Statistical Field Theory: Example Sheet 1

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1. The purpose of this question is to solve the 1d Ising model with N spins, using periodic boundary conditions such that $s_{N+1} = s_1$. The partition function is given by

$$Z = \sum_{s_1=\pm 1} \dots \sum_{s_N=\pm 1} \prod_i \exp\left(\beta J s_i s_{i+1} + \frac{\beta B}{2}(s_i + s_{i+1})\right)$$

Show that the partition can be written in terms of the 2×2 *transfer matrix* T as

$$Z = \mathrm{tr}\, T^N \quad \text{with} \quad T = \begin{pmatrix} e^{\beta J - \beta B} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J + \beta B} \end{pmatrix}$$

[*Hint:* Think of the i^{th} transfer matrix in the string T^N using index notation, written as $T_{s_i, s_{i+1}} = \exp\left(\beta J s_i s_{i+1} + \frac{\beta B}{2}(s_i + s_{i+1})\right)$. Here $s_i = \pm 1$ labels the column and $s_{i+1} = \pm 1$ labels the row.]

Show that the eigenvalues of T are given by

$$\lambda_\pm = e^{\beta J} \cosh \beta B \pm \sqrt{e^{2\beta J} \cosh^2 \beta B - 2 \sinh 2\beta J}$$

and hence that, as $N \to \infty$, the partition function is $Z \approx \lambda_+^N$. Use this to show that, for $B = 0$, the magnetisation always vanishes and hence that there is no phase transition as a function of temperature.

2. This question provides an alternative, mean field approach to the Ising model. (This is, in fact, the original approach, first suggested by Weiss.) Write the interaction terms in the Ising model as

$$s_i s_j = (s_i - m)(s_j - m) + m(s_j - m) + m(s_i - m) + m^2$$

where m is the equilibrium magnetisation. The mean field "approximation" hinges on assuming that the deviation from equilibrium, $(s_i - m)$, is small and so the first term above can be neglected. With this assumption, show that the partition function of the Ising model can be written as

$$Z = \sum_{\{s_i\}} e^{-\frac{1}{2}\beta J N q m^2 + \beta(Jqm + B)\sum_i s_i} = e^{-\frac{1}{2}\beta J N q m^2} 2^N \cosh^N(\beta J q m + \beta B)$$

with q the number of nearest neighbour pairs. Use this partition function to determine the equilibrium magnetisation and show that it obeys the self-consistency condition

$$m = \tanh\left(\beta B + \beta J q m\right)$$

For $B = 0$, show that the form of the solution differs for $T > T_c$ and $T < T_c$ where $T_c = Jq$. [*Hint:* Think graphically.] For $B \neq 0$, find the asymptotic form of the solution as $T \to \infty$.

3. The long range Ising model, in which each spin interacts with all other spins, including itself, provides an example where mean field theory is exact. The energy is

$$E = -B \sum_i s_i - \frac{J}{2N} \left(\sum_i s_i \right) \left(\sum_j s_j \right)$$

Why is there an extra factor of N in the coupling? First, show that

$$e^{\beta J \alpha^2/2N} = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{+\infty} dx\ e^{-N\beta J x^2/2 + \alpha \beta J x}$$

Use this identity to write the partition function as

$$Z = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{+\infty} dx\ e^{-NS(x)} \quad \text{where} \quad S(x) = \frac{\beta J x^2}{2} - \log\Big(2\cosh\beta(B + Jx)\Big)$$

In the large N limit, the partition function can be evaluated by saddle point so that, ignoring a pre-factor, $Z \approx e^{-NS(x_\star)}$ where $x_\star$ is the minimum of $S(x)$. Find an equation for $x_\star$. Show that the equilibrium magnetisation is given by $m = x_\star$.

4*. Consider the free energy

$$f(m) = \alpha_2(T) m^2 + \alpha_4 m^4 + \alpha_6 m^6$$

where $\alpha_4 < 0$ and $\alpha_6 > 0$ and $\alpha_2(T)$ varies from positive to negative as the temperature is lowered. Sketch $f(m)$ for various values of the temperature. Show that the system undergoes a first order phase transition at $\alpha_2 = \alpha_4^2/4\alpha_6$. What is the jump in the magnetisation? Draw the phase diagram in the $\alpha_2 - \alpha_4$ plane.

When $\alpha_4 = 0$, the phase transition occurs at $\alpha_2(T_c) = 0$. This is said to be a *tri-critical point*. Compute the mean field critical exponents α, β, γ and δ. (You will need to add a Bm term to the free energy to compute the latter two.)

5. A superfluid has complex order parameter ψ and free energy given by the so-called XY-model

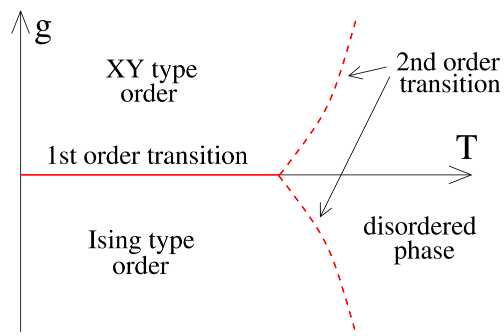
$$f(\psi) = \alpha_2|\psi|^2 + \alpha_4|\psi|^4$$

Sketch the free energy for $\alpha_2 < 0$ and $\alpha_4 > 0$. What are the possible ground states when $\alpha_2 < 0$? What is the meaning of "spontaneous symmetry breaking" in this context? Show that the critical point at $\alpha_2 = 0$ has the same mean field exponents α and β as the Ising model.

6. Consider a lattice model where there lives, on each site, a unit 3-vector $\mathbf{s} = (s^x, s^y, s^z)$ with $|\mathbf{s}| = 1$. The energy is given by

$$E = -J\sum_{\langle ij\rangle} s_i s_j + g\sum_i \left((s_i^z)^2 - \frac{1}{2}\left((s_i^x)^2 + (s_i^y)^2\right)\right)$$

with $J > 0$. Use mean field theory to give plausibility arguments for why the phase diagram takes the form shown below:



[*Hint:* You do not need to do any detailed manipulations of the path integral. Instead, think physically about how we expect the spins to behave for different parameters. In particular, consider the ground states, and the phase transitions, between them in the following regimes:

- At low T, as g changes from negative to positive.
- With $g > 0$ as T changes from low to high.
- With $g < 0$ as T changes from low to high.]

7. Consider the Landau-Ginzburg free energy for a complex scalar ψ,

$$F = \int dx \left(\alpha_2(T)|\psi|^2 + \alpha_4|\psi|^4 - \gamma \left|\frac{d\psi}{dx}\right|^2 + \kappa \left|\frac{d^2\psi}{dx^2}\right|^2 \right)$$

with $\alpha_4, \gamma, \kappa > 0$. Consider an ansatz in which a single Fourier mode $\psi_k = A_k e^{ikx}$ with $k = \pm k_0$ is non-vanishing. What value of k_0 minimises the free energy? Show that the system undergoes a phase transition to a spatially modulated phase when $\alpha_2 = \gamma^2/4\kappa$. What symmetries are broken in the ordered phase?

8. A multi-critical point arises from the free energy

$$f(m) = \alpha_2 m^2 + \alpha_{2n} m^{2n} \qquad , \; n \in \mathbf{Z}^+$$

Determine the mean field critical exponent $m \sim (T_c - T)^\beta$ in the ordered phase. Substitute this back into the free energy to show that the mean field critical exponent $c \sim (T_c - T)^{-\alpha}$ in the ordered phase is given by $\alpha = 1 - 2\beta$. Compare this with the contribution to the free energy from fluctuations in the Gaussian path integral. Show that the mean field contribution to the free energy dominates at the critical point provided $d > d_c = 2n/(n-1)$.