6. Edge Modes

If a quantum Hall fluid is confined to a finite region, there will be gapless modes that live on the edge. We've already met these in Section 2.1 for the integer quantum Hall states where we noticed that they are chiral: they propagate only in one direction. This is a key property shared by all edge modes.

In this section we'll describe the edge modes for the fractional quantum Hall states. At first glance it may seem like this is quite an esoteric part of the story. However, there's a surprise in store. The edge modes know much more about the quantum Hall states than you might naively imagine. Not only do they offer the best characterisation of these states, but they also provide a link between the Chern-Simons approach and the microscopic wavefunctions.

6.1 Laughlin States

We start by looking at edge modes in the $\nu = 1/m$ Laughlin states. The basic idea is that the ground state forms an incompressible disc. The low-energy excitations of this state are deformations which change its shape, but not its area. These travel as waves around the edge of the disc, only in one direction. In what follows, we will see this picture emerging from several different points of view.

6.1.1 The View from the Wavefunction

Let's first return to the description of the quantum Hall state in terms of the microscopic wavefunction. Recall that when we were discussing the toy Hamiltonians in Section 3.1.3, the Hamiltonian $H_{\rm toy}$ that we cooked up in (3.15) had the property that the zero energy ground states are

$$\psi(z_i) = s(z_i) \prod_{i<j} (z_i - z_j)^m \, e^{-\sum_i |z_i|^2/4l_B^2} \tag{6.1}$$

for any symmetric polynomial $s(z_i)$. The Laughlin wavefunction with $s(z_i) = 1$ has the property that it is the most compact of these states. Equivalently, it is the state with the lowest angular momentum. We can pick this out as the unique ground state by adding a placing the system in a confining potential which we take to be the angular momentum operator J,

$$V_{\rm confining} = \omega J$$

The Laughlin state, with $s(z_i) = 1$, then has ground state energy

$$E_0 = \frac{\omega}{2} m N(N-1)$$

where N is the number of electrons. What about the excited states? We can write down a basis of symmetric polynomials

$$s_n(z_i) = \sum_i z_i^n$$

The most general state (6.1) has polynomial

$$s(z_i) = \sum_{n=1}^{\infty} s_n(z_i)^{d_n}$$

which has energy

$$E = E_0 + \omega \sum_{n=1}^{\infty} n d_n \tag{6.2}$$

We see that each polynomial s_n contributes an energy

$$E_n = \omega n$$

We're going to give an interpretation for this. Here we'll simply pull the interpretation out of thin air, but we'll spend the next couple of sections providing a more direct derivation. The idea is to interpret this as the Kaluza-Klein spectrum as a gapless $d = 1 + 1$ scalar field. We'll think of this scalar as living on the edge of the quantum Hall droplet. Recall that the Laughlin state has area $A = 2\pi m N l_B^2$ which means that the boundary is a circle of circumference $L = 2\pi\sqrt{2mN} l_B$. The Fourier modes of such a scalar field have energies

$$E_n = \frac{2\pi n v}{L}$$

where v is the speed of propagation the excitations. (Note: don't worry if this formula is unfamiliar: we'll derive it below). Comparing the two formulae, we see that the speed of propagation depends on the strength of the confining potential,

$$v = \frac{L\omega}{2\pi}$$

To see that this is a good interpretation of the spectrum (6.2), we should also check that the degeneracies match. There's a nice formula for the number of quantum Hall states with energy $q\omega$ with $q \in \mathbf{Z}^+$. To see this, let's look at some examples. There is, of course, a unique ground state. There is also a unique state with $\Delta E = \omega$ which has $d_1 = 1$ and $d_n = 0$ for $n \geq 2$. However, for $\Delta E = 2\omega$ there are two states: $d_1 = 2$ or

$d_2 = 1$. And for $\Delta E = 3\omega$ there are 3 states: $d_1 = 3$, or $d_1 = 1$ and $d_2 = 2$, or $d_3 = 1$. In general, the number of states at energy $\Delta E = q\omega$ is the number of *partitions* of the integer q. This is the number of ways of writing q as a sum of positive integers. It is usually denoted as $P(q)$,

$$\left.\begin{array}{c}\text{Degeneracy of states}\\ \text{with } \Delta E = a\omega\end{array}\right\} \; = \; P(q) \tag{6.3}$$

Now let's compare this to the Fourier modes of a scalar field. Suppose that we focus on the modes that only move one way around the circle, labelled by the momenta $n > 0$. Then there's one way to create a state with energy $E = 2\pi v/L$: we excite the first Fourier mode once. There are two ways to create a state with energies $E = 4\pi v/L$: we excite the first Fourier mode twice, or we excite the second Fourier mode once. And so on. What we're seeing is that the degeneracies match the quantum Hall result (6.3) if we restrict the momenta to be positive. If we allowed the momenta to also be negative, we would not get the correct degeneracy of the spectrum. This is our first hint that the edge modes are described by a chiral scalar field, propagating only in one direction.

6.1.2 The View from Chern-Simons Theory

Let's see how this plays out in the effective Chern-Simons theory. We saw in Section 5.2 that the low-energy effective action for the Laughlin state is

$$S_{CS}[a] = \frac{m}{4\pi}\int d^3x\; \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \tag{6.4}$$

where we're working in units in which $e = \hbar = 1$.

We'll now think about this action on a manifold with boundary. Ultimately we'll be interested in a disc-shaped quantum Hall droplet. But to get started it's simplest to think of the boundary as a straight line which we'll take to be at $y = 0$. The quantum Hall droplet lies at $y < 0$ while at $y > 0$ there is only the vacuum.

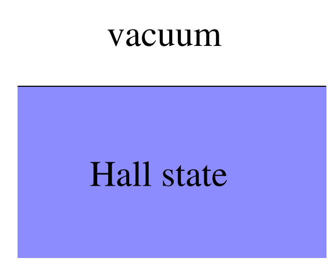

Figure 46:

There are a number of things to worry about in the presence of a boundary. The first is important for any field theory. When we derive the equations of motion from the action, we always integrate by parts and discard the boundary term. But now there's a boundary, we have to be more careful to make sure that this term vanishes. This is simply telling us that we should specify some boundary condition if we want to make the theory well defined. For our Chern-Simons theory, a variation of the fields gives

$$\begin{aligned}\delta S_{CS} &= \frac{m}{4\pi}\int d^3x\ \epsilon^{\mu\nu\rho}\left[\delta a_\mu \partial_\nu a_\rho + a_\mu \partial_\nu \delta a_\rho\right] \\ &= \frac{m}{4\pi}\int d^3x\ \epsilon^{\mu\nu\rho}\left[\delta a_\mu f_{\nu\rho} + \partial_\mu(a_\nu \delta a_\rho)\right]\end{aligned}$$

Minimising the action gives the required equation of motion $f_{\mu\nu} = 0$ only if we can set the last term to zero. We can do this if either by setting $a_t(y=0) = 0$ on the boundary, or by setting $a_x(y=0) = 0$. Alternatively, we can take a linear combination of these. We choose

$$(a_t - v a_x)\Big|_{y=0} = 0 \tag{6.5}$$

Here we've introduced a parameter v; this will turn out to be the velocity of excitations on the boundary. Note that the Chern-Simons theory alone has no knowledge of this speed. It's something that we have to put in by hand through the boundary condition.

The next issue is specific to Chern-Simons theory. As we've mentioned before, the action (6.4) is only invariant up to a total derivative. Under a gauge transformation

$$a_\mu \to a_\mu + \partial_\mu \omega$$

we have

$$S_{CS} \to S_{CS} + \frac{m}{4\pi}\int_{y=0} dxdt\ \omega(\partial_t a_x - \partial_x a_t)$$

and the Chern-Simons action is no longer gauge invariant. We're going to have to deal with this. One obvious strategy is simply to insist that we only take gauge transformations that vanish on the boundary, so that $w(y=0) = 0$. This has the happy corrolary that gauge transformations don't change our chosen boundary condition for the gauge fields. However, this approach has other consequences. Recall that the role of gauge transformations is to identify field configurations, ensuring that they are physically indistinguishable. Said another way, gauge transformations kill would-be degrees of freedom. This means that restricting the kinds of gauge transformations will resurrect some these degrees of freedom from the dead.

To derive an action for these degrees of freedom, we choose a gauge. The obvious one is to extend the boundary condition (6.5) into the bulk, requiring that

$$a_t - v a_x = 0 \tag{6.6}$$

everywhere. The easiest way to deal with this is to work in new coordinates

$$t' = t \quad , \quad x' = x + vt \quad , \quad y' = y \tag{6.7}$$

The Chern-Simons action is topological and so invariant under such coordinate transformations if we also transform the gauge fields as

$$a'_{t'} = a_t - va_x \quad , \quad a'_{x'} = a_x \quad , \quad a'_{y'} = a_y \tag{6.8}$$

so the gauge fixing condition (6.6) becomes simply

$$a'_{t'} = 0 \tag{6.9}$$

But now this is easy to deal with. The constraint imposed by the gauge fixing condition is simply $f'_{x'y'} = 0$. Solutions to this are simply

$$a'_i = \partial_i \phi$$

with $i = x', y'$. Of course, usually such solutions would be pure gauge. But that's what we wanted: a mode that was pure gauge which becomes dynamical. To see how this happens, we simply need to insert this solution back into the Chern-Simons action which, having set $a'_{t'} = 0$, is

$$\begin{aligned} S_{CS} &= \frac{m}{4\pi} \int d^3x' \; \epsilon^{ij} a'_i \partial_{t'} a'_j \\ &= \frac{m}{4\pi} \int d^3x' \; \partial_{x'}\phi \, \partial_{t'} \partial_{y'} \phi - \partial_{y'}\phi \, \partial_{t'} \partial_{x'} \phi \\ &= \frac{m}{4\pi} \int_{y=0} d^2x' \; \partial_{t'}\phi \partial_{x'}\phi \end{aligned}$$

Writing this in terms of our original coordinates, we have

$$S = \frac{m}{4\pi} \int d^2x \; \partial_t \phi \partial_x \phi - v(\partial_x \phi)^2 \tag{6.10}$$

This is sometimes called the Floreanini-Jackiw action. It looks slightly unusual, but it actually describes something very straightforward. The equations of motion are

$$\partial_t \partial_x \phi - v \partial_x^2 \phi = 0 \tag{6.11}$$

If we define a new field,

$$\rho = \frac{1}{2\pi} \frac{\partial \phi}{\partial x}$$

then the equation of motion is simply

$$\partial_t \rho(x,t) - v\partial_x \rho(x,t) = 0 \tag{6.12}$$

This is the expression for a chiral wave propagating at speed v. The equation has solutions of the form $\rho(x + vt)$. However, waves propagating in the other direction, described by $\rho(x - vt)$ are not solutions. The upshot of this analysis is that the $U(1)$ Chern-Simons theory has a chiral scalar field living on the boundary. This, of course, is the same conclusion that we came to by studying the excitations above the Laughlin state.

The Interpretation of ρ

There's a nice physical interpretation of the chiral field ρ. To see this, recall that our Chern-Simons theory is coupled to a background gauge field A_μ through the coupling

$$S_J = \int d^3x\ A_\mu J^\mu = \frac{1}{2\pi}\int d^3x\ \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho$$

This is invariant under gauge transformations of a_μ but, in the presence of a boundary, is not gauge invariant under transformations of A_μ. That's not acceptable. While a_μ is an emergent gauge field, which only exists within the sample, A_μ is electromagnetism. It doesn't stop just because the sample stops and there's no reason that we should only consider electromagnetic gauge transformations that vanish on the boundary. However, there's a simple fix to this. We integrate the expression by parts and throw away the boundary term. We then get the subtly different coupling

$$S_J = \frac{1}{2\pi}\int d^3x\ \epsilon^{\mu\nu\rho} a_\mu \partial_\nu A_\rho$$

This is now invariant under electromagnetic gauge transformations and, as we saw above, under the restricted gauge transformations of a_μ. This is the correct way to couple electromagnetism in the presence of a boundary.

We'll set $A_y = 0$ and turn on background fields A_t and A_x, both of which are independent of the y direction. Then, working in the coordinate system (6.7), (6.8), and the gauge (6.9), the coupling becomes

$$\begin{aligned} S_J &= \frac{1}{2\pi}\int d^3x\ a'_{y'}(\partial_{t'}A'_{x'} - \partial_{x'}A'_{t'}) \\ &= \frac{1}{2\pi}\int d^3x\ \partial_{y'}\phi(\partial_{t'}A'_{x'} - \partial_{x'}A'_{t'}) \\ &= \frac{1}{2\pi}\int_{y=0} d^2x\ \phi(\partial_{t'}A'_{x'} - \partial_{x'}A'_{t'}) \end{aligned}$$

Integrating the first term by parts gives $\partial_{t'}\phi = \partial_t\phi - v\partial_x\phi$. (Recall that $\partial_{t'}$ transforms like $a'_{t'}$ and so is not the same thing as ∂_t). But this vanishes or, at least, is a constant by the equation of motion (6.11). We'll set this term to zero. We're left with

$$S_J = \frac{1}{2\pi}\int_{y=0} dtdx\ (A_t - vA_x)\partial_x\phi$$

The coupling to A_t tells us that the field

$$\rho = \frac{1}{2\pi}\frac{\partial\phi}{\partial x}$$

is the charge density along the boundary. The coupling to A_x tells us that $-v\rho$ also has the interpretation as the current. The same object is both charge density and current reflects the fact that the waves propagate in a chiral manner with speed v. The current is conserved by virtue of the chiral wave equation (6.12)

There is a simple intuitive way to think about ρ. Consider the edge of the boundary as shown in the figure. The excitations that we're considering are waves in which the boundary deviates from a straight line. If the height of these waves is $h(x,t)$, then the charge density is $\rho(x,t) = nh(x,t)$ where $n = 1/2\pi m l_B^2$ is the density of the Laughlin state at filling fraction $\nu = 1/m$.

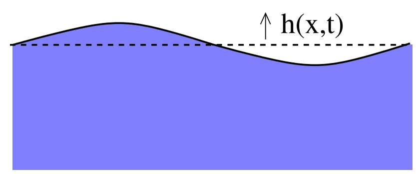

Figure 47:

Towards an Interpretation of ϕ

There's one important property of ϕ that we haven't mentioned until now: it's periodic. This follows because the emergent gauge $U(1)$ gauge group is compact. When we write the flat connection $a_\mu = \partial_\mu\phi$, what we really mean is

$$a_\mu = ig^{-1}\partial_\mu g \quad \text{with} \quad g = e^{-i\phi}$$

This tells us that ϕ should be thought of as a scalar with period 2π. It is sometimes called a *compact* boson.

As an aside: sometimes in the literature, people work with the rescaled field $\phi \to \sqrt{m}\phi$. This choice is made so that the normalisation of the action (6.10) becomes $1/2\pi$ for all filling fractions. The price that's paid is that the periodicity of the boson becomes $2\pi\sqrt{m}$. In these lectures, we'll work with the normalisation (6.10) in which ϕ has period 2π.

This possibility allows us to capture some new physics. Consider the more realistic situation where the quantum Hall fluid forms a disc and the boundary is a circle $\mathbf{S}^1$ of circumference L. We'll denote the coordinate around the boundary as $\sigma \in [0, L)$. The total charge on the boundary is

$$Q = \int_0^L d\sigma \ \rho = \frac{1}{2\pi} \int_0^L d\sigma \ \frac{\partial \phi}{\partial \sigma} \tag{6.13}$$

It's tempting to say that this vanishes because it's the integral of a total derivative. But if ϕ is compact, that's no longer true. We have the possibility that ϕ winds some number of times as we go around the circle. For example, the configuration $\phi = 2\pi p\sigma/L$ is single valued for any integer p. Evaluated on this configuration, the charge on the boundary is $Q = p$. Happily, the charge is quantised even though we haven't needed to invoke quantum mechanics anywhere: it's quantised for topological reasons.

Although we've introduced Q as the charge on the boundary, it's really capturing the charge in the bulk. This is simply because the quantum Hall fluid is incompressible. If you add p electrons to the system, the boundary has to swell a little bit. That's what Q is measuring. This is our first hint that the boundary knows about things that happen in the bulk.

There's one other lesson to take from the compact nature of ϕ. Observables should be single valued. This means that ϕ itself is not something we can measure. One way around this is to look at $\partial_x \phi$ which, as we have seen, gives the charge density. However, one could also consider the exponential operators $e^{i\phi}$. What is the interpretation of these? We will answer this in Section 6.1.4 where we will see that $e^{i\phi}$ describes quasi-holes in the boundary theory.

6.1.3 The Chiral Boson

We've seen that the edge modes of the quantum Hall fluid are described by a chiral wave. From now on, we'll think of the quantum Hall droplet as forming a disc, with the boundary a circle of circumference $L = 2\pi\sqrt{2mN} l_B$. We'll parameterise the circle by $\sigma \in [0, L)$. The chiral wave equation obeyed by the density is

$$\partial_t \rho(\sigma, t) - v\partial_\sigma \rho(\sigma, t) = 0 \tag{6.14}$$

which, as we've seen, arises from the action for a field

$$S = \frac{m}{4\pi} \int_{\mathbf{R}\times\mathbf{S}^1} dt d\sigma \ \partial_t \phi \, \partial_\sigma \phi - v(\partial_\sigma \phi)^2 \tag{6.15}$$

The original charge density is related to ϕ by $\rho = \partial_\sigma \phi/2\pi$.

In this section, our goal is to quantise this theory. It's clear from (6.15) that the momentum conjugate to ϕ is proportional to $\partial_\sigma\phi$. If you just naively go ahead and write down canonical commutation relations then there's an annoying factor of 2 that you'll get wrong, arising from the fact that there is a constraint on phase space. To avoid this, the simplest thing to do is to work with Fourier modes in what follows. Because these modes live on a circle of circumference L, we can write

$$\phi(\sigma,t) = \frac{1}{\sqrt{L}}\sum_{n=-\infty}^{\infty}\phi_n(t)\,e^{2\pi i n\sigma/L}$$

and

$$\rho(\sigma,t) = \frac{1}{\sqrt{L}}\sum_{n=-\infty}^{\infty}\rho_n(t)\,e^{2\pi i n\sigma/L}$$

The Fourier modes are related by

$$\rho_n = \frac{ik_n}{2\pi}\phi_n$$

with k_n the momentum carried by the $n^{\rm th}$ Fourier mode given by

$$k_n = \frac{2\pi n}{L}$$

The condition on ϕ and ρ means that $\phi_n^\star = \phi_{-n}$ and $\rho_n^\star = \rho_{-n}$. Note that the zero mode ρ_0 vanishes according to this formula. This reflects the fact that the corresponding zero mode ϕ_0 decouples from the dynamics since the action is written using $\partial_\sigma\phi$. The correct treatment of this zero mode is rather subtle. In what follows, we will simply ignore it and set $\phi_0 = 0$. Using these Fourier modes, the action (6.15) becomes

$$\begin{aligned} S &= \frac{m}{4\pi}\int dt\sum_{n=-\infty}^{\infty}\Big(ik_{-n}\dot{\phi}_n\phi_{-n} + vk_nk_{-n}\phi_n\phi_{-n}\Big) \\ &= -\frac{m}{2\pi}\int dt\sum_{n=0}^{\infty}\Big(ik_n\dot{\phi}_n\phi_{-n} + vk_n^2\phi_n\phi_{-n}\Big) \end{aligned}$$

This final expression suggests that we treat the Fourier modes ϕ_n with $n>0$ as the "coordinates" of the problem. The momenta conjugate to ϕ_n is then proportional to ϕ_{-n}. This gives us the Poisson bracket structure for the theory or, passing to quantum mechanics, the commutators

$$\begin{aligned} [\phi_n,\phi_{n'}] &= \frac{2\pi}{m}\frac{1}{k_n}\delta_{n+n'} \\ [\rho_n,\phi_{n'}] &= \frac{i}{m}\delta_{n+n'} \\ [\rho_n,\rho_{n'}] &= \frac{k_n}{2\pi m}\delta_{n+n'} \end{aligned}$$

This final equation is an example of a $U(1)$ *Kac-Moody algebra.* It's a provides a powerful constraint on the dynamics of conformal field theories. We won't have much use for this algebra in the present context, but its non-Abelian extension plays a much more important role in WZW conformal field theories. These commutation relations can be translated back to equal-time commutation relations for the continuum fields. They read

$$[\phi(\sigma), \phi(\sigma')] = \frac{\pi i}{m}\,\mathrm{sign}(\sigma - \sigma') \tag{6.16}$$

$$[\rho(\sigma), \phi(\sigma')] = \frac{i}{m}\delta(\sigma - \sigma') \tag{6.17}$$

$$[\rho(\sigma), \rho(\sigma')] = -\frac{i}{2\pi m}\partial_\sigma \delta(\sigma - \sigma') \tag{6.18}$$

The Hamiltonian

We can easily construct the Hamiltonian from the action (6.14). It is

$$H = \frac{mv}{2\pi}\sum_{n=0}^{\infty} k_n^2 \phi_n \phi_{-n} = 2\pi m v \sum_{n=0}^{\infty} \rho_n \rho_{-n}$$

where, in the quantum theory, we've chosen to normal order the operators. The time dependence of the operators is given by

$$\dot{\rho}_n = i[H, \rho_n] = i v k_n \rho_n$$

One can check that this is indeed the time dependence of the Fourier modes that follows from the equation of motion (6.14).

Our final Hamiltonian is simply that of a bunch of harmonic oscillators. The ground state $|0\rangle$ satisfies $\rho_{-n}|0\rangle = 0$ for $n > 0$. The excited states can then be constructed by acting with

$$|\psi\rangle = \sum_{n=1}^{\infty} \rho_n^{d_n} |0\rangle \quad \Rightarrow \quad H|\psi\rangle = \frac{2\pi v}{L}\sum_{n=1}^{\infty} n d_n |\psi\rangle$$

We've recovered exactly the spectrum and degeneracy of the excited modes of the Laughlin wavefunction that we saw in Section 6.1.1.

6.1.4 Electrons and Quasi-Holes

All of the excitations that we saw above describe ripples of the edge. They do not change the total charge of the system. In this section, we'll see how we can build new operators in the theory that carry charge. As a hint, recall that we saw in (6.13) that any object that changes the charge has to involve ϕ winding around the boundary. This suggests that it has something to do with the compact nature of the scalar field ϕ.

We claim that the operator describing an electron in the boundary is

$$\Psi = :e^{im\phi}: \tag{6.19}$$

where the dots denote normal ordering, which means that all ϕ_{-n}, with n positive, are moved to the right. In the language of conformal field theory, exponentials of this type are called *vertex operators*. To see that this operator carries the right charge, we can use the commutation relation (6.17) to show that

$$[\rho(\sigma), \Psi^\dagger(\sigma')] = \Psi^\dagger(\sigma')\,\delta(\sigma - \sigma') \quad \text{and} \quad [\rho(\sigma), \Psi(\sigma')] = -\Psi(\sigma')\,\delta(\sigma - \sigma')$$

which tells us that $\Psi^\dagger$ inserts an object of unit charge while Ψ removes an object of unit charge. This looks good. However, there's something rather surprising about the formula (6.19). The field ϕ is a boson, but if Ψ is really the electron operator then it should be a fermion. To see that this is indeed the case, we use the Baker-Campbell-Hausdorff formula to get

$$\Psi(\sigma)\Psi(\sigma') = e^{-m^2[\phi(\sigma),\phi(\sigma')]}\Psi(\sigma')\Psi(\sigma)$$

The commutator of ϕ was given in (6.16). We find that when $\sigma \neq \sigma'$,

$$\begin{aligned} [\Psi(\sigma), \Psi(\sigma')] &= 0 \qquad m \text{ even} \\ \{\Psi(\sigma), \Psi(\sigma')\} &= 0 \qquad m \text{ odd} \end{aligned}$$

We see that the field Φ acts like a boson if m is even and a fermion if m is odd. But we know from the Laughlin wavefunction that the objects underlying the quantum Hall state are bosons when m is even and fermions when m is odd. Miraculously, our edge theory knows about the nature of the underlying constituents in the bulk. The formula (6.19) is one of the key formulas in the subject of *bosonisation*, in which fermions in $d = 1 + 1$ dimensions can be written in terms of bosons and vice versa.

It should be clear that the electron operator (6.19) is not the simplest operator that we can construct in our theory. Since ϕ has periodicity 2π, it also makes sense to look at the operator

$$\Psi_{qp} = :e^{i\phi}: \tag{6.20}$$

No prizes are awarded for guessing that this corresponds to the quasi-particle excitations in the quantum Hall fluid. The commutator with ρ

$$[\rho(\sigma), \Psi^\dagger_{qp}(\sigma')] = \frac{1}{m}\Psi^\dagger_{qp}(\sigma')\,\delta(\sigma - \sigma') \quad \text{and} \quad [\rho(\sigma), \Psi_{qp}(\sigma')] = -\frac{1}{m}\Psi_{qp}(\sigma')\,\delta(\sigma - \sigma')$$

tells us that these operators create particles with charge $\pm 1/m$. The statistics of these operators can be seen by commuting

$$\Psi_{qp}(\sigma)\Psi_{qp}(\sigma') = e^{-[\phi(\sigma),\phi(\sigma')]}\Psi_{qp}(\sigma')\Psi_{qp}(\sigma) = e^{\pm\pi i/m}\Psi_{qp}(\sigma')\Psi_{qp}(\sigma)$$

We see that the particles are anyons, with statistical phase $e^{\pm\pi i/m}$ as expected. In this approach, the sign of the phase depends on the $\mathrm{sign}(\sigma - \sigma')$. This is analogous to the way the sign depends on whether to do a clockwise or anti-clockwise rotation in the bulk.

Propagators

Let's now turn to the propagators, starting with the compact boson ϕ. Deriving the propagator directly from the action (6.10) involves a fiddly contour integral. However, the answer is straightforward and simple to understand intuitively: it is simply the left-moving part of the propagator for a normal boson. Let's start from action

$$S = \frac{m}{8\pi}\int d^2x\ \partial_i\varphi\partial^i\varphi$$

The propagator for a free boson is simple to work out: it is

$$\langle\, \varphi(x,t)\varphi(0,0)\,\rangle = -\frac{1}{m}\log(v^2t^2 - x^2)$$

where, as usual, there is an implicit time ordering in all correlation functions of this kind, and there should really be a UV cut-off in the log which we've dropped. Of course, this action describes a scalar field which can propagate in both left-moving and right-moving directions. The equation of motion $(\partial_t^2 - v^2\partial_x^2)\varphi = 0$ ensures that all solutions decompose as $\varphi(x,t) = \varphi_L(x+vt) + \varphi_R(x-vt)$ (although there is, once again a subtlety with the zero mode which does not split into left- and right-moving pieces). The propagator above has a simple decomposition into left- and right- moving parts, with

$$\langle\, \varphi_L(x+vt)\varphi_L(0)\,\rangle = -\frac{1}{m}\log(x+vt) + \text{const.}$$

Our chiral boson ϕ is precisely this left-moving boson φ_L, albeit without the accompanying right-moving partner. The propagator. Indeed, one can show the correct propagator derived from (6.10) is equal to that found above

$$\langle\, \phi(x+vt)\phi(0,0)\,\rangle = -\frac{1}{m}\log(x+vt) + \text{const.} \tag{6.21}$$

(An aside: there is a seeming factor of 2 discrepancy between the normalisation of the boson action above and the normalisation of (6.10). This can be traced to the Jacobian in going between Euclidean coordinates and the light-cone coordinates $X^\pm = \sigma \pm vt$ which are appropriate for the chiral boson).

The logarithmic dependence seen in (6.21) reflects the fact that there are infra-red divergences if we work with massless scalar fields in $d = 1 + 1$. It's telling us that the physical information is carried by other fields. The propagator for the charge density follows immediately from differentiating (6.21),

$$\langle \rho(x + vt)\rho(0) \rangle = -\frac{1}{(2\pi)^2 m} \frac{1}{(x + vt)^2}$$

However, more interesting for us is the electron propagator.

$$G_F(x, t) = \langle \Psi^\dagger(x, t)\Psi(0, 0) \rangle \tag{6.22}$$

To compute this, we need to learn how to compute expectation values of normal ordered exponentials (6.19). Since the field ϕ is free, this must ultimately reduce to a problem in terms of harmonic oscillators. Because this is a calculation that we'll need to use again later, we pause briefly to explain how this works for the harmonic oscillator. . We'll then pick up our thread and compute the electron propagator (6.22).

An Aside: Coherent States in the Harmonic Oscillator

Consider a harmonic oscillator with the usual creation and annihilation operators satisfying $[a, a^\dagger] = 1$ and a vacuum $|0\rangle$ obeying $a|0\rangle = 0$. A *coherent state* is defined as the exponential

$$|z\rangle = e^{za^\dagger}|0\rangle$$

with $z \in \mathbf{C}$. It's simple to show that $[a, e^{za^\dagger}] = ze^{za^\dagger}$ from which we see that $|z\rangle$ is the eigenstate of the annihilation operator: $a|z\rangle = z|z\rangle$.

Now consider some linear combination of creation and annihilation operators,

$$A_i = \alpha_i a + \beta_i a^\dagger$$

The analog of the electron vertex operator (6.19) is the normal ordered exponential

$$: e^{A_i} : = e^{\beta_i a^\dagger} e^{\alpha_i a}$$

Our goal is to compute the vacuum expectation value of a string of these vertex operators,

$$\langle 0| : e^{A_1} :: e^{A_2} : \ \ldots \ : e^{A_N} : |0\rangle \tag{6.23}$$

To do this, we need to move all the $e^{i\alpha_i a}$ to the right, commuting them past the $e^{i\beta_j a^\dagger}$ with $j > i$ as they go. By the Baker-Campbell-Hausdorff formula, this is achieved by

$$e^{\alpha a}e^{\beta a^\dagger} = e^{\beta a^\dagger}e^{\alpha a}e^{\alpha\beta[a,a^\dagger]} = e^{\beta a^\dagger}e^{\alpha a}e^{\alpha\beta}$$

Applying this to the whole string of operators in (6.23), we have

$$\begin{aligned} :e^{A_1}::e^{A_2}: \ \ldots \ :e^{A_N}: \ &= e^{(\beta_1+\ldots+\beta_N)a^\dagger} e^{(\alpha_1+\ldots+\alpha_N)a} \, e^{\sum_{i<j}\alpha_i\beta_j} \\ &= \ :e^{A_1+\ldots+A_N}: \ e^{\sum_{i<j}\langle 0|A_iA_j|0\rangle} \end{aligned} \tag{6.24}$$

Taking the expectation value of both sides, we have our final result

$$\langle 0| :e^{A_1}::e^{A_2}: \ \ldots \ :e^{A_N}: |0\rangle = \exp\left(\sum_{i<j}\langle 0|A_iA_j|0\rangle\right) \tag{6.25}$$

This is the result that we want. Let's now see what it means for our electrons on the edge.

The Electron Propagator

Because the free field ϕ is simply a collection of harmonic oscillators, we can apply the formula (6.25) to vertex operators like (6.19). We have

$$G_F(x+vt) = \langle\, \Psi^\dagger(x,t)\Psi(0,0)\,\rangle = \exp\Big(m^2\langle\, \phi(x,t)\phi(0,0)\,\rangle\Big)$$

Using (6.21), we find that the electron Green's function is given by

$$G_F(x,t) \sim \frac{1}{(x+vt)^m} \tag{6.26}$$

This is interesting because it's not the usual expression for an electron Green's function in $d=1+1$.

To explain this, let's first review some condensed matter field theory. There's a simple theory that describes Fermi surfaces in $d=1+1$ dimensions (where they are really just Fermi points). Unlike in higher dimensions, these electrons are typically interacting, but in a way that is under control. The resulting theory is known as the *Luttinger liquid*. One of its key results is that the electron propagator for left-moving modes scales as $G_{\text{Luttinger}} \sim 1/(x+vt)$.

Comparing to our propagator (6.26), we see that it coincides with the Luttinger liquid result when $m=1$. This should not be surprising: $m=1$ describes a fully-filled Landau level which does not exhibit topological order. In contrast, in the fractional quantum Hall states with $m \neq 1$, the electrons on the edge of the sample do not follow the standard lore. This reflects the fact that they are strongly coupled. What we are calling an "electron" in not the same thing as an electron in the Standard Model. Instead, it is some collective excitation that carries the same quantum numbers as the electron in the Standard model. The resulting theory usually goes by the name of the *chiral Luttinger liquid*[61].

The most important information to take from the propagator (6.26) comes from some simple dimensional analysis. Comparing both sides, we learn that the electron operator Ψ has dimension $m/2$. This should be contrasted with the usual value of $1/2$ The fact that electrons are fermions means that m has to be odd. But this means that the exponent in the propagator can't change continuously as the Hamiltonian underlying the quantum Hall state varies. For this reason, the dimension of the edge operator can be viewed as a characterisation of the bulk state. It can only change if the bulk goes through a phase transition.

6.1.5 Tunnelling

The electron propagator (6.26) has some surprisingly physical consequences. There is a long and detailed literature on this subject. Here we provide only a baby version to explain the basic physics.

Suppose we connect the edge of the quantum Hall fluid to a wire, but put a small insulating material in between. This kind of set-up goes by the name of a *tunnel junction*. It means that if electrons want to get from the one side to the other, they have to tunnel. The way to model this in our theory is to add the interaction

$$S_{\text{tunnel}} = \tau \int dt\; e^{im\phi(0,t)}\, \Psi_e^\dagger(0,t) + \text{h.c.}$$

where $\Psi_e^\dagger$ is the creation operator for the electron in the wire. Here we've inserted the junction at the point $\sigma = 0$ on the edge.

The strength of the tunnelling is governed by the coupling constant τ. The action must be dimensionless (in units with $\hbar = 1$). We learned above that $e^{im\phi}$ has dimension $m/2$. Meanwhile Ψ_e refers to a "common or garden" electron in a wire and has dimension $1/2$. This means that the dimension of τ must be

$$[\tau] = \frac{1-m}{2}$$

We learn that for $m > 1$, the tunnelling is an irrelevant interaction in the language of the renormalisation group. The tunnelling will be suppressed at low energies or low

[61]These ideas were pioneered by Xiao-Gang Wen in a series of papers, starting with "*Chiral Luttinger Liquid and the Edge Excitations in the Fractional Quantum Hall State*", Phys. Rev. **B41** 12838 (1990) which can be downloaded here. A review can be found in "*Chiral Luttinger Liquids at the Fractional Quantum Hall Edge*" by A. M. Chang, Rev. Mod. Phys. **75**, 1449 (2003) which can be found here.

temperature where we can work perturbatively. We can use dimensional analysis to determine the way various quantities scale. In $d = 1+1$, the conductivity has dimension $[\sigma] = -1$, but this means that the conductance G is dimensionless: $[G] = 0$.

Fermi's golden rule tells us that the lowest order contribution to the tunnelling conductance G scales as τ^2. The deficit in dimensions must be made up by temperature T, simply because there's no other scale in the game. We have

$$G \sim \tau^2 T^{m-1}$$

Alternatively, if we're at zero temperature then the current is driven by a voltage V. We have $[I] = 1$ and $[V] = 1$, so we

$$I \sim \tau^2 V^m \tag{6.27}$$

This final result is particularly striking as violates the form of Ohm's law, $V = IR$, that we all learned in high school. This prediction has been successfully tested for the $\nu = 1/3$ quantum Hall state. The data shown in the figure[62] fits the solid line which matches (6.27) with $m \approx 2.7$.

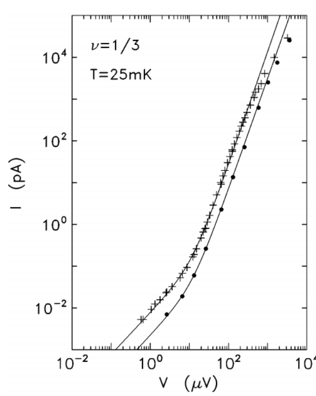

Figure 48:

We can also play variants on this game. For example, suppose that we add a tunnel junction between two Hall fluids of the same type. Now the interaction is

$$S_{\text{tunnel}} = \tau \int dt \; e^{im\phi_1(0,t)} \, e^{-im\phi_2(0,t)} + \text{h.c.}$$

This time we have $[\tau] = 1 - m$ and, correspondingly, we have

$$G \sim \tau^2 T^{2m-2} \quad \text{and} \quad I \sim \tau^2 V^{2m-1}$$

Quasi-Particle Tunnelling

It's also possible to set up a situation where the quasi-particles can tunnel. We do this by taking a single Hall fluid and putting in a *constriction* as shown in the figure. Because the bulk supports quasi-particles, these can tunnel from the top edge to the bottom. The tunnelling interaction is now

$$S_{\text{tunnel}} = \tau \int dt \; e^{i\phi_1(0,t)} \, e^{-i\phi_2(0,t)} + \text{h.c.}$$

[62]This plot is from A. M. Chang, L. N. Pfeiffer, and K. W. West, "*Observation of Chiral Luttinger Behavior in Electron Tunneling into Fractional Quantum Hall Edges*," Phys. Rev. Lett. 77, 2538 (1996).

To figure out the dimension of τ in this case, we first need the dimension of the quasi-particle operator. Repeating the calculation that led to (6.26) tells us that $[e^{i\phi}] = 1/2m$, so now we have

$$[\tau] = 1 - \frac{1}{m}$$

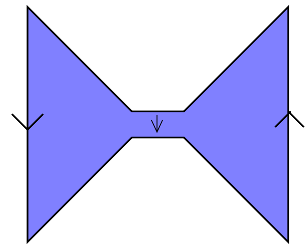

Figure 49: A Constriction

Now this is a relevant interaction. It becomes strong at low temperatures and our naive analysis does not work. (For example, the dimensions of operators at this point may be driven to something else at low temperatures). Instead, the scaling is valid at high temperatures or high voltages, where "high" means compared to the scale set by τ but, obviously not too high as to destroy the Hall state itself. When this scaling is valid, we get

$$G \sim \frac{\tau^2}{T^{2-2/m}} \quad \text{and} \quad I \sim \frac{\tau^2}{V^{1-2/m}}$$

Again, we see a striking difference from the usual form of Ohm's law.

6.2 The Bulk-Boundary Correspondence

We've seen that the theory of the edge modes know about the spectrum of quasi-holes in the bulk. However, it turns out that the edge knows somewhat more than this. Remarkably, it's possible to reconstruct the Laughlin wavefunction itself purely from knowledge about what's happening on the edge. In this section, we see how.

6.2.1 Recovering the Laughlin Wavefunction

We'll work with the chiral boson theory that we introduced in the previous section. To make these arguments, we need to do some simply gymnastics. First, we set the speed of propagation $v = 1$. Next, we Wick rotate to Euclidean space, defining the complex variables

$$w = \frac{2\pi}{L}\sigma + it \quad \text{and} \quad \bar{w} = \frac{2\pi}{L}\sigma - it \tag{6.28}$$

The complex coordinate w parameterises the cylinder that lies at the edge of the Hall sample, with $\text{Re}(w) \in [0, 2\pi)$. The final step is to work with single-valued complex coordinates

$$z = e^{-iw} \quad \text{and} \quad \bar{z} = e^{+i\bar{w}}$$

This can be thought of as a map from the cylinder to the plane as shown in the figure. If you know some conformal field theory, what we've done here the usual conformal transformation that implements the state-operator map. (You can learn more about this in the introduction to conformal field theory in the *String Theory* lecture notes).

In this framework, the fact that the boson is chiral translates to the statement that ϕ is a holomorphic function of z, so $\phi = \phi(z)$. One can check that the propagator (6.21) takes the same form, which now reads

$$\langle\, \phi(z)\phi(w)\,\rangle = -\frac{1}{m}\log(z-w) + \text{const.}$$

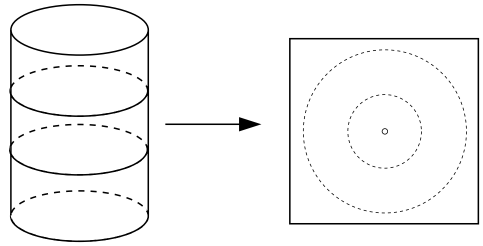

Figure 50:

The basis idea is to look at correlation functions involving insertions of electron operators of the form

$$\Psi = \,:e^{im\phi}:$$

Let's start by looking at something a little more general. We consider the correlation function involving a string of different vertex operators. Using (6.25), it looks like we should have

$$\begin{aligned}\langle :e^{im_1\phi(z_1)}::e^{im_2\phi(z_2)}: \ \ldots \ :e^{im_N\phi(z_N)}:\rangle &= \exp\left(-\sum_{i<j} m_i m_j \langle \phi(z_i)\phi(z_j)\rangle\right) \\ &\sim \prod_{i<j}(z_i - z_j)^{m_i m_j/m} \qquad (6.29)\end{aligned}$$

For a bunch of electron operators, with $m_i = m$, this looks very close to the pre-factor of the Laughlin wavefunction. However, the result (6.29) is not quite right. What we missed was a subtle issue to do with the zero mode ϕ_0 which we were hoping that we could ignore. Rather than deal with this zero mode, let's just see why the calculation above must be wrong[63]. Our original theory was invariant under the shift $\phi \to \phi + \alpha$ for any constant α. This means that all correlation functions should also be invariant under this shift. But the left-hand side above transforms picks up a phase $e^{i\alpha(m_1+\ldots+m_N)}$. This means that the correlation function can only be non-zero if

$$\sum_{i=1}^{N} m_i = 0$$

Previously we computed the electron propagator $\langle \Psi^\dagger \Psi\rangle$ which indeed satisfies this requirement. In general the the correct result for the correlation function is

$$\langle :e^{im_1\phi(z_1)}::e^{im_2\phi(z_2)}: \ \ldots \ :e^{im_N\phi(z_N)}:\rangle \sim \prod_{i<j}(z_i - z)^{m_i m_j/m}\,\delta(\sum_i m_i)$$

[63]A correct treatment of the zero mode can be found in the lecture notes on *String Theory* where this same issue arises when computing scattering amplitudes and is ultimately responsible for momentum conservation in spacetime.

The upshot of this argument is that a correlation function involving only electron operators does not give us the Laughlin wavefunction. Instead, it vanishes.

To get something non-zero, we need to insert another operator into the correlation function. We will look at

$$G(z_i, \bar{z}_i) = \langle\, \Psi(z_1) \ldots \Psi(z_N)\, \exp\Big(- \rho_0 \int_\gamma d^2z'\, \phi(z')\Big) \,\rangle \tag{6.30}$$

This is often said to be inserting a background charge into the correlation function. We take $\rho_0 = 1/2\pi l_B^2$. Note that this is the same as the background charge density (3.10) that we found when discussing the plasma analogy. Meanwhile, γ is a disc-shaped region of radius R, large enough to encompass all point z_i. Now the requirement that the correlation function is invariant under the shift $\phi \to \phi + \alpha$ tells us that it can be non-zero only if

$$mN = \rho_0 \int_\gamma d^2z' = \pi R^2 \rho_0$$

Using $\rho_0 = 1/2\pi l_B^2$, we see that we should take $R = \sqrt{2mN} l_B$ which we recognise as the radius of the droplet described by the quantum Hall wavefunction.

Using (6.25), the correlation function (6.30) can be written as

$$G(z_i, \bar{z}_i) \sim \prod_{i<j} (z_i - z_j)^m \exp\left(-\rho_0 \sum_{i=1}^N \int_\gamma d^2z'\, \log(z_i - z')\right)$$

We're still left with an integral to do. The imaginary part of this integral is ill-defined because of the branch cuts inherent in the logarithm. However, as its only a phase, it can be undone by a (admittedly very singular) gauge transformation. Omitting terms the overall constant, and terms that are suppressed by $|z_i|/R$, the final result for the correlation function is

$$G(z_i, \bar{z}_i) \sim \prod_{i<j} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4l_B^2} \tag{6.31}$$

This, of course, is the Laughlin wavefunction.

We can extend this to wavefunctions that involve quasi-holes. We simply need to insert some number of quasi-hole operators (6.20) into the correlation function

$$\tilde{G}(z_i, \bar{z}_i; \eta_a, \bar{\eta}_a) = \langle\, \Psi_{qh}(\eta_1) \ldots \Psi_{qh}(\eta_p) \Psi(z_1) \ldots \Psi(z_N)\, \exp\Big(- \rho_0 \int_\gamma d^2z'\, \phi(z')\Big) \,\rangle$$

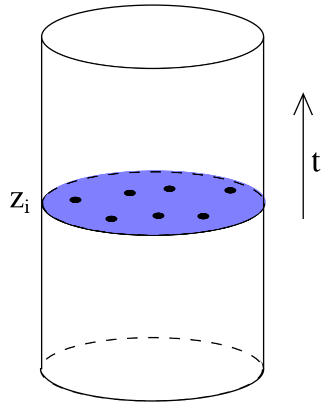

Figure 51: The wavefunction lives here

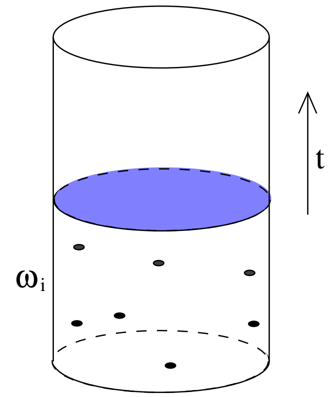

Figure 52: The correlation function lives here

where the size of the disc γ must now be extended so that the system remains charge neutral. The same calculations as above now yield

$$\tilde{G}(z_i, \bar{z}_i; \eta_a, \bar{\eta}_a) = \prod_{a<b} (\eta_a - \eta_b)^{1/m} \prod_{a,i} (z_i - \eta_a) \prod_{k<l} (z_k - z_l)^m \, e^{-\sum_i |z_i|^2/4l_B^2 - \sum_a |\eta_a|^2/4ml_B^2}$$

This is the Laughlin wavefunction for the quasi-hole excitations. Note that we've recovered the wavefunction in the form (3.30) where the Berry phase vanishes. Instead the correlation function is not single valued and all the statistical phases that arise from braiding the quasi-hole positions are explicit.

What the Hell Just Happened?

It's been a long journey. But finally, after travelling through Chern-Simons theories and the theory of edge states, we've come right back to where we started: the Laughlin wavefunction[64]. How did this happen? It seems like magic!

The most glaring issue in identifying the correlation function with the wavefunction is that the two live in different spaces. Our quantum Hall fluid lives on a disc, so spacetime is a cylinder as shown in the figures. The wavefunction is defined on a spatial slice at a fixed time; this is the blue disc in the figure. In the wavefunction, the positions z_i lies within this disc as shown in the left-hand figure. Meanwhile, the conformal field theory lives on the boundary. The operators inserted in the correlation

[64]The connection between correlation functions and quantum Hall wavefunctions was first noticed by Greg Moore and Nick Read in the "*Nonabelions in the Fractional Quantum Hall Effect*", Nucl. Phys. **B360**, 362 (1991) which can be downloaded here. This was also the paper where they first proposed the Moore-Read wavefunction. This is not coincidence: they arrived at the wavefunction by thinking about correlation functions in different conformal field theories.

function sit at positions $w_i = 2\pi\sigma/L + it$ which are subsequently mapped to the plane by $z = e^{-i\omega}$. Why should we identify the positions in these two different spaces?

The answer is that there are actually two different ways in which the Chern-Simons theory is related to the CFT. This arises because the bulk Chern-Simons theory is topological, which means that you can cut it in different way and get the same answer. Above we've considered cutting the bulk along a timelike boundary to give a CFT in $d = 1 + 1$ dimensions. This, of course, is what happens in a physical system. However, we could also consider an alternative way to slice the bulk along a spacelike section, as in the left-hand figure above. This gives the same CFT, but now Wick rotated to $d = 2 + 0$ dimensions.

The next question we should ask is: why does the very high-point correlation function in the CFT capture the ground state wavefunction in the bulk? The way to think about this is as follows: after the Wick rotation, the insertion of operators $\Psi(w_i)$ should be thought of as annihilating a bulk electron as it hits the surface at position w_i. In this way, we compute the overlap of the vacuum with a specific state on the spacelike slice, which is prepared by the insertion of these operators. This overlap of matrix elements is the vacuum wavefunction. We will make this more precise imminently in Section 6.2.2.

Let me also mention a separate surprise about the relationship between correlation functions and the Laughlin wavefunction. Our original viewpoint in Section 3 was that there was nothing particularly special about the Laughlin wavefunction; it is simply a wavefunction that is easy to write down which lives in the right universality class. Admittedly it has good overlap with the true ground state for low number of electrons, but it's only the genuine ground state for artificial toy Hamiltonians. But now we learn that there is something special about this state: it is the correlation function of primary operators in the boundary theory. I don't understand what to make of this.

Practically speaking, the connection between bulk wavefunctions and boundary correlation functions has proven to be a very powerful tool. It is conjectured that this correspondence extends to all quantum Hall states. First, this means that you don't need to guess quantum Hall wavefunctions anymore. Instead you can just guess a boundary CFT and compute its correlation functions. But there's a whole slew of CFTs that people have studied. We'll look at another example in Section 6.3. Second, it turns out that the CFT framework is most useful for understanding the properties of quantum Hall states, especially those with non-Abelian anyons. The braiding properties of anyons are related to well-studied properties of CFTs. We'll give some flavour of this in Section 6.4.

6.2.2 Wavefunction for Chern-Simons Theory

Above we saw how the boundary correlation functions of the CFT capture the bulk Laughlin wavefunctions. As we described above, the key is to consider a different cut of the Chern-Simons theory. With this in mind, we will place Chern-Simons theory on $\mathbf{R} \times \mathbf{S}^2$ where $\mathbf{R}$ is time and $\mathbf{S}^2$ is a compact spatial manifold which no longer has a boundary. Instead, we will consider the system at some fixed time. But in any quantum system, the kind of object that sits at a fixed time is a wavefunction. We will see how the wavefunction of Chern-Simons theory is related to the boundary CFT.

We're going to proceed by implementing a canonical quantisation of $U(1)_m$ Chern-Simons theory. We already did this for Abelian Chern-Simons theory in Section 5.2.3. Working in $a_0 = 0$ gauge, the canonical commutation relations (5.49)

$$[a_i(\mathbf{x}), a_j(\mathbf{y})] = \frac{2\pi i}{m}\,\epsilon_{ij}\,\delta^2(\mathbf{x}-\mathbf{y})$$

subject to the constraint $f_{12} = 0$.

At this stage, we differ slightly from what went before. We introduce complex coordinates z and $\bar{z}$ on the spatial $\mathbf{S}^2$. As an aside, I should mention that if we were working on a general spatial manifold Σ then there is no canonical choice of complex structure, but the end result is independent of the complex structure you pick. This complex structure can also be used to complexify the gauge fields, so we have a_z and $a_{\bar{z}}$ which obey the commutation relation

$$[a_z(z,\bar{z}), a_{\bar{z}}(w,\bar{w})] = \frac{4\pi}{m}\,\delta^2(\mathbf{z}-\mathbf{w}) \tag{6.32}$$

The next step is somewhat novel. We're going to write down a Schrödinger equation for the theory. That's something very familiar in quantum mechanics, but not something that we tend to do in field theory. Of course, to write down a Schrödinger equation, we first need to introduce a wavefunction which depends only on the "position" degrees of freedom and not on the momentum. This means that we need to make a choice on what is position and what is momentum. The commutation relations (6.32) suggest that it's sensible to choose $a_{\bar{z}}$ as "position" and a_z as "momentum". This kind of choice goes by the name of *holomorphic quantisation*. This means that we describe the state of the theory by a wavefunction

$$\Psi(a_{\bar{z}}(z,\bar{z}))$$

Meanwhile, the a_z act as a momentum type operator,

$$a_z^a = \frac{4\pi}{k}\frac{\delta}{\delta a_{\bar{z}}^a}$$

The Hamiltonian for the Chern-Simons theory vanishes. Instead, what we're calling the Schrödinger equation arises from imposing the constraint $f_{z\bar{z}} = 0$ as an operator equation on Ψ. Replacing a_z with the momentum operator, this reads

$$\left(\partial_{\bar{z}}\frac{\delta}{\delta a_{\bar{z}}} - \frac{m}{4\pi}\partial_z a_{\bar{z}}\right)\Psi(a_{\bar{z}}) = 0 \tag{6.33}$$

This is our Schrödinger equation.

The Partition Function of the Chiral Boson

We'll now show that this same equation arises from the conformal field theory of a chiral boson. The key idea is to couple the current in the CFT to a background gauge field. We will call this background gauge field a.

Recall from our discussion in Section 6.1.2 that the charge density is given by $\rho \sim \partial\phi/\partial x$ and, for the chiral action (6.10), the associated current density is simply $-v\rho$, reflecting the fact that charge, like all excitations, precesses along the edge.

Here we want to think about the appropriate action in the Euclidean theory. It's simplest to look at the action for a massless boson and subsequently focus on the chiral part of it. This means we take

$$S[\phi] = \frac{m}{2\pi}\int d^2x\ \partial_{\bar{z}}\phi\,\partial_z\phi$$

Now the charge becomes

$$\rho = \frac{1}{2\pi}\frac{\partial\phi}{\partial z}$$

The chiral conservation law is simply $\partial_{\bar{z}}\rho \sim \partial_{\bar{z}}\partial_z\phi = 0$ by virtue of the equation of motion.

We want to couple this charge to a background gauge field. We achieve this by writing

$$S[\phi; a] = \frac{m}{2\pi}\int d^2x\ \mathcal{D}_{\bar{z}}\phi\,\partial_z\phi \tag{6.34}$$

where

$$\mathcal{D}_{\bar{z}}\phi = \partial_{\bar{z}}\phi - a_{\bar{z}}$$

The extra term in this action takes the form $a_{\bar{z}}\rho$, which is what we wanted. Moreover, the form of the covariant derivative tells us that we've essentially gauged the shift symmetry $\phi \to \phi + \text{constant}$ which was responsible for the existence of the charge in the first place. Note that, although we've given the gauge field the same name as in the Chern-Simons calculation above, they are (at this stage) rather different objects. The Chern-Simons gauge field is dynamical but, in the equation above, $a_{\bar{z}}(z, \bar{z})$ is some fixed function. We will see shortly why it's sensible to give them the same name.

The action (6.34) looks rather odd. We've promoted $\partial_{\bar{z}}$ into a covariant derivative $\mathcal{D}_{\bar{z}}$ but not ∂_z. This is because we're dealing with a chiral boson rather than a normal boson. It has an important consequence. The equation of motion from (6.34) is

$$\partial_{\bar{z}}\partial_z\phi = \frac{1}{2}\partial_z a_{\bar{z}} \tag{6.35}$$

This tells us that the charge ρ is no longer conserved! That's quite a dramatic change. It is an example of an *anomaly* in quantum field theory.

If you've heard of anomalies in the past, it is probably in the more familiar (and more subtle) context of chiral fermions. The classical chiral symmetry of fermions is not preserved at the quantum level, and the associated charge can change in the presence of a background field. The anomaly for the chiral boson above is much simpler: it appears already in the classical equations of motion. It is related to the chiral fermion anomaly through bosonization.

Now consider the partition function for the chiral boson. It is a function of the background field.

$$Z[a_{\bar{z}}] = \int \mathcal{D}\phi \, e^{-S[\phi,a]}$$

This, of course, is the generating function for the conformal field theory. The partition function in the present case obeys a rather nice equation,

$$\left(\partial_{\bar{z}}\frac{\delta}{\delta a_{\bar{z}}} - \frac{m}{4\pi}\partial_z a_{\bar{z}}\right) Z(a_{\bar{z}}) = 0 \tag{6.36}$$

To see this, simply move the $\delta/\delta a_{\bar{z}}$ into the path integral where it brings down a factor of $\partial_z\phi$. The left-hand side of the above equation is then equivalent to computing the expectation value $\langle \partial_{\bar{z}}\partial_z\phi - \frac{1}{2}\partial_z a_{\bar{z}}\rangle_a$, where the subscript a is there to remind us that we evaluate this in the presence of the background gauge field. But this is precisely the equation of motion (6.35) and so vanishes.

Finally, note that we've seen the equation (6.36) before; it is the Schrödinger equation (6.33) for the Chern-Simons theory. Because they solve the same equation, we can equate

$$\Psi(a_{\bar{z}}) = Z[a_{\bar{z}}] \tag{6.37}$$

This is a lovely and surprising equation. It provides a quantitative relationship between the boundary correlation functions, which are generated by $Z[a]$, and the bulk Chern-Simons wavefunction.

The relationship (6.37) says that the bulk vacuum wavefunction $a_{\bar{z}}$ is captured by correlation functions of $\rho \sim \partial\phi$. This smells like what we want, but it isn't quite the same. Our previous calculation looked at correlation functions of vertex operators $e^{im\phi}$. One might expect that these are related to bulk wavefunctions in the presence of Wilson lines. This is what we have seen coincides with our quantum Hall wavefunctions.

The bulk-boundary correspondence that we've discussed here is reminiscent of what happens in gauge/gravity duality. The relationship (6.37) is very similar to what happens in the ds/CFT correspondence (as opposed to the AdS/CFT correspondence). In spacetimes which are asymptotically de Sitter, the bulk Hartle-Hawking wavefunction at spacelike infinity is captured by a boundary Euclidean conformal field theory.

Wavefunction for Non-Abelian Chern-Simons Theories

The discussion above generalises straightforwardly to non-Abelian Chern-Simons theories. Although we won't need this result for our quantum Hall discussion, it is important enough to warrant comment. The canonical commutation relations were given in (5.48) and, in complex coordinates, read

$$[a_z^a(z,\bar{z}), a_{\bar{z}}^b(w,\bar{w})] = \frac{4\pi}{k}\,\delta^{ab}\,\delta^2(\mathbf{z}-\mathbf{w})$$

with a, b the group indices and k the level. The constraint $f_{zz'} = 0$ is once again interpreted as an operator equation acting on the wavefunction $\Psi(a_{\bar{z}})$. The only difference is that there is an extra commutator term in the non-Abelian $f_{zz'}$. The resulting Schrödinger equation is now

$$\left(\partial_{\bar{z}}\frac{\delta}{\delta a_{\bar{z}}} + [a_{\bar{z}}, \frac{\delta}{\delta a_{\bar{z}}}]\right)\Psi(a_{\bar{z}}) = \frac{k}{4\pi}\partial_z a_{\bar{z}}\Psi(a_{\bar{z}})$$

As before, this same equation governs the partition function $Z[a_{\bar{z}}]$ boundary CFT, with the gauge field $a_{\bar{z}}$ coupled to the current. In this case, the boundary CFT is a WZW model about which we shall say (infinitesimally) more in Section 6.4.

6.3 Fermions on the Boundary

In this section we give another example of the bulk/boundary correspondence. However, we're not going to proceed systematically by figuring out the edge modes. Instead, we'll ask the question: what happens when you have fermions propagating on the edge? We will that this situation corresponds to the Moore-Read wavefunction. We'll later explain the relationship between this and the Chern-Simons effective theories that we described in Section 5.

6.3.1 The Free Fermion

In $d = 1+1$ dimensions, a Dirac fermion ψ is a two-component spinor. The action for a massless fermion is

$$S = \frac{1}{4\pi}\int d^2x\ i\psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi$$

In Minkowski space we take the gamma matrices to be $\gamma^0 = i\sigma^2$ and $\gamma^1 = \sigma^1$ with σ^i the Pauli matrices. These obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. We can decompose the Dirac spinor into chiral spinors by constructing the other "γ^5" gamma matrix. In our chosen basis this is simply σ^3 and the left-moving and right-moving spinors, which are eigenstates of σ^3, are simply

$$\psi = \begin{pmatrix}\chi_L \\ \chi_R\end{pmatrix}$$

Written in the terms of these one-component Weyl spinors, the action is

$$S = -\frac{1}{4\pi}\int d^2x\ i\chi_L^\dagger(\partial_t - \partial_x)\chi_L + i\chi_R^\dagger(\partial_t + \partial_x)\chi_R$$

The solutions to the equations of motion are $\chi_L = \chi_L(x+t)$ and $\chi_R = \chi_R(x-t)$.

There's something rather special about spinors in $d = 1+1$ dimensions (and, indeed in $d = 4k+2$ dimensions): they can be both Weyl and Majorana at the same time. We can see this already in our gamma matrices which are both real and in a Weyl basis. From now on, we will be interested in a single left-moving Majorana-Weyl spinor. We will denote this as χ. The Majorana condition simply tells us that $\chi = \chi^\dagger$.

Fermions on a Circle

The edge of our quantum Hall state is a cylinder. We'll take the spatial circle to be parameterised by $\sigma \in [0, L)$. If the fermion is periodic around the circle, so $\chi(\sigma + L) =$

$\chi(\sigma)$, then it can be decomposed in Fourier modes as

$$\chi(\sigma) = \sqrt{\frac{2\pi}{L}} \sum_{n \in \mathbf{Z}} \chi_n \, e^{2\pi i n \sigma / L} \tag{6.38}$$

The Majorana condition is $\chi_n^\dagger = \chi_{-n}$. However, for fermions there is a different choice: we could instead ask that they are anti-periodic around the circle. In this case $\chi(\sigma + L) = -\chi(\sigma)$, and the modes n get shifted by $1/2$, so the decomposition becomes

$$\chi(\sigma) = \sqrt{\frac{2\pi}{L}} \sum_{n \in \mathbf{Z}+\frac{1}{2}} \chi_n \, e^{2\pi i n \sigma / L} \tag{6.39}$$

The periodic case is known as *Ramond* boundary conditions; the anti-periodic case as *Neveu-Schwarz* (NS) boundary conditions. In both cases, the modes have canonical anti-commutation relations

$$\{\chi_n, \chi_m\} = \delta_{n+m} \tag{6.40}$$

Fermions on the Plane

At this stage, we play the same games that we saw at the beginning of Section 6.2.1; we Wick rotate, define complex coordinates $w = 2\pi\sigma/L + it$ as in (6.38), and then map to the complex plane by introducing $z = e^{-iw}$. However, something new happens for the fermion that didn't happen for the boson: it picks up an extra contribution in the map from the cylinder to the plane:

$$\chi(w) \to \sqrt{\frac{2\pi z}{L}} \chi(z)$$

In the language of conformal field theory, this arises because χ has dimension $1/2$. However, one can also see the reason behind this if we look at the mode expansion on the plane. With Ramond boundary conditions we get

$$\chi(z) = \sum_{n \in \mathbf{Z}} \chi_n \, z^{-n-1/2} \quad \Rightarrow \quad \chi(e^{2\pi i} z) = -\chi(z)$$

We see that the extra factor of $1/2$ in the mode expansion leads to the familiar fact that fermions pick up a minus sign when rotated by 2π.

In contrast, for NS boundary conditions we have

$$\chi(z) = \sum_{n \in \mathbf{Z}+\frac{1}{2}} \chi_n \, z^{-n-1/2} \quad \Rightarrow \quad \chi(e^{2\pi i} z) = +\chi(z)$$

As we will see, various aspects of the physics depend on which of these boundary conditions we use. This is clear already when compute the propagators. These are simplest for the NS boundary condition, where χ is single valued on the plane. The propagator can be computed from the anti-commutation relations (6.40),

$$\begin{aligned}\langle\, \chi(z)\chi(w)\,\rangle &= \sum_{n,m\in\mathbf{Z}+\frac{1}{2}} z^{-n}w^{-m}\langle\chi_n\chi_m\rangle \\ &= \sum_{n=0}^{\infty}\frac{1}{z}\left(\frac{w}{z}\right)^n \\ &= \frac{1}{z-w}\end{aligned} \tag{6.41}$$

Meanwhile, in the Ramond sector, the result is more complicated as we get an extra contribution from $\langle\chi_0^2\rangle$. This time we find

$$\begin{aligned}\langle\, \chi(z)\chi(w)\,\rangle &= \sum_{n,m\in\mathbf{Z}} z^{-n-1/2}w^{-m}\langle\chi_n\chi_m\rangle \\ &= \frac{1}{2\sqrt{zw}}+\sum_{n=1}^{\infty} z^{-n-1/2}w^{n-1/2} \\ &= \frac{1}{\sqrt{zw}}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{w}{z}\right)^n\right) \\ &= \frac{1}{2}\frac{\sqrt{z/w}+\sqrt{w/z}}{z-w}\end{aligned}$$

We see that there propagator inherits some global structure that differs from the Ramond case.

This is the Ising Model in Disguise!

The free fermion that we've described provides the solution to one of the classic problems in theoretical physics: it is the critical point of the 2d Ising model! We won't prove this here, but will sketch the extra ingredient that we need to make contact with the Ising model. It is called the *twist operator* $\sigma(z)$. It's role is to switch between the two boundary conditions that we defined above. Specifically, if we insert a twist operator at the origin and at infinity then it relates the correlation functions with different boundary conditions,

$$\langle \mathrm{NS}|\,\sigma(\infty)\chi(z)\chi(w)\sigma(0)\,|\mathrm{NS}\rangle = \langle \mathrm{Ramond}|\,\chi(z)\chi(w)\,|\mathrm{Ramond}\rangle$$

With this definition, one can show that the dimension of the twist operator is $h_\sigma = 1/16$. This is identified with the spin field of the Ising model. Meanwhile, the fermion χ is related to the energy density.

One reason for mentioning this connection is that it finally explains the name "*Ising anyons*" that we gave to the quasi-particles of the Moore-Read state. In particular, the "fusion rules" that we met in Section 4.3 have a precise analog in conformal field theories. (What follows involves lots of conformal field theory talk that won't make much sense if you haven't studied the subject.) In this context, a basic tool is the operator product expansion (OPE) between different operators. Every operator lives in a conformal family determined by a primary operator. The fusion rules are the answer to the question: if I know the family that two operators live in, what are the families of operators that can appear in the OPE?

For the Ising model, there are two primary operators other than the identity: these are χ and σ. The fusion rules for the associated families are

$$\sigma \star \sigma = 1 \oplus \chi \quad , \quad \sigma \star \chi = \sigma \quad , \quad \chi \star \chi = 1$$

But we've seen these equations before: they are precisely the fusion rules for the Ising anyons (4.24) that appear in the Moore-Read state (although we've renamed ψ in (4.24) as χ).

Of course, none of this is coincidence. As we will now see, we can reconstruct the Moore-Read wavefunction from correlators in a $d = 1 + 1$ field theory that includes the free fermion.

6.3.2 Recovering the Moore-Read Wavefunction

Let's now see how to write the Moore-Read wavefunction

$$\psi_{MR}(z_i, \bar{z}_i) = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^m \, e^{-\sum |z_i|^2/4l_B^2}$$

as a correlation function of a $d = 1 + 1$ dimensional field theory. The new ingredient is obviously the Pfaffian. But this is easily built from a free, chiral Majorana fermion. As we have seen, armed with NS boundary conditions such a fermion has propagator

$$\langle\, \chi(z)\chi(w)\, \rangle = \frac{1}{z - w}$$

Using this, we can then employ Wick's theorem to compute the general correlation function. The result is

$$\langle\, \chi(z_1) \ldots \chi(z_N)\, \rangle = \text{Pf}\left(\frac{1}{z_i - z_j}\right)$$

which is just what we want. The piece that remains is simply a Laughlin wavefunction and we know how to build this from a chiral boson with propagator

$$\langle\, \phi(z)\phi(w)\,\rangle = -\frac{1}{m}\log(z-w) + \text{const.} \tag{6.42}$$

The net result is that the Moore-Read wavefunction can be constructed from the product of correlation functions

$$\psi_{MR}(z_i,\bar{z}_i) = \langle\, \chi(z_1)\ldots\chi(z_N)\,\rangle\,\langle :e^{im\phi(z_1)}: \;\ldots :e^{im\phi(z_N)}:\; e^{-\rho_0\int_\gamma d^2z'\,\phi(z')}\,\rangle$$

From this expression, it's clear that we should identify the electron operator as the combination

$$\Psi(z) = \chi(z) :e^{im\phi(z)}:$$

These are fermions for m even and bosons for m odd.

What about the quasi-holes of the theory? We won't give details but will instead just state the answer: the quasi-hole operator is related to the twist operator

$$\Psi_{qh} = \sigma(z) :e^{i\phi(z)/2}:$$

Note that the bosonic vertex operator has a charge which would be illegal in the pure bosonic theory. However, the multi-valued issues are precisely compensated by similar properties of the twist, so their product is single valued. This factor of 1/2 explains how the quasi-holes have half the charge than in the Laughlin state. One can show that inserting Ψ_{qh} results in an ambiguity. There are a number of different correlation functions. These are precisely the different quasi-hole wavefunctions (4.6) that we met in Section 4.2.

Finally, the theory also has the elementary excitation that we started with: the fermion χ. This corresponds to a fermionic, neutral excitation of the Moore-Read state.

Relationship to Chern-Simons Theory

In this section, we just conjured the fermion theory out of thin air and showed that one can reconstruct the Moore-Read state. It would be nice to do better and show that it arises as the boundary theory of the corresponding Chern-Simons theory. This is (fairly) straightforward for the case of the bosonic, $m = 1$ Moore-Read state. Again, we won't be able to describe the details without getting into a lot more conformal field theory, but here's a sketch of the basics.

When $m = 1$, the propagator (6.42) for the chiral boson has no fractional piece in its normalisation. Or, said another way, if we normalise the action canonically, so we rescale $\phi \to \sqrt{m}\phi$, then the radius of the chiral boson remains $R = 1$. However, a chiral boson at this radius has the nice property that it is equivalent to a chiral Dirac fermion. This, in turn, is the same as two Majorana fermions. The upshot is that the conformal field theory for $m = 1$ is really three Majorana fermions: the χ that we started with and two more that come from ϕ. There is an $SU(2)$ symmetry which rotates these three fermions among themselves. Indeed, it's known that this is the theory that arises on the edge of the $SU(2)$ Chern-Simons theory at level $k = 2$.

As we discussed in Section 5.4, for $m > 1$ the corresponding Chern-Simons theories are less clear. Instead, it's better to think of the quantum Hall states as characterised by the conformal field theories on the edge. It is conjectured that, in general, the correct edge theory is precisely the one whose correlation functions reproduce bulk wavefunctions. Moreover, there are many powerful techniques that have been developed for conformal field theory which allow one to determine the properties of the wavefunctions, in particular the braiding of non-Abelian anyons. In the final section, we paint a cartoon picture of these techniques.

6.4 Looking Forwards: More Conformal Field Theory

In the last few sections, we've seen an increasing need to import results from conformal field theory. This doesn't improve moving forward! To make progress, we would really need to first pause and better understand the structure of conformal field theories. However, this is a large subject which we won't cover in these lectures. Instead, we will just attempt to paint a picture with a broad brush while stating a few facts. At the very least, this will hopefully provide some vocabulary that will be useful if you want to pursue these ideas further.

Fusion, Braiding and Conformal Blocks

The key idea is that the formal structure underlying non-Abelian anyons that we described in Section 4.3 also appears in conformal field theory (CFT). Indeed, it was first discovered in this context[65].

The role of the different kinds of anyons is now played by the different representations of the conformal algebra (by which we mean either the Virasoro algebra, or something

[65]See the paper "*Classical and Quantum Conformal Field Theory*", Comm. Math. Phys 123, 177 (1989) by Greg Moore and Nati Seiberg, or their subsequent "*Lectures on RCFT*" which can be downloaded here.

larger, such as a current algebra) that appear in a given conformal field theory. Each of these representations can be labelled by a highest-weight state called a *primary operator*, $\mathcal{O}_i$. A rational conformal field theory is one which has a finite number of these primary operators.

Next up, we need to define fusion. We already met this briefly in the previous section in the context of the Ising model. If you have two operators which live within representations associated to the primary operators $\mathcal{O}_i$ and $\mathcal{O}_j$ respectively, then the operator product expansion can contain operators in other representations associated to $\mathcal{O}_k$. We write these fusion rules, following (4.20), as

$$\mathcal{O}_i \star \mathcal{O}_j = \sum_k N_{ij}^k \; \mathcal{O}_k$$

where N_{ij}^k are integers.

Similarly, we can define braiding matrices for a CFT. The general idea of the braiding is as follows. Consider a CFT which has both left-moving and right-moving modes. In general, correlation functions of primary operators can be decomposed as

$$\langle \prod_{i=1}^{N} \mathcal{O}_i(z_i, \bar{z}_i) \rangle = \sum_p |\mathcal{F}_p(z_i)|^2$$

Here the $F_p(z_i)$ are multi-branched analytic functions of the z_i which depend on the set of list of operators inserted on the left-hand-side. They are known as *conformal blocks*. In a rational conformal field theory (which is defined to have a finite number of primary operators) the sum over p runs over a finite range.

Now vary the z_i, which has the effect of exchanging the particles. (In the context of the quantum Hall wavefunctions, we would exchange the positions of the quasi-hole insertions.) The conformal blocks will be analytically continued onto different branches. However, the final answer can be written in terms of some linear combination of the original function. This linear map is analogous to the braiding of anyons. One of the main results of Moore and Seiberg is that there are consistency relations on the kinds of braiding that can arise. These are precisely the pentagon and hexagon relations that we described in Section 4.3.

We've already seen two examples of this. For the Laughlin states with quasi-holes, there is a single conformal block but it is multi-valued due to the presence of the factor $\prod(\eta_i - \eta_j)^{1/m}$ involving the quasi-hole positions η. Meanwhile, for the Moore-Read state there are multiple conformal blocks corresponding to the different wavefunctions (4.6).

In both these cases, the conformal field theory gave the wavefunction in a form in which all the monodromy properties are explicit and there is no further contribution from the Berry phase. (Recall the discussion at the end of Section 3.2.3.) It is conjectured that this is always the case although, to my knowledge, there is no proof of this.

WZW Models

The most important conformal field theories for our purposes are known as WZW models. (The initials stand for Wess, Zumino and Witten. Sometimes Novikov's name breaks the symmetry and they are called WZNW models.) Their importance stems in large part from their relationship to non-Abelian Chern-Simons theories. These models describe the modes which live at the edge of a non-Abelian Chern-Simons theory with boundary. Further, it turns out that the braiding of their conformal blocks coincides with the braiding of Wilson lines in the Chern-Simons theory that we briefly described in Section 5.4.4.

The WZW models are defined by the choice of gauge group G, which we will take to be $SU(N)$, and a level $k \in \mathbf{Z}$. These theories are denoted as $SU(N)_k$. The CFT for a compact boson that we met in Section 6.2.1 is a particularly simple example of a WZW model model with $U(1)_m$.

Unusually for conformal field theories, WZW models have a Lagrangian description which can be derived using the basic method that we saw in Section 6.1.2 for $U(1)$ Chern-Simons theories. The Lagrangian is

$$S = \frac{k}{4\pi}\int d^2x \ \mathrm{tr}\left(g^{-1}\partial_t g\, g^{-1}\partial_x g - v(g^{-1}\partial_x g)^2\right) + 2\pi k\, w(g)$$

Here $g \in G$ is a group valued field in $d = 1+1$ dimensions. The first term describes a chiral sigma model whose target space is the group manifold G. If we're working with a quantum Hall fluid on a disc then this theory lives on the $\mathbf{R} \times \mathbf{S}^1$ boundary.

The second term is more subtle. It is defined as the integral over the full three-dimensional manifold $\mathcal{M}$ on which the quantum Hall fluid lives,

$$w(g) = \frac{1}{24\pi^2}\int_{\mathcal{M}} d^3x \ \epsilon^{\mu\nu\rho}\mathrm{tr}\,(g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\, g^{-1}\partial_\rho g)$$

which we recognise as the winding (5.47) that we saw earlier. Although the quantum Hall fluid provides us with a natural 3-manifold $\mathcal{M}$, taking the level $k \in \mathbf{Z}$ ensures that the two-dimensional theory on the boundary is actually independent of our choice of $\mathcal{M}$. In this way, the WZW model is, despite appearances, an intrinsically two-dimensional theory.

The central charge of the $SU(N)_k$ WZW model is

$$c = \frac{k(N^2-1)}{k+N}$$

The theories are weakly coupled as $k \to \infty$ where the central charge is equal to the dimension of the group $SU(N)$. Theories becomes strongly coupled as k gets smaller. In particular, for $k = 0$ we have $c = 0$. This reflects that the fact that the sigma-model on the group manifold without any topological term flows to a gapped theory in the infra-red.

The WZW models have a large symmetry G known as a current algebra. Usually in quantum field theory, a symmetry implies a current J^μ which obeys $\partial_\mu J^\mu = 0$. The symmetry of the WZW model is much stronger as the left-moving and right-moving parts of the current are independently conserved. In terms of complex coordinates, this means that we have holomorphic and anti-holomorphic currents $J = \partial g\, g^{-1}$ and $\bar{J} = g^{-1}\bar{\partial} g$ obeying

$$\bar{\partial} J(z) = 0 \quad \text{and} \quad \partial \bar{J}(\bar{z}) = 0$$

This is very similar to the conditions on the stress-tensor that you first meet in the study of CFT. In that case, one writes the stress tensor in a Laurent expansion and the resulting modes obey the Virasoro algebra. Here we do the same thing. This time the resulting modes obey

$$[J^a_n, J^b_m] = if^{ab}{}_c J^c_{n+m} + kn\delta^{ab}\delta_{n+m} \tag{6.43}$$

Here a, b label the different generators of the Lie algebra associated to G and $f^{ab}{}_c$ are the structure constants of the Lie algebra. Meanwhile, n, m label the modes of the current algebra. Note that if we restrict to the $n, m = 0$ sector then this is contains the Lie algebra. Including all the modes gives an infinite dimensional generalisation of the Lie algebra known as the *Kac-Moody algebra*.

Both the Kac-Moody algebra and the Virasoro algebra are infinite. But the Kac-Moody algebra should be thought of as bigger. Indeed, one can build the generators of the Virasoro algebra from bi-linears of the current using what's known as the *Sugawara construction*. We therefore work with representations of (6.43), each of which splits into an infinite number of representations of the Virasoro algebra.

The representations of (6.43) are characterised by their highest weight state, a primary operator. Each of these can be characterised by the way it transforms under the zero modes. In other words, the primary operators of the Kac-Moody algebra are labelled by representations of the underlying Lie algebra. The question that remains is: what are the primary operators?

In fact, we've already seen the answer to this in Section 5.4.4: the primary operators are the same as the non-trivial Wilson lines allowed in the bulk. For $G = SU(2)$, this means that the primary operators are labelled by their spin $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$. For $G = SU(N)$, the primary operators are labelled by Young diagrams whose upper row has no more than k boxes.

Armed with this list of primary operators, we can start to compute correlation functions and their braiding. However, there are a number of powerful tools that aid in this, not least the *Knizhnik-Zamolodchikov equations*, which are a set of partial differential equations which the conformal blocks must obey. In many cases, these tools allow one to determine completely the braiding properties of the conformal blocks.

To end, we will simply list some of the theories that have been useful in describing fractional quantum Hall states

- $SU(2)_1$: The WZW models at level $k = 1$ have Abelian anyons. For $SU(2)_1$, the central charge is $c = 1$ which is just that of a free boson. It turns out that theory describes the Halperin $(2, 2, 1)$ spin-singlet state that we met in Section 3.3.4

- $SU(2)_2$: The central charge is $c = 3/2$, which is the same as that of a free boson and a free Majorana fermion. But this is precisely the content that we needed to describe the Moore-Read states. The $SU(2)_2$ theory describes the physics of the state at filling fraction $\nu = 1$. For filling fraction $\nu = 1/2$, we should resort to the description of the CFT that we met in the last section as $U(1)_2 \times \text{Ising}$.

- $SU(2)_k/U(1)$: One can use the WZW models as the starting point to construct further conformal field theories known as coset models. Roughly, this means that you mod out by a $U(1)$ symmetry. These are sometimes referred to as $\mathbf{Z}_k$ parafermion theories. They are associated to the $p = k$-clustered Read Rezayi states that we met in Section 4.2.3. In particular, the $\mathbf{Z}_3$ theory exhibits Fibonacci anyons.