Preprint typeset in JHEP style - HYPER VERSION

Lent Term, 2013

Dynamics and Relativity

University of Cambridge Part IA Mathematical Tripos

David Tong

Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences,
Wilberforce Road,
Cambridge, CB3 OBA, UK

http://www.damtp.cam.ac.uk/user/tong/relativity.html
d.tong@damtp.cam.ac.uk

Recommended Books and Resources

- Tom Kibble and Frank Berkshire, "*Classical Mechanics*"
- Douglas Gregory, "*Classical Mechanics*"

Both of these books are well written and do an excellent job of explaining the fundamentals of classical mechanics. If you're struggling to understand some of the basic concepts, these are both good places to turn.

- S. Chandrasekhar, "*Newton's Principia (for the common reader)*"

Want to hear about Newtonian mechanics straight from the horse's mouth? This is an annotated version of the Principia with commentary by the Nobel prize winning astrophysicist Chandrasekhar who walks you through Newton's geometrical proofs. Although, in fairness, Newton is sometimes easier to understand than Chandra.

- A.P. French, "*Special Relativity*"

A clear introduction, covering the theory in some detail.

- Wolfgang Pauli, "*Theory of Relativity*"

Pauli was one of the founders of quantum mechanics and one of the great physicists of the last century. Much of this book was written when he was just 21. It remains one of the most authoritative and scholarly accounts of special relativity. It's not for the faint of heart. (But it is cheap).

A number of excellent lecture notes are available on the web. Links can be found on the course webpage: http://www.damtp.cam.ac.uk/user/tong/relativity.html

Contents

Acknowledgements

I inherited this course from Stephen Siklos. His excellent set of printed lecture notes form the backbone of these notes and can be found at:

http://www.damtp.cam.ac.uk/user/stcs/dynamics.html

I'm grateful to the students, and especially Henry Mak, for pointing out typos and corrections. My thanks to Alex Considine for putting up with the lost weekends while these lectures were written.

1. Newtonian Mechanics

Classical mechanics is an ambitious theory. Its purpose is to predict the future and reconstruct the past, to determine the history of every particle in the Universe.

In this course, we will cover the basics of classical mechanics as formulated by Galileo and Newton. Starting from a few simple axioms, Newton constructed a mathematical framework which is powerful enough to explain a broad range of phenomena, from the orbits of the planets, to the motion of the tides, to the scattering of elementary particles. Before it can be applied to any specific problem, the framework needs just a single input: a force. With this in place, it is merely a matter of turning a mathematical handle to reveal what happens next.

We start this course by exploring the framework of Newtonian mechanics, understanding the axioms and what they have to tell us about the way the Universe works. We then move on to look at a number of forces that are at play in the world. Nature is kind and the list is surprisingly short. Moreover, many of forces that arise have special properties, from which we will see new concepts emerging such as energy and conservation principles. Finally, for each of these forces, we turn the mathematical handle. We turn this handle many many times. In doing so, we will see how classical mechanics is able to explain large swathes of what we see around us.

Despite its wild success, Newtonian mechanics is not the last word in theoretical physics. It struggles in extremes: the realm of the very small, the very heavy or the very fast. We finish these lectures with an introduction to special relativity, the theory which replaces Newtonian mechanics when the speed of particles is comparable to the speed of light. We will see how our common sense ideas of space and time are replaced by something more intricate and more beautiful, with surprising consequences. Time goes slow for those on the move; lengths get smaller; mass is merely another form of energy.

Ultimately, the framework of classical mechanics falls short of its ambitious goal to tell the story of every particle in the Universe. Yet it provides the basis for all that follows. Some of the Newtonian ideas do not survive to later, more sophisticated, theories of physics. Even the seemingly primary idea of force will fall by the wayside. Instead other concepts that we will meet along the way, most notably energy, step to the fore. But all subsequent theories are built on the Newtonian foundation.

Moreover, developments in the past 300 years have confirmed what is perhaps the most important legacy of Newton: the laws of Nature are written in the language of mathematics. This is one of the great insights of human civilisation. It has ushered in scientific, industrial and technological revolutions. It has given us a new way to look at the Universe. And, most crucially of all, it means that the power to predict the future lies in hands of mathematicians rather than, say, gypsy astrologers. In this course, we take the first steps towards grasping this power.

1.1 Newton's Laws of Motion

Classical mechanics is all about the motion of particles. We start with a definition.

Definition: A *particle* is an object of insignificant size. This means that if you want to say what a particle looks like at a given time, the only information you have to specify is its position.

During this course, we will treat electrons, tennis balls, falling cats and planets as particles. In all of these cases, this means that we only care about the position of the object and our analysis will not, for example, be able to say anything about the look on the cat's face as it falls. However, it's not immediately obvious that we can meaningfully assign a single position to a complicated object such as a spinning, mewing cat. Should we describe its position as the end of its tail or the tip of its nose? We will not provide an immediate answer to this question, but we will return to it in Section 5 where we will show that any object can be treated as a point-like particle if we look at the motion of its centre of mass.

To describe the position of a particle we need a *reference frame*. This is a choice of origin, together with a set of axes which, for now, we pick to be Cartesian. With respect to this frame, the position of a particle is specified by a vector $\mathbf{x}$, which we denote using bold font. Since the particle moves, the position depends on time, resulting in a *trajectory* of the particle described by

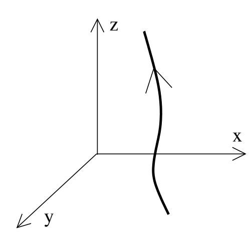

Figure 1:

$$\mathbf{x} = \mathbf{x}(t)$$

In these notes we will also use both the notation $\mathbf{x}(t)$ and $\mathbf{r}(t)$ to describe the trajectory of a particle.

The *velocity* of a particle is defined to be

$$\mathbf{v} \equiv \frac{d\mathbf{x}(t)}{dt}$$

Throughout these notes, we will often denote differentiation with respect to time by a "dot" above the variable. So we will also write $\mathbf{v} = \dot{\mathbf{x}}$. The acceleration of the particle is defined to be

$$\mathbf{a} \equiv \ddot{\mathbf{x}} = \frac{d^2\mathbf{x}(t)}{dt^2}$$

A Comment on Vector Differentiation

The derivative of a vector is defined by differentiating each of the components. So, if $\mathbf{x} = (x_1, x_2, x_3)$ then

$$\frac{d\mathbf{x}}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\right)$$

Geometrically, the derivative of a path $\mathbf{x}(\mathbf{t})$ lies tangent to the path (a fact that you will see in the Vector Calculus course).

In this course, we will be working with vector differential equations. These should be viewed as three, coupled differential equations – one for each component. We will frequently come across situations where we need to differentiate vector dot-products and cross-products. The meaning of these is easy to see if we use the chain rule on each component. For example, given two vector functions of time, $\mathbf{f}(t)$ and $\mathbf{g}(t)$, we have

$$\frac{d}{dt}(\mathbf{f} \cdot \mathbf{g}) = \frac{d\mathbf{f}}{dt} \cdot \mathbf{g} + \mathbf{f} \cdot \frac{d\mathbf{g}}{dt}$$

and

$$\frac{d}{dt}(\mathbf{f} \times \mathbf{g}) = \frac{d\mathbf{f}}{dt} \times \mathbf{g} + \mathbf{f} \times \frac{d\mathbf{g}}{dt}$$

As usual, it doesn't matter what order we write the terms in the dot product, but we have to be more careful with the cross product because, for example, $d\mathbf{f}/dt \times \mathbf{g} = -\mathbf{g} \times d\mathbf{f}/dt$.

1.1.1 Newton's Laws

Newtonian mechanics is a framework which allows us to determine the trajectory $\mathbf{x}(t)$ of a particle in any given situation. This framework is usually presented as three axioms known as Newton's laws of motion. They look something like:

- **N1:** Left alone, a particle moves with constant velocity.
- **N2:** The acceleration (or, more precisely, the rate of change of momentum) of a particle is proportional to the force acting upon it.

- **N3:** Every action has an equal and opposite reaction.

While it is worthy to try to construct axioms on which the laws of physics rest, the trite, minimalistic attempt above falls somewhat short. For example, on first glance, it appears that the first law is nothing more than a special case of the second law. (If the force vanishes, the acceleration vanishes which is the same thing as saying that the velocity is constant). But the truth is somewhat more subtle. In what follows we will take a closer look at what really underlies Newtonian mechanics.

1.2 Inertial Frames and Newton's First Law

Placed in the historical context, it is understandable that Newton wished to stress the first law. It is a rebuttal to the Aristotelian idea that, left alone, an object will naturally come to rest. Instead, as Galileo had previously realised, the natural state of an object is to travel with constant speed. This is the essence of the law of inertia.

However, these days we're not bound to any Aristotelian dogma. Do we really need the first law? The answer is yes, but it has a somewhat different meaning.

We've already introduced the idea of a frame of reference: a Cartesian coordinate system in which you measure the position of the particle. But for most reference frames you can think of, Newton's first law is obviously incorrect. For example, suppose the coordinate system that I'm measuring from is rotating. Then, everything will appear to be spinning around me. If I measure a particle's trajectory in my coordinates as $\mathbf{x}(t)$, then I certainly won't find that $d^2\mathbf{x}/dt^2 = 0$, even if I leave the particle alone. In rotating frames, particles do not travel at constant velocity.

We see that if we want Newton's first law to fly at all, we must be more careful about the kind of reference frames we're talking about. We define an *inertial reference frame* to be one in which particles do indeed travel at constant velocity when the force acting on it vanishes. In other words, in an inertial frame

$$\ddot{\mathbf{x}} = 0 \quad \text{when} \quad \mathbf{F} = 0$$

The true content of Newton's first law can then be better stated as

- **N1 Revisited:** Inertial frames exist.

These inertial frames provide the setting for all that follows. For example, the second law — which we shall discuss shortly — should be formulated in inertial frames.

One way to ensure that you are in an inertial frame is to insist that you are left alone yourself: fly out into deep space, far from the effects of gravity and other influences, turn off your engines and sit there. This is an inertial frame. However, for most purposes it will suffice to treat axes of the room you're sitting in as an inertial frame. Of course, these axes are stationary with respect to the Earth and the Earth is rotating, both about its own axis and about the Sun. This means that the Earth does not quite provide an inertial frame and we will study the consequences of this in Section 6.

1.2.1 Galilean Relativity

Inertial frames are not unique. Given one inertial frame, S, in which a particle has coordinates $\mathbf{x}(t)$, we can always construct another inertial frame S' in which the particle has coordinates $\mathbf{x}'(t)$ by any combination of the following transformations,

- Translations: $\mathbf{x}' = \mathbf{x} + \mathbf{a}$, for constant $\mathbf{a}$.
- Rotations: $\mathbf{x}' = R\mathbf{x}$, for a 3×3 matrix R obeying $R^T R = 1$. (This also allows for reflections if $\det R = -1$, although our interest will primarily be on continuous transformations).
- Boosts: $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$, for constant velocity $\mathbf{v}$.

It is simple to prove that all of these transformations map one inertial frame to another. Suppose that a particle moves with constant velocity with respect to frame S, so that $d^2\mathbf{x}/dt^2 = 0$. Then, for each of the transformations above, we also have $d^2\mathbf{x}'/dt^2 = 0$ which tells us that the particle also moves at constant velocity in S'. Or, in other words, if S is an inertial frame then so too is S'. The three transformations generate a group known as the *Galilean group*.

The three transformations above are not quite the unique transformations that map between inertial frames. But, for most purposes, they are the only interesting ones! The others are transformations of the form $\mathbf{x}' = \lambda\mathbf{x}$ for some $\lambda \in \mathbf{R}$. This is just a trivial rescaling of the coordinates. For example, we may choose to measure distances in S in units of meters and distances in S' in units of parsecs.

We have already mentioned that Newton's second law is to be formulated in an inertial frame. But, importantly, it doesn't matter which inertial frame. In fact, this is true for all laws of physics: they are the same in any inertial frame. This is known as the *principle of relativity*. The three types of transformation laws that make up the Galilean group map from one inertial frame to another. Combined with the principle of relativity, each is telling us something important about the Universe

- Translations: There is no special point in the Universe.
- Rotations: There is no special direction in the Universe.
- Boosts: There is no special velocity in the Universe

The first two are fairly unsurprising: position is relative; direction is relative. The third perhaps needs more explanation. Firstly, it is telling us that there is no such thing as "absolutely stationary". You can only be stationary *with respect* to something else. Although this is true (and continues to hold in subsequent laws of physics) it is not true that there is no special speed in the Universe. The speed of light is special. We will see how this changes the principle of relativity in Section 7.

So position, direction and velocity are relative. But acceleration is not. You do not have to accelerate relative to something else. It makes perfect sense to simply say that you are accelerating or you are not accelerating. In fact, this brings us back to Newton's first law: if you are not accelerating, you are sitting in an inertial frame.

The principle of relativity is usually associated to Einstein, but in fact dates back at least as far as Galileo. In his book, "*Dialogue Concerning the Two Chief World Systems*", Galileo has the character Salviati talk about the relativity of boosts,

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still.
>
> *Galileo Galilei, 1632*

Absolute Time

There is one last issue that we have left implicit in the discussion above: the choice of time coordinate t. If observers in two inertial frames, S and S', fix the units – seconds, minutes, hours – in which to measure the duration time then the only remaining choice they can make is when to start the clock. In other words, the time variable in S and S' differ only by

$$t' = t + t_0$$

This is sometimes included among the transformations that make up the Galilean group.

The existence of a uniform time, measured equally in all inertial reference frames, is referred to as *absolute time*. It is something that we will have to revisit when we discuss special relativity. As with the other Galilean transformations, the ability to shift the origin of time is reflected in an important property of the laws of physics. The fundamental laws don't care when you start the clock. All evidence suggests that the laws of physics are the same today as they were yesterday. They are time translationally invariant.

Cosmology

Notably, the Universe itself breaks several of the Galilean transformations. There was a very special time in the Universe, around 13.7 billion years ago. This is the time of the Big Bang (which, loosely translated, means "we don't know what happened here").

Similarly, there is one inertial frame in which the background Universe is stationary. The "background" here refers to the sea of photons at a temperature of 2.7 K which fills the Universe, known as the Cosmic Microwave Background Radiation. This is the afterglow of the fireball that filled all of space when the Universe was much younger. Different inertial frames are moving relative to this background and measure the radiation differently: the radiation looks more blue in the direction that you're travelling, redder in the direction that you've come from. There is an inertial frame in which this background radiation is uniform, meaning that it is the same colour in all directions.

To the best of our knowledge however, the Universe defines neither a special point, nor a special direction. It is, to very good approximation, homogeneous and isotropic.

However, it's worth stressing that this discussion of cosmology in no way invalidates the principle of relativity. All laws of physics are the same regardless of which inertial frame you are in. Overwhelming evidence suggests that the laws of physics are the same in far flung reaches of the Universe. They were the same in first few microseconds after the Big Bang as they are now.

1.3 Newton's Second Law

The second law is the meat of the Newtonian framework. It is the famous "$F = ma$", which tells us how a particle's motion is affected when subjected to a force $\mathbf{F}$. The correct form of the second law is

$$\frac{d}{dt}(m\dot{\mathbf{x}}) = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) \tag{1.1}$$

This is usually referred to as the *equation of motion*. The quantity in brackets is called the *momentum*,

$$\mathbf{p} \equiv m\dot{\mathbf{x}}$$

Here m is the mass of the particle or, more precisely, the *inertial mass*. It is a measure of the reluctance of the particle to change its motion when subjected to a given force $\mathbf{F}$. In most situations, the mass of the particle does not change with time. In this case, we can write the second law in the more familiar form,

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) \tag{1.2}$$

For much of this course, we will use the form (1.2) of the equation of motion. However, in Section 5.3, we will briefly look at a few cases where masses are time dependent and we need the more general form (1.1).

Newton's second law doesn't actually tell us anything until someone else tells us what the force $\mathbf{F}$ is in any given situation. We will describe several examples in the next section. In general, the force can depend on the position $\mathbf{x}$ and the velocity $\dot{\mathbf{x}}$ of the particle, but does not depend on any higher derivatives. We could also, in principle, consider forces which include an explicit time dependence, $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$, although we won't do so in these lectures. Finally, if more than one (independent) force is acting on the particle, then we simply take their sum on the right-hand side of (1.2).

The single most important fact about Newton's equation is that it is a *second order* differential equation. This means that we will have a unique solution only if we specify two initial conditions. These are usually taken to be the position $\mathbf{x}(t_0)$ and the velocity $\dot{\mathbf{x}}(t_0)$ at some initial time t_0. However, exactly what boundary conditions you must choose in order to figure out the trajectory depends on the problem you are trying to solve. It is not unusual, for example, to have to specify the position at an initial time t_0 and final time t_f to determine the trajectory.

The fact that the equation of motion is second order is a deep statement about the Universe. It carries over, in essence, to all other laws of physics, from quantum mechanics to general relativity to particle physics. Indeed, the fact that all initial conditions must come in pairs — two for each "degree of freedom" in the problem — has important ramifications for later formulations of both classical and quantum mechanics.

For now, the fact that the equations of motion are second order means the following: if you are given a snapshot of some situation and asked "what happens next?" then there is no way of knowing the answer. It's not enough just to know the positions of the particles at some point of time; you need to know their velocities too. However, once both of these are specified, the future evolution of the system is fully determined for all time.

1.4 Looking Forwards: The Validity of Newtonian Mechanics

Although Newton's laws of motion provide an excellent approximation to many phenomena, when pushed to extreme situation they are found wanting. Broadly speaking, there are three directions in which Newtonian physics needs replacing with a different framework: they are

- When particles travel at speeds close to the speed light, $c \approx 3 \times 10^8\ ms^{-1}$, the Newtonian concept of absolute time breaks down and Newton's laws need modification. The resulting theory is called special relativity and will be described in Section 7. As we will see, although the relationship between space and time is dramatically altered in special relativity, much of the framework of Newtonian mechanics survives unscathed.

- On very small scales, much more radical change is needed. Here the whole framework of classical mechanics breaks down so that even the most basic concepts, such as the trajectory of a particle, become ill-defined. The new framework that holds on these small scales is called quantum mechanics. Nonetheless, there are quantities which carry over from the classical world to the quantum, in particular energy and momentum.

- When we try to describe the forces at play between particles, we need to introduce a new concept: the *field*. This is a function of both space and time. Familiar examples are the electric and magnetic fields of electromagnetism. We won't have too much to say about fields in this course. For now, we mention only that the equations which govern the dynamics of fields are always second order differential equations, similar in spirit to Newton's equations. Because of this similarity, field theories are again referred to as "classical".

Eventually, the ideas of special relativity, quantum mechanics and field theories are combined into *quantum field theory*. Here even the concept of particle gets subsumed into the concept of a field. This is currently the best framework we have to describe the world around us. But we're getting ahead of ourselves. Let's firstly return to our Newtonian world....