1. Draw the integral curves corresponding to a vector field in \( \mathbb{R}^2 \) with components, in Cartesian coordinates, given by
\[
\begin{align*}
X^\mu &= (y, -x) \\
X^\mu &= (x - y, x + y)
\end{align*}
\]
[Hint: you might find life easier in polar coordinates.]

2. Let \( \hat{H} : T_p(M) \to T_p^*(M) \) be a linear map. Define
\[
H(X, Y) = [\hat{H}(Y)](X)
\]
Show that this map is linear in both arguments (e.g. \( H(fX + gY, Z) = fH(X, Z) + gH(Y, Z) \) for \( f, g \in \mathbb{C} \) and \( X, Y, Z \in T_p(M) \)) and hence defines a rank \((0, 2)\) tensor.

Similarly, show that a linear map \( T_p(M) \to T_p(M) \) defines a tensor of rank \((1, 1)\). What tensor \( \delta \) arises from the identity map?

3*. Let \( M \) be a manifold and \( f : M \to \mathbb{R} \) be a smooth function such that \( df = 0 \) at some point \( p \in M \). Let \( x^\mu \) be a coordinate chart defined in a neighbourhood of \( p \). Define
\[
F_{\mu\nu} = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu}.
\]
By considering the transformation law for components show that \( F_{\mu\nu} \) defines a rank \((0, 2)\) tensor. (This is called the Hessian of \( f \) at \( p \).) Construct also a coordinate-free definition and demonstrate its tensorial properties.

4. Let \( g_{\mu\nu} \) be a rank \((0, 2)\) tensor. In a basis, one can regard the components \( g_{\mu\nu} \) as elements of an \( n \times n \) matrix, so that one may define the determinant \( g = \det(g_{\mu\nu}) \). How does \( g \) transform under a change of basis?
5*. Use the Leibniz rule to derive the formula for the Lie derivative of a 1-form $\omega$, valid in any coordinate basis:

$$(\mathcal{L}_X \omega)_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu$$

[Hint: consider $(\mathcal{L}_X \omega)(Y)$ for a vector field $Y$.] Show that the Lie derivative of a $(0,2)$ tensor $g$ is

$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu X^\rho + g_{\nu\rho} \partial_\mu X^\rho$$

For a $p$-form $\eta$, define $\iota_X \eta$ to be the $(p-1)$-form that results from contracting a vector field $X$ with the first index of $\eta$. Show that for a 1-form $\omega$,

$$\mathcal{L}_X \omega = \iota_X (d\omega) + d(\iota_X \omega)$$

6. Let $\omega$ be a $p$-form and $\eta$ a $q$-form. Show that the exterior derivative satisfies the properties

- $d(d\omega) = 0$
- $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d\eta$
- $d(\varphi^* \omega) = \varphi^* (d\omega)$ where $\varphi : M \to N$ for some manifolds $M$ and $N$

7. The exterior derivative of $\omega \in \Lambda^p(M)$ can be defined as

$$d\omega(X_1, \ldots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j-1} X_j(\omega(X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{p+1}))$$

$$+ \sum_{j<k} (-1)^{j+k} \omega([X_j, X_k], X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{p+1})$$

In a coordinate basis $\omega = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$, use this definition to determine the components of $d\omega$ in the cases $p = 1$ and $p = 2$.

8. A three-sphere can be parameterized by Euler angles $(\theta, \phi, \psi)$ where $0 < \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$. Define the following 1-forms

$$\sigma_1 = -\sin \psi \, d\theta + \cos \psi \sin \theta \, d\phi \quad \sigma_2 = \cos \psi \, d\theta + \sin \psi \sin \theta \, d\phi \quad \sigma_3 = d\psi + \cos \theta \, d\phi$$

Show that $d\sigma_1 = \sigma_2 \wedge \sigma_3$ with analogous results for $d\sigma_2$ and $d\sigma_3$. 


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9. [Optional] Let \( \{e_\mu\} \) be a basis of vectors fields, with

\[
[e_\mu, e_\nu] = \gamma^\rho_{\mu\nu} e_\rho.
\]

The functions \( \gamma^\rho_{\mu\nu} \) are known as \textit{commutator components}. For a choice of coordinates \( \{x^\mu\} \), we often work with the \textit{coordinate induced} basis \( e_\mu = \{\partial_\mu\} \). Show that, in this case, \( [e_\mu, e_\nu] = 0 \).

The purpose of this question is to show the converse: that \( [e_\mu, e_\nu] = 0 \) only for a coordinate induced basis. Consider a general basis \( \{e_\mu\} \) and the dual basis \( \{f^\mu\} \) of one-forms. In general, these can be expanded as

\[
e_\mu = e_\mu^\rho \frac{\partial}{\partial x^\rho} \quad \text{and} \quad f^\mu = f^\mu_\rho dx^\rho
\]

where \( e_\mu^\rho f^\nu_\rho = \delta^\nu_\mu \). Show that

\[
e_\mu^\sigma \frac{\partial e_\nu^\lambda}{\partial x^\sigma} - e_\nu^\sigma \frac{\partial e_\mu^\lambda}{\partial x^\sigma} = \gamma^\rho_{\mu\nu} e_\rho^\lambda
\]

Hence deduce that

\[
e_\mu^\sigma e_\nu^\lambda \frac{\partial f^\rho_\lambda}{\partial x^\sigma} - e_\nu^\sigma e_\mu^\lambda \frac{\partial f^\rho_\lambda}{\partial x^\sigma} = -\gamma^\rho_{\mu\nu},
\]

and finally that

\[
\frac{\partial f^\rho_\sigma}{\partial x^\lambda} - \frac{\partial f^\rho_\lambda}{\partial x^\sigma} = -\gamma^\rho_{\mu\nu} f^\mu_\lambda f^\nu_\sigma
\]

Use this result, together with the Poincaré lemma, to show that if \( [e_\mu, e_\nu] = 0 \ \forall \ \mu, \nu \) then the basis is coordinate induced.