The Standard Model: Example Sheet 3

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1. Consider the $SU(2)$ gauge configuration

$$A_\mu = \frac{1}{g}\frac{1}{x^2+\rho^2}\,\eta^a_{\mu\nu}x^\nu\sigma^a$$

with ρ parameter and $\eta^a_{\mu\nu}$ a collection of three 4×4 *'t Hooft matrices* given by

$$\eta^1_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\ ,\ \ \eta^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\ ,\ \ \eta^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Show that the corresponding field strength is given by

$$F_{\mu\nu} = -\frac{1}{g}\frac{2\rho^2}{(x^2+\rho^2)^2}\eta^a_{\mu\nu}\sigma^a$$

Why does this solve the Euclidean Yang-Mills equation of motion $\mathcal{D}_\mu F^{\mu\nu}=0$? Compute the action

$$S = \frac{1}{2}\int d^4x\ \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$

Note: You will need the identity

$$\epsilon^{abc}\eta^a_{\mu\rho}\eta^b_{\nu\sigma} = \eta^c_{\mu\nu}\delta_{\rho\sigma} + \eta^c_{\rho\sigma}\delta_{\mu\nu} - \eta^c_{\mu\sigma}\delta_{\rho\nu} - \eta^c_{\rho\nu}\delta_{\mu\sigma}$$

2. Show that there is no consistent *chiral* $U(1)$ gauge theory with $N=4$ Weyl fermions with integer charges that can be coupled to gravity.

Find an example of a consistent $U(1)$ gauge theory with $N=5$ Weyl fermions with integer charges that can be coupled to gravity.

3*. An $SU(N)$ gauge theory is coupled to a single left-handed Weyl fermion χ in the symmetric $\square\!\square$ representation and p left-handed Weyl fermions ψ_i, $i = 1, \ldots, p$, in the anti-fundamental $\overline{\square}$ representation. For what value of p is the quantum theory consistent?

Classically, the theory has a $G_F = SU(p)$ global symmetry that acts on the ψ_i. In addition, there are two $U(1)$ symmetries

$$\begin{aligned} U(1)_\chi : &\qquad \chi \to e^{i\alpha}\chi \quad \text{and} \quad \psi_i \to \psi_i \\ U(1)_\psi : &\qquad \chi \to \chi \quad \text{and} \quad \psi_i \to e^{i\beta}\psi_i \end{aligned}$$

Show that each of the $U(1)$ symmetries suffers a chiral anomaly and so is not a symmetry of the quantum theory. Find a linear combination of these symmetries that that does survive in the quantum theory.

Compute the $SU(p)^3$, $SU(p)^2\,U(1)$, $U(1)^3$ and $U(1)$ (i.e. mixed gauge-gravity) 't Hooft anomalies for the symmetries of the quantum theory.

It is conjectured that this theory confines without spontaneously breaking any global symmetry. The massless degrees fields are thought to be a collection of gauge singlet fermions

$$\lambda^{ij} = \psi^{[i}(\chi\psi^{j]})$$

transforming in the anti-symmetric $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ representation of the $SU(p)$ global symmetry. What is the $U(1)$ charge of this fermion? Show that the 't Hooft anomalies of the fermion λ match those of the original gauge theory.

[You may find the following data helpful: for $SU(N)$, the fundamental $\square$, adjoint, symmetric $\square\!\square$ and anti-symmetric $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ have the dimension, Dynkin index I and anomaly coefficient A given by

R	$\square$	adj	$\square\!\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\dim(R)$	N	N^2-1	$\frac{1}{2}N(N+1)$	$\frac{1}{2}N(N-1)$
$I(R)$	1	$2N$	$N+2$	$N-2$
$A(R)$	1	0	$N+4$	$N-4$

with $I(\bar{R}) = I(R)$ and $A(\bar{R}) = -A(R)$.]

4a. Consider $SU(N)$ Yang-Mills coupled to a single, massless left-handed Weyl fermion λ in the adjoint representation. Why does this make sense as a quantum theory? (As an aside: this theory happens to enjoy $\mathcal{N}=1$ supersymmetry, although this fact is not needed for this question.)

Classically this theory has a $U(1)$ global symmetry which acts as $\lambda \to e^{i\alpha}\lambda$. By considering how this transformation affects the θ angle, show that the quantum theory that a $\mathbb{Z}_{2N}$ subgroup survives in the quantum theory.

The theory is expected to confine, develop a mass gap, and form a condensate

$$\langle \mathrm{Tr}\,\lambda\lambda \rangle \sim \Lambda_{\mathrm{QCD}}^3 \ . \tag{1}$$

How does the $\mathbb{Z}_{2N}$ global symmetry act on this condensate? How many ground states does the theory have?

b. Consider $SU(N)$ Yang-Mills (with $N>2$) coupled to a single *Dirac* fermion ψ in the symmetric $\square\!\square$ representation. What are the classical global symmetries? What are the quantum global symmetries?

This theory is expected to confine, develop a mass gap, and form a condensate

$$\langle \bar{\psi}\psi \rangle \sim \Lambda_{\mathrm{QCD}}^3$$

How many ground states does the theory have? How do your answers change if the Dirac fermion is in the anti-symmetric $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ representation of $SU(N)$?

[Note: You may need to refer to the table in Question 3.]

5*. Consider one generation of the Standard Model, with left- and right-handed Weyl fermions transforming in the following representations of $U(1)\times SU(2)\times SU(3)$:

$$Q_L : (\mathbf{2},\mathbf{3})_q\,, \quad L_L : (\mathbf{2},\mathbf{1})_l\,, \quad u_R : (\mathbf{1},\mathbf{3})_u\,, \quad d_R : (\mathbf{1},\mathbf{3})_d\,, \quad e_R : (\mathbf{1},\mathbf{1})_x\ .$$

This is the usual set of representations (ignoring the right-handed neutrino), but with the hypercharges q, l, u, d, and x left arbitrary. Assume that all hypercharges are real numbers, i.e. $q, l, u, d, x \in \mathbb{R}$

i) Write down all the conditions for anomaly cancellation, including the mixed $U(1)$-gravitational anomaly.

ii) Show that these equations have two solutions, one with $u=-d$ and the other the hypercharges of the Standard Model (up to scaling).

6. Repeat question 5, this time including a right-handed neutrino transforming as $(\mathbf{1},\mathbf{1})_z$ with $z\in\mathbb{R}$.

7. What are the global symmetries of the classical Lagrangian of the Standard Model assuming:

i) there are no right-handed neutrinos

ii) right-handed neutrinos exist, with all possible relevant and marginal interactions included?

How do these conclusions change if you include the effect of anomalies?