the Hall conductivity, leaving us with (8.43) with $\tilde{k} = 0$. We see that the scalar field does reproduce the topological phases of the the fermion theory as promised. This requires the map,

$$m^2 \quad \longleftrightarrow \quad -m' \quad \Rightarrow \quad \phi^\dagger \phi \quad \longleftrightarrow \quad -\bar{\psi}\psi$$

The agreement between the topological phases is promising, but a long way from demonstrating the claimed duality between Theory A (8.38) and the free fermion (8.41). There are a number of other routes which lead us to the duality (including large N methods, holography, lattice constructions and supersymmetry) but we will not discuss them here. Instead we will assume that bosonization duality holds and ask: what can we do with it?

8.6.3 The Beginning of a Duality Web

We will now show how, starting from the bosonization duality, we can derive further equivalences between quantum field theories. First, some conventions. We will revert to form notation for the gauge fields, and write the Chern-Simons terms as

$$\frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho = \frac{1}{4\pi}ada$$
$$\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho = \frac{1}{2\pi}Ada = \frac{1}{2\pi}adA$$

Both of these are correctly normalised as explained in Section 8.4: they can be added to the action only with integer-valued coefficients. We will denote the gauge field under which matter is charged by adding a subscript to the covariant derivative like this,

$$\mathcal{D}_a\phi = \partial\phi - ia\phi$$

The spacetime index on the derivatives will be suppressed. In what follows, the distinction between dynamical gauge fields and background gauge fields will be crucial. As we mentioned previously, they are distinguished by case. Lower case gauge fields, $a, b, c, \ldots$ will always be dynamical; upper case gauge fields $A, B, C, \ldots$ will always be background.

In this notation, we write the 3d bosonization duality that we described above as an equivalence between two theories

$$|\mathcal{D}_b\phi|^2 - |\phi|^4 + \frac{1}{4\pi}bdb + \frac{1}{2\pi}Adb \quad \longleftrightarrow \quad i\bar{\psi}\not{D}_A\psi - \frac{1}{2}\frac{1}{4\pi}AdA \tag{8.44}$$

Much of this expression is shorthand. First, we have set the mass terms to zero on both sides. This really means that we tune to the critical point. On the fermionic side this is obvious, but the scalar side includes a $|\phi|^4$ term which is taken to mean that we flow to the Wilson-Fisher fixed point of the theory, rather than the free fixed point. Of course, we don't literally get to the Wilson-Fisher by simply setting $m^2 = 0$; instead we must tune m^2, or more generally the coefficient of the relevant operator, as we flow to the IR to hit the critical point. All of this is buried in the notation above.

Second, we reiterate that the scalar ϕ in the above expression is charged under a dynamical gauge field, which we have called b to prepare us for some manipulations ahead. This means that we integrate over (gauge equivalent) configurations of b in the path integral. In contrast, the fermion ψ is charged under the background field A. We can read off the duality map (8.42) between currents by seeing which terms on both side are coupled to A. Finally, we've omitted nearly all the details of the regularisation of the field theory, with one exception: the level $-1/2$ Chern-Simons term on the right-hand-side can be thought of as coming from integrating out a Pauli-Villars regulator. This was explained in Section 8.5. (A warning: some places in the literature adopt a different convention where this level $-1/2$ Chern-Simons term remains hidden in the regulator.)

At this point we start to play with these two theories. Both sides of the duality (8.44) have a background $U(1)$ gauge field A. The key idea is to promote this to a dynamical gauge field. This is misleadingly easy in our notation: we simply write a instead of A. As we explained in Section 8.1, gauging a $U(1)$ symmetry in $d = 2 + 1$ results in a new global symmetry,

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho$$

We couple this to a background gauge field C. This means that we add $\frac{1}{2\pi}AdC$ to both sides of (8.44), and then make $A \to a$ dynamical. This results in a new duality,

$$|\mathcal{D}_b\phi|^2 - |\phi|^4 + \frac{1}{4\pi}bdb + \frac{1}{2\pi}adb + \frac{1}{2\pi}adC \quad\longleftrightarrow\quad i\bar{\psi}\not{D}_a\psi - \frac{1}{2}\frac{1}{4\pi}ada + \frac{1}{2\pi}adC$$

The number of gauge fields on the left-hand side are proliferating. But, at this point, something nice happens: the gauge field a only appears linearly in the action. This means that it acts as a Lagrange multiplier, setting $db = -dC$. But, this, in turn, freezes the first dynamical gauge field b to be equal, up to gauge connection, to the new background field $-C$. The upshot is that we end up with a scalar field theory with no dynamical gauge fields at all, and the duality

$$|\mathcal{D}_C\phi|^2 - |\phi|^4 + \frac{1}{4\pi}CdC \quad\longleftrightarrow\quad i\bar{\psi}\not{D}_a\psi - \frac{1}{2}\frac{1}{4\pi}ada + \frac{1}{2\pi}adC \tag{8.45}$$

This is a new equivalence between two, seemingly very different looking, theories. The left-hand-side is something very familiar: it is the XY Wilson-Fisher fixed point. In contrast, the right-hand side is the a strongly coupled $U(1)$ gauge theory. The claim is that these two fixed points are the same, so

$$\text{XY Wilson-Fisher} \quad \longleftrightarrow \quad U(1)_{-1/2} \text{ coupled to a Dirac fermion}$$

From our first bosonization duality, we have derived another. Similarly, we can go in reverse: starting from the equality of partition functions (8.45), it is not hard to derive the original (8.44).

We can continue in this vein, adding different matter fields and gauging global symmetries, to derive an infinite number of dualities between different 3d Abelian theories with Chern-Simons terms. This is referred to as the *duality web*. Below we give just a handful of interesting examples.

8.6.4 Particle-Vortex Duality Revisited

Our second bosonization duality (8.45) includes a Chern-Simons coupling for the background field C on the left-hand-side. Since we don't integrate over the background field, there is nothing to stop us taking this term onto the other side of the equation. We will also take this opportunity to rename some of the variables. The duality (8.45) is equivalent to

$$|\mathcal{D}_A\phi|^2 - |\phi|^4 \quad \longleftrightarrow \quad i\bar{\psi}\not{D}_b\psi - \frac{1}{2}\frac{1}{4\pi}bdb + \frac{1}{2\pi}bdA - \frac{1}{4\pi}AdA \tag{8.46}$$

Having moved the background Chern-Simons term to the other side, we now play the same game as before: we add a term $\frac{1}{2\pi}AdC$, and then again promote A to a dynamical field, $A \to a$. We now have

$$|\mathcal{D}_a\phi|^2 - |\phi|^4 + \frac{1}{2\pi}adC \quad \longleftrightarrow \quad i\bar{\psi}\not{D}_b\psi - \frac{1}{2}\frac{1}{4\pi}bdb + \frac{1}{2\pi}bda - \frac{1}{4\pi}ada + \frac{1}{2\pi}adC$$

Again, there's a lot of gauge fields on the right-hand-side. Now a does not appear linearly as a Lagrange multiplier, but quadratically. Still, it is begging to be integrated out by imposing the equation of motion $a = b + C$, leaving us with

$$|\mathcal{D}_a\phi|^2 - |\phi|^4 + \frac{1}{2\pi}adC \quad \longleftrightarrow \quad i\bar{\psi}\not{D}_b\psi + \frac{1}{2}\frac{1}{4\pi}bdb + \frac{1}{2\pi}bdC + \frac{1}{4\pi}CdC \tag{8.47}$$

This is still a bosonization duality, relating a scalar theory to a fermionic theory. But the right-hand-side is very nearly the same expression that we started with in (8.46), but with one important difference: two of the Chern-Simons have their sign flipped. In fact, we we send $C \to -C$, all of the Chern-Simons terms have their sign flipped. In other words, this partition function describes the time reversal of the theory in (8.46).

As we have seen, Chern-Simons terms break time reversal, so one would not naively expect that $U(1)_{1/2}$ coupled to a Dirac fermion is time reversal invariant. However, if we take the time reversal of the duality (8.46), we have

$$|\mathcal{D}_{-C}\phi|^2 - |\phi|^4 \quad \longleftrightarrow \quad i\bar{\psi}\not{D}_b\psi + \frac{1}{2}\frac{1}{4\pi}bdb - \frac{1}{2\pi}bd(-C) + \frac{1}{4\pi}CdC \tag{8.48}$$

By charge conjugation we can replace $\mathcal{D}_{-C}\phi \to \mathcal{D}_C\phi$. The left-hand-side is once again the XY critical point. It is clearly time-reversal invariant. The duality tells us that $U(1)_{1/2}$ coupled to a massless fermion must be secretly time reversal invariant: it must emerge as a discrete symmetry of the quantum theory.

Combining (8.47) together with (8.48) gives us yet another duality. It is

$$|\mathcal{D}_a\phi|^2 - |\phi|^4 + \frac{1}{2\pi}adC \quad \longleftrightarrow \quad |\mathcal{D}_C\phi|^2 - |\phi|^4$$

But this is precisely the statement of particle vortex duality that we discussed in Section 8.2.1: the left-hand-side is the Abelian Higgs model while the right-hand-side is the XY model. We learn that particle-vortex duality = bosonization^2.

8.6.5 Fermionic Particle-Vortex Duality

Above we have managed to use 3d bosonization to derive a duality between purely bosonic theories. We might ask: can we do something similar to derive a duality between purely fermionic theories? The answer is yes. But, there will be a new subtlety that we have to address.

We can see this subtlety by retracing the steps above. To derive bosonic particle-vortex duality, we started with the bosonization dual (8.45), moved the background Chern-Simons term to the other side, and then promoted the background gauge field to a dynamical one. To derive a fermionic particle-vortex duality, it is natural to attempt the same manoeuvres for our original bosonization duality (8.44),

$$|\mathcal{D}_b\phi|^2 - |\phi|^4 + \frac{1}{4\pi}bdb + \frac{1}{2\pi}Adb \quad \longleftrightarrow \quad i\bar{\psi}\not{D}_A\psi - \frac{1}{2}\frac{1}{4\pi}AdA \tag{8.49}$$

But we immediately run into a stumbling block: we can't move the background Chern-Simons term to the other side because it is half-integer valued. It is needed on the right-hand-side to ensure that the fermion partition function is gauge invariant.

To get around this, we will stipulate that the background gauge field A only admits flux quantised as

$$\frac{1}{2\pi}\int dA \in 2\mathbf{Z}$$

This is twice the usual requirement. We can then write

$$A = 2C$$

with C a background gauge field whose flux is correctly quantised. The duality (8.49) is then

$$|\mathcal{D}_b\phi|^2 - |\phi|^4 + \frac{1}{4\pi}bdb + \frac{2}{2\pi}Cdb \quad\longleftrightarrow\quad i\bar{\psi}\not{D}_{2C}\psi - \frac{2}{4\pi}CdC \tag{8.50}$$

All Chern-Simons terms are now properly quantised. But the fermion on the right-hand-side has charge 2 under the gauge field C. If we give a fermion of charge q a mass m and integrate it out, it will generate a Chern-Simons term with level $\frac{1}{2}q^2\text{sign}(m)$. (This follows from the fact that the one-loop diagram in Section 8.5 has two insertions of the photon-fermion vertex.) So integrating out a fermion of charge 2 generates an integer-valued Chern-Simons level and there is no problem with the parity anomaly.

Now let us play games with this theory. We will move the CdC background Chern-Simons term to the other side, add $\frac{1}{2\pi}BdC$ to both sides, and finally breath life into C to make it dynamical, $C \to a$. We have

$$|\mathcal{D}_b\phi|^2 - |\phi|^4 + \frac{1}{4\pi}bdb + \frac{2}{2\pi}adb + \frac{2}{4\pi}ada + \frac{1}{2\pi}adB \quad\longleftrightarrow\quad i\bar{\psi}\not{D}_{2a}\psi + \frac{1}{2\pi}adB$$

The mess of mixed Chern-Simons terms on the left-hand-side is easily dealt with: we simply define the new linear combination

$$\hat{a} = a + b$$

Then we find

$$|\mathcal{D}_b\phi|^2 - |\phi|^4 - \frac{1}{4\pi}bdb - \frac{1}{2\pi}bdB + \frac{2}{4\pi}\hat{a}d\hat{a} + \frac{1}{2\pi}\hat{a}dB \quad\longleftrightarrow\quad i\bar{\psi}\not{D}_{2a}\psi + \frac{1}{2\pi}adB$$

But the first four terms in this expression – those which involve ϕ and b — coincide with the time-reversal of the left-hand-side of (8.49). We can then use the duality (8.49) to replace them, leaving us with the promised fermion-fermion duality,

$$i\bar{\psi}\not{D}_A\psi + \frac{1}{2}\frac{1}{4\pi}AdA + \frac{2}{4\pi}\hat{a}d\hat{a} + \frac{1}{2\pi}\hat{a}dA \quad\longleftrightarrow\quad i\bar{\psi}\not{D}_{2a}\psi + \frac{1}{2\pi}adA$$

where we've taken this opportunity to rename the background field A.

What is this final expression telling us? The right-hand-side is a $U(1)$ gauge theory coupled to a single Dirac fermion of charge 2. The left-hand-side is very almost a free fermion. But it also includes a decoupled topological theory, $U(1)_2$, described by the dynamical gauge field $\hat{a}$. We learn that

$$U(1) \text{ with Dirac fermion of charge 2} \quad \longleftrightarrow \quad \text{Free Dirac fermion} + U(1)_2$$

This is the fermionic version of particle-vortex duality, with the monopole operators of the gauge theory identified with the fermion. A closely related duality was first suggested by Son in the context of the half-filled Landau level. It has also been invoked in the context of topological insulators.

8.7 Further Reading

Quantum field theories in $d = 2 + 1$ dimensions have a rather special relation to the real world because, after a Wick rotation, many of them (but not all of them!) can be viewed as statistical field theories in $d = 3 + 0$ dimensions, where they describe systems near critical points. For example, ϕ^4 scalar field theory in $d = 3$ dimensions describes the water boiling in your kettle. (Admittedly, you might need to put a fairly tight lid on the kettle.)

From the high energy perspective, $d = 2 + 1$ dimensions offer another arena to study questions about gauge theories that seemed too challenging in $d = 3 + 1$. Polyakov's demonstration of confinement [158, 159], driven by the proliferation of instantons (monopoles), was a highlight in this regard. Similarly, particle vortex duality was first introduced by Peskin [151], in an attempt to see whether a similar duality in $d = 3 + 1$ could help explain confinement. This was subsequently rediscovered in the condensed matter community by Dasgupta and Halperin, who also performed numerics to find convincing evidence of a second order phase transition [37]. Both of these papers originally expressed the duality in terms of lattice theories; the continuum version that we described here was first proposed in [60].

Chern-Simons theory was introduced by Deser, Jackiw and Templeton [42, 43], initially as a surprising, gauge invariant mechanism to give the three dimensional photon a mass. The depth of the theory became apparent with Witten's Fields medal winning work on knot invariants [224], and the connection to WZW models [52]. The interplay between massive fermions and Chern-Simons terms was discovered in [148] and [166, 167]; a more modern perspective was provided by Witten in [226]. A very clear discussion of the properties of Chern-Simons theories can be found in the lectures by Dunne [48].

The story of 3d bosonization has a long and complicated history. The idea that one can use Chern-Simons terms to transmute the statistics of non-relativistic particles from bosons to fermions was pointed out by Wilczek and Zee [207]. Polyakov was the first to conjecture that there might be a relativistic version of bosonization, but he missed the need to bosonize at the Wilson-Fisher fixed point [160]. The full story came by bringing together a wonderfully diverse set of ideas from both high energy and condensed matter physics. These include dualities in supersymmetric theories [109], large N bosonization and its relation to holography [75, 5, 6, 7], and physics associated to superfluids [13], the half-filled Landau level [183] and topological insulators [197, 135]. The web of dualities among Abelian gauge theories, relating bosonization and particle-vortex duality, was first described [118, 173].

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