Quantum Field Theory: Example Sheet 2

Dr David Tong, October 2007

1. A string has classical Hamiltonian given by

$$H = \sum_{n=1}^{\infty} \left(\tfrac{1}{2} p_n^2 + \tfrac{1}{2} \omega_n^2 \, q_n^2 \right) \tag{1}$$

where ω_n is the frequency of the nth mode. (Compare this Hamiltonian to the Lagrangian (3) in Example Sheet 1. We have set the mass per unit length in that question to $\sigma = 1$ to simplify some of the formulae a little). After quantization, q_n and p_n become operators satisfying

$$[q_n, q_m] = [p_n, p_m] = 0 \quad \text{and} \quad [q_n, p_m] = i\delta_{nm} \tag{2}$$

Introduce creation and annihilation operators a_n and $a_n^\dagger$,

$$a_n = \sqrt{\frac{\omega_n}{2}}\, q_n + \frac{i}{\sqrt{2\omega_n}}\, p_n \quad \text{and} \quad a_n^\dagger = \sqrt{\frac{\omega_n}{2}}\, q_n - \frac{i}{\sqrt{2\omega_n}}\, p_n \tag{3}$$

Show that they satisfy the commutation relations

$$[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0 \quad \text{and} \quad [a_n, a_m^\dagger] = \delta_{nm} \tag{4}$$

Show that the Hamiltonian of the system can be written in the form

$$H = \sum_{n=1}^{\infty} \tfrac{1}{2}\, \omega_n \left(a_n a_n^\dagger + a_n^\dagger a_n \right) \tag{5}$$

Given the existence of a ground state $|0\rangle$ such that $a_n |0\rangle = 0$, explain how, after removing the vacuum energy, the Hamiltonian can be expressed as

$$H = \sum_{n=1}^{\infty} \omega_n a_n^\dagger a_n \tag{6}$$

Show further that $[H, a_n^\dagger] = \omega_n\, a_n^\dagger$ and hence calculate the energy of the state

$$|l_1, l_2, \ldots, l_N\rangle = \left(a_1^\dagger\right)^{l_1} \left(a_2^\dagger\right)^{l_2} \ldots \left(a_N^\dagger\right)^{l_N} |0\rangle \tag{7}$$

2. The Fourier decomposition of a real scalar field and its conjugate momentum in the Schrödinger picture is given by

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \right] \tag{8}$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left[a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \right] \tag{9}$$

Show that the commutation relations

$$[\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \quad \text{and} \quad [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \tag{10}$$

imply that

$$[a_{\vec{p}}, a_{\vec{q}}] = [a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0 \quad \text{and} \quad [a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{11}$$

3. Consider a real scalar field with the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 \tag{12}$$

Show that, after normal ordering, the conserved four-momentum $P^\mu = \int d^3x\, T^{0\mu}$ takes the operator form

$$P^\mu = \int \frac{d^3p}{(2\pi)^3}\, p^\mu\, a_{\vec{p}}^\dagger a_{\vec{p}} \tag{13}$$

where $p^0 = E_{\vec{p}}$ in this expression. From this expression for P^μ verify that if $\phi(x)$ is now in the Heisenberg picture, then

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x) \tag{14}$$

4. Show that in the Heisenberg picture,

$$\dot{\phi}(x) = i[H, \phi(x)] = \pi(x) \quad \text{and} \quad \dot{\pi}(x) = i[H, \pi(x)] = \nabla^2\phi(x) - m^2\phi(x) \tag{15}$$

Hence show that the operator $\phi(x)$ satisfies the Klein-Gordon equation.

5. Let $\phi(x)$ be a real scalar field in the Heisenberg picture. Show that the relativistically normalized one-particle states $|p\rangle = \sqrt{2E_{\vec{p}}}\, a_{\vec{p}}^\dagger\, |0\rangle$ satisfy

$$\langle 0|\, \phi(x)\, |p\rangle = e^{-ip\cdot x} \tag{16}$$

6. In Example Sheet 1, you showed that the classical angular momentum of field is given by

$$Q_i = \epsilon_{ijk} \int d^3x \; (x^j T^{0k} - x^k T^{0j}) \tag{17}$$

Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian (12). Show that, after normal ordering, the quantum operator Q_i can be written as

$$Q_i = -i\,\epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3}\, a_{\vec{p}}^\dagger \left(p^j \frac{\partial}{\partial p_k} - p^k \frac{\partial}{\partial p_j} \right) a_{\vec{p}} \tag{18}$$

Hence confirm that the quanta of the scalar field have spin zero (i.e. a stationary one-particle state $|\vec{p} = 0\rangle$ has zero angular momentum).

7. The purpose of this question is to introduce you to non-relativistic quantum field theory. This is the only place you will encounter such a thing in this course. Consider the Lagrangian for a complex scalar field ψ given by

$$\mathcal{L} = +i\psi^\star \partial_0 \psi - \frac{1}{2m}\, \nabla\psi^\star \cdot \nabla\psi \tag{19}$$

Determine the equation of motion, the energy-momentum tensor and the conserved current arising from the symmetry $\psi \to e^{i\alpha}\psi$. Show that the momentum conjugate to ψ is $i\psi^\star$ and compute the classical Hamiltonian.

We now wish to quantize this theory. We will work in the Schrödinger picture. Explain why the correct commutation relations are

$$[\psi(\vec{x}), \psi(\vec{y})] = [\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})] = 0 \quad \text{and} \quad [\psi(\vec{x}), \psi^\dagger(\vec{y})] = \delta^{(3)}(\vec{x} - \vec{y}) \tag{20}$$

Expand the fields in a Fourier decomposition as

$$\begin{aligned} \psi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\, a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} \\ \psi^\dagger(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\, a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \end{aligned} \tag{21}$$

Determine the commutation relations obeyed by $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$. Why do we have only a single set of creation and annihilation operators $a_{\vec{p}}$, $a_{\vec{p}}^\dagger$ even though ψ is complex? What is the physical significance of this fact? Show that one particle states have the energy appropriate to a free non-relativistic particle of mass m.

8. Show that the time ordered product $\mathrm{T}\left(\phi(x_1)\phi(x_2)\right)$ and the normal ordered product $:\phi(x_1)\phi(x_2):$ are both symmetric under the interchange of x_1 and x_2. Deduce that the Feynman propagator $\Delta_F(x_1 - x_2)$ has the same symmetry property.

9. Verify Wick's theorem for the case of three scalar fields:

$$\begin{aligned} T\left(\phi(x_1)\phi(x_2)\phi(x_3)\right) &= :\phi(x_1)\phi(x_2)\phi(x_3): + \phi(x_1)\Delta_F(x_2 - x_3) \\ &\quad + \phi(x_2)\Delta_F(x_3 - x_1) + \phi(x_3)\Delta_F(x_1 - x_2) \end{aligned} \tag{22}$$

10. Consider the scalar Yukawa theory given by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^\star \partial^\mu \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - M^2 \psi^\star \psi - \frac{1}{2} m^2 \phi^2 - g \psi^\star \psi \phi \tag{23}$$

Compute the amplitude for

- "Nucleon-Anti-Nucleon" annihilation $\psi + \bar{\psi} \rightarrow \phi$ at order g
- "Nucleon-Meson" scattering $\phi + \psi \rightarrow \phi + \psi$ at order g^2