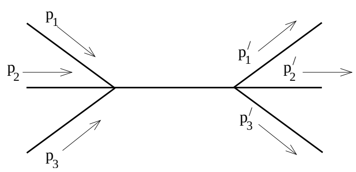
Quantum Field Theory: Example Sheet 4

Dr David Tong, November 2007

1. A real scalar field with ϕ^4 interaction has the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

Use Dyson's formula and Wick's theorem to show that the leading order contribution to 3-particle → 3-particle scattering includes the amplitude

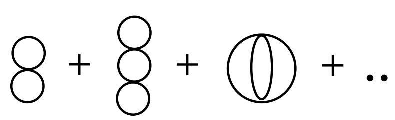


$$= (-i\lambda)^2 \frac{i}{(p_1+p_2+p_3)^2 - m^2} \tag{2}$$

Check that this result is consistent with the Feynman rules for the theory. What other diagrams also contribute to this process?

2. Examine $\langle 0|S|0\rangle$ to order λ^2 in ϕ^4 theory. Identify the different diagrams with the different contributions arising from an application of Wick's theorem. Confirm that to order λ^2, the combinatoric factors work out so that the the vacuum to vacuum amplitude is given by the exponential of the sum of distinct vacuum bubble types,

$$\langle 0|\, S\, |0\rangle = \exp \Big(\quad + \quad + \quad + \dots \Big) \tag{3}$$



3. Consider the Lagrangian for 3 scalar fields ϕ_i, $i = 1, 2, 3$, given by

$$\mathcal{L} = \sum_{i=1}^{3} \frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m^2(\sum_{i=1}^{3}\phi_i^2) - \frac{\lambda}{8}(\sum_{i=1}^{3}\phi_i^2)^2 \tag{4}$$

Show that the Feynman propagator for the free field theory (i.e. $\lambda = 0$) is of the form

$$\langle 0|T\phi_i(x)\phi_j(y)|0\rangle = \delta_{ij}D_F(x-y) \tag{5}$$

where $D_F(x-y)$ is the usual scalar propagator. Write down the Feynman rules of the theory. Compute the amplitude for the scattering $\phi_i\phi_j \to \phi_k\phi_l$ to lowest order in λ.

4. The Lagrangian for Yukawa theory is given by

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\,\partial\!\!\!/ - m)\psi - \lambda\phi\bar{\psi}\psi \tag{6}$$

a) Consider $\psi\psi \to \psi\psi$ scattering, with the initial and final states given by,

$$\begin{aligned} |i\rangle &= \sqrt{4E_{\vec{p}}E_{\vec{q}}}\, b_{\vec{p}}^{s\,\dagger}\, b_{\vec{q}}^{r\,\dagger}\, |0\rangle \\ |f\rangle &= \sqrt{4E_{\vec{p}'}E_{\vec{q}'}}\, b_{\vec{p}'}^{s'\,\dagger}\, b_{\vec{q}'}^{r'\,\dagger}\, |0\rangle \end{aligned} \tag{7}$$

Show using Dyson's formula and Wick's theorem that the scattering amplitude at order λ^2 is given by,

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p}\,') \cdot u^s(\vec{p})]\,[\bar{u}^{r'}(\vec{q}\,') \cdot u^r(\vec{q})]}{(p'-p)^2 - \mu^2} - \frac{[\bar{u}^{s'}(\vec{p}\,') \cdot u^r(\vec{q})]\,[\bar{u}^{r'}(\vec{q}\,') \cdot u^s(\vec{p})]}{(q'-p)^2 - \mu^2} \right)$$

Draw the two Feynman diagrams that correspond to these two terms.

b) Consider now $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering, with initial and final states given by

$$\begin{aligned} |i\rangle &= \sqrt{4E_{\vec{p}}E_{\vec{q}}}\, b_{\vec{p}}^{s\,\dagger}\, c_{\vec{q}}^{r\,\dagger}\, |0\rangle \\ |f\rangle &= \sqrt{4E_{\vec{p}'}E_{\vec{q}'}}\, b_{\vec{p}'}^{s'\,\dagger}\, c_{\vec{q}'}^{r'\,\dagger}\, |0\rangle \end{aligned} \tag{8}$$

Show that the amplitude is this time given by

$$\mathcal{A} = -(-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p}\,') \cdot u^s(\vec{p})]\,[\bar{v}^{r}(\vec{q}) \cdot v^{r'}(\vec{q}\,')]}{(p-p')^2 - \mu^2} - \frac{[\bar{v}^{r}(\vec{q}) \cdot u^s(\vec{p})]\,[\bar{u}^{s'}(\vec{p}\,') \cdot v^{r'}(\vec{q}\,')]}{(p+q)^2 - \mu^2} \right)$$

(Be careful with minus signs!!). What are the Feynman diagrams that now contribute?

5. The Lagrangian for a pseudoscalar Yukawa interaction is given by

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\,\partial\!\!\!/ - m)\psi - \lambda\phi\bar{\psi}\gamma^5\psi \tag{9}$$

Write down the Feynman rules for this theory. Use this to write down the amplitude at order λ^2 for $\psi\psi \to \psi\psi$ scattering and $\psi\bar{\psi} \to \psi\bar{\psi}$ scattering.

6. Any vector function $\mathbf{f}(\mathbf{x})$ has a decomposition into a sum of transverse (zero divergence) and longitudinal (zero curl) parts, namely

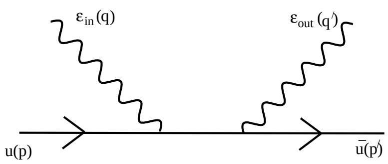
$$\mathbf{f} = \nabla \times \mathbf{g} + \nabla h \equiv \mathbf{f}^T + \mathbf{f}^L \tag{10}$$

where $\mathbf{g}$ and h are unique if one imposes the additional constraint $\nabla \cdot \mathbf{g} = 0$ and certain vanishing conditions at infinity. By taking the divergence and curl of equation (10), determine $\mathbf{g}$ and h in terms of $\mathbf{f}$. Show formally that

$$\mathbf{f}^T = \mathbf{f} - \nabla(\nabla^2)^{-1}\nabla \cdot \mathbf{f} \tag{11}$$

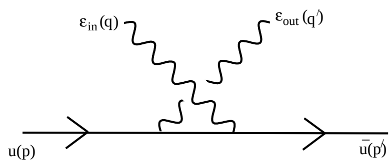
Use this result to comment on the commutation relations of the quantized electromagnetic gauge potential in Coulomb gauge.

7. Consider the Compton scattering process $e^-\gamma \to e^-\gamma$ in QED. Let the incoming photon have polarization vector $\epsilon_{\rm in}^\mu$, and the outgoing photon have polarization $\epsilon_{\rm out}^\mu$. Use the Feynman rules to derive the following amplitude associated to the lowest order diagram,



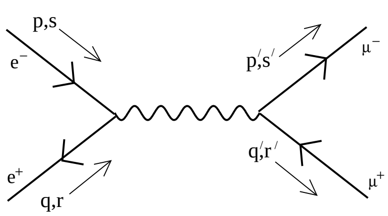
$$= i(-ie)^2 \bar{u}^{r'}(\vec{p}\,') \; \not\epsilon_{\rm out} \frac{(\not p + \not q + m)}{(p+q)^2 - m^2} \; \not\epsilon_{\rm in} u^s(\vec{p})$$

Compute also the contribution from the diagram



The complete amplitude at order e^2 is the sum of these two contributions. Show that the total amplitude vanishes if $\epsilon_{\rm in}$ is replaced by the incoming photon momentum q then the amplitude vanishes. Check that the same holds true if $\epsilon_{\rm out}$ is replaced by q' . (Note that it will be helpful to recall the equation $(\not p - m)u(\vec{p}) = 0$ satisfied by the spinor).

8. Use the Feynman rules to show that the QED amplitude for $e^-e^+ \to \mu^-\mu^+$ is given at lowest order in e by,



$$= (-ie)^2 \frac{[\bar{v}_e^r(\vec{q})\gamma_\mu u_e^s(\vec{p})]\,[\bar{u}_m^{s'}(\vec{p}\,')\gamma^\mu v_m^{r'}(\vec{q}\,')]}{(p+q)^2} \tag{12}$$

where the subscripts e and m denote whether the spinors satisfy the Dirac equation for electrons or for muons.

9. Viki Weisskopf is one of the more charming characters from the history of quantum field theory. This from his autobiography:

> "Pauli asked me to calculate the amplitude for pair creation of scalar particles by photons. It was only a short time after Bethe and Heitler had solved the same problem for electrons and positrons. I met Bethe in Copenhagen at a conference and asked him to tell me how he did the calculations. I also inquired how long it would take to perform this task; he answered, "It would take me three days, but you will need about three weeks." He was right, as usual; furthermore, the published result was wrong by a factor of two."

Can you do better?

10. Now you understand the role played by fields in Nature, why do you think classical physicists such as Faraday and Maxwell found it useful to introduce the concept of the electric and magnetic field, but never fields for the electron or other particles?