

6. Non-Inertial Frames

We stated, long ago, that inertial frames provide the setting for Newtonian mechanics. But what if you, one day, find yourself in a frame that is not inertial? For example, suppose that every 24 hours you happen to spin around an axis which is 2500 miles away. What would you feel? Or what if every year you spin around an axis 36 million miles away? Would that have any effect on your everyday life?

In this section we will discuss what Newton's equations of motion look like in non-inertial frames. Just as there are many ways that an animal can be not a dog, so there are many ways in which a reference frame can be non-inertial. Here we will just consider one type: reference frames that rotate. We'll start with some basic concepts.

6.1 Rotating Frames

Let's start with the inertial frame S drawn in the figure with coordinate axes x , y and z . Our goal is to understand the motion of particles as seen in a non-inertial frame S' , with axes x' , y' and z' , which is rotating with respect to S . We'll denote the angle between the x -axis of S and the x' -axis of S' as θ . Since S' is rotating, we clearly have $\theta = \theta(t)$ and $\dot{\theta} \neq 0$.

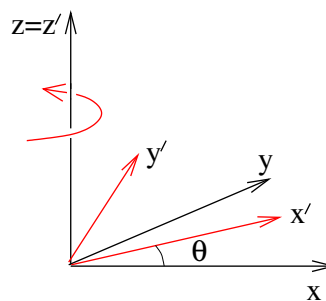


Figure 31:

Our first task is to find a way to describe the rotation of the axes. For this, we can use the angular velocity vector $\boldsymbol{\omega}$ that we introduced in the last section to describe the motion of particles. Consider a particle that is sitting stationary in the S' frame. Then, from the perspective of frame S it will appear to be moving with velocity

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$$

where, in the present case, $\boldsymbol{\omega} = \dot{\theta}\hat{\mathbf{z}}$. Recall that in general, $|\boldsymbol{\omega}| = \dot{\theta}$ is the angular speed, while the direction of $\boldsymbol{\omega}$ is the axis of rotation, defined in a right-handed sense.

We can extend this description of the rotation of the axes of S' themselves. Let \mathbf{e}'_i , $i = 1, 2, 3$ be the unit vectors that point along the x' , y' and z' directions of S' . Then these also rotate with velocity

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega} \times \mathbf{e}'_i$$

This will be the main formula that will allow us to understand motion in rotating frames.

6.1.1 Velocity and Acceleration in a Rotating Frame

Consider now a particle which is no longer stuck in the S' frame, but moves on some trajectory. We can measure the position of the particle in the inertial frame S , where, using the summation convention, we write

$$\mathbf{r} = r_i \mathbf{e}_i$$

Here the unit vectors \mathbf{e}_i , with $i = 1, 2, 3$ point along the axes of S . Alternatively, we can measure the position of the particle in frame S' , where the position is

$$\mathbf{r} = r'_i \mathbf{e}'_i$$

Note that the position vector \mathbf{r} is the same in both of these expressions: but the coordinates r_i and r'_i differ because they are measured with respect to different axes. Now, we can compute an expression for the velocity of the particle. In frame S , it is simply

$$\dot{\mathbf{r}} = \dot{r}_i \mathbf{e}_i \tag{6.1}$$

because the axes \mathbf{e}_i do not change with time. However, in the rotating frame S' , the velocity of the particle is

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{r}'_i \mathbf{e}'_i + r'_i \dot{\mathbf{e}}'_i \\ &= \dot{r}'_i \mathbf{e}'_i + r'_i \boldsymbol{\omega} \times \mathbf{e}'_i \\ &= \dot{r}'_i \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{r} \end{aligned} \tag{6.2}$$

We'll introduce a slightly novel notation to help highlight the physics hiding in these two equations. We write the velocity of the particle as seen by an observer in frame S as

$$\left(\frac{d\mathbf{r}}{dt} \right)_S = \dot{r}_i \mathbf{e}_i$$

Similarly, the velocity as seen by an observer in frame S' is just

$$\left(\frac{d\mathbf{r}}{dt} \right)_{S'} = \dot{r}'_i \mathbf{e}'_i$$

From equations (6.1) and (6.2), we see that the two observers measure different velocities,

$$\left(\frac{d\mathbf{r}}{dt} \right)_S = \left(\frac{d\mathbf{r}}{dt} \right)_{S'} + \boldsymbol{\omega} \times \mathbf{r} \tag{6.3}$$

This is not completely surprising: the difference is just the relative velocity of the two frames.

What about acceleration? We can play the same game. In frame S , we have

$$\ddot{\mathbf{r}} = \ddot{r}_i \mathbf{e}_i$$

while in frame S' , the expression is a little more complicated. Differentiating (6.2) once more, we have

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{r}'_i \mathbf{e}'_i + \dot{r}'_i \dot{\mathbf{e}}'_i + \dot{r}'_i \boldsymbol{\omega} \times \mathbf{e}'_i + r'_i \dot{\boldsymbol{\omega}} \times \mathbf{e}'_i + r'_i \boldsymbol{\omega} \times \dot{\mathbf{e}}'_i \\ &= \ddot{r}'_i \mathbf{e}'_i + 2\dot{r}'_i \boldsymbol{\omega} \times \mathbf{e}'_i + \dot{\boldsymbol{\omega}} \times \mathbf{r} + r'_i \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{e}'_i) \end{aligned}$$

As with velocities, the acceleration seen by the observer in S is $\ddot{r}_i \mathbf{e}_i$ while the acceleration seen by the observer in S' is $\ddot{r}'_i \mathbf{e}'_i$. Equating the two equations above gives us

$$\left(\frac{d^2 \mathbf{r}}{dt^2} \right)_S = \left(\frac{d^2 \mathbf{r}}{dt^2} \right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt} \right)_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad (6.4)$$

This equation contains the key to understanding the motion of particles in a rotating frame.

6.2 Newton's Equation of Motion in a Rotating Frame

With the hard work behind us, let's see how a person sitting in the rotating frame S' would see Newton's law of motion. We know that in the inertial frame S , we have

$$m \left(\frac{d^2 \mathbf{r}}{dt^2} \right)_S = \mathbf{F}$$

So, using (6.4), in frame S' , we have

$$m \left(\frac{d^2 \mathbf{r}}{dt^2} \right)_{S'} = \mathbf{F} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt} \right)_{S'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad (6.5)$$

In other words, to explain the motion of a particle an observer in S' must invoke the existence of three further terms on the right-hand side of Newton's equation. These are called *fictitious forces*. Viewed from S' , a free particle doesn't travel in a straight line and these fictitious forces are necessary to explain this departure from uniform motion. In the rest of this section, we will see several examples of this.

The $-2m\boldsymbol{\omega} \times \dot{\mathbf{r}}$ term in (6.5) is the *Coriolis force*; the $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ term is called the *centrifugal force*; the $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ term is called the *Euler force*.

The most familiar non-inertial frame is the room you are sitting in. It rotates once per day around the north-south axis of the Earth. It further rotates once a year about the Sun which, in turn, rotates about the centre of the galaxy. From these time scales, we can easily compute $\omega = |\boldsymbol{\omega}|$.

The radius of the Earth is $R_{\text{Earth}} \approx 6 \times 10^3 \text{ km}$. The Earth rotates with angular frequency

$$\omega_{\text{rot}} = \frac{2\pi}{1 \text{ day}} \approx 7 \times 10^{-5} \text{ s}^{-1}$$

The distance from the Earth to the Sun is $a_e \approx 2 \times 10^8 \text{ km}$. The angular frequency of the orbit is

$$\omega_{\text{orb}} = \frac{2\pi}{1 \text{ year}} \approx 2 \times 10^{-7} \text{ s}^{-1}$$

It should come as no surprise to learn that $\omega_{\text{rot}}/\omega_{\text{orb}} = T_{\text{orb}}/T_{\text{rot}} \approx 365$.

In what follows, we will see the effect of the centrifugal and Coriolis forces on our daily lives. We will not discuss the Euler force, which arises only when the speed of the rotation changes with time. Although this plays a role in various funfair rides, it's not important in the frame of the Earth. (The angular velocity of the Earth's rotation does, in fact, have a small, but non-vanishing, $\dot{\boldsymbol{\omega}}$ due to the precession and nutation of the Earth's rotational axis. However, it is tiny, with $\dot{\boldsymbol{\omega}} \ll \boldsymbol{\omega}^2$ and, as far as I know, the resulting Euler force has no consequence).

Inertial vs Gravitational Mass Revisited

Notice that all the fictitious forces are proportional to the inertial mass m . There is no mystery here: it's because they all originated from the "ma" side of " $\mathbf{F}=\mathbf{ma}$ " rather than "F" side. But, as we mentioned in Section 2, experimentally the gravitational force also appears to be proportional to the inertial mass. Is this evidence that gravity too is a fictitious force? In fact it is. Einstein's theory of general relativity recasts gravity as the fictitious force that we experience due to the curvature of space and time.

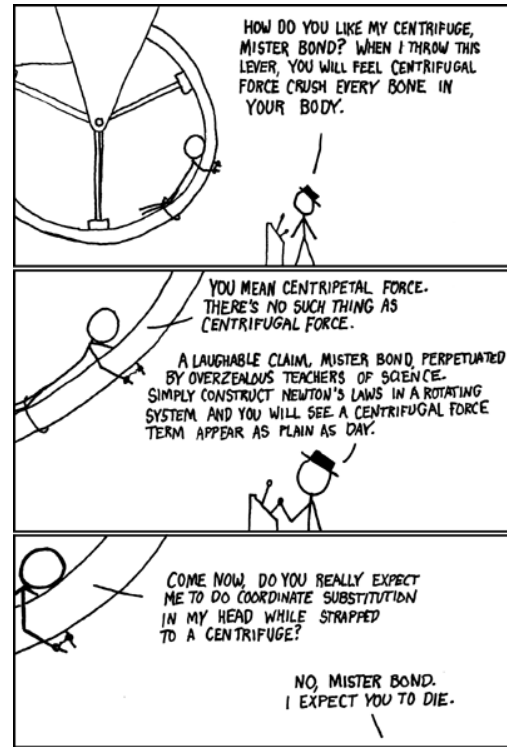


Figure 32: xkcd.com

6.3 Centrifugal Force

The centrifugal force is given by

$$\begin{aligned}\mathbf{F}_{\text{cent}} &= -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= -m(\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega} + m\omega^2\mathbf{r}\end{aligned}$$

We can get a feel for this by looking at the figure. The vector $\boldsymbol{\omega} \times \mathbf{r}$ points into the page, which means that $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ points away from the axis of rotation as shown. The magnitude of the force is

$$|\mathbf{F}_{\text{cent}}| = m\omega^2 r \cos\theta = m\omega^2 d \quad (6.6)$$

where d is the distance to the axis of rotation as shown in the figure.

The centrifugal force does not depend on the velocity of the particle. In fact, it is an example of a conservative force. We can see this by writing

$$\mathbf{F}_{\text{cent}} = -\nabla V \quad \text{with} \quad V = -\frac{m}{2}|\boldsymbol{\omega} \times \mathbf{r}|^2 \quad (6.7)$$

In a rotating frame, V has the interpretation of the potential energy associated to a particle. The potential V is negative, which tells us that particles want to fly out from the axis of rotation to lower their energy by increasing $|\mathbf{r}|$.

6.3.1 An Example: Apparent Gravity

Suspend a piece of string from the ceiling. You might expect that the string points down to the centre of the Earth. But the effect of the centrifugal force due to the Earth's rotation means that this isn't the case. A somewhat exaggerated picture of this is shown in the figure. The question that we would like to answer is: what is the angle ϕ that the string makes with the line pointing to the Earth's centre? As we will now show, the angle ϕ depends on the latitude, θ , at which we're sitting.

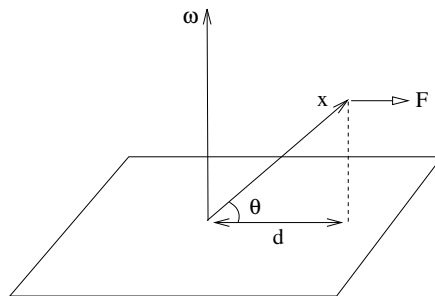


Figure 33:

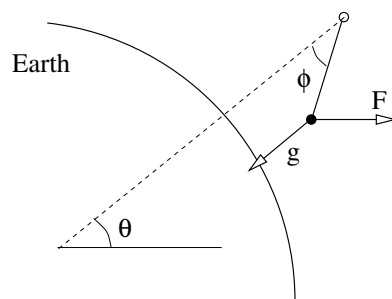


Figure 34:

The effective acceleration, due to the combination of gravity and the centrifugal force, is

$$\mathbf{g}_{\text{eff}} = \mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

It is useful to resolve this acceleration in the radial and southerly directions by using the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. The centrifugal force \mathbf{F} is resolved as

$$\begin{aligned} \mathbf{F} &= |\mathbf{F}| \cos \theta \hat{\mathbf{r}} - |\mathbf{F}| \sin \theta \hat{\boldsymbol{\theta}} \\ &= m\omega^2 r \cos^2 \theta \hat{\mathbf{r}} - m\omega^2 r \cos \theta \sin \theta \hat{\boldsymbol{\theta}} \end{aligned}$$

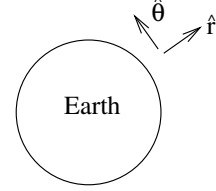


Figure 35:

where, in the second line, we have used the magnitude of the centrifugal force computed in (6.6). Notice that, at the pole $\theta = \pi/2$ and the centrifugal force vanishes as expected. This gives the effective acceleration

$$\mathbf{g}_{\text{eff}} = -g\hat{\mathbf{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = (-g + \omega^2 R \cos^2 \theta)\hat{\mathbf{r}} - \omega^2 R \cos \theta \sin \theta \hat{\boldsymbol{\theta}}$$

where R is the radius of the Earth.

The force $m\mathbf{g}_{\text{eff}}$ must be balanced by the tension \mathbf{T} in the string. This too can be resolved as

$$\mathbf{T} = T \cos \phi \hat{\mathbf{r}} + T \sin \phi \hat{\boldsymbol{\theta}}$$

In equilibrium, we need $m\mathbf{g}_{\text{eff}} + \mathbf{T} = 0$, which allows us to eliminate T to get an equation relating ϕ to the latitude θ ,

$$\tan \phi = \frac{\omega^2 R \cos \theta \sin \theta}{g - \omega^2 R \cos^2 \theta}$$

This is the answer we wanted. Let's see at what latitude the angle ϕ is largest. If we compute $d(\tan \phi)/d\theta$, we find a fairly complicated expression. However, if we take into account the fact that $\omega^2 R \approx 3 \times 10^{-2} \text{ ms}^{-2} \ll g$ then we can neglect the term in which we differentiate the denominator. We learn that the maximum departure from the vertical occurs more or less when $d(\cos \theta \sin \theta)/d\theta = 0$. Or, in other words, at a latitude of $\theta \approx 45^\circ$. However, even at this point the deflection from the vertical is tiny: an order of magnitude gives $\phi \approx 10^{-4}$.

When we sit at the equator, with $\theta = 0$, then $\phi = 0$ and the string hangs directly towards the centre of the Earth. However, gravity is somewhat weaker due to the centrifugal force. We have

$$g_{\text{eff}}|_{\text{equator}} = g - \omega^2 R$$

Based on this, we expect $g_{\text{eff}} - g \approx 3 \times 10^{-2} \text{ m.s}^{-2}$ at the equator. In fact, the experimental result is more like $5 \times 10^{-2} \text{ m.s}^{-2}$. The reason for this discrepancy can also be traced to the centrifugal force which means that the Earth is not spherical, but rather bulges near the equator.

A Rotating Bucket

Fill a bucket with water and spin it. The surface of the water will form a concave shape like that shown in the figure. What is the shape?

We assume that the water spins with the bucket. The potential energy of a water molecule then has two contributions: one from gravity and the other due to the centrifugal force given in (6.7)

$$V_{\text{water}} = mgz - \frac{1}{2}m\omega^2 r^2$$

Now we use a somewhat slick physics argument. Consider a water molecule on the surface of the fluid. If it could lower its energy by moving along the surface, then it would. But we're looking for the equilibrium shape of the surface, which means that each point on the surface must have equal potential energy. This means that the shape of the surface is a parabola, governed by the equation

$$z = \frac{\omega^2 r^2}{2g} + \text{constant}$$

6.4 Coriolis Force

The Coriolis force is given by

$$\mathbf{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \mathbf{v}$$

where, from (6.5), we see that $\mathbf{v} = (d\mathbf{r}/dt)_{S'}$ is the velocity of the particle measured in the rotating frame S' . The force is velocity dependent: it is only felt by moving particles. Moreover, it is independent on the position.

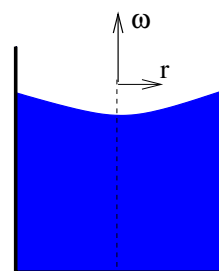


Figure 36:

6.4.1 Particles, Baths and Hurricanes

The mathematical form of the Coriolis force is identical to the Lorentz force (2.19) describing a particle moving in a magnetic field. This means we already know what the effect of the Coriolis force will be: it makes moving particles turn in circles.

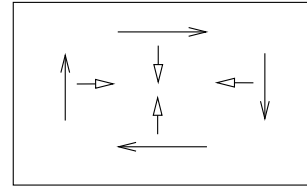


Figure 37:

We can easily check that this is indeed the case. Consider a particle moving on a spinning plane as shown in the figure, where ω is coming out of the page. In the diagram we have drawn various particle velocities, together with the Coriolis force experienced by the particle. We see that the effect of the Coriolis force is that a free particle travelling on the plane will move in a clockwise direction.

There is a similar force — at least in principle — when you pull the plug from your bathroom sink. But here there’s a subtle difference which actually reverses the direction of motion!

Consider a fluid in which there is a region of low pressure. This region could be formed in a sink because we pulled the plug, or it could be formed in the atmosphere due to random weather fluctuations. Now the particles in the fluid will move radially towards the low pressure region. As they move, they will be deflected by the Coriolis force as shown in the figure. The direction of the deflection is the same as that of a particle moving in the plane. But the net effect is that the swirling fluid moves in an anti-clockwise direction.

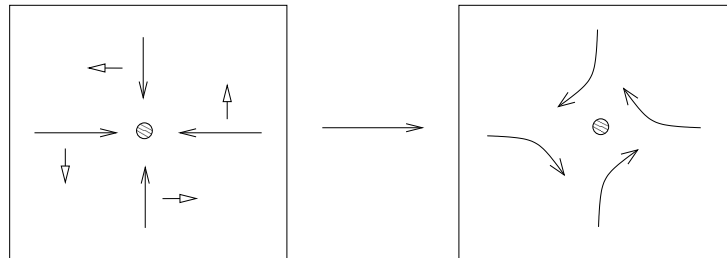


Figure 38:

The Coriolis force is responsible for the formation of hurricanes. These rotate in an anti-clockwise direction in the Northern hemisphere and a clockwise direction in



Figure 39: Cyclone Catarina which hit Brazil in 2004

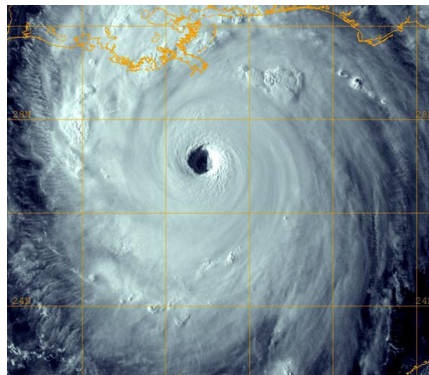


Figure 40: Hurricane Katrina, which hit New Orleans in 2005

the Southern hemisphere. However, don't spend too long staring at the rotation in your bath water. Although the effect can be reproduced in laboratory settings, in your bathroom the Coriolis force is too small: it is no more likely to make your bath water change direction than it is to make your CD change direction. (An aside: CDs are what people used before phones. Some towns have museums – they used to be called record stores – that display examples of CD cases for people to look at.)

Our discussion above supposed that objects were moving on a plane which is perpendicular to the angular velocity $\boldsymbol{\omega}$. But that's not true for hurricanes: they move along the surface of the Earth, which means that their velocity has a component parallel to $\boldsymbol{\omega}$. In this case, the effective magnitude of the Coriolis force gets a geometric factor,

$$|\mathbf{F}_{\text{cor}}| = 2m\omega v \sin \theta \quad (6.8)$$

It's simplest to see the $\sin \theta$ factor in the case of a particle travelling North. Here the Coriolis force acts in an Easterly direction and a little bit of trigonometry shows that the force has magnitude $2m\omega v \sin \theta$ as claimed. This is particularly clear at the equator where $\theta = 0$. Here a particle travelling North has \mathbf{v} parallel to $\boldsymbol{\omega}$ and so the Coriolis force vanishes.

It's a little more tricky to see the $\sin \theta$ factor for a particle travelling in the Easterly direction. In this case, \mathbf{v} is perpendicular to $\boldsymbol{\omega}$, so the magnitude of the force is actually $2m\omega v$, with no trigonometric factor. However, the direction of the force no longer lies parallel to the Earth's surface: it has a component which points directly upwards. But we're not interested in this component; it's certainly not going to be big enough to compete with gravity. Projecting onto the component that lies parallel to the Earth's surface (in a Southerly direction in this case), we again get a $\sin \theta$ factor.

The factor of $\sin \theta$ in (6.8) has an important meteorological consequence: the Coriolis force vanishes when $\theta = 0$, which ensures that hurricanes do not form within 500 miles of the equator.

6.4.2 Balls and Towers

Climb up a tower and drop a ball. Where does it land? Since the Earth is rotating under the tower, you might think that the ball lands behind you. In fact, it lands in front! Let's see where this somewhat counterintuitive result comes from.

The equation of motion in a rotating frame is

$$\ddot{\mathbf{r}} = \mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2\boldsymbol{\omega} \times \dot{\mathbf{r}}$$

We've already seen in Section 6.3.1 that the effect of the centrifugal force is to change the effective direction of gravity. But we've also seen that this effect is small. In what follows we will neglect the centrifugal term. In fact, we will ignore all terms of order $\mathcal{O}(\omega^2)$ (there will be one more coming shortly!). We will therefore solve the equation of motion

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} \tag{6.9}$$

The first step is easy: we can integrate this once to give

$$\dot{\mathbf{r}} = \mathbf{g}t - 2\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r}_0)$$

where we've introduced the initial position \mathbf{r}_0 as an integration constant. If we now substitute this back into the equation of motion (6.9), we get a messy, but manageable, equation. Let's, however, make our life easier by recalling that we've already agreed to drop terms of order $\mathcal{O}(\omega^2)$. Then, upon substitution, we're left with

$$\ddot{\mathbf{r}} \approx \mathbf{g} - 2\boldsymbol{\omega} \times \mathbf{g}t$$

which we can easily integrate one last time to find

$$\mathbf{r} \approx \mathbf{r}_0 + \frac{1}{2}\mathbf{g}t^2 - \frac{1}{3}\boldsymbol{\omega} \times \mathbf{g}t^3$$

We'll pick a right-handed basis of vectors so that \mathbf{e}_1 points North, \mathbf{e}_2 points West and $\mathbf{e}_3 = \hat{\mathbf{r}}$ points radially outward as shown in the figure. However, we'll also make life easier for ourselves and assume that the tower sits at the equator. (This means that we don't have to worry about the annoying $\sin \theta$ factor that we saw in (6.8) and we will see again in the next section). Then

$$\mathbf{g} = -g\mathbf{e}_3 \quad , \quad \boldsymbol{\omega} = \omega\mathbf{e}_1 \quad , \quad \mathbf{r}_0 = (R + h)\mathbf{e}_3$$

where R is the radius of the Earth and h is the height of the tower. Our solution reads

$$\mathbf{r} \approx \left(R + h - \frac{1}{2}gt^2 \right) \mathbf{e}_3 - \frac{1}{3}\omega gt^3 \mathbf{e}_2$$

The first term tells us the familiar result that the particle hits the ground in time $t^2 = 2h/g$. The last term gives the displacement, d ,

$$d = -\frac{1}{3}\omega g \left(\frac{2h}{g} \right)^{3/2} = -\frac{2\omega}{3} \sqrt{\frac{2h^3}{g}}$$

Recall that \mathbf{e}_2 points West, so that the fact that d is negative means that the displacement is in the Easterly direction. But the Earth rotates West to East. This means that the ball falls in front of the tower as promised.

In fact, there is a simple intuitive way to understand this result. Although we have presented it as a consequence of the Coriolis force, it follows from the conservation of angular momentum. When dropped, the angular momentum (per unit mass) of the particle is

$$l = \omega(R + h)^2$$

This can't change as the ball falls. This means that the ball's final speed in the Easterly direction is

$$Rv = (R + h)^2\omega \quad \Rightarrow \quad v = \frac{(R + h)^2\omega}{R} > v_{\text{Earth}} = R\omega$$

So its tangential velocity is greater than that of the Earth's surface. This is the reason that it falls in front of the tower.

6.4.3 Foucault's Pendulum

A pendulum placed at the North pole will stay aligned with its own inertial plane while the Earth rotates beneath. An observer on the Earth would attribute this rotation of the pendulum's axis to the Coriolis force. What happens if we place the pendulum at some latitude θ ?

Let's call the length of the pendulum l . As in the previous example, we'll work with a right-handed orthonormal basis of vectors so that \mathbf{e}_1 points North, \mathbf{e}_2 points West and $\mathbf{e}_3 = \hat{\mathbf{r}}$ point radially outward from the earth. We place the origin a distance l below the pivot, so that when the pendulum hangs directly downwards the bob at the end sits on the origin. Finally, we ignore the centrifugal force.

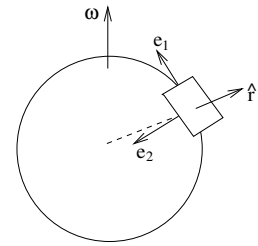


Figure 41:

The equation of motion for the pendulum, including the Coriolis force, is

$$m\ddot{\mathbf{r}} = \mathbf{T} + m\mathbf{g} - 2m\boldsymbol{\omega} \times \dot{\mathbf{x}}$$

Notice that we've reverted to calling the position of the particle \mathbf{x} instead of \mathbf{r} . This is to (hopefully) avoid confusion: our basis vector $\hat{\mathbf{r}}$ does not point towards the particle; it points radially out from the earth. This is in a different direction to $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ which is the position of the bob shown in the figure. Because the bob sits at the end of the pendulum, the coordinates are subject to the constraint

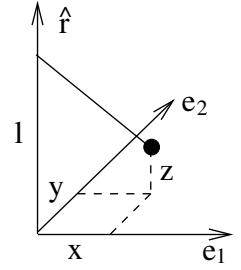


Figure 42:

$$x^2 + y^2 + (l - z)^2 = l^2 \quad (6.10)$$

At latitude θ , the rotation vector is

$$\boldsymbol{\omega} = \omega \cos \theta \mathbf{e}_1 + \omega \sin \theta \hat{\mathbf{r}}$$

while the acceleration due to gravity is $\mathbf{g} = -g\hat{\mathbf{r}}$. We also need an expression for the tension \mathbf{T} , which points along the direction of the pendulum. Again consulting the figure, we can see that the tension is given by

$$\mathbf{T} = -\frac{Tx}{l} \mathbf{e}_1 - \frac{T y}{l} \mathbf{e}_2 + \frac{T(l - z)}{l} \hat{\mathbf{r}}$$

Resolving the equation of motion along the axes gives us three equations,

$$m\ddot{x} = -\frac{xT}{l} + 2m\omega\dot{y} \sin \theta \quad (6.11)$$

$$m\ddot{y} = -\frac{yT}{l} + 2m\omega(\dot{z} \cos \theta - \dot{x} \sin \theta) \quad (6.12)$$

$$m\ddot{z} = -mg + \frac{T(l - z)}{l} - 2m\omega\dot{y} \cos \theta \quad (6.13)$$

These equations, together with the constraint (6.10), look rather formidable. To make progress, we will assume that $x/l \ll 1$ and $y/l \ll 1$ and work to leading order in this small number. This is not as random as it may seem: Foucault's original pendulum hangs in the Pantheon in Paris and is 67 meters long, with the amplitude of the swing a few meters. The advantage of this approximation becomes apparent when we revisit the constraint (6.10) which tells us that z/l is second order,

$$l - z = l\sqrt{1 - \frac{x^2}{l^2} - \frac{y^2}{l^2}} \approx l - \frac{x^2}{2l} - \frac{y^2}{2l} + \dots$$

This means that, to leading order, we can set z , \dot{z} and \ddot{z} all to zero. The last of the equations (6.13) then provides an equation that will soon allow us to eliminate T

$$T \approx mg + 2m\omega\dot{y} \cos \theta \quad (6.14)$$

Meanwhile, we rewrite the first two equations (6.11) and (6.12) using the same trick we saw in our study of Larmor circles in Section (2.4.2): we introduce $\xi = x + iy$ and add (6.11) to i times (6.12) to get

$$\ddot{\xi} \approx -\frac{g}{l} \xi - 2\omega i \dot{\xi} \sin \theta$$

Here we have substituted $T \approx mg$ since the second term in (6.14) contributes only at sub-leading order. This is the equation of motion for a damped harmonic oscillator, albeit with a complex variable. We can solve it in the same way: the ansatz $\xi = e^{\beta t}$ results in the quadratic equation

$$\beta^2 + 2i\omega\beta \sin \theta + \frac{g}{l} = 0$$

which has solutions

$$\beta_{\pm} = -i\omega \sin \theta \pm i\sqrt{\frac{1}{4}\omega^2 \sin^2 \theta + \frac{g}{l}} \approx -i\left(\omega \sin \theta \pm \sqrt{\frac{g}{l}}\right)$$

From this we can write the general solution as

$$\xi = e^{-i\omega t \sin \theta} \left(A \cos \sqrt{\frac{g}{l}} t + B \sin \sqrt{\frac{g}{l}} t \right)$$

Without the overall phase factor, $e^{-i\omega t \sin \theta}$, this equation describes an ellipse. The role of the phase factor is to make the orientation of the ellipse slowly rotate in the $x - y$ plane. Viewed from above, the rotation is clockwise in the Northern hemisphere; anti-clockwise in the Southern hemisphere. Notice that the period of rotation is not 24 hours unless the pendulum is suspended at the poles. Instead the period is $24/\sin \theta$ hours. In Paris, this is 32 hours.

6.4.4 Larmor Precession

The transformation to rotating frames can also be used as a cute trick to solve certain problems. Consider, for example, a charged particle orbiting around a second, fixed particle under the influence of the Coulomb force. Now add to this a constant magnetic field \mathbf{B} . The resulting equation of motion is

$$m\ddot{\mathbf{r}} = -\frac{k}{r^2}\hat{\mathbf{r}} + q\dot{\mathbf{r}} \times \mathbf{B}$$

where $k = qQ/4\pi\epsilon_0$. When $\mathbf{B} = 0$, this is the central force problem that we solved in Section 4 and we know the orbit of the particle is an ellipse. But what about when $\mathbf{B} \neq 0$?

Let's look at the problem in a rotating frame. Using (6.3) and (6.4), we have

$$m(\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) = -\frac{k}{r^2}\hat{\mathbf{r}} + q(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B}$$

where now \mathbf{r} describes the position of the coordinate in the rotating frame. Now we do something clever: we pick the angular velocity of rotation $\boldsymbol{\omega}$ so that the $\dot{\mathbf{r}}$ terms above cancel. This works for

$$\boldsymbol{\omega} = -\frac{q\mathbf{B}}{2m}$$

Then the equation of motion becomes

$$m\ddot{\mathbf{r}} = -\frac{k}{r^2}\hat{\mathbf{r}} + \frac{q^2}{2m}\mathbf{B} \times (\mathbf{B} \times \mathbf{r})$$

This is almost of the form that we studied in Section 4. In fact, for suitably small magnetic fields we can just ignore the last term. This holds as long as $B^2 \ll 2mk/q^2r^3$. In this limit, we can just adopt our old solution of elliptic motion. However, transforming back to the original frame, the ellipse will appear to rotate — or precess — with angular speed

$$\omega = \frac{qB}{2m}$$

This is known as the *Larmor frequency*. It is half of the cyclotron frequency that we met in 2.4.2.