4 Anomalies

Our goal in this section is to understand the beautiful and subtle phenomenon known as an *anomaly*[8]. This is one of the deepest ideas in quantum field theory and, as we will see in Section 5, underpins much of the structure of the Standard Model.

Before we jump in, here are two motivating comments.

We already met the theories of QED and QCD in the previous section. Both are described by Lagrangians in which a gauge field is coupled to a bunch of Dirac fermions. But Dirac fermions are not the simplest kind of fermion. Or, said differently, Dirac fermions are not irreducible representations of the Lorentz group. Instead, a Dirac fermion decomposes into two Weyl fermions. So why doesn't nature make use of this more minimal Weyl fermion? And why don't we study the seemingly simpler theory of, say, Yang-Mills coupled to a single Weyl fermion?

The answer, it turns out, is that Yang-Mills coupled to a single Weyl fermion is an inconsistent quantum theory! This is an important and striking statement. There's no problem in writing down a classical Lagrangian, nor indeed a classical Hamiltonian, for this system. But there's no corresponding quantum theory. As we will explain, this is one manifestation of the anomaly.

Here's a second motivation. In the theory of massless QCD, we mentioned that there is a classical $U(1)_A$ axial symmetry which, naively, appears to be spontaneously broken like the non-Abelian chiral symmetry. But there is no associated light meson. The meson that carries the right quantum numbers is the η' and its mass is almost 1 GeV, significantly more than the other pseudo-Goldstone bosons. What's going on?

The answer, it turns out, is that the axial $U(1)_A$ symmetry in massless QED and QCD is a good symmetry of the classical theory, but it is not a symmetry of the quantum theory. This, too, is a manifestation of the anomaly.

Our purpose is to understand these statements and more. There are various ways to understand these features, but the most revealing is through the path integral. As we will see, both of the issues above, and several others, arise from trying to carefully define the path integral for Weyl fermions.

[8]Because these are lectures on the Standard Model, I should mention that there is another, very different meaning to the word "anomaly" in the particle physics community, which is when an experimental result that deviates slightly from the prediction of the Standard Model. Typically, this leads to approximately 10^4 papers being written before the whole things fades away 3 years later. That's not what we're talking about here.

Our First Anomaly

There are a number of different manifestations of anomalies in quantum field theory. Indeed, understanding when such effects arise remains a vibrant research area. Here we will discuss just the simplest kind of anomaly, associated to Weyl fermions.

To set the scene, recall that a Dirac fermion ψ splits into two Weyl fermions

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \ . \tag{4.1}$$

For our story, we want to take just a single Weyl fermion. We will take a left-handed spinor ψ_L, but everything we're about to say also holds for a single right-handed spinor. The action for a massless Weyl spinor is

$$S = \int d^4x \ i\bar{\psi}_L \bar{\sigma}^\mu \partial_\mu \psi_L \tag{4.2}$$

with $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$. This action is clearly invariant under the $U(1)$ global symmetry $\psi_L \to e^{i\alpha}\psi_L$, with the corresponding current $j^\mu = \psi_L^\dagger \bar{\sigma}^\mu \psi_L$. To illustrate the anomaly, we will couple this current to a gauge field A_μ with charge $q \in \mathbb{Z}$. The action is now

$$S = \int d^4x \ i\bar{\psi}_L \bar{\sigma}^\mu \mathcal{D}_\mu \psi_L \tag{4.3}$$

where the covariant derivative contains the coupling to the gauge field $\mathcal{D}_\mu \psi_L = \partial_\mu \psi_L - ieqA_\mu \psi_L$. This action is now invariant under the gauge symmetry

$$\psi \to e^{ieq\alpha(x)}\psi \quad \text{and} \quad A_\mu \to A_\mu + \partial_\mu \alpha \ . \tag{4.4}$$

Before we proceed, I should mention that there are two distinct ways to think about the gauge field A_μ and this distinction will be important when we come to look at the various implications of anomalies. They are:

- A_μ could be a *dynamical gauge field*. In the classical theory, this means that we treat it as a dynamical variable, with its own equation of motion, typically after adding a Maxwell term to the action. In the quantum theory, it means that we integrate over A_μ in the path integral.

- A_μ could be a *background gauge field*. This means that it is something fixed, under our control, and should be viewed as a parameter of the theory. Turning it on typically breaks Lorentz symmetry, but could be useful to explore how our system responds to the presence of an electric or magnetic field. In the quantum theory, A_μ appears as a source on which the partition function depends.

We will consider gauge fields of both types in what follows. However, for now, we will consider A_μ to be a background gauge field, whose value is something that we get to decide.

While the classical theory is clearly invariant under the gauge transformation (4.4), the question that we really want to ask is: what happens in the quantum theory? For this, we should turn to the path integral, with the partition function in Euclidean space defined as

$$Z[A] = \int D\psi_L D\bar{\psi}_L \ \exp\left(- \int d^4x \ i\bar{\psi}_L \bar{\sigma}^\mu \mathcal{D}_\mu \psi_L \right) \ . \tag{4.5}$$

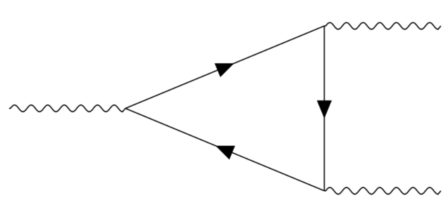
The action in the exponent is designed so that it is invariant under gauge transformations. But now we must also worry about the measure in the path integral and this takes some care to define. The statement of the anomaly is that the measure is *not* invariant under gauge transformations. Instead, it turns out that the measure, and hence the partition function, changes by a phase

$$Z[A] \to \exp\left(\frac{ie^3q^3}{32\pi^2} \int d^4x \ \alpha F_{\mu\nu} {}^\star F^{\mu\nu} \right) \ Z[A] \tag{4.6}$$

with ${}^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

This subtlety only happens for fermions. If we have scalar fields charged under a symmetry, then the measure is perfectly invariant. At heart, this is related to the fact that there is no difficulty in giving masses to scalar fields while preserving symmetries, but giving masses for fermions necessarily breaks certain symmetries.

I won't prove the anomaly (4.6) here, but a detailed derivation is given in the lectures on Gauge Theory. In fact, there are two such derivations. The first involves a careful definition of the measure in the path integral to see that it does indeed transform as (4.6). The second derivation works with more conventional perturbation theory. In particular, the anomaly is associated to the following triangle diagram



The external legs are currents associated to the $U(1)$ symmetry, while the fermion runs in the loop. Like most one-loop diagrams, the resulting integral is divergent and has to be regulated. The subtlety arises because of the interplay between regulating the divergence and preserving the $U(1)$ symmetry. It turns out that only diagrams of this kind suffer from this subtlety, and the fact that there are three legs is reflected in the q^3 prefactor of the anomaly in (4.6). Although we won't compute these triangle diagrams here, they will be a useful mnemonic as we describe different kinds of anomalies.

Rather than derive the anomaly, we will instead focus on its implications. Broadly, there are three different implications, depending on whether we think of the gauge field A_μ as background or dynamical. We will address these in turn in Sections 4.1, 4.2, and 4.3.

4.1 Gauge Anomalies

The first implication of the anomaly (4.6) is that it is an obstruction to gauging. Although the action is invariant under the gauge symmetry, the measure is not and neither is the partition function. That means that we cannot promote the gauge field A_μ to a dynamical field, where we integrate over it in the path integral. If we attempted to do this, we would get a sick theory. (Sick as in bad, not sick as in good.)

There are a number of ways to see why the theory is sick but here is a simple one. Recall that when we first attempted to quantise the gauge field A_μ in the lectures on Quantum Field Theory we had some work to do to decouple the negative norm states that arise from quantising A_0. That work ultimately boiled down to using the gauge invariance to remove these states. But in an anomalous theory, we no longer have that gauge invariance at our disposal and the Hilbert space will involve negative norm states. That's bad.

The upshot is that a $U(1)$ gauge theory, coupled to a single Weyl fermion, is a sick theory. If we want to write down a consistent gauge theory, then we must have multiple Weyl fermions so that, combined, the anomaly cancels.

Typically, we think of a given theory in terms of a bunch of left-handed fermions and another bunch of right-handed fermions. But, given a right-handed fermion of charge q, its complex conjugation is a left-handed fermion of charge $-q$. So, we're always at liberty to talk only about left-handed fermions. If we have a bunch of left-handed Weyl fermions $(\psi_L)_i$, each carrying charge q_i under a $U(1)$ gauge field, then the phase in (4.6) is proportional to the sum of q_i^3. The theory is consistent only if

$$\sum_i q_i^3 = 0 \ . \tag{4.7}$$

Alternatively, if we keep the theory written in terms of left-handed and right-handed Weyl fermions, then the anomaly cancellation condition (4.7) becomes

$$\sum_{\text{left}} q_i^3 = \sum_{\text{right}} q_i^3 \ . \tag{4.8}$$

There is a simple way to satisfy (4.7): we just take pairs of Weyl fermions with charges $\pm q$. If we conjugate one of these, then we can equivalently think of one left-handed and one right-handed Weyl fermion, each with charge q. Or, equivalently, we have a single Dirac fermion of charge q. Theories of this kind are called *vector-like*. They enjoy a parity symmetry (at least among the gauge interactions) which, as we saw in Section 1.4, exchanges left- and right-handed fermions. The simplest example is QED.

There are, however, more interesting solutions to (4.7) that do involve $\pm$ pairs. These are known as *chiral gauge theories*. These theories necessarily break parity.

Abelian Chiral Gauge Theories

Can we write down a consistent, Abelian chiral gauge theory? In fact, I'll ask for one more criterion: can we write down a consistent chiral gauge theory with integer charges

$$q_i \in \mathbb{Z} \ . \tag{4.9}$$

I'll say some words below about why we might want to require this.

First, it's clear that for $N = 2$ Weyl fermions, charges obeying (4.7) must come in $\pm$ pairs which is a vector-like theory. What about for $N = 3$ fermions? We must have two positive charges and one negative (or the other way round). Set $q_i = (x, y, -z)$ with x, y, z positive integers. The condition for anomaly cancellation (4.7) then becomes

$$x^3 + y^3 = z^3 \ . \tag{4.10}$$

Rather famously, this equation has no positive integer solutions. (This is the baby version of Fermat's last theorem, proven by Euler.)

What about chiral gauge theories with $N = 4$ Weyl fermions? Now we have two options: we could take three positive charges and one negative and look for positive integers satisfying

$$x^3 + y^3 + z^3 = w^3 \ . \tag{4.11}$$

The simplest integers satisfying this are 3,4,5 and 6. We can also construct chiral gauge theories with $N = 4$ Weyl fermions by having two of positive charge and two of negative charge, so that

$$x^3 + y^3 = z^3 + w^3 \ . \tag{4.12}$$

This equation is closely associated to Ramanujan and the famous story of Hardy's visit to his hospital bed. Struggling for small talk, Hardy commented that the number of his taxicab was particularly uninteresting: 1729. Ramanujan responded that, far from being uninteresting, this corresponds to the simplest four dimensional chiral gauge theory, since it is the first number that can be expressed as the sum of two cubes in two different ways: $1^3 + 12^3 = 9^3 + 10^3$.

There is one further condition that we've not yet met. As we will explain shortly, if you want to be able to couple your theory to gravity (and, let's face it, we do) then the condition (4.7) should be augmented by the requirement

$$\sum_i q_i = 0 \ . \tag{4.13}$$

None of the examples with $N = 4$ Weyl fermions above obey this. The simplest Abelian chiral gauge theory that can be coupled to gravity has $N = 5$ Weyl fermions. For example, the charges $q_i = \{1, 5, -7, -8, 9\}$ do the job.

We see that restricting to integer valued charges $q_i \in \mathbb{Z}$ means that we have to solve Diophantine equations and this breathes a little number theory into the proceedings. But why do we require that $q_i \in \mathbb{Z}$? The answer to this is a little subtle.

Strictly, there are two different Abelian gauge groups. The first is $G = U(1)$ which has only integer charges $q_i \in \mathbb{Z}$. Sometimes, it's useful to rescale the charges (and the Standard Model will be an example) so that you take the charges to be rational, $q_i \in \mathbb{Q}$, but that doesn't change the fact that the charges are quantised. The second is $G = \mathbb{R}$ which have charges that can take any value $q_i \in \mathbb{R}$ so you could have, for example, $q_1 = 1$ and $q_2 = \sqrt{2}$.

The gauge groups $U(1)$ and $\mathbb{R}$ have other differences, beyond the allowed electric charges. In particular, the gauge group $U(1)$ admits magnetic monopoles while the gauge group $\mathbb{R}$ does not (essentially because you can't respect the Dirac quantisation condition with respect to all charges). So one obvious question is: which of these gauge groups describes our world?

Irrep	$\square$	adj	$\square\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
dim	N	N^2-1	$\frac{1}{2}N(N+1)$	$\frac{1}{2}N(N-1)$
$I(R)$	1	$2N$	$N+2$	$N-2$
$A(R)$	1	0	$N+4$	$N-4$

Table 8. Some group theoretic properties of $SU(N)$ representations. Here $\square\square$ is the symmetric representation and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ the anti-symmetric. Conjugate representations have $I(\bar{R}) = I(R)$ and $A(\bar{R}) = -A(R)$.

The experimental evidence strongly points to $U(1)$ because all electric charges (and, as we will see in Section 5, all hypercharges) are quantised. Moreover, there are arguments that invoke quantum gravity that we won't describe that are reasonably compelling, but far from rigorous, for why the gauge group in any quantum field theory should be $U(1)$, and not $\mathbb{R}$.

4.1.1 Non-Abelian Gauge Anomalies

So far we've only discussed anomalies for an Abelian gauge field. There is an analogous result for non-Abelian gauge symmetry G. Suppose that we have a single Weyl fermion in the representation R of a group G, with generator T^A_R so that, under a gauge transformation, we have

$$\psi_L \to e^{ig\alpha^A(x)T^A_R}\psi_L \quad \text{and} \quad A_\mu \to \Omega A_\mu \Omega^{-1} + \frac{i}{g}\Omega\partial_\mu\Omega^{-1} \tag{4.14}$$

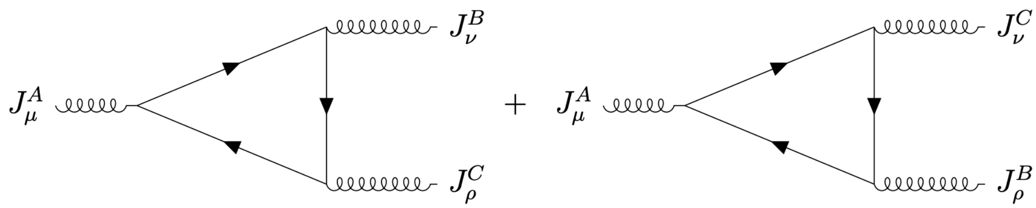
where $\Omega = e^{i\alpha^A T^A}$ with T^A in the fundamental representation. We can define the partition function just as (4.5), but where various fields are now viewed as their non-Abelian avatars. Then, under a gauge transformation, the partition function again changes by a phase

$$Z[A] \to \exp\left(\frac{ig^3 A(R)}{16\pi^2}\int d^4x\ \mathrm{Tr}\left(\alpha F_{\mu\nu}{}^\star F^{\mu\nu}\right)\right)\ Z[A]\ . \tag{4.15}$$

Here $A(R)$ is a group theoretic factor. For the fundamental representation, we have $A(R) = 1$ while, for all other representations, this is defined to be

$$\mathrm{Tr}\, T^A_R\,\{T^B_R, T^C_R\} = A(R)\,\mathrm{Tr}\, T^A\,\{T^B, T^C\}\ . \tag{4.16}$$

The emergence of the anti-commutator can be traced to the requirement to sum over different indices in the triangle diagrams



Some examples of $A(R)$ for $SU(N)$ representations are collected in Table 8. To be consistent, a non-Abelian gauge theory coupled to a bunch of left-handed Weyl fermions must obey

$$\sum_i A(R_i) = 0 \tag{4.17}$$

which is the non-Abelian version of (4.7).

For Abelian anomalies, we could always ensure that things work by taking fermions to come in pairs with charges $\pm q$. A similar result holds for non-Abelian anomalies. This follows from the following result.

Claim: If R is a complex representation, then the conjugate representation $\bar{R}$ has $A(\bar{R}) = -A(R)$.

Proof: If we write a group element as $e^{i\alpha^A T_R^A}$ then, in the conjugate representation, the same group element is given by the complex conjugate $e^{-i\alpha^A T_R^{A\star}}$. This means that the generators for the conjugate representation are $\bar{T}_R^A = -T_R^{A\star} = -(T_R^A)^T$ where the last equality holds because our generators are Hermitian, so $T_R^A = (T_R^A)^\dagger$. Now we have

$$\mathrm{Tr}\, \bar{T}_R^A \{\bar{T}_R^B, \bar{T}_R^C\} = -\mathrm{Tr}\, (T_R^A)^T \{(T_R^B)^T, (T_R^C)^T\} = -\mathrm{Tr}\, T_R^A \{T_R^B, T_R^C\} \quad . \tag{4.18}$$

Here the last equality holds because $\mathrm{Tr}\, A = \mathrm{Tr}\, A^T$. (It's important that we have the anti-commutator inside the trace, because the two terms get exchanged but, happily, they come with a relative plus sign rather than a minus sign.) $\square$

The fact that $A(\bar{R}) = -A(R)$ means that we can always satisfy the anomaly by coupling our gauge field to left-handed fermions that come in R and $\bar{R}$ pairs. Alternatively, instead of working with left-handed fermions in the $\bar{R}$ representation, we could instead view them as right-handed fermions in the R representation. This means that the anomaly cancellation condition (4.17) is satisfied whenever we have a Dirac fermion. That, of course, is what happens for QCD.

One consequence of the relation $A(\bar{R}) = -A(R)$ is that $A(R) = 0$ for any real representation. This means that there is no obstacle to coupling a single Weyl fermion in a real representation to a non-Abelian gauge group. For example, $SU(N)$ coupled to a single adjoint Weyl fermion is a perfectly good field theory. (In fact, it is a very well studied field theory known as *super-Yang-Mills*.) But $SU(N)$ coupled to a single fundamental Weyl fermion does not make sense as a quantum theory.

This highlights a property of anomalies that will become increasingly important as we proceed: only massless fermions contribute to anomalies. Or, said differently, the contribution to the anomaly from any massive fermions will always cancel.

For example, to write down a Dirac mass for a fermion in a complex representation that preserves a symmetry, we need a left-handed ψ_L and a right-handed ψ_R, both transforming in the same representation, so that we can construct the mass term $\bar{\psi}_L \psi_R$. But the contribution to the anomaly from these two Weyl fermions cancels. Meanwhile, if we have a fermion in a real representation, like the adjoint, then we can always write down a Majorana mass $\text{Tr}\, \psi_L \psi_L$ that preserves the symmetry. But now the contribution to the anomaly vanishes. The upshot is that only fermions that cannot get a mass preserving G contribute to the anomaly for G.

The story above also means that the only gauge groups that suffer from potential anomalies are those with complex representations. This already limits the possibilities: we need only worry about gauge anomalies in simply laced groups when

$$G = \begin{cases} SU(N) \text{ with } N \geq 3 \\ SO(4N+2) \\ E_6 \end{cases} . \tag{4.19}$$

We should also add $G = U(1)$ to this list which we discussed previously.

This list is short, but it turns out to be shorter still because all anomaly coefficients $\text{Tr}\, T^A \{T^B, T^C\}$ vanish for E_6 and for $SO(4N+2)$ with $N \geq 2$. (Note that the Lie algebra $so(6) \cong su(4)$ so this one remains.) This means that, when it comes to perturbative anomalies discussed above, we only need to worry when we have gauge groups $G = SU(N)$ with $N \geq 3$.

There is, however, a "non-perturbative anomaly", usually called the *Witten anomaly* that rears its head for $SU(2)$ and, indeed, for all $Sp(N)$. We'll discuss this briefly below.

Non-Abelian Chiral Gauge Theories

We could try to write down chiral non-Abelian gauge theories, in which left-handed and right-handed fermions transform in different representations. This is straightforward to do. For gauge group $G = SU(N)$, from Table 8, the anomaly coefficients for the symmetric $\square\square$ and anti-symmetric $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ representations are

$$A(\square\square) = N + 4 \quad \text{and} \quad A\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) = N - 4\ . \tag{4.20}$$

Meanwhile, for the anti-fundamental representation $\bar{\mathbf{N}}$, which we denote as $\overline{\square}$, we have $A(\overline{\square}) = -1$. This means that we can construct a chiral gauge theory by taking, for example $G = SU(N)$ with a $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $N - 4$ $\overline{\square}$ left-handed Weyl fermions. The simplest of these theories is $G = SU(5)$ with a $\mathbf{10}$ and a $\bar{\mathbf{5}}$.

Alternatively, we could build a chiral gauge theory by taking either E_6 or $SO(4N + 2)$ with complex representations, for which the anomaly coefficients all vanish. The simplest such example is $SO(10)$ with a single Weyl fermion in the $\mathbf{16}$ representation. This is the spinor representation of $SO(10)$. (Strictly, we should be talking about the double cover $Spin(10)$ as the gauge group, rather than $SO(10)$.) Rather strikingly, both this $SO(10)$ example and the $SU(5)$ example above are prominent candidates for grand unified theories.

One key feature of chiral gauge theories – both non-Abelian and Abelian – is that it's not possible to write down mass terms for fermions. Any such mass term should be of the form $\chi_L \psi_L$ or, equivalently, $\bar{\chi}_R \psi_L$, but these quadratic terms are not gauge invariant.

4.1.2 Mixed Anomalies

Again consider a single Weyl fermion, now coupled to a background non-Abelian gauge field A_μ in some representation R of the global symmetry $G = SU(N)$ and an Abelian gauge field that, for the purposes of this argument, we will call a_μ. The partition function is

$$Z[A; a] = \int D\psi_L D\bar{\psi}_L \ \exp\left(-\int d^4x \ i\bar{\psi}_L \bar{\sigma}^\mu \mathcal{D}_\mu \psi_L\right) \tag{4.21}$$

now with

$$\mathcal{D}_\mu \psi_L = \partial_\mu \psi_L - igA_\mu^A T_R^A \psi_L - ieqa_\mu \psi_L\ . \tag{4.22}$$

Now when we do a $U(1)$ gauge transformation $\psi_L \to e^{ieq\alpha}\psi_L$, the partition function picks up two contributions: one is the phase (4.6) that depends on the $U(1)$ field strength $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, but there is another that depends on the $SU(N)$ field strength,

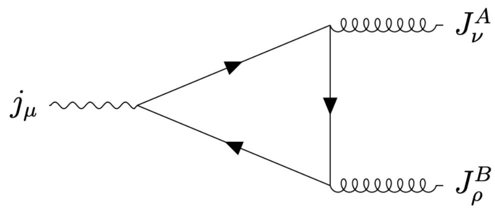
$$Z[A;a] \rightarrow \exp\left(\frac{ie^3q^3}{32\pi^2}\int d^4x\ \alpha f_{\mu\nu}{}^\star f^{\mu\nu} + \frac{ieg^2qI(R)}{16\pi^2}\int d^4x\ \alpha \mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}\right) Z[A;a] \quad (4.23)$$

Here $I(R)$ is another group theoretic quantity, known as the *Dynkin index*, defined as

$$\mathrm{Tr}\ T^A_R\, T^B_R = \frac{1}{2}I(R)\,\delta^{AB}\ . \quad (4.24)$$

The Dynkin index is related to the quadratic Casimir $C(R)$, which we previously defined in (3.27) by $T^A_R T^A_R = C(R)\,\mathbb{1}$. You can take the trace of both sides to get $I(R)\dim(G) = 2C(R)\dim(R)$. The fundamental representation has $I(\Box) = 1$ and the Dynkin index of the conjugate representation is $I(\bar{R}) = I(R)$. The Dynkin indices for some other common representations of $SU(N)$ are given in Table 8.

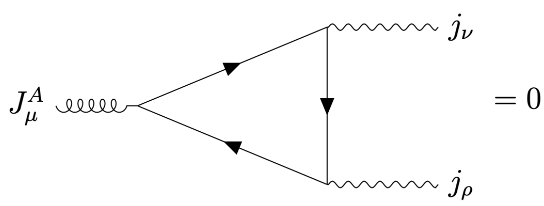
The second term in (4.23) is known as a *mixed anomaly*. It is again cubic in the charges, but this is shared between a single $U(1)$ charge q and two non-Abelian charges. In perturbation theory, it arises from the triangle diagram:



To have a consistent gauge theory, any mixed anomalies must also cancel. For a bunch of left-handed fermions with $U(1)$ charge q_i, sitting in $SU(N)$ representations R_i, the requirement of anomaly cancellation is

$$\sum_i q_i I(R_i) = 0\ . \quad (4.25)$$

You might wonder what happens if we have a single non-Abelian current, and two Abelian currents,



But this vanishes automatically, because it's proportional to the trace of the generator $\mathrm{Tr}\, T^A = 0$.

The Mixed Gauge-Gravitational Anomaly

Something similar plays out if we couple a quantum field theory to gravity. We needn't be bold and talk about quantum gravity here: it's enough just to think about a quantum field theory on a curved spacetime with metric g.

To motivate this, let's first review how to couple spinors to a curved spacetime. The starting point is to decompose the metric in terms of vierbeins,

$$g_{\mu\nu}(x) = e^a_\mu(x)\, e^b_\nu(x) \; . \tag{4.26}$$

There is an arbitrariness in our choice of vierbein, and this arbitrariness introduces an $SO(1,3)$ gauge symmetry into the game. The associated gauge field ω^{ab}_μ is called the *spin connection*. It is determined by the requirement that the vierbeins are covariantly constant

$$\mathcal{D}_\mu e^a_\nu \equiv \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu} e^a_\lambda + \omega^{a}_{\mu\, b} e^b_\nu = 0 \tag{4.27}$$

where $\Gamma^\rho_{\mu\nu}$ are the usual Christoffel symbols. This language makes general relativity look very much like any other gauge theory. In particular, the field strength of the spin connection is

$$(R_{\mu\nu})^a{}_b = \partial_\mu \omega^a_{\nu\, b} - \partial_\nu \omega^a_{\mu\, b} + [\omega_\mu, \omega_\nu]^a{}_b \; . \tag{4.28}$$

This is related to the usual Riemann tensor by $(R_{\mu\nu})^a{}_b = e^a_\rho e^\sigma_b \, R_{\mu\nu}{}^\rho{}_\sigma$.

This machinery is just what we need to couple a Dirac spinor to a background curved spacetime. The appropriate covariant derivative is

$$\mathcal{D}_\mu \psi_\alpha = \partial_\mu \psi_\alpha + \frac{1}{2}\omega^{ab}_\mu (S_{ab})^\beta{}_\alpha \psi_\beta \tag{4.29}$$

where $S_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ is the generator of the Lorentz group in the spinor representation. Written in this way, the coupling of spinors to a curved spacetime looks very similar to the coupling to any other gauge field.

This manifests itself in the path integral measure. If we assign the Weyl fermion a charge q and couple it to a $U(1)$ gauge field a transformation, the partition function shifts as

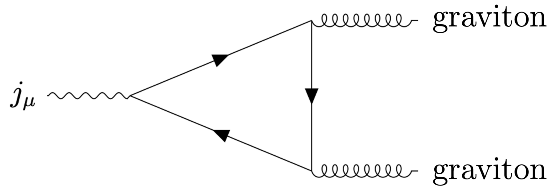
$$Z[a] \rightarrow \exp\left(\frac{eq}{192\pi^2}\int d^4x \; \alpha \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\tau} R_{\rho\sigma}{}^{\lambda\tau}\right) \, Z[a] \; . \tag{4.30}$$

with $R_{\mu\nu\lambda\tau}$ the Riemann tensor. This is a mixed $U(1)$-gravitational anomaly. The equivalence principle means that everything couples the same to gravity, so there's no analog of the Dynkin index in (4.25) and the requirement that a $U(1)$ gauge theory is consistent when placed on a curved spacetime becomes

$$\sum_i q_i = 0 \; . \tag{4.31}$$

This is the condition (4.13) that we advertised previously.

Again, this result can also be seen in perturbation theory, this time by a suitable regularisation of the triangle diagram,



This mixed gauge-gravitational anomaly only arises for Abelian gauge groups. There's no corresponding requirement for non-Abelian gauge theories, essentially because $\text{Tr}\, T^A = 0$ for any generator of a simply connected Lie algebra.

It turns out that there is no purely gravitational anomaly, with gravitons on all three legs, in $d = 3 + 1$ dimensions. Such gravitational anomalies do exist in $d = 2$ mod 8 dimensions, and there are important implications in $d = 1 + 1$ for condensed matter physics and in $d = 9 + 1$ for string theory.

4.1.3 The Witten Anomaly

Among the $G = SU(N)$ gauge groups, the smallest $G = SU(2)$ stands out as special. This is because all representations of $G = SU(2)$ are either real or pseudoreal. (A *pseudoreal* representation means that, while not actually real, the representation is isomorphic to its complex conjugate.) This means that there are no perturbative gauge anomalies of the kind described above for $G = SU(2)$.

You can check this explicitly for the fundamental representation. This has generators $T^A = \frac{1}{2}\sigma^A$ with σ^A the Pauli matrices. But a little matrix multiplication will convince you that

$$\text{Tr}\, \sigma^A \left\{\sigma^B, \sigma^C\right\} = 0 \tag{4.32}$$

for all $A, B, C = 1, 2, 3$. That's the statement that there's no anomaly.

Taken at face value, this suggests that $SU(N)$ coupled to a single fundamental Weyl fermion is inconsistent for all $N \geq 3$ but is fine for $N = 2$. That's a slightly odd state of affairs, not least because the $SU(2)$ theory has a number of strange and hard-to-interpret properties. (The instanton has an odd number of fermion zero modes for example.) However, there's something else at play that we've missed. It turns out that the $SU(2)$ theory suffers from a different kind of anomaly. This is known as the *Witten anomaly*, or sometimes just as the $SU(2)$ anomaly.

The Witten anomaly doesn't show up in perturbation theory. Instead it can be traced to some strange field configurations that we must sum over in the path integral that wind in a non-trivial way around Euclidean spacetime. Mathematically, this follows from the homotopy group

$$\Pi_4(SU(2)) = \mathbb{Z}_2 \ . \tag{4.33}$$

For this anomaly to cancel, an $SU(2)$ gauge theory must have an even number of fundamental Weyl fermions to be consistent. Again, you can find details of this calculation in the lectures on Gauge Theory.

4.2 Chiral (or ABJ) Anomalies

As we stressed at the beginning of this section, the anomaly for a symmetry group G has various avatars depending on whether the symmetry is global or gauged. So far, we've seen one of these avatars: the anomaly provides a collection of consistency conditions on any gauge theory: the charges, or representations, must obey (4.7) and (4.17) and, for mixed anomalies, (4.25) and (4.31).

In this section we discuss the second avatar of anomalies: a perfectly good global symmetry of the classical theory, can fail to be a symmetry of the quantum theory. This was the first place in which anomalies in quantum field theories were discovered. This phenomenon is known as the *ABJ anomaly*, after its discoverer's Adler, Bell and Jackiw, and sometimes as the *chiral anomaly* and sometimes, confusingly, just as the *anomaly*.

The ABJ anomaly can be viewed as a mixed anomaly between a $U(1)$ global symmetry and a gauge symmetry G. As an example, suppose that we have a bunch of left-handed Weyl fermions, transforming in the representation R_i under a $G = SU(N)$ gauge symmetry. Suppose, in addition, that there is a global $U(1)$ symmetry of the classical action, under which the fermions have charges q_i.

The full Euclidean partition function for this theory is, schematically,

$$\mathcal{Z} = \int \mathcal{D}A \ \exp\left(-\frac{1}{2}\int d^4x \ \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}\right) Z[A] \tag{4.34}$$

where A is the non-Abelian gauge field and $Z[A]$ is the partition function for the fermions, which are coupled to this gauge field

$$Z[A] = \int D\psi_{L\,i} D\bar{\psi}_{L\,i} \ \exp\left(-\int d^4x \ i\sum_i \bar{\psi}_L \bar{\sigma}^\mu \mathcal{D}_\mu \psi_L\right) \ . \tag{4.35}$$

Note that, in contrast to the previous section, we haven't introduced a background gauge field for the $U(1)$ global symmetry. (This is what we called a_μ in (4.23).)

Now we do a global $U(1)$ transformation

$$\psi_{L\,i} \to e^{i\alpha q_i}\psi_{L\,i} \tag{4.36}$$

for some $\alpha \in \mathbb{R}$. The mixed anomaly (4.23) means that the partition function is not invariant. Instead, the fermionic part of the partition function transforms as

$$Z[A] \to \exp\left(\frac{i\alpha}{16\pi^2}\sum_i q_i I(R_i)\int d^4x \ \mathrm{Tr}\, F_{\mu\nu}{}^{\star}F^{\mu\nu}\right) Z[A] \ . \tag{4.37}$$

We see that, although the classical action may be invariant under the global $U(1)$ symmetry, for this to persist as a symmetry of the quantum theory we also need the fermionic measure to be invariant. This is true only if

$$\sum_i q_i I(R_i) = 0 \ . \tag{4.38}$$

If this condition does not hold, then the classical symmetry is not a symmetry of the quantum theory. It is said to be *anomalous*.

An Example: The Axial Anomaly in QCD

The most familiar example of this kind of anomaly arises for the (approximate) $U(1)_A$ axial symmetry of QCD. Consider the generalised theory, in which we have a $G = SU(N_c)$, coupled to N_f massless Dirac fermions. The action is

$$S = \int d^4x \ \left(-\frac{1}{2}\mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu} + i\sum_{i=1}^{N_f}\bar{\psi}_i\gamma^\mu\mathcal{D}_\mu\psi_i\right) \ . \tag{4.39}$$

We studied this theory in some detail in Section 3.2 where we learned about the implications of chiral symmetry breaking. Recall that the classical action 4.39 has an $U(N_f)_L \times U(N_f)_R$ global symmetry, with each factor rotating q_L and q_R independently. The $SU(N_f)_L \times SU(N_f)_R$ subgroup is the main character in the story of chiral symmetry breaking. Here we are more interested in the two $U(1)$ subgroups, which we take to act as

$$U(1)_V: \ \psi_{L\,i} \to e^{i\alpha}\psi_{L\,i} \quad \text{and} \quad \psi_{R\,i} \to e^{i\alpha}\psi_{R\,i}$$
$$U(1)_A: \ \psi_{L\,i} \to e^{i\alpha}\psi_{L\,i} \quad \text{and} \quad \psi_{R\,i} \to e^{-i\alpha}\psi_{R\,i}\ . \tag{4.40}$$

Here $U(1)_V$ is the "vector-like" symmetry, meaning that it acts the same on left- and right-handed spinors. In the context of the Standard Model, this is also referred to as *baryon number* because it counts the number of baryons in a given state. Meanwhile, the axial symmetry $U(1)_A$ acts differently on the left and right-handed spinors.

The left-handed spinors ψ_L transform in the $\mathbf{N_c}$ of $SU(N_c)$ while the conjugated right-handed spinors $\bar{\psi}_R$ (which, due to the conjugation, are themselves left-handed) transform in the $\overline{\mathbf{N}}_\mathbf{c}$. For both of these, the Dynkin index is $I(\mathbf{N_c}) = I(\overline{\mathbf{N}}_\mathbf{c}) = 1$.

Under $U(1)_V$, the ψ_L have charge $+1$ and the $\bar{\psi}_R$ charge -1, which means that the anomaly (4.38) vanishes. Hence, $U(1)_V$ is a good symmetry of the quantum theory. In contrast, under $U(1)_A$, the ψ_L have charge $+1$ while the $\bar{\psi}_R$ also have charge $+1$. This means that the anomaly (4.38) does not vanish, and $U(1)_A$ is *not* a symmetry of the quantum theory.

We've already seen one consequence of the QCD axial anomaly in Section 3.2: the chiral condensate would naively seem to spontaneously break the $U(1)_A$ axial symmetry, but there's no associated light Goldstone boson in the QCD spectrum. Indeed, the would-be Goldstone boson is the η' which is significantly heavier than the pions. The reason is that $U(1)_A$ was never a symmetry of the quantum theory in the first place and wasn't available to be spontaneously broken.

4.2.1 The Theta Term Revisited

There is another way to think about the chiral anomaly. We see from (4.37), that acting with an anomalous $U(1)$ global symmetry adds a term to the path integral that is proportional to $\text{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}$.

But we've met a term like this before. We can always add to the Yang-Mills action (or, indeed, to the Maxwell action) a theta term that takes the form

$$S_\theta = \frac{\theta g^2}{16\pi^2}\int d^4x\ \text{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu}\ . \tag{4.41}$$

We discussed some properties of this term in Section 3.4. Comparing with the form of the chiral anomaly (4.37), we can interpret the anomaly as saying that the theta parameter is shifted by a $U(1)$ transformation,

$$U(1)_A: \;\; \theta \to \theta + \alpha \sum_i q_i I(R_i) \; . \tag{4.42}$$

But if a parameter (as opposed to a field) changes under a symmetry, then that means that the symmetry is explicitly broken. This is another way to frame the anomaly.

For example, if we return to our generalised QCD with $G = SU(N_c)$ gauge group and N_f massless Dirac fermions then, under the axial transformation (4.40), the theta angle transforms as

$$U(1)_A : \theta \to \theta + 2N_f\alpha \; . \tag{4.43}$$

Thinking about things in this way makes certain aspects of the physics more transparent. For example, suppose that we have a theory with a single massive Dirac fermion ψ. There are two different Dirac masses that we could write down:

$$\mathcal{L}_{\rm mass} = m_1\bar{\psi}\psi + im_2\bar{\psi}\gamma^5\psi \; . \tag{4.44}$$

If we decompose the Dirac fermion into Weyl fermions, $\psi = (\psi_L, \psi_R)$, then these masses become

$$\mathcal{L}_{\rm mass} = m\bar{\psi}_L\psi_R + m^\star\bar{\psi}_R\psi_L \quad \text{with} \quad m = m_1 + im_2 \; . \tag{4.45}$$

Now suppose that we do an axial rotation, $\psi_L \to e^{i\alpha}\psi_L$ and $\psi_R \to e^{-i\alpha}\psi_R$. Then the theory isn't invariant because the mass term shifts by a phase. But, from (4.42), so too does the theta angle. We have

$$U(1)_A : \quad m \to e^{-2i\alpha}m \quad \text{and} \quad \theta \to \theta + 2\alpha \; . \tag{4.46}$$

However, rotating the phase of the fermion can't change the physics of the theory. For example, if we have a free massive fermion (not coupled to a gauge field) then for every value of the mass $m \in \mathbb{C}$ in (4.45), the physical excitation always has mass $|m|$. Now when we couple the fermion to the gauge field, rotating the phase of the fermion changes both the phase of m and the value of θ. This means that the physics depends only on the invariant combination $\theta + \arg(m)$. More generally, with N_f fermions we can have a complex mass matrix M and the quantity $\theta + \arg(\det M)$ remains invariant under chiral rotations.

This, ultimately, is the way in which the strong CP problem in QCD gets its teeth: it's not quite true to say that $\theta = 0$ in QCD. It's more accurate to say that $\theta +$ a bunch of phases of masses $= 0$. And, as we will see in Section 5, those phases of the masses come from rather different physics of the Yukawa couplings.

There is one further observation that follows from the discussion above. Suppose that we have a gauge theory coupled to one, or more, massless fermions. Then rotating the phase of that massless fermion shouldn't affect the physics of the theory, but acts to shift theta as in (4.42). This means that, in a theory with massless fermions, the theta angle isn't physical: it can just be shifted away by an axial rotation. This suggests a rather cute solution to the strong CP problem: perhaps the mass of the up quark is actually zero! In that case, the physics would be independent of the value of θ. Sadly, as numerical simulations have got better, we're now pretty confident that the mass of the up quark is non-zero, and this idea is not a viable solution to the strong CP problem.

4.2.2 Noether's Theorem for Anomalous Symmetries

If a theory has a continuous symmetry, then Noether's theorem tells us that there will be a corresponding conserved current J^μ, obeying the continuity equation

$$\partial_\mu J^\mu = 0 \ . \tag{4.47}$$

What happens if the symmetry is anomalous, so that it's a symmetry of the classical action, but not of the full quantum theory? How does this show up in the conservation of the current?

To answer this, let's first recall how to derive Noether's theorem. To start, we'll work with scalar fields, even though our ultimate interest is in fermions. Consider the transformation of a scalar field ϕ

$$\delta\phi = \alpha X(\phi) \ . \tag{4.48}$$

Here α is a constant, infinitesimally small parameter. This transformation is a *symmetry* if the change in the Lagrangian is

$$\delta\mathcal{L} = 0 \ . \tag{4.49}$$

We can actually be more relaxed than this and allow the Lagrangian to change by a total derivative; this won't change our conclusions below.

The quick way to prove Noether's theorem is to allow the constant α to depend on spacetime: $\alpha = \alpha(x)$. Now the Lagrangian is no longer invariant, but changes as

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu(\delta\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi \\
&= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu(\alpha X(\phi)) + \frac{\partial\mathcal{L}}{\partial\phi}\,\alpha X(\phi) \\
&= (\partial_\mu\alpha)\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,X(\phi) + \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu X(\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\,X(\phi)\right]\alpha\ .
\end{aligned} \tag{4.50}$$

But we know that $\delta\mathcal{L} = 0$ when α is constant, which means that the term in square brackets must vanish. We're left with the expression

$$\delta\mathcal{L} = (\partial_\mu\alpha)J^\mu \quad \text{with} \quad J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,X(\phi)\ . \tag{4.51}$$

The action $S = \int d^4x\ \mathcal{L}$ then changes as

$$\delta S = \int d^4x\ \delta\mathcal{L} = \int d^4x\ (\partial_\mu\alpha)J^\mu = -\int d^4x\ \alpha\,\partial_\mu J^\mu \tag{4.52}$$

where we pick $\alpha(x)$ to decay asymptotically so that we can safely discard the surface term.

The expression (4.52) holds for any field configuration ϕ with the specific change $\delta\phi$. However, when ϕ obeys the classical equations of motion then $\delta S = 0$ for *any* $\delta\phi$, including the symmetry transformation (4.48) with $\alpha(x)$ a function of spacetime. This means that *when* the equations of motion are satisfied we have the conservation law

$$\partial_\mu J^\mu = 0\ . \tag{4.53}$$

This is Noether's theorem.

An Example: the Free Fermion

We can apply all of the above ideas to the fermions that we're really interested in. As a warm-up, consider a free, massless Dirac fermion ψ with action

$$S = -\int d^4x\ i\bar{\psi}\gamma^\mu\partial_\mu\psi \tag{4.54}$$

with $\bar{\psi} = \psi^\dagger\gamma^0$. This theory has two symmetries, the vector and axial symmetries of (4.40). Written in terms of the Dirac fermion, the vector symmetry acts as $\psi \to e^{i\alpha}\psi$ and, infinitesimally, this becomes

$$U(1)_V : \delta\psi = i\alpha\psi \quad \text{and} \quad \delta\bar{\psi} = -i\alpha\bar{\psi}\ . \tag{4.55}$$

We can read off the associated current from (4.51): it is

$$J_V^\mu = \bar{\psi}\gamma^\mu\psi \ . \tag{4.56}$$

Meanwhile, the axial symmetry acts as $\psi \to e^{i\alpha\gamma^5}\psi$ and, infinitesimally, this becomes

$$U(1)_A : \delta\psi = i\alpha\gamma^5\psi \quad \text{and} \quad \delta\bar{\psi} = i\alpha\bar{\psi}\gamma^5 \ . \tag{4.57}$$

Here there's an extra minus sign that rears its head in the transformation of $\delta\bar{\psi}$ which arises because the γ^5 has to sneak past the γ^0 that sits in the definition of $\bar{\psi}$. Again, we can read off the associated current from (4.51): this time it is

$$J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \ . \tag{4.58}$$

As a warm-up to understand the effect of the anomaly, we can see how the currents are affected when we turn on a mass term for the fermion, so

$$S = -\int d^4x \ i\bar{\psi}\gamma^\mu\partial_\mu\psi + m\bar{\psi}\psi \ . \tag{4.59}$$

The action remains invariant under the vector symmetry, and so the current J_V^μ continues to obey $\partial_\mu J_V^\mu = 0$. But the mass term is not invariant under the axial symmetry. Nonetheless, that doesn't mean that we can't say anything. Let's return to our derivation of Noether's theorem and do a transformation with the constant α again promoted to a function of spacetime $\alpha(x)$. We can repeat the steps we did before, except that we need to include an extra term because the action is no longer invariant under the symmetry. Instead, we have

$$\delta S = \int d^4x \ (\partial_\mu\alpha)J_A^\mu + 2im\alpha\bar{\psi}\gamma^5\psi \tag{4.60}$$

with J_A^μ given in (4.58). Now the argument proceeds as before: when the equations of motion are obeyed, we must have $\delta S = 0$ for all transformations, including those with $\alpha(x)$. So whenever the equations of motion are obeyed, the axial current satisfies

$$\partial_\mu J_A^\mu = 2im\bar{\psi}\gamma^5\psi \ . \tag{4.61}$$

This tells us how conservation of axial charge fails when the fermion has a mass.

The Conservation Law for Anomalous Symmetries

Now we can reframe our original question: how is conservation of axial charge affected by the anomaly? We'll consider N_f massless Dirac fermions, coupled to a Yang-Mills theory, with action

$$S_\theta = \int d^4x \ \Big(-\frac{1}{2g^2}\text{Tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{\theta g_s^2}{16\pi^2}\text{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu} - i\sum_{i=1}^{N_f}\bar{\psi}_i\gamma^\mu\mathcal{D}_\mu\psi_i\Big) \ .$$

We've seen that we can capture the effect of the anomaly by shifting the theta angle, as in (4.43)

$$U(1)_A : \theta \to \theta + 2N_f\alpha \ . \tag{4.62}$$

But now we can think of this as a shift of the classical action, and we're in the same boat as when we looked at massive fermions above. In particular, we find that the axial current obeys

$$\partial_\mu J^\mu_A = \frac{N_f g^2}{8\pi^2}\,\mathrm{Tr}\,F_{\mu\nu}{}^\star F^{\mu\nu} \ . \tag{4.63}$$

This is the effect of the anomaly.

Above, we have derived the anomaly equation (4.63) by thinking about the classical action. But one can also show that this holds as an operator equation in quantum field theory, what's known as a Ward identity. You can read about this in the lectures on Gauge Theory.

The anomaly equation (4.63) tells us that the axial symmetry is not conserved. However, at first glance, it appears that there might be a loophole in this statement. This is because, as we saw in (3.109), the term $\mathrm{Tr}\,F_{\mu\nu}{}^\star F^{\mu\nu}$ is actually a total derivative, with

$$\mathrm{Tr}\,F_{\mu\nu}{}^\star F^{\mu\nu} = 2\partial_\mu K^\mu \quad \text{with} \quad K^\mu = \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(A_\nu\partial_\rho A_\sigma - \frac{2i}{3}A_\nu A_\rho A_\sigma\right) \ . \tag{4.64}$$

This suggests that we can define a combination of J^μ_A and K^μ to construct a current that is conserved. Indeed that is naively possible, but it's not legal because K^μ is not gauge invariant, even though $\partial_\mu K^\mu$ is.

We can also ask: under what circumstances does the axial charge change? The axial charge is measured by integrating over a spatial slice

$$Q_A = \int d^3x\ J^0_A \ . \tag{4.65}$$

The change in axial charge from time $t \to -\infty$ to time $t \to +\infty$ is (assuming that things drop off suitably fast at spatial infinity)

$$\Delta Q_A = \int dt d^3x\ \frac{\partial J^0_A}{\partial t} = \int d^4x\ \partial_\mu J^\mu_A = \frac{N_f g_s^2}{8\pi^2}\int d^4x\ \mathrm{Tr}\,F_{\mu\nu}{}^\star F^{\mu\nu} \ . \tag{4.66}$$

But we've already seen in section 3.4 that the integral of $\mathrm{Tr}\,F_{\mu\nu}{}^\star F^{\mu\nu}$ is quantised. This means that Q_A can jump by integer amounts. At weak coupling, the violation of axial charge is mediated by instantons.

There is a similar story for the mixed gauge-gravitational anomaly that we discussed previously. For example we saw that a single, free Weyl fermion has a $U(1)$ symmetry that suffers a mixed gravitational anomaly. This shows up because the current for this $U(1)$ is no longer conserved when the theory is placed in a curved background. Instead, it obeys

$$\nabla_\mu j^\mu_A = -\frac{N_f}{384\pi^2}\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\lambda\tau}R_{\rho\sigma}{}^{\lambda\tau} \tag{4.67}$$

where ∇_μ is the appropriate covariant derivative from differential geometry.

4.2.3 Neutral Pion Decay

The neutral pion, $\pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$ has a substantially shorter lifespan than its charged cousin. It lasts only around $\sim 10^{-16}$ seconds, decaying primarily to

$$\pi^0 \to \gamma\gamma \ . \tag{4.68}$$

There is an interesting story associated to this. Indeed, it was the effort to understand why this decay occurs at all that first led to the discovery of the anomaly.

To set the scene, first note that, although we've focused on massless QCD above, the axial anomaly also arises in QED coupled to massless fermions. Suppose that we have N_f Dirac fermions ψ_i, each with charge Q_i under a $U(1)$ gauge symmetry. Then the axial symmetry $\psi_i \to e^{i\alpha\gamma^5 q_i}\psi_i$ suffers an ABJ anomaly, and the associated current obeys

$$\partial_\mu J^\mu_A = \left(\sum_i q_i Q_i^2\right)\frac{1}{16\pi^2}F_{\mu\nu}{}^\star F^{\mu\nu} \ . \tag{4.69}$$

Again, this follows from a triangle diagram with one J^μ_A leg, and two photon legs. This is reflected in the charges, which are linear in the axial charge q_i and quadratic in the gauge charge Q_i.

Now let's see the implications of this for QCD. We'll take $N_f = 2$ light quarks, corresponding to the up and down. If we assume that these are massless, we know that the QCD action has a $U(1)_V \times SU(2)_L \times SU(2)_R$ symmetry. Now we introduce the coupling to the photon with charges

$$Q_1 = \frac{2}{3} \quad \text{and} \quad Q_2 = -\frac{1}{3} \ . \tag{4.70}$$

Because the quarks have different electric charges, this breaks the flavour symmetry down to $U(1)_L \times U(1)_R \subset SU(2)_L \times SU(2)_R$. We can combine these into a new vector symmetry $U(1)'_V$ and a new axial symmetry $U(1)'_A$, under which the quarks transform as

$$\begin{aligned} U(1)'_V : &\quad u \to e^{i\alpha} u \quad \text{and} \quad d \to e^{-i\alpha} d \ . \\ U(1)'_A : &\quad u \to e^{i\alpha\gamma^5} u \ \text{ and } \ d \to e^{-i\alpha\gamma^5} d \ . \end{aligned} \tag{4.71}$$

The vector symmetry $U(1)'_V$ is anomaly-free, while the axial symmetry $U(1)'_A$ does not suffer an anomaly due to the QCD gauge field because there is a cancellation between the $q_1 = +1$ charge of the up quark and the $q_2 = -1$ charge of the down quark. However, the axial $U(1)'_A$ *does* suffer an anomaly with the QED gauge field. To compute this, we need to remember that, from the perspective of electromagnetism, each quark comes in $N_c = 3$ different varieties, due to the fact that they also transform under the $SU(3)$ gauge group. This means that the ABJ anomaly (4.69) is

$$\partial_\mu J'^{\mu}_A = N_c \left(\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2 \right) \frac{1}{16\pi^2} F_{\mu\nu} {}^\star F^{\mu\nu} = \frac{N_c}{48\pi^2} F_{\mu\nu} {}^\star F^{\mu\nu} \ . \tag{4.72}$$

where we've left the value of $N_c = 3$ in this formula to highlight that the anomaly coefficient depends on the number of quark colours.

This additional axial current is $J'^{\mu}_A = \bar{u}\gamma^\mu\gamma^5 u - \bar{d}\gamma^\mu\gamma^5 d$ and, from (3.68), is precisely the current that creates the neutral pion π^0,

$$\langle 0 | J'^{\mu}_A(x) | \pi^0(p) \rangle = -i f_\pi \delta^{ab} \, p^\mu e^{-ix\cdot p} \ . \tag{4.73}$$

The anomaly equation then gives an amplitude for $\pi^0 \to \gamma\gamma$. This amplitude is proportional to N_c, the number of colours, and gives an experimental method to determine $N_c = 3$.

There is more to this story which we mention only briefly. This amplitude for $\pi^0 \to \gamma\gamma$ is the same as that which would arise from the coupling in the Lagrangian

$$\mathcal{L} = \frac{N_c e^2}{48\pi^2 f_\pi} \pi^0 \, F_{\mu\nu} {}^\star F^{\mu\nu} \ . \tag{4.74}$$

In other words, the neutral pion field π^0 acts very much like a dynamical theta term! There's something odd in this because π^0 is a Goldstone boson and, as such, should only appear in the action with derivative couplings. But, after an integration by parts, the pion is derivatively coupled in (4.74) if we remember that $F_{\mu\nu} {}^\star F^{\mu\nu} = 2\partial_\mu K^\mu$ as in (4.64). There is a much longer story here, involving the beautiful Wess-Zumino-Witten (WZW) term that you can read about in the lectures on Gauge Theory.

4.2.4 Surviving Discrete Symmetries

Thinking of the anomalous symmetry as shifting the theta angle reveals something novel. That's because the theta angle is, as the name suggests, an angle with $\theta \in [0, 2\pi)$. This means that if we transform by an anomalous $U(1)$ symmetry that maps $\theta \to \theta + 2\pi$, then that hasn't actually changed the value of θ at all. In this way, some discrete subgroup of the $U(1)$ may remain.

We can see this in the case of QCD, although the end result turns out to be a little fiddly and not particularly interesting. From (4.43), we see that a $U(1)_A$ transformation of the form $e^{-i\alpha} = e^{2\pi i/2N_f}$ will send $\theta \to \theta + 2\pi$. By acting with a compensating $U(1)_V$ transformation, there is a surviving $\mathbb{Z}_{N_f}$ subgroup which acts as

$$\mathbb{Z}_{N_f}: \quad \psi_{L\,i} \to e^{2\pi i/N_f} \psi_{L\,i} \quad \text{and} \quad \psi_{R\,i} \to \psi_{R\,i} \ . \tag{4.75}$$

But we recognise this as the centre of the $SU(N_f)_L$ global symmetry. So in this case, the surviving discrete symmetry doesn't tell us anything new.

Here's a different example where things are more interesting. Consider $SU(N)$ Yang-Mills coupled to a single, massless Weyl spinor λ in the adjoint representation. We've already seen that the adjoint representation is real, so this theory doesn't suffer from a gauge anomaly. Indeed, it's a rather famous theory because it secretly has a supersymmetry, exchanging the gauge field and fermion. This theory is known as *super Yang-Mills*. Thankfully, we won't need to know anything about supersymmetry for our discussion. (You can read more in the lectures on Supersymmetry.)

Classically this theory has a global $U(1)$ symmetry which rotates the phase of λ

$$U(1): \quad \lambda \to e^{i\alpha}\lambda \ . \tag{4.76}$$

But quantum mechanically, this theory suffers an anomaly. We need the fact, from Table 8, that $I(\text{adj}) = 2N$ for the adjoint representation. Then, from (4.42), we see that the theta angle shifts under this $U(1)$ symmetry as

$$U(1): \quad \theta \to \theta + 2N\alpha \ . \tag{4.77}$$

This is telling us that the $U(1)$ symmetry is anomalous. But, by the argument above, a discrete $\mathbb{Z}_{2N}$ survives since this shifts $\theta \to \theta + 2\pi$, while the fermion transforms as

$$\mathbb{Z}_{2N}: \quad \lambda \mapsto e^{2\pi i/2N}\lambda \ . \tag{4.78}$$

This discrete symmetry becomes particularly interesting because this theory, like many other non-Abelian gauge theories, flows to strong coupling at some scale $\Lambda_{\rm QCD}$ where it exhibits confinement and the formation of a fermion condensate,

$$\langle \lambda\lambda \rangle \sim \Lambda_{\rm QCD}^3 \ . \tag{4.79}$$

In actual QCD, such a condensate breaks the chiral symmetry. And the same is true here, but with the important difference that the chiral symmetry in question is not $U(1)$ but instead just the surviving $\mathbb{Z}_{2N}$. The condensate breaks this to $\mathbb{Z}_{2N} \rightarrow \mathbb{Z}_2$, where $\mathbb{Z}_2 : \lambda \mapsto -\lambda$. But we know from our discussion in Section 2.1 that, when a discrete symmetry is spontaneously broken, it means that the theory has multiple, degenerate ground states. Indeed, that's the case here: $SU(N)$ gauge theory, with a single adjoint Weyl fermion, has N degenerate ground states, distinguished by the phase of the fermion condensate $\langle \lambda\lambda \rangle$.

4.3 't Hooft Anomalies

So far we have discussed two manifestations of the anomaly:

- For a gauge symmetry, the anomaly better cancel. Or else.
- A mixed anomaly between a global symmetry and gauge symmetry means that the global symmetry isn't.

But what if we have an anomaly just for a global symmetry? What are the consequences? From what we've discussed above, we know that the symmetry isn't conserved if we couple it to background gauge fields. But nothing compels us to do so. Indeed, if we're in the realm of particle physics then it's a little odd to do so because we're usually interested in relativistic physics in Minkowski space, while turning on a constant background electric or magnetic field breaks Lorentz invariance. So what else can we learn from this?

The answer is both subtle and powerful. The basic idea is that the anomaly provides a way to classify different quantum field theories: two quantum field theories with the same global symmetry group G_F can only be deformed into each other if they share the same anomaly. This is particularly useful when thinking about how theories flow to strong coupling, where we often don't know what happens. The anomalies provide constraints on what the theory can do. Such anomalies in global symmetries are referred to as *'t Hooft anomalies.*

We can flesh out this idea some more. Suppose that we've got some theory with a global symmetry that, for the sake of this argument, I'll call G_F. We can compute the anomaly for this symmetry. This is just a number – say $\sum_i Q_i^3$ if the symmetry is $G_F = U(1)$, or the generalisation if G_F is non-Abelian. As we will now argue, this anomaly is a way to characterise the theory and, provided that the symmetry is not broken, the anomaly remains unchanged under any deformation of the theory. In particular, the anomaly remains unchanged if the theory flows to strong coupling. In fact, this anomaly is one of the few handles that we have on the strong coupling physics of quantum field theories.

We will first explain the basic idea and then give a concrete example. Suppose that we have some quantum field theory – typically a non-Abelian gauge theory – that is weakly coupled in the UV, but flows to strong coupling in the IR. The most important example is, of course, QCD. We will abstractly call the UV theory $\mathcal{T}_{UV}$. We assume that it has some global symmetry G_F. This should be a true symmetry of the quantum theory meaning, in particular, that it has no mixed anomalies with the gauge symmetry.

This UV theory may have a 't Hooft anomaly for G_F. This anomaly is just a number. If G_F is Abelian, this anomaly is simply $\sum Q_i^3$ as in (4.7); if it is non-Abelian the anomaly is $\sum A(R_i)$ as in (4.17). Either way, we will denote this anomaly as $\mathcal{A}_{UV}$ and assume $\mathcal{A}_{UV} \neq 0$.

The theory now flows under RG to a theory $\mathcal{T}_{IR}$ in the IR which will typically be very different. For QCD this is the theory of mesons and baryons. For other quantum field theories, the infra-red physics may be quite mysterious. We have the following result:

Claim: Either the symmetry G_F is spontaneously broken, or the anomalies match meaning

$$\mathcal{A}_{UV} = \mathcal{A}_{IR} \ . \tag{4.80}$$

This is a wonderfully powerful result. If G_F is spontaneously broken then we necessarily have massless Goldstone bosons. But if G_F is unbroken then we must have massless fermions that reproduce the anomaly. This is known as *'t Hooft anomaly matching.*

Proof: The argument for 't Hooft anomaly matching is very slick. Suppose that $\mathcal{A}_{UV} \neq 0$ then we know from the discussion above that we're not allowed to couple G_F to dynamical gauge fields. That would lead to a sick theory.

To proceed, we introduce a bunch of extra massless Weyl fermions transforming under G_F. We call these *spectator fermions*. These won't interact directly with our original fields in $\mathcal{T}_{UV}$, but they are designed so that the total anomaly of the original fields and these new fermions vanishes:

$$\mathcal{A}_{UV} + \mathcal{A}_{\text{spectator}} = 0 \ . \tag{4.81}$$

Now that the anomaly cancels, there's nothing to stop us introducing dynamical gauge fields for G_F. We do so, but with a very (very!) small coupling constant.

Now let's go back to our original theory $\mathcal{T}_{UV}$. It will flow to strong coupling at some scale Λ_{QCD} and we'd like to understand the physics $\mathcal{T}_{IR}$ below this scale. If the gauge coupling for G_F is small enough, then this RG flow takes place entirely unaffected by the presence of the G_F gauge fields. This means that one of two things could have happened. It may be that the strong coupling dynamics of $\mathcal{T}_{UV}$ spontaneously breaks the symmetry G_F. (For example, as we've seen, this is expected to happen if we take G_F to be the chiral symmetry of QCD.) This was the first possibility of our claim. Alternatively, G_F may be unbroken at low-energies. In this case, we're left with $\mathcal{T}_{IR}$, together with the spectator fermions, all coupled to the G_F gauge fields. But this can only be consistent if

$$\mathcal{A}_{IR} + \mathcal{A}_{\text{spectator}} = 0 \ . \tag{4.82}$$

Clearly, this means that we must have $\mathcal{A}_{IR} = \mathcal{A}_{UV}$. □

4.3.1 Confinement Implies Chiral Symmetry Breaking

Anomaly matching has many uses. But the most important is a statement about QCD.

Recall from Section 3 that there are two strong coupling effects that arise in QCD. The first is confinement, the second chiral symmetry breaking. We will now use 't Hooft anomalies to argue that the former implies the latter.

We can work more generally with an $SU(N_c)$ gauge theory, coupled to N_f massless Dirac fermions q_i, each in the fundamental representation. This is a vector-like theory, so doesn't suffer any gauge anomaly. We've already seen that the $U(1)_A$ axial symmetry suffers an ABJ anomaly, so the global symmetry of the theory is

$$G_F = U(1)_V \times SU(N_f)_L \times SU(N_f)_R \ . \tag{4.83}$$

We want to compute the 't Hooft anomalies of this global symmetry group.

This is straightforward if we work in the UV where the theory is weakly coupled. In this case, we can just pretend that the fermions are essentially free and read off the result. There is no 't Hooft anomaly for $U(1)_V^3$, where the subscript 3 means that all three legs in the triangle diagram have $U(1)_V$ currents, because this is a vector-like symmetry. In contrast, there is a 't Hooft anomaly associated to the chiral, $SU(N_f)$ factors. In fact, there are two. The first is the purely non-Abelian anomaly,

$$[SU(N_f)_L]^3 : \quad \mathcal{A} = \sum A(\Box) = N_c \ . \tag{4.84}$$

Here the anomaly arises because each left-handed quark q_L transforms in the fundamental $\Box$ of $SU(N_f)_L$ and $A(\Box) = 1$. But the quarks also come with a colour index which means that there are N_c such fermions. (More generally, you have to sum over any other indices that the fermion carries that aren't themselves involved in the anomaly.) Hence the result $\mathcal{A} = N_c A(\Box) = N_c$. There is a similar anomaly for $SU(N_f)_R$.

In addition, there is a mixed 't Hooft anomaly between $U(1)_V$ and $SU(N_f)$. This is

$$[SU(N_f)_L]^2 \times U(1)_V : \quad \mathcal{A}' = \sum qI(\Box) = N_c \tag{4.85}$$

which again simply counts the number of quark colours.

Now the question is: what happens in the infra-red? For suitably low N_f, we've seen in Section 3 that we expect the chiral symmetry G_F to be broken down to $U(1)_V \times SU(N_f)_{\rm diag}$, but proving this remains an open problem. Here we will shed some insight. We will assume that the theory confines and, moreover, that in the infra-red, the physics is described by weakly interacting mesons and baryons. (This is in contrast to the conformal field theories that we see at larger N_f.) In such a situation, 't Hooft anomaly matching shows that the chiral symmetry *must* be broken.

Here is the argument. Suppose that G_F is unbroken in the infra-red. Then there must be massless fermions around that can reproduce the anomalies $\mathcal{A}$ and $\mathcal{A}'$. Moreover, by assumption, these massless fermions must be bound states of quarks, either mesons or baryons.

Mesons certainly can't do the job because these are bosons. Baryons, meanwhile, contain N_c quarks so these too are bosons when N_c is even. This is telling us that when N_c is even, a confining theory contains no fermions at low-energies and so certainly can't reproduce the anomalies. We learn that chiral symmetry breaking must occur when N_c is even.

What about N_c odd? Now baryons are fermions. Is it possible that some of these baryons could be massless and reproduce the 't Hooft anomalies? Of course, this doesn't happen in our world: the simplest baryons are the proton and neutron which are certainly not massless. But might it be a theoretical possibility? The answer, it turns out, is no. The basic argument is to figure out what representations of G_F the putative massless baryons must sit in, and then to show that there's no possible combination of baryons that can reproduce the 't Hooft anomalies $\mathcal{A}$ and $\mathcal{A}'$. This means that if QCD confines into weakly interacting colour singlets, then chiral symmetry is necessarily broken. We now present this argument in more detail.

The Representations of Massless Baryons

It turns out that we can make the argument for any number of colours N_c, but it is simplest if we restrict to $N_c = 3$. Which, happily, is the case we care about for QCD.

If the $SU(3)$ gauge group confines, then any massless fermion must be a colour singlet. The only possibility is baryons, comprised of three quarks. Each constituent quark can be either left-handed or right-handed. Under $SU(N_f)_L \times SU(N_f)_R \subset G_F$, the left-handed fermions transform as $(\mathbf{N}_f, \mathbf{1})$, while the right-handed fermions transform as $(\mathbf{1}, \mathbf{N}_f)$. Both of these Weyl fermions have charge $+1$ under $U(1)_V$.

We've already seen in Section 3.3 that baryons in QCD can have either spin $\frac{1}{2}$ or spin $\frac{3}{2}$, depending on how the constituent spins of the quarks are aligned. You might imagine that the same can be true for our putative massless baryons, but there is a theorem by Weinberg and Witten which says that one cannot form massless bound states with helicity $\lambda \geq 1$. So if the massless baryons above do indeed form then they must have helicity $\pm\frac{1}{2}$.

So what representations of $G_F = U(1)_V \times SU(N_f)_L \times SU(N_f)_R$ do the colour singlet baryons sit in? Well, to form a helicity $\frac{1}{2}$ baryon, we should contract the spin indices of two fermions of the same handedness, and then leave the third spinor degree of freedom hanging. There are different ways to do this. For example, we could have three left-handed spinors, so that the indices combine to leave us with a left-handed spinor. In this case, the resulting bound state will transform in one of three possible representations of the $SU(N_f)_L$ symmetry which, in the language of Young diagrams, read

$$\begin{array}{|c|c|c|}\hline L & L & L \\ \hline\end{array} \quad , \quad \begin{array}{|c|}\hline L \\ \hline L \\ \hline L \\ \hline\end{array} \quad , \quad \begin{array}{|c|c|}\hline L & L \\ \hline L & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{4.86}$$

The first representation is the totally symmetric, the second the totally anti-symmetric, and the final is some representation whose name I don't know. Some properties of these representations are listed in Table 9. We've labelled the boxes with L to remind us that these are constructed out of three left-handed quarks.

But, alternatively, we could get ourselves a left-handed spinor by combining the indices on two right-handed spinors, and then leaving the final left-handed spinor hanging. These baryons would transform in representations of $SU(N_f)_L \times SU(N_f)_R$ that take the form

$$\begin{array}{|c|}\hline L \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline R & R \\ \hline\end{array} \quad , \quad \begin{array}{|c|}\hline L \\ \hline\end{array} \otimes \begin{array}{|c|}\hline R \\ \hline R \\ \hline\end{array} \tag{4.87}$$

Each of these transforms in the fundamental $\square$ of $SU(N_f)_L$, while the first transforms in the symmetric $\square\!\square$ of $SU(N_f)_R$ and the second transforms in the anti-symmetric $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of $SU(N_f)_R$.

So (4.86) and (4.87) are the possible representations for massless left-handed baryons. But there's also the option for massless right-handed baryons which we get by simply exchanging $L \leftrightarrow R$,

$$\begin{array}{|c|c|c|}\hline R & R & R \\ \hline\end{array} \quad , \quad \begin{array}{|c|}\hline R \\ \hline R \\ \hline R \\ \hline\end{array} \quad , \quad \begin{array}{|c|c|}\hline R & R \\ \hline R \\ \cline{1-1}\end{array} \quad , \quad \begin{array}{|c|c|}\hline L & L \\ \hline\end{array} \otimes \begin{array}{|c|}\hline R \\ \hline\end{array} \quad , \quad \begin{array}{|c|}\hline L \\ \hline L \\ \hline\end{array} \otimes \begin{array}{|c|}\hline R \\ \hline\end{array} \tag{4.88}$$

So these are our options for forming massless baryons. Now the question is: which combination of these massless baryons will reproduce the 't Hooft anomalies of the UV theory?

We started with a vector-like theory, in which all fermions came in left/right pairs to make a Dirac fermion. So it seems reasonable to assume that we end up with a vector-like theory. Indeed, a strong constraint comes from the $U(1)_V^3$ anomaly which vanishes. We will assume that we reproduce this by taking left/right pairs, so that if one of the massless baryons in (4.86) or (4.87) arises in the spectrum, then so too does its counterpart from (4.88).

So now we have a well-defined problem on our hands. We take some number $p_\alpha \geq 0$ of each of the $\alpha = 1, 2, 3, 4, 5$ possible baryons above and then see which values of p_α can reproduce the 't Hooft anomalies $\mathcal{A}$ and $\mathcal{A}'$.

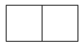
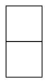
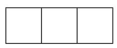
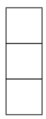
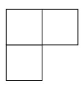
R	$\dim(R)$	$I(R)$	$A(R)$
	N_f	1	1
	$\frac{1}{2}N_f(N_f+1)$	N_f+2	N_f+4
	$\frac{1}{2}N_f(N_f-1)$	N_f-2	N_f-4
	$\frac{1}{6}N_f(N_f+1)(N_f+2)$	$\frac{1}{2}(N_f+2)(N_f+3)$	$\frac{1}{2}(N_f+3)(N_f+6)$
	$\frac{1}{6}N_f(N_f-1)(N_f-2)$	$\frac{1}{2}(N_f-2)(N_f-3)$	$\frac{1}{2}(N_f-3)(N_f-6)$
	$\frac{1}{3}N_f(N_f^2-1)$	N_f^2-3	N_f^2-9

Table 9. Properties of some representations of $SU(N_f)$

Actually, at this point a subtlety raises its head. Above, we confidently asserted that (4.86) and (4.87) were left-handed spinors, while (4.88) were right-handed spinors. That's certainly true if we're dealing with a weakly interacting theory where we can just read off the representations from contracting indices. But things could be more complicated in a strongly interacting theory. In particular, it may be that a massless spin 1 gluon binds with one of the baryons to flip its helicity from $+\frac{1}{2}$ to $-\frac{1}{2}$. So it may be that some of the baryons that we listed in (4.86) and (4.87) are actually right-handed instead of left-handed.

In fact, it's easy to take this subtlety into account. We'll assign an *index*, $p_\alpha \in \mathbf{Z}$, with $\alpha = 1, \ldots, 5$ to each of the five baryons in (4.86) and (4.87). The magnitude $|p_\alpha|$ denotes the number of species of baryon that arise in the massless spectrum. If these baryons are left-handed then we take $p_\alpha > 0$; if they are right-handed then we take $p_\alpha < 0$. Our task is to find which values of p_α will satisfy anomaly matching and reproduce (4.84) and (4.85).

Next, we need a little group theory. For a representation $\mathbf{R}$ of $SU(N_f)$, we will need to know the dimension $\dim(R)$, the anomaly coefficient $A(R)$, as well as the Dynkin index $I(R)$ that we already met in (4.24). The relevant data is shown in Table 9.

We can now compute the infra-red anomalies, assuming that we have p_α massless baryons of each type. For $SU(N_f)^3_L$ with $N_f \geq 3$, the anomaly is

$$\mathcal{A} = \frac{1}{2}(N_f+3)(N_f+6)p_1 + \frac{1}{2}(N_f-3)(N_f-6)p_2 + (N_f^2-9)p_3 + \left(\frac{1}{2}N_f(N_f+1) - N_f(N_f+4)\right)p_4 + \left(\frac{1}{2}N_f(N_f-1) - N_f(N_f-4)\right)p_5 \ . \tag{4.89}$$

Note that the baryons with numbers p_4 and p_5 arise from tensor products and have two terms. For example, for p_4 the first term comes from the left-handed baryon $\boxed{L}\otimes\boxed{R}\boxed{R}$, and the second — with the minus sign — from the right-handed baryon $\boxed{R}\otimes\boxed{L}\boxed{L}$.

Meanwhile, for the $SU(N_f)^2 \times U(1)_V$ anomaly, each baryon has charge 3 under the $U(1)_V$. Dividing through by this, we get a contribution proportional to the Dynkin index $I(R)$,

$$\frac{\mathcal{A}'}{3} = \frac{1}{2}(N_f+2)(N_f+3)p_1 + \frac{1}{2}(N_f-2)(N_f-3)p_2 + (N_f^2-3)p_3 + \left(\frac{1}{2}N_f(N_f+1) - N_f(N_f+2)\right)p_4 + \left(\frac{1}{2}N_f(N_f-1) - N_f(N_f-2)\right)p_5 \ . \tag{4.90}$$

To match the anomalies, we need to find p_α such that $\mathcal{A} = \mathcal{A}' = 3$.

To start, let's look at $N_f = 3$. Anomaly matching gives

$$\mathcal{A} = 27p_1 - 15p_4 = 3 \quad \text{and} \quad \frac{\mathcal{A}'}{3} = 15p_1 + 6p_3 - 9p_4 = 1 \ . \tag{4.91}$$

We can immediately see that there can be no solutions to the second of these equations since $\mathcal{A}'/3$ in the infra-red theory is necessarily a multiple of 3 and cannot reproduce the ultra-violet anomaly $\mathcal{A}'/3 = 1$. We learn that $G = SU(3)$ gauge theory with $N_f = 3$ massless fermions must spontaneously break the G_F flavour symmetry, as long as the theory confines. You can check that the same argument works whenever N_f is a multiple of 3.

Decoupling Massive Quarks

When N_f is not a multiple of 3, things are not quite so simple. Indeed, we will need one further ingredient to complete the argument. To see this, let's look at the anomaly matching conditions for $G = SU(3)$ gauge theory with $N_f = 4$ flavours. They are:

$$\mathcal{A} = 35p_1 - p_2 + 7p_3 - 22p_4 + 6p_5 = 3$$
$$\frac{\mathcal{A}'}{3} = 21p_1 + p_2 + 13p_3 - 14p_4 - 2p_5 = 1 \ . \tag{4.92}$$

Now there are solutions. For example $p_2 = 3$ and $p_5 = 1$ with $p_1 = p_3 = p_4 = 0$ does the job. This corresponds to four massless baryons in the representations

$$[3(\bar{\mathbf{4}},\mathbf{1}) \oplus (\mathbf{4},\mathbf{6})]_L \oplus [3(\mathbf{1},\bar{\mathbf{4}}) \oplus (\mathbf{6},\mathbf{4})]_R \tag{4.93}$$

where the L and R subscripts denote the chirality of these Weyl spinors. Note that the left-handed baryons now transform under both $SU(4)_L$ and $SU(4)_R$ of the chiral flavour symmetry.

Naively, the existence of the solution (4.93) suggests that there is a phase with massless baryons and the chiral symmetry left unbroken. In fact, this cannot happen. The problem comes when we think about giving one of the quarks a mass. We will make the following assumption: when we give a quark a mass, any baryon that contains this quark will also become massive. It is not obvious that this happens, but it turns out to be true, a result known as the Vafa-Witten theorem. (It's one of a number of Vafa-Witten theorems.)

If we give one of the quarks a mass, then the symmetry group is explicitly broken to

$$G_F = U(1)_V \times SU(4)_L \times SU(4)_R \;\longrightarrow\; G'_F = U(1)_V \times SU(3)_L \times SU(3)_R\ . \tag{4.94}$$

What happens to our putative massless spectrum (4.93)? A little group decomposition tells us that under G'_F, the left-handed baryons transform as

$$3(\bar{\mathbf{4}},\mathbf{1}) \rightarrow 3(\bar{\mathbf{3}},\mathbf{1}) \oplus 3(\mathbf{1},\mathbf{1}) \quad \text{and} \quad (\mathbf{4},\mathbf{6}) \rightarrow (\mathbf{3},\bar{\mathbf{3}}) \oplus (\mathbf{3},\mathbf{3}) \oplus (\mathbf{1},\bar{\mathbf{3}}) \oplus (\mathbf{1},\mathbf{3})\ . \tag{4.95}$$

The right-handed baryons have their $SU(3)_L \times SU(3)_R$ representations reversed. Of these, the $(\mathbf{1},\mathbf{1})$ and the $(\mathbf{3},\bar{\mathbf{3}})$ do not contain the massive fourth quark. By our assumption above, the remainder should become massive.

There is a further constraint however: all of the baryons that contain the fourth quark should become massive while leaving the surviving symmetry G'_F intact. This is because, as the mass becomes large, we should return to the theory with $N_f = 3$ flavours and the symmetry group G'_F. Although we now know that G'_F will ultimately be spontaneously broken by the strong coupling dynamics, this should happen at the scale $\Lambda_{\rm QCD}$ and not at the much higher scale of the fourth quark mass.

So what G'_F-singlet mass terms can we write for the baryons that contain the fourth quark? The left-handed spinors transform as $3(\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{3},\mathbf{3}) \oplus (\mathbf{1},\bar{\mathbf{3}}) \oplus (\mathbf{1},\mathbf{3})$. Of these, $(\mathbf{3},\mathbf{3})$ can happily pair up with its right-handed counterpart. Further, one of the $(\bar{\mathbf{3}},\mathbf{1})$ representations can pair up with the right-handed counterpart of $(\mathbf{1},\bar{\mathbf{3}})$. But that still leaves us with $2(\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$ and these have nowhere to go. Any mass term will necessarily break the remaining G'_F chiral symmetry and, as we argued above, this is unacceptable.

The result above should not be surprising. Any baryon that can get a mass without breaking G'_F does not change the 't Hooft anomaly for G'_F. If it were possible for all the baryons containing the massive quark to get a mass without breaking G'_F then the remaining massless baryons should satisfy anomaly matching. Yet we've seen that no such solution is possible for N_f.

The upshot of this argument is that there exists no solution to anomaly matching for $N_f = 4$ which is consistent with the decoupling of massive quarks. It is simple to extend this to all N_f and, indeed, to all N_c. 't Hooft anomaly matching then tells us that the chiral symmetry must be broken for all $N_c \geq 2$ and all $N_f \geq 3$.

Massless Baryons when $N_f = 2$?

There is one situation where it is possible to satisfy the anomaly matching: this is when $N_f = 2$. Since there is no triangle anomaly for $SU(2)$, we need only worry about the mixed $SU(2)_L^2 \times U(1)_V$ 't Hooft anomaly. We can import our results from earlier, although we should be a little bit careful: the anti-symmetric representation $\begin{array}{|c|}\hline R \\ \hline R \\ \hline \end{array}$ is the singlet of $SU(2)$ while the representation $\begin{array}{|c|c|}\hline L & L \\ \hline L \\ \cline{1-1} \end{array}$ does not exist. The 't Hooft matching condition for gauge group $SU(3)$ now gives

$$\frac{\mathcal{A}'}{3} = 10p_1 - 5p_4 + p_5 = 1 \ . \tag{4.96}$$

This has many solutions. The simplest possibility is $p_1 = p_4 = 0$ and $p_5 = 1$. This means that we can match the anomaly if there are massless baryons which transform under $SU(2)_L \times SU(2)_R \times U(1)_V$ as

$$(\mathbf{2}, \mathbf{1})_3 \oplus (\mathbf{1}, \mathbf{2})_3 \ . \tag{4.97}$$

So for $N_f = 2$ we cannot use 't Hooft anomaly matching to rule out the existence of massless baryons. But it does not mean that they actually arise. To understand what happens, we need to look more carefully at the actual dynamics. The only real tool we have at our disposal is the lattice and this strongly suggests that even for $N_f = 2$ the chiral symmetry is broken and there are no massless baryons.