The Standard Model: Example Sheet 3

David Tong, February 2025

1. Show that there is no consistent *chiral* $U(1)$ gauge theory with $N = 4$ Weyl fermions with integer charges that can be coupled to gravity.

Find an example of a consistent $U(1)$ gauge theory with $N = 5$ Weyl fermions with integer charges that can be coupled to gravity.

2*. An $SU(N)$ gauge theory is coupled to a single left-handed Weyl fermion χ in the symmetric $\square\!\square$ representation and p left-handed Weyl fermions ψ_i, $i = 1, \ldots, p$, in the anti-fundamental $\overline{\square}$ representation. For what value of p is the quantum theory consistent?

Classically, the theory has a $G_F = SU(p)$ global symmetry that acts on the ψ_i. In addition, there are two $U(1)$ symmetries

$$
\begin{aligned}
U(1)_\chi : &\qquad \chi \to e^{i\alpha}\chi \quad \text{and} \quad \psi_i \to \psi_i \\
U(1)_\psi : &\qquad \chi \to \chi \quad \text{and} \quad \psi_i \to e^{i\beta}\psi_i
\end{aligned}
$$

Show that each of the $U(1)$ symmetries suffers a chiral anomaly and so is not a symmetry of the quantum theory. Find a linear combination of these symmetries that does survive in the quantum theory.

Compute the $SU(p)^3$, $SU(p)^2\,U(1)$, $U(1)^3$ and $U(1)$ (i.e. mixed gauge-gravity) 't Hooft anomalies for the symmetries of the quantum theory.

It is conjectured that this theory confines without spontaneously breaking any global symmetry. The massless fields are thought to be a collection of gauge singlet fermions

$$\lambda^{ij} = \psi^{[i}(\chi\psi^{j]})$$

transforming in the anti-symmetric $\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$ representation of the $SU(p)$ global symmetry. What is the $U(1)$ charge of this fermion? Show that the 't Hooft anomalies of the fermion λ match those of the original gauge theory.

Note: You may find the following data helpful. For $SU(N)$, the fundamental $\square$, adjoint, symmetric $\square\!\square$ and anti-symmetric $\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$ have the dimension, Dynkin index $I(R) = I(\bar{R})$ and anomaly coefficient $A(R) = -A(\bar{R})$ given by

R	$\square$	adj	$\square\!\square$	$\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$
$\dim(R)$	N	N^2-1	$\frac{1}{2}N(N+1)$	$\frac{1}{2}N(N-1)$
$I(R)$	1	$2N$	$N+2$	$N-2$
$A(R)$	1	0	$N+4$	$N-4$

3*. Consider one generation of the Standard Model, with left- and right-handed Weyl fermions transforming in the following representations of $U(1) \times SU(2) \times SU(3)$:

$$Q_L : (\mathbf{2}, \mathbf{3})_q \,, \quad L_L : (\mathbf{2}, \mathbf{1})_l \,, \quad u_R : (\mathbf{1}, \mathbf{3})_u \,, \quad d_R : (\mathbf{1}, \mathbf{3})_d \,, \quad e_R : (\mathbf{1}, \mathbf{1})_x \,.$$

This is the usual set of representations (ignoring the right-handed neutrino), but with the hypercharges q, l, u, d, and x left arbitrary. Assume that all hypercharges are real numbers, i.e. $q, l, u, d, x \in \mathbb{R}$

i) Write down all the conditions for anomaly cancellation, including the mixed $U(1)$-gravitational anomaly.

ii) Show that these equations have two solutions, one with $u = -d$ and the other the hypercharges of the Standard Model (up to scaling).

4. This is a repeat of question 4, but with the additional assumption that hypercharges are integers, i.e. $q, l, u, d, x \in \mathbb{Z}$.

i) Write down all conditions for anomaly cancellation, this time omitting the requirement that the theory can be coupled to gravity.

ii) Show that there is a unique solution to these equations that automatically satisfies the mixed gauge-gravitational anomaly. (You may invoke, without proof, any pure mathematics result that your grandmother has heard of.)

[Hint: First argue that you can write $u - d = 2y$ for $y \in \mathbb{Z}$. Then use the change of variables

$$x = -\frac{6}{v+w} \quad \text{and} \quad y = \frac{3(v-w)}{v+w} \ . \,]$$

5. What are the global symmetries of the classical Lagrangian of the Standard Model assuming:

i) there are no right-handed neutrinos

ii) right-handed neutrinos exist, with all possible relevant and marginal interactions included?

How do these conclusions change if you include the effect of anomalies?

6a. [This question is optional. I didn't discuss surviving discrete symmetries in class this year because it's something of a distraction from our main interest, namely the Standard Model. Still, this is a nice question that you might try if you are interested in more formal aspects of physics.]

Consider $SU(N)$ Yang-Mills coupled to a single, massless left-handed Weyl fermion λ in the adjoint representation. Why does this make sense as a quantum theory? (As an aside: this theory happens to enjoy $\mathcal{N} = 1$ supersymmetry, although this fact is not needed for this question.)

Classically this theory has a $U(1)$ global symmetry which acts as $\lambda \to e^{i\alpha}\lambda$. By considering how this transformation affects the θ angle, show that a $\mathbb{Z}_{2N}$ subgroup survives in the quantum theory.

The theory is expected to confine, develop a mass gap, and form a condensate

$$\langle \mathrm{Tr}\, \lambda\lambda \rangle \sim \Lambda_{\mathrm{QCD}}^3 \; . \tag{1}$$

How does the $\mathbb{Z}_{2N}$ global symmetry act on this condensate? How many ground states does the theory have?

b. Consider $SU(N)$ Yang-Mills (with $N > 2$) coupled to a single *Dirac* fermion ψ in the symmetric $\square\square$ representation. What are the classical global symmetries? What are the quantum global symmetries?

This theory is expected to confine, develop a mass gap, and form a condensate

$$\langle \bar{\psi}\psi \rangle \sim \Lambda_{\mathrm{QCD}}^3$$

How many ground states does the theory have? How do your answers change if the Dirac fermion is in the anti-symmetric $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ representation of $SU(N)$?

[Note: You may need to refer to the table in Question 2.]