5 Electroweak Interactions

In this section, we turn to the weak force. But, in contrast to the strong force, if we want to understand the weak force then we really need to take a step back and take in the full structure of the Standard Model. This is because of the single most important feature of the weak force: it breaks parity.

The weak force breaks parity because it is a chiral gauge theory. This means that the gauge bosons interact differently with the left- and right-handed fermions. And, as we saw in Section 4, this forces us to grapple with the issue of gauge anomalies. And this, in turn, means that we must look at all the fermions to check consistency.

5.1 The Structure of the Standard Model

As we advertised in the introduction, the Standard Model is built on the gauge group

$$G = U(1) \times SU(2) \times SU(3) \ . \tag{5.1}$$

Here $U(1)$ is a force known as *hypercharge*. It is not electromagnetism. We will see how electromagnetism emerges from the Standard Model in Section 5.2 when we discuss electroweak symmetry breaking. The group for hypercharge is sometimes denoted as $U(1)_Y$ to distinguish it from electromagnetism. Correspondingly, the charges are usually denoted as Y.

There are a collection of fermions that are charged under this gauge group. The fermions for a single generation are:

	$U(1)$	$SU(2)$	$SU(3)$
Q_L	$\frac{1}{6}$	$\mathbf{2}$	$\mathbf{3}$
L_L	$-\frac{1}{2}$	$\mathbf{2}$	$\mathbf{1}$
u_R	$\frac{2}{3}$	$\mathbf{1}$	$\mathbf{3}$
d_R	$-\frac{1}{3}$	$\mathbf{1}$	$\mathbf{3}$
e_R	-1	$\mathbf{1}$	$\mathbf{1}$

(5.2)

What a weird collection of charges and representations! Why these? We'll answer this question below. First some comments.

The hypercharges are taken to be fractional. In some sense, this is merely a convention: we could just have well rescaled the charges so that Q_L has charge $+1$ and e_R charge -6. However, as we will see, the slightly odd fractional scaling above will reproduce our familiar convention for electric charges, in which the electron has charge -1, the up quark charge $\frac{2}{3}$ and the down quark charge $-\frac{1}{3}$.

Each of the fields transforms in either the fundamental representations of $SU(2)$ or $SU(3)$, denoted by $\mathbf{2}$ and $\mathbf{3}$ respectively, or in the singlet representation denoted by $\mathbf{1}$. This means that a bold $\mathbf{1}$ for a non-Abelian group is telling us that a field doesn't experience that force. (In contrast, a charge 1 for the $U(1)$ means that the field very much experiences that force; only charge 0 fields are neutral under $U(1)$.) We will sometimes denote the representations as $(\mathbf{R}_2, \mathbf{R}_3)_Y$, with $\mathbf{R}_2$ and $\mathbf{R}_3$ the representations of $SU(2)$ and $SU(3)$ respective, and Y the hypercharge. So, for example, the field Q_L transforms as $(\mathbf{2}, \mathbf{3})_{1/6}$.

Each of the fields in the table is a Weyl fermion, either left-handed or right-handed as denoted by the L and R subscripts. As we saw in Section 1, the conjugate fermion has the opposite handedness. So, for example, $\bar{Q}_L$ is a right-handed fermion that transforms as $(\mathbf{2}, \bar{\mathbf{3}})_{-1/6}$. (You might have thought that we should have written $\bar{\mathbf{2}}$ but the doublet of $SU(2)$ is pseudoreal, meaning that $\bar{\mathbf{2}} \cong \mathbf{2}$.)

The fermions that transform in the $\mathbf{3}$ of $SU(3)$ are the quarks that we met in Section 3.1. That statement is straightforwardly true for the right-handed quarks, which we've labelled u_R and d_R for the up quark and down quark. But there is just a single left-handed quark Q_L, albeit one that transforms in the $\mathbf{2}$ of $SU(2)$. Indeed, it's only the left-handed fermions that transform in the $\mathbf{2}$ of $SU(2)$. How should we think of the associated $a = 1, 2$ index? In other words, what's the analog of colour for the $SU(2)$ gauge group?

It turns out that the $SU(2)$ index is the names that we give to different particles. We often write the $SU(2)$ gauge structure of the left-handed fermions as

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \text{and} \quad L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \,. \tag{5.3}$$

For Q_L, we interpret the $SU(2)$ doublet components as the left-handed up quark and left-handed down quark. For L_L, which we refer to as the left-handed lepton, we interpret the $SU(2)$ doublet as the left-handed neutrino ν_L and left-handed electron e_L.

This part of the story is very surprising. For the strong force, the $SU(3)$ gauge symmetry rotates different colours into each other. That's intuitive: we think that the red quark behaves very much like the blue quark. The analogous statement of (5.3) is that the $SU(2)$ gauge symmetry rotates, say, the left-handed neutrino into the left-handed electron. But these particles are nothing like each other, neither in mass nor their interactions! How can they possibly be related by a gauge symmetry? The answer, as we shall see, is that the Higgs field spontaneously breaks the $SU(2)$ gauge symmetry and, when the dust settles, leaves ν_L and e_L with very different properties. Indeed, at this point it's really misleading to write (5.3) because, before we talk about spontaneous symmetry breaking, there's really no sense in which the top component of Q_L is related to the up quark and the second component to the down quark. These properties will only manifest themselves after the Higgs mechanism (and, even then, only when we've made an arbitrary choice of vacuum structure).

Including the gauge degrees of freedom, there are a total of 15 fermions listed above. (The left-handed quark Q_L has $2 \times 3 = 6$. The total number is then 6+2+3+3+1=15.) It is possible that we should augment these 15 fermions with one additional one. This is a right-handed neutrino

	$U(1)$	$SU(2)$	$SU(3)$
ν_R	0	$\mathbf{1}$	$\mathbf{1}$

(5.4)

Unfortunately, we don't yet know if the right-handed neutrino ν_R exists or not! This is deeply unsatisfactory and the situation will hopefully change in the near future. The main reason for our ignorance is that, as shown above, ν_R doesn't interact with any of the forces. That makes it hard to detect and it is sometimes referred to as a *sterile neutrino*. It's interactions with the other particles are only through the Higgs field and it manifests itself in the way in which neutrinos get masses. We will describe this in Section **??**. On aesthetic grounds, things look marginally nicer if ν_R exists, in the sense that each particle has a right-handed fermion and a left-handed counterpart sitting in the doublet of $SU(2)$. But this is not a particularly compelling argument and the situation should ultimately be determined by experiment.

There is one final field in the Standard Model: this is the Higgs boson which we denote as H. It is the only spin 0 particle in the Standard Model and has quantum numbers

	$U(1)$	$SU(2)$	$SU(3)$
H	$\frac{1}{2}$	$\mathbf{2}$	$\mathbf{1}$

(5.5)

These are the same quantum numbers as $\bar{L}_L$. As we will see, it turns out that there is something magical about this choice which allows the whole jigsaw to fit together.

5.1.1 Anomaly Cancellation

The Standard Model is a chiral gauge theory. The first thing that we have to do is check that it makes sense! As we've seen in Section 4.1, there are a number of stringent consistency checks that any chiral gauge theory must pass. You will probably not be surprised to hear that the Standard Model, and hence our universe, is mathematically consistent. But it should give you a warm fuzzy feeling to check this explicitly.

Only the charged fermions (5.2) contribute to the anomalies. We can go through each anomaly in turn and check that it cancels. Some of these are straightforward. For example, for the $SU(3)^3$ anomaly, we require

$$\sum_{\rm left-handed} A(R) = \sum_{\rm right-handed} A(R) \ . \tag{5.6}$$

All fermions are either singlets with $A(\mathbf{1}) = 0$ or sit in the fundamental representation with $A(\mathbf{3}) = 1$. Clearly there are two right-handed quarks u_R and d_R. There is only the single left-handed quark Q_L but, when computing the anomaly, we should sum over the $SU(2)$ gauge index. (From the perspective of the $SU(3)$ gauge field, the anomaly doesn't know if Q_L is two distinct fields, or a single field transforming as an $SU(2)$ doublet.) The upshot is that $\sum A(R) = 2$ for both left-handed and right-handed quarks.

As we mentioned in Section 4.1, there is no perturbative $SU(2)^3$ anomaly, only the more subtle Witten anomaly which means that we must have an even number of $SU(2)$ doublets. This is achieved because there are three in Q_L (when computing the $SU(2)$ anomaly, we should sum over $SU(3)$ indices) and a single doublet in L_L. Note that the Witten anomaly ties together the quarks and leptons: the theory doesn't make sense with just Q_L alone: we must also have L_L.

The remaining gauge anomalies involve the $U(1)$ factor and are even more intricate. The $U(1)^3$ anomaly requires matching between the sum of the cubes of the charges

$$\sum_{\rm left-handed} Y^3 = \sum_{\rm right-handed} Y^3 \ . \tag{5.7}$$

As above, in all of these calculations, we must remember to multiply by the dimension of the representation of the non-Abelian factors. We have

$$\begin{aligned} \sum_{\rm left-handed} Y^3 &= 6 \times \left(\frac{1}{6}\right)^3 + 2 \times \left(-\frac{1}{2}\right)^3 = -\frac{2}{9} \\ \sum_{\rm right-handed} Y^3 &= 3 \times \left(\frac{2}{3}\right)^3 + 3 \times \left(-\frac{1}{3}\right)^3 + (-1)^3 = -\frac{2}{9} \ . \end{aligned} \tag{5.8}$$

So that works.

We also have to check the mixed anomalies between two factors of the gauge group. The $SU(2)^2 \times U(1)$ anomaly requires that

$$\sum_{\text{left-handed}} Y = \sum_{\text{right-handed}} Y \tag{5.9}$$

where the sum is only over those fermions that sit in the **2** of $SU(2)$. This is satisfied by virtue of

$$SU(2)^2 \times U(1): \quad 3 \times \left(\frac{1}{6}\right) + \left(-\frac{1}{2}\right) = 0\ . \tag{5.10}$$

Meanwhile, the $SU(3)^2 \times U(1)$ anomaly requires that (5.9) holds when we sum over the quarks that sit in the **3** of $SU(3)$ which also holds, by virtue of

$$SU(3)^2 \times U(1): \quad 2 \times \left(\frac{1}{6}\right) = \frac{2}{3} - \frac{1}{3}\ . \tag{5.11}$$

Finally, we want to be able to couple our theory consistently to gravity. This requires that (5.9) holds when we sum over all fermions. We have

$$\begin{aligned} \sum_{\text{left-handed}} Y &= 6 \times \frac{1}{6} + 2 \times \left(-\frac{1}{2}\right) = 0 \\ \sum_{\text{right-handed}} Y &= 3 \times \frac{2}{3} + 3 \times \left(-\frac{1}{3}\right) - 1 = 0\ . \end{aligned} \tag{5.12}$$

The sum over left- and right-handed fermions vanish individually, which is stronger than is needed for anomaly cancellation. We see that, happily, our universe makes sense. This is cause for celebration.

This also explains a statement that we made in the introduction to these lectures: there is a remarkable unification in the Standard Model. It is not the usual kind of unification, where seemingly different phenomena are seen to have the same underlying cause. Instead, it is something more subtle: the quarks, electron and neutrino are unified by the need for mathematical consistency. If you remove one of them, then the delicate cancellations that we saw above fail. The whole collection of fermions (5.2) is needed for our theory to hold together.

There are variations on this calculation that we could play. For example, we could keep the matter content of (5.2), but allow the hypercharges Y to be arbitrary. We could then ask: what values of hypercharge are consistent? It turns out that there are two possibilities: one gives a non-chiral theory, the other is (up to rescaling) the world you inhabit. You will be offered the opportunity to do this, and a related calculation, on the examples sheet.

5.1.2 Yukawa Interactions

Because the Standard Model is a chiral gauge theory, it's not possible to write down gauge invariant mass terms for the fermions. That would need left- and right-handed fermions to transform the same way under the gauge symmetry which, as shown in (5.2), they do not. This is striking: it means that all the fermions in the Standard Model are naturally massless! Needless to say, that's not our everyday experience and something must happen along the way to change the situation.

What happens is that all fermions interact with the Higgs boson. We will tell the full story of how they get mass later, but for now we can look at the form of these interactions.

The Higgs field plays no role in the anomaly cancellation story above. But its quantum numbers $(\mathbf{2},\mathbf{1})_{1/2}$ under the gauge group restrict its couplings to the fermions. And, as we now show, the quantum numbers (5.5) are such that it can couple to *all* fermions through Yukawa couplings.

First, consider the quarks. We can form fermion bilinears which are Lorentz scalars and singlets under $SU(3)$ by contracting $\bar{Q}_L$ with either u_R or d_R. From (5.2), we see that $\bar{Q}_L u_R$ has gauge quantum numbers $(\bar{\mathbf{2}},\mathbf{1})_{+1/2}$ and $\bar{Q}_L d_r$ has $(\bar{\mathbf{2}},\mathbf{1})_{-1/2}$. We can then form a gauge invariant Yukawa term by contracting these with either H or $H^\dagger$.

At this point, we need to say a word about how the $SU(2)$ representations combine. Given two $SU(2)$ vectors x^a and z^a, with $a = 1,2$, each of which transform in the $\mathbf{2}$ of $SU(2)$, there are two ways to form singlets. We can either write $x^\dagger z = \bar{x}_a z^a$ which is what we would call a "meson" in the context of the strong force. Or we can write $xz = \epsilon_{ab} x^a z^b$, making use of the epsilon symbol. This is what we would call a "baryon" for the strong force. The group $SU(2)$ is special because you get to make singlets in two different ways out of just two vectors. More mathematically, this is the statement that the representation $\mathbf{2}$ is pseudoreal because given x^a in the $\mathbf{2}$, we can always form $\epsilon_{ab} x^b$ in the $\bar{\mathbf{2}}$.

For us, $\bar{Q}_L$ naturally sits in the $\bar{\mathbf{2}}$ so we can contract it with H which sits in the $\mathbf{2}$. But we need that epsilon symbol if we are to contract it with $H^\dagger$. To this end, it's common to define

$$\tilde{H}^a = \epsilon^{ab} H_b^\dagger \tag{5.13}$$

with $a, b = 1, 2$ the $SU(2)$ gauge indices. We can then construct gauge invariant Yukawa couplings with the quarks of the form

$$\mathcal{L}_{\text{Yuk}} = -y^d\, \bar{Q}_L H d_R - y^u\, \bar{Q}_L \tilde{H} u_R + \text{h.c.}\ . \tag{5.14}$$

Here y^d and y^u are Yukawa coupling constants. Both of these terms are neutral under hypercharge and, by construction, also singlets under $SU(2) \times SU(3)$.

We can also write down Yukawa interactions with the leptons. This time we have the bilinears $\bar{L}_L e_R$ with quantum numbers $(\bar{\mathbf{2}}, \mathbf{1})_{-1/2}$ and, if the right-handed neutrino exists, $\bar{L}_L \nu_R$ with quantum numbers $(\bar{\mathbf{2}}, \mathbf{1})_{+1/2}$. We can see that both of these also have gauge invariant Yukawa interactions with the Higgs

$$\mathcal{L}_{\text{Yuk}} = -y^e \, \bar{L}_L H e_R - y^\nu \, \bar{L}_L \tilde{H} \nu_R + \text{h.c.} \; . \tag{5.15}$$

Again, y^e and y^ν are Yukawa coupling constants and, as above, the neutrino Yukawa term with $H^\dagger$ should have the $SU(2)$ gauge indices contracted with an ϵ_{ab}.

If we have a right-handed neutrino ν_R, then there is one further term that we can add. This is a Majorana mass of the kind we introduced in (1.59). It's possible only for ν_R because this fermion isn't charged under the gauge group,

$$\mathcal{L}_{\text{Maj}} = M \, \nu_R \nu_R + \text{h.c.} \; . \tag{5.16}$$

We'll discuss this further in Section **??**.

5.1.3 Three Generations

For reasons that remain mysterious, the pattern of fermions presented in (5.2) is repeated twice over. Mathematically, it is straightforward to incorporate this: we just add a flavour index $i = 1, 2, 3$ to each of the fermions. We ascribe these additional fields names that we met in the introduction: strange and charm, and bottom and top for the quarks. We write these as

$$\begin{aligned} d_R^i &= \left\{ d_R \; , \; s_R \; , \; b_R \; \right\} \quad &:& \quad (\mathbf{1}, \mathbf{3})_{-1/3} \\ u_R^i &= \left\{ u_R \; , \; c_R \; , \; t_R \; \right\} \quad &:& \quad (\mathbf{1}, \mathbf{3})_{2/3} \end{aligned} \tag{5.17}$$

and, writing the $SU(2)$ doublets explicitly,

$$Q_L^i = \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix} \; , \; \begin{pmatrix} c_L \\ s_L \end{pmatrix} \; , \; \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\} \quad : \quad (\mathbf{2}, \mathbf{3})_{1/6} \tag{5.18}$$

As before, it's really premature to write this: the labelling only makes sense after we have taken into account the Higgs mechanism.

The names that we give to the leptons are the electron, muon, and tau. We write

$$e_R^i = \{e_R \ , \ \mu_R \ , \ \tau_R \} \qquad : \quad (\mathbf{1}, \mathbf{1})_{-1} \tag{5.19}$$

and

$$L_L^i = \left\{ \begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix} , \begin{pmatrix} \nu_L^\mu \\ \mu_L \end{pmatrix} , \begin{pmatrix} \nu_L^\tau \\ \tau_L \end{pmatrix} \right\} \qquad : \quad (\mathbf{2}, \mathbf{1})_{-1/2} \tag{5.20}$$

where, again, the labelling is premature and should be taken with a grain of salt before the Higgs mechanism does its thing.

Meanwhile, the Higgs itself is unaffected by this increase in generations: there is just a unique Higgs.

The fate of the right-handed neutrino ν_R is less certain. It seems tempting to also add an $i = 1, 2, 3$ index to this field too,

$$\nu_R^i = \{\nu_R^e \ , \ \nu_R^\mu \ , \ \nu_R^\tau \} \qquad : \quad (\mathbf{1}, \mathbf{1})_0. \tag{5.21}$$

Because each of these is sterile, meaning uncharged under the gauge group, they do not interact directly with any of the forces, nor contribute to anomaly cancellation. It is quite possible there are no right-handed neutrinos or, indeed, any number!

As far as the gauge interactions are concerned, each generation experiences the same forces as the others. In particular, anomaly cancellation happens within each individual generation. There is, as far as we can tell, no necessity to introduce three generations rather than, say, one or seventeen.

The place where the additional generations really add a level of complexity and richness is in the Yukawa couplings. In contrast to the gauge couplings, the Yukawa couplings involve a great deal of inter-generational mixing. The most general Yukawa interactions that we can write down replace each of the coupling constants y^u, y^d, y^e and y^ν with 3×3 matrices,

$$\mathcal{L}_{\text{Yuk}} = -y_{ij}^d \, \bar{Q}_L^i H d_R^j - y_{ij}^u \, \bar{Q}_L^i \tilde{H} u_R^j - y_{ij}^e \, \bar{L}_L^i H e_R^j - y_{ij}^\nu \, \bar{L}_L^i \tilde{H} \nu_R^j + \text{h.c.} \ . \tag{5.22}$$

We will devote Section 6 to understanding the structure of these Yukawa couplings.

5.1.4 The Lagrangian

Usually when introducing a quantum field theory, the first thing that we do is write down an action. But that's not the case here: instead, we've discussed the symmetry structure of the theory. The reason this is sensible is because the symmetries are entirely sufficient to determine the structure of the action.

The game that we play is to write down all possible marginal and relevant terms. These terms must be Lorentz invariant and gauge invariant, but otherwise you write down anything that you want. Despite the plethora of fields, there isn't too much freedom. The full Lagrangian takes the form

$$\mathcal{L}_{\rm SM} = \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm fermi} + \mathcal{L}_{\rm Higgs} + \mathcal{L}_{\rm Yuk} \ . \tag{5.23}$$

The first two of these are simply kinetic terms for our fields. We will need to give our gauge fields some names. Back in Section 3, we already dubbed the $SU(3)$ gluon field strength $G_{\mu\nu}$. We will call the $SU(2)$ gauge field strength $W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ig[W_\mu, W_\nu]$ and the $U(1)$ hypercharge field strength $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The gauge field kinetic terms are then

$$\mathcal{L}_{\rm gauge} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}{\rm Tr}\, W_{\mu\nu}W^{\mu\nu} - \frac{1}{2}{\rm Tr}\, G_{\mu\nu}G^{\mu\nu} \ . \tag{5.24}$$

The kinetic terms for the fermions are

$$\mathcal{L}_{\rm fermi} = -i\sum_{i=1}^{3}\Big(\bar{Q}^i_L\bar{\sigma}^\mu\mathcal{D}_\mu Q^i_L + \bar{L}^i_L\bar{\sigma}^\mu\mathcal{D}_\mu L^i_L + \bar{u}^i_R\sigma^\mu\mathcal{D}_\mu u^i_R \\ + \bar{d}^i_R\sigma^\mu\mathcal{D}_\mu d^i_R + \bar{e}^i_R\sigma^\mu\mathcal{D}_\mu e^i_R + \bar{\nu}^i_R\sigma^\mu\partial_\mu\nu^i_R\Big) \ . \tag{5.25}$$

The exact form of these kinetic terms depends on the representation of the fermion field. So, for example, Q_L is charged under each of the three gauge fields and has kinetic term

$$\mathcal{D}_\mu Q_L = \partial_\mu Q_L - ig_s G_\mu Q_L - igW_\mu Q_L - \frac{i}{6}g' B_\mu Q_L \ . \tag{5.26}$$

There are similar expressions for all other fields. Buried within these covariant derivatives are the coupling constants: g_s for the $SU(3)$ strong force, g for the $SU(2)$ weak force, and g' for the $U(1)$ hypercharge.

The Lagrangian for the Higgs term includes both its kinetic term and potential

$$\mathcal{L}_{\rm Higgs} = \mathcal{D}_\mu H^\dagger\, \mathcal{D}^\mu H - \lambda\left(H^\dagger H - \frac{v^2}{2}\right)^2 \ . \tag{5.27}$$

The potential is written to emphasise that the minimum will lie away from $H = 0$. We will explore the consequences of this shortly. The Higgs kinetic term also follows from its gauge quantum numbers,

$$\mathcal{D}_\mu H = \partial_\mu H - igW_\mu H - \frac{i}{2}g' B_\mu H \ . \tag{5.28}$$

Finally, the Yukawa terms are given in (5.22).

We can start to count the parameters in the Standard Model. There are three gauge couplings, g_s, g and g', one for each gauge group. And there are two parameters λ and v^2 in the Higgs potential. Then there are the plethora of Yukawa couplings that we will explore further (and count!) in Section 6.

I've omitted two possible terms from the Lagrangian (5.23). One is the theta term for the strong force that we met in Section 3.4. This is omitted on the grounds that, experimentally, $\theta \approx 0$. Still, if we're accounting for parameters of the Standard Model then we should certainly include this one. The second term that I've omitted is the Majorana masses for the right-handed neutrinos, on the slightly weaker grounds that we don't know if they're there or not. We'll discuss this more in Section **??**.

There's a lot of repetition in the Standard Model Lagrangian as written. I think that you could be forgiven for advertising it in the more compact form

$$\mathcal{L} = -\frac{1}{4}\sum_a F^a_{\mu\nu}F^{a\,\mu\nu} + i\sum_i \bar{\psi}_i\bar{\sigma}^\mu\mathcal{D}_\mu\psi_i + |\mathcal{D}H|^2 - V(H) - y\psi H\psi + \text{h.c.}\ . \tag{5.29}$$

Admittedly, there's a lot of heavy lifting going on in that $\sum_a$ and $\sum_i$. Still, it's remarkable that everything we know about the universe can be distilled in such a way.

You can sometimes find the Standard Model Lagrangian written out in full component form, in which case it looks something like what's shown in Figure 16. This is usually done by someone trying to convince you that the theory is inelegant (typically because they have their own wares to sell). This always strikes me as being deliberately obtuse, like writing out haiku in binary in an attempt to argue that its beauty is over-rated. The beauty of the Standard Model isn't in the form of the Lagrangian: it's in the consistency conditions inherent in anomaly cancellation that we have taken pains to explain in these lectures.

5.1.5 Global Symmetries

We've built the Standard Model around the gauge group $G = U(1) \times SU(2) \times SU(3)$. But it's natural to ask: what are the global symmetries of the Standard Model?

In the absence of Yukawa terms, this is an easy question to answer: the classical theory has a $U(3)^5$ global symmetry if there are no right-handed neutrinos, and a $U(3)^6$ global symmetry if there are right-handed neutrinos. Here the 3 corresponds to the three generations, and we get a global symmetry group acting on each of Q_L, L_L, u_R, d_R, e_R and (possibly) ν_R.

Figure 16. If you want to write the Standard Model Lagrangian like this, then you should probably write the Einstein-Hilbert action by expanding out $\mathcal{L} = \sqrt{-g}R$ in terms of the metric $g_{\mu\nu}$.

But the Yukawa terms (5.22) break this symmetry. As we will see later, the values of the Yukawas are different for different generations, ultimately resulting in their different masses. There are some approximate symmetries remaining, like isospin or the eightfold way, but when the dust settles the classical theory has just two exact global symmetries. This is $U(1)_B \times U(1)_L$, corresponding to *baryon number* and *lepton number* respectively. The charges of the various fields under these two $U(1)'s$ are

$$\begin{array}{c|cccccc} & Q_L & L_L & u_R & d_R & e_R & \nu_R \\ \hline U(1)_B & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ U(1)_L & 0 & 1 & 0 & 0 & 1 & 1 \end{array} \tag{5.30}$$

You can see that $U(1)_B$ acts only on quarks and $U(1)_L$ acts only on leptons. (In fact, $U(1)_B$ is essentially the same as the vector symmetry $U(1)_V$ that we saw when discussing QCD in Section 3.) The normalisation of $\frac{1}{3}$ for the charge of the quarks is just convention: it guarantees that the proton and neutron each have baryon number

+1. These symmetries $U(1)_B$ and $U(1)_L$ act the same on each generation. (The Yukawa interactions include couplings between generations which means that there's no global symmetry which acts on one generation, leaving the others untouched.)

Note that we didn't impose either of these global symmetries $U(1)_L$ and $U(1)_B$ from the outset. Instead, we just wrote down all possible terms consistent with the gauge symmetry and discovered that the end result has $U(1)_L \times U(1)_B$ as a global symmetry. In this sense, we view these symmetries as *accidental*. There is no particular reason to think that they survive to arbitrarily high energies (and, indeed, some reasonably good reasons that we shall explain shortly to think that they do not survive). This means, in particular, that if we were to add irrelevant terms to the Standard Model in an attempt to capture the high energy physics then we should include such terms that break $U(1)_B$ and $U(1)_L$.

ABJ Anomalies Revisited

As we saw in Section 4.2, just because a $U(1)$ symmetry is a good symmetry of the classical theory, doesn't mean that it is necessarily a symmetry of the quantum theory. This is because it may suffer from an ABJ anomaly. And, indeed, both $U(1)_B$ and $U(1)_L$ suffer ABJ anomalies. There is an ABJ anomaly with $SU(2)$ gauge group (because only left-handed fermions carry $SU(2)$ charge), and also with $U(1)$ hypercharge. For the latter, the anomaly for a single generation is given by

$$\sum_{\text{left}} BY^2 - \sum_{\text{right}} BY^2 = \frac{1}{3}\left(6 \times \left(\frac{1}{6}\right)^2 - 3 \times \left(\frac{2}{3}\right)^2 - 3 \times \left(-\frac{1}{3}\right)^2\right) = -\frac{1}{2} \quad (5.31)$$

and

$$\sum_{\text{left}} LY^2 - \sum_{\text{right}} LY^2 = \left(2 \times \left(-\frac{1}{2}\right)^2 - (-1)^2\right) = -\frac{1}{2} \; . \quad (5.32)$$

So neither $U(1)_B$ nor $U(1)_L$ are good symmetries of the quantum theory. However, in contrast to the ABJ anomaly of the axial symmetry of the strong force, these ABJ anomalies are associated to the gauge fields of the weak force. And the weak force is, as we shall see, weak. The upshot is that although neither $U(1)_B$ nor $U(1)_L$ are strictly symmetries of the Standard Model, they are both *extremely* good approximate symmetries. Indeed, neither has been observed to be violated!

We can quantify this. If we focus just on the $SU(2)$ anomaly, then the conservation of baryon number picks up a term analogous to (4.62),

$$\partial_\mu J^\mu_B = \frac{12g^2}{8\pi^2} \text{Tr}\, W^\star_{\mu\nu} W^{\mu\nu} \; . \quad (5.33)$$

where the factor of 12 arises because there are four $SU(2)$ doublets in each of the three generations, and $3 \times 4 = 12$. There is a similar contribution from $B_{\mu\nu}$.

The kind of process that can violate baryon number is an electroweak instanton. There is a story of fermion zero modes that we will not tell but the end result is that electroweak instantons cannot, for example, allow a proton to decay into a positron: the proton is absolutely stable in the Standard Model. Instead, these instantons can allow a collection of three baryons to decay, where the "three" arises because it's the number of generations. This means, for example, that a ^{3}He nucleus could decay. But the decay is due to instantons and these come with a characteristic suppression factor of $e^{-8\pi^2/g^2}$, as in (3.120). For electroweak instantons, this turns out to give a lifetime of around 10^{173} years! (The age of our universe is roughly 10^{10} years.) That's why baryons seem stable.

All of which means that, for all practical purposes, both baryon number and lepton number are good symmetries. But, if you're a purist (and willing to wait 10^{173} years) then you should accept that neither are good symmetries.

Importantly, however, the ABJ anomalies for both $U(1)_B$ and $U(1)_L$ are the same. This is true both for the mixed anomaly with $U(1)_Y$ shown in (5.31) and (5.32) and also for the mixed anomaly with $SU(2)$. This means that the combination $B - L$ is non-anomalous. This is the one exact global symmetry of the Standard Model.

We still have to check if there is a gravitational contribution to the $B - L$ anomaly. You can check that this vanishes only if there is a right-handed neutrino.

The Weak Theta Term

For the strong force, we can write down a theta term. As we discussed in Section 3.4, this leads to a mystery because, experimentally, $\theta \approx 0$ and we don't know why. This is the strong CP problem.

What about the theta term for the other two gauge groups, $U(1)$ and $SU(2)$?

For Abelian gauge theories, we can write down a theta term but it doesn't affect the local dynamics, such as masses or cross-sections or decay rates. (This is essentially because there are no $U(1)$ instantons.) Instead, the effects are much more subtle. For example, this term would endow magnetic monopoles with electric charge through the Witten effect. We don't have any experimental insight into these features of the theory and so the $U(1)$ theta term remains unknown to us.

That leaves the $SU(2)$ theta term which takes the form

$$S_\theta = \frac{g^2\theta_W}{16\pi^2}\int d^4x \,\,\text{Tr}\, W_{\mu\nu}{}^\star W^{\mu\nu} \,\,. \tag{5.34}$$

Is this another term that we should add to the Standard Model action? The answer is no. And the reason is because of the global $U(1)_L$ (or, equivalently $U(1)_B$) ABJ anomaly. As shown in (4.41), if we act with a $U(1)_L$ transformation of $e^{i\alpha L}$, where L is the charge of each fermion, then the anomaly can be re-interpreted as shifting the theta term

$$U(1)_B : \theta_W \rightarrow \theta_W + 3\alpha \tag{5.35}$$

where the factor of 3 comes from the existence of three generations. This means that the value of θ_W is unphysical and does not affect the physics. Said differently, we can always use the anomalous $U(1)_L$ symmetry to set $\theta_W = 0$. There is no weak CP problem. In contrast, this mechanism doesn't work for the strong force.

Black Holes

We have seen that the Standard Model has just a single $U(1)$ global symmetry, namely $B-L$. But the standard lore is that there are no global symmetries in the fundamental laws of physics. The main argument for this is black holes.

Black holes aren't black. Hawking taught us long ago that they slowly emit radiation due to quantum effects. While there is much that we don't understand about quantum gravity, the existence of Hawking radiation stands out as one of the few robust and trustworthy calculations that we can do. The prediction of this radiation follows from the known laws of physics and doesn't rely on any speculative ideas about what lies beyond.

If we wait long enough (and we're talking ridiculously long times here), then any black hole will eventually evaporate and disappear. So we can ask: what became of the stuff that we threw in?

First, the black hole can't destroy electric charge. If you throw, say, an electron into a black hole then the black hole itself now carries the electric charge. Moreover, this is visible outside of the event horizon because the black hole emits an electric field and we can detect the electric field by Gauss' law. (This is the Reissner-Nordström solution that we described in the lectures on General Relativity.) That electric field can't just disappear. So, as the black hole evaporates, it must eventually spit out a charged particle – maybe an electron, maybe an anti-proton – which carries the electric charge. The process of black hole evaporation must respect conservation of electric charge.

In contrast, there is nothing to prevent black holes from destroying baryons and leptons. When a black hole forms from the collapse of a star, it will typically contain around 10^{57} protons, and roughly the same number of electrons. But there's no way to detect the baryons from outside the black hole. Furthermore, as the black hole evaporates there's no reason that it should spit back these particles in tact. In fact, the vast majority of the mass of a black hole will be emitted in gravitational and electromagnetic radiation rather than baryons or leptons. In this way, we expect black hole evaporation to respect neither baryon number nor lepton number conservation.

This means that, in a full theory of quantum gravity, one doesn't expect any global conservation laws, since one can always construct states in the theory in which the symmetry is violated. What does this mean for our parochial Standard Model? The usual answer is that we shouldn't view $B - L$ as something sacrosanct, but rather just a symmetry that emerges in the infra-red simply because there are no relevant or marginal operators that we can write down that violate it. When we get to high energy scales – and certainly by the time we get to the Planck scale – we expect it to be violated.

5.1.6 What is the Gauge Group of the Standard Model?

The title of this section seems a little daft. After all, we've been running through these lectures safe in the knowledge that the gauge group of the Standard Model is

$$G = U(1) \times SU(2) \times SU(3) \ . \tag{5.36}$$

Or is it?! In fact, there's a subtlety here.

To see this subtlety, consider the action on all fermions by the centre $(-1) \in SU(2)$ and $e^{2\pi i/3} \in SU(3)$. A quick check will confirm that

$$Q_L \to \omega^{-1} Q_L \ , \quad L_L \to \omega^3 L_L \ , \quad u_R \to \omega^2 u_R \ , \quad d_R \to \omega^2 d_R \ , \quad e_R \to e_R \tag{5.37}$$

with $\omega = e^{2\pi i/6}$. If we simultaneously act with the $U(1)$ hypercharge transformation $e^{2\pi i Y}$, then the result is that every fermion is either left unchanged, or picks up a minus sign. But a minus sign on a fermion is just part of the Lorentz group. The upshot is that there is a $\mathbb{Z}_6$ subgroup of G that does not act on the fermions (or, indeed, on the Higgs).

This means that it's tempting to say that the gauge group of the Standard Model is

$$G = \frac{U(1) \times SU(2) \times SU(3)}{\Gamma} \tag{5.38}$$

where $\Gamma = \mathbb{Z}_6$. But this too is overly hasty! The honest answer is that we don't know what the gauge group of the Standard Model is. There are four different choices, given by (5.38) where Γ is a subgroup of $\mathbb{Z}_6$, meaning $\Gamma = \mathbb{Z}_6$, $\mathbb{Z}_3$, $\mathbb{Z}_2$ or nothing at all. Strictly, these are all different quantum field theories, although the differences between them are rather subtle and don't show up in correlation functions of local operators. This means, among other things, that the differences between them won't show up in particle colliders like the LHC. Instead, one has to look to more formal aspects of the theories to see the difference, like the spectrum of allowed magnetic monopoles or what happens when the theory is placed on a manifold with non-trivial topology[9] .

5.2 Electroweak Symmetry Breaking

We now have the full Standard Model laid out before us in (5.23). The next question is: how does it give rise to the physics that we know and love? The answer largely lies in the role that the Higgs plays.

The dynamics of the Higgs boson is governed by the action (5.27)

$$\mathcal{L}_{\text{Higgs}} = \mathcal{D}_\mu H^\dagger \, \mathcal{D}^\mu H - \lambda \left(H^\dagger H - \frac{v^2}{2} \right)^2 \ . \tag{5.39}$$

The potential is such that it causes the Higgs to condense. This breaks the $U(1) \times SU(2)$ gauge symmetry under which the Higgs is charged, giving masses to the gauge bosons in the way we saw in Section 2.3. And, through the Yukawa interactions (5.22), it also gives masses to the fermions. In this section, we describe these effects.

Including the Maxwell and Yang-Mills terms for the $U(1) \times SU(2)$ gauge fields, we have the Lagrangian

$$\mathcal{L} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2} \text{Tr}\, W_{\mu\nu} W^{\mu\nu} + \mathcal{L}_{\text{Higgs}} \ . \tag{5.40}$$

To understand the physics, we need the Higgs covariant derivative which is given by

$$\mathcal{D}_\mu H = \partial_\mu H - i g W_\mu H - \frac{i}{2} g' B_\mu H \ . \tag{5.41}$$

This reflects the charges (5.5).

[9]For more details on these ideas, see Ofer Aharony, Nati Seiberg, and Yuji Tachikawa's Reading Between the Lines paper. Applications of these ideas to the Standard Model were given in Line Operators in the Standard Model.

In the ground state of the potential (5.27), we have $H^\dagger H = v^2/2$. As usual, we have to pick a direction for the Higgs vacuum expectation value to point in. We choose

$$\langle H\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} . \tag{5.42}$$

Then we parameterise the fluctuations of the Higgs as

$$H = e^{i\xi^A(x)T^A}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+h(x)\end{pmatrix} . \tag{5.43}$$

Here $h(x)$ is a real scalar field, $T^A = \frac{1}{2}\sigma^A$ with $A = 1, 2, 3$ are the generators of $SU(2)$ and $\xi^A(x)$ are the would-be Goldstone bosons. As usual, they are eaten by the gauge bosons as part of the Higgs mechanism. A quick way to say this is to observe that we can just eliminate the factor of $e^{i\xi^A T^A}$ in (5.43) through a gauge transformation. Alternatively, to make contact with the what we saw in Section 2.3, we can look at the covariant derivative. If we write $\Omega(x) = e^{i\xi^A(x)T^A}$, then we have

$$\mathcal{D}_\mu H = \frac{1}{\sqrt{2}}\Omega\left(\begin{pmatrix}0\\ \partial_\mu h\end{pmatrix} - i\left[g\left(\Omega^{-1}W_\mu\Omega + \frac{i}{g}\Omega^{-1}\partial_\mu\Omega\right) + \frac{g'}{2}B_\mu\right]\begin{pmatrix}0\\ v+h\end{pmatrix}\right) \tag{5.44}$$

Here we see that the overall field Ω sits in a way that can be eliminated by a gauge transformation (1.82).

We can always choose to work in unitary gauge in which, through a judicious $SU(2)$ rotation, we simply take $\xi^A(x) = 0$ or, equivalently, $\Omega = \mathbb{1}$. In this case, the Lagrangian (5.40) becomes

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}\mathrm{Tr}\, W_{\mu\nu}W^{\mu\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h - \lambda h^2\left(v + \frac{h}{2}\right)^2 \\ - \frac{1}{8}(v+h)^2\left(g^2(W^1_\mu)^2 + g^2(W^2_\mu)^2 + (gW^3_\mu - g'B_\mu)^2\right) .\end{aligned} \tag{5.45}$$

To get the second line, we expanded out $SU(2)$ gauge boson fields W_μ in terms of the generators $T^A = \frac{1}{2}\sigma^A$, and contracted them with the Higgs field. From this we can read off the masses from the quadratic term. There is a $\lambda v^2 h^2$ term that gives a mass for h. This is the particle that, experimentally, we call *the* Higgs boson. It's mass is measured to be

$$M_h = \sqrt{2\lambda}v \approx 125 \text{ GeV} . \tag{5.46}$$

We see that this mass is a combination of the Higgs vev v and the dimensionless coupling λ.

We can also read off the masses of the gauge bosons from the second line in (5.45). Both W^1_μ and W^2_μ have the same mass $m_W = vg/2$. It will prove fruitful to combine them into the complex combination

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^2_\mu) \; . \tag{5.47}$$

Note the flip of the $\pm$ sign on the right-hand side. We will see shortly that this ensures that $W^\pm$ has electric charge ± 1. The experimentally measured mass of these spin 1 bosons is

$$M_W = \frac{gv}{2} \approx 80 \text{ GeV} \; . \tag{5.48}$$

This mass is set by the Higgs vev v and the $SU(2)$ gauge coupling g.

The final massive gauge boson is slightly more interesting. We see from (5.45) that it is a linear combination of the W^3_μ which is part of $SU(2)$ and B_μ which is associated to the fundamental $U(1)$ hypercharge gauge symmetry. The relevant linear combination is set by the two coupling constants, g and g'. To this end, we define the *Weinberg angle*, also known as the *weak mixing angle*

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}} \quad \Longleftrightarrow \quad \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \; . \tag{5.49}$$

We then define the two linear combinations of gauge fields

$$\begin{aligned} Z_\mu &= \cos\theta_W \, W^3_\mu - \sin\theta_W \, B_\mu \\ A_\mu &= \sin\theta_W \, W^3_\mu + \cos\theta_W \, B_\mu \; . \end{aligned} \tag{5.50}$$

The first of these has a mass from (5.45) which is experimentally measured to be

$$M_Z = \frac{v}{2}\sqrt{g^2+g'^2} \approx 91 \text{ GeV} \; . \tag{5.51}$$

We don't have any way to determine any of these masses from first principles. They are combinations of the Higgs vev v, the Higgs coupling λ and the gauge couplings g and g', none of which we know without going out and measuring them. However, the theoretical framework does ensure the mild inequality

$$M_W = M_Z \cos\theta_W < m_Z \tag{5.52}$$

which is indeed observed.

We can do some simple counting here. Our original Higgs boson H was a doublet of $SU(2)$. This means that it has two complex degrees of freedom or, equivalently, four real degrees of freedom. One of these remains as the real scalar h that we call the Higgs boson. The other three got eaten by the three gauge bosons W^1_μ, W^2_μ and Z_μ.

The discovery of the Higgs boson h was announced at CERN in 2013. But in a very real sense, 3/4 of the more fundamental Higgs boson H were discovered when the massive W and Z bosons were first seen in 1983. As we've seen, they get their mass only by eating three of the components of H.

The scales of the masses of the Higgs h and the W and Z bosons are all set by the Higgs expectation value v, multiplied by some dimensionless coupling constant. This is a theme that will continue shortly when we discuss matter particles. These couplings can all be measured directly, through cross-sections or decay rates. We learn that the only dimensionful parameter in the classical Standard Model Lagrangian takes the value

$$v \approx 250 \text{ GeV} . \tag{5.53}$$

We will later see that this is directly related to the *Fermi constant* that governs the strength of weak decays. The dimensionless parameters are

$$\lambda \approx 0.35 \quad \text{and} \quad g \approx 0.64 \quad \text{and} \quad g' \approx 0.34 . \tag{5.54}$$

Each of these runs under RG; the values above are given at the scale $\mu = M_Z$. We also have the Weinberg angle (5.49) which takes the value

$$\cos\theta_W \approx 0.88 \quad \Longrightarrow \quad \theta_W \approx 29^\circ . \tag{5.55}$$

It's common to quote the value $\sin^2\theta_W \approx 0.223$.

5.2.1 Electromagnetism

There is one of the $U(1) \times SU(2)$ gauge bosons that escapes the clutches of the Higgs and remains massless. This is the field A_μ defined in (5.50) and it is the most famous gauge boson of all: the photon.

We can look at this more closely. From a group theoretic perspective, the photon remains massless because the Higgs induces the symmetry breaking

$$U(1)_Y \times SU(2) \to U(1)_{\text{EM}} . \tag{5.56}$$

This is why the $U(1) \times SU(2)$ sector of the Standard Model is referred to as *electroweak theory*.

We can identify this unbroken $U(1)$ symmetry by looking at how the Higgs vev (5.42) transforms under a general $U(1) \times SU(2)$ transformation, with parameters α^A and β,

$$\langle H \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \longrightarrow e^{gi\alpha^A T^A} e^{ig'\beta Y} \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{5.57}$$

The Higgs has hypercharge $Y = \frac{1}{2}$ so, writing the $SU(2)$ generators $T^A = \frac{1}{2}\sigma^A$, we have

$$g\alpha^A T^A + g'\beta Y = \frac{g}{2} \begin{pmatrix} \alpha^3 + g'\beta/g & \alpha^1 - i\alpha^2 \\ \alpha^1 + i\alpha^2 & -\alpha^3 + g'\beta/g \end{pmatrix} . \tag{5.58}$$

We see that the choice of parameters that leaves $\langle H \rangle$ invariant is $\alpha^1 = \alpha^2 = 0$ and $g\alpha^3 = g'\beta$. This means that the unbroken generator is the combination

$$Q = T^3 + Y . \tag{5.59}$$

We identify this with the generator of the unbroken $U(1)_{\rm EM}$ subgroup which, in more everyday terms, means that Q determines the electric charge of the fields. We'll see how this works in practice for all the fermion fields below.

The electroweak theory also sets the electromagnetic coupling constant e. This is simplest to see if we look at the general covariant derivative for a field that transforms in the fundamental of $SU(2)$ and with hypercharge Y,

$$\mathcal{D}_\mu = \partial_\mu - igW_\mu^A T^A - ig'YB_\mu . \tag{5.60}$$

We work with the fields $W_\mu^\pm$ defined in (5.47) and the corresponding generators $T^\pm = (T^1 \pm iT^2)/\sqrt{2}$. We also work with the fields Z_μ and A_μ defined in (5.50) to get

$$\mathcal{D}_\mu = \partial_\mu - ig(W_\mu^+ T^+ + W_\mu^- T^-) - i(g\cos\theta_W T^3 - g'\sin\theta_W Y)Z_\mu - ieQA_\mu . \tag{5.61}$$

For our immediate interests, it's that last term that's important. It involves the charge Q, together with the coupling

$$e = g\sin\theta_W = g'\cos\theta_W . \tag{5.62}$$

The electromagnetic coupling takes value

$$e \approx 0.30 . \tag{5.63}$$

This particular coupling constant is better known in the form $\alpha = e^2/4\pi$ which is called the *fine structure constant* and takes the famous value $\alpha \approx 1/137$.

The bosons of the electroweak sector are the Higgs, and the W and Z bosons. The Higgs h is electrically neutral. This must be the case simply because it's a real scalar field, but we can check explicitly by noting that it sits in the lower component of the doublet (5.43) which has $T^3 = \frac{1}{2}\sigma^3$ eigenvalue $-\frac{1}{2}$. The Higgs also has hypercharge $Y = +\frac{1}{2}$ ensuring that $Q = T^3 + Y = 0$.

The Z boson is similarly neutral. Again, this must be the case because it is a real field. Operationally, this follows because it carries no hypercharge and commutes with the $SU(2)$ generator T^3.

That leaves us with the W bosons. Under an $SU(2)$ transformation with $\alpha^1 = \alpha^2 = 0$ and α^3 constant, we have, from (1.87),

$$\delta W_\mu = -ig[W_\mu, \alpha^3 T^3] = g\alpha^3(-W_\mu^1 T^2 + W_\mu^2 T^1) \tag{5.64}$$

We can write this as $\delta W_\mu^1 = g\alpha^3 W_\mu^2$ and $\delta W_\mu^2 = -g\alpha^3 W_\mu^1$. We think of this $SU(2)$ transformation as part of the $U(1)_{\rm EM}$ transformation, with $g\alpha^3 = e\alpha$. Then, written in terms of our fields $W_\mu^\pm$ defined in (5.47), we have

$$\delta W_\mu^\pm = \pm ie\alpha W_\mu^\pm \ . \tag{5.65}$$

This is telling us that the W boson $W_\mu^\pm$ has electric charge $Q = \pm 1$.

5.2.2 Running of the Weak Coupling

The gauge couplings of the electroweak sector run with energy scale. Because hypercharge is a $U(1)$ gauge theory, the associated coupling g' gets smaller as we flow to the infra-red.

But for the non-Abelian $SU(2)$ gauge symmetry, we have to be more careful. We gave the general formula for $SU(N_c)$ gauge theory coupled to N_f massless Dirac fermions in (3.11) when discussing QCD. Now we need the generalisation to include N_s scalars in the fundamental representation. The result is

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} - \frac{b_0}{(4\pi)^2}\log\frac{\Lambda_{UV}^2}{\mu^2} \tag{5.66}$$

with the coefficient given by

$$b_0 = \frac{11}{3}N_c - \frac{2}{3}N_f - \frac{1}{3}N_s \ . \tag{5.67}$$

Applied to electroweak theory, we clearly have $N_c = 2$ and $N_s = 1$, corresponding to the Higgs doublet. But what about N_f? We saw in (5.2) that each generation of fermions has an $SU(2)$ doublet of quarks Q_L and a doublet of leptons L_L. This is $3 + 1 = 4$ Weyl fermions. But the N_f in (5.67) counts *Dirac* fermions, so each generation has $N_f = 2$ Dirac fermions as far as the beta function is concerned. And, of course, we have three generations. So the coefficient of the one-loop beta function for the weak force is $b_0 = b_{\text{weak}}$ with

$$b_{\text{weak}} = \frac{11}{3} \times 2 - \frac{2}{3} \times 6 - \frac{1}{3} = 3 \; . \tag{5.68}$$

With $b_{\text{weak}} > 0$, we see that the $SU(2)$ sector of the Standard Model is, like QCD, asymptotically free. It flows to strong coupling in the infra-red.

This begs the question: do we have to worry about strong coupling effects in the weak sector, like we did for QCD? The answer is no. And the reason is that the Higgs mechanism gives masses to the gauge bosons and, in doing so, freezes the running of the coupling g at the scale $\mu \sim M_W$. This is where the quoted value of $g \approx 0.64$ in (5.54) is measured.

It's worth commenting that, although we call the weak nuclear force "weak", the actual value of the coupling is not small. Indeed, $\alpha_W = g^2/4\pi \approx 1/30$, which is almost 5 times bigger than the fine structure constant! The reason that the weak force is actually weak has nothing to do with the strength of the coupling and everything to do with the mass of the W and Z bosons (or, equivalently, the scale of the Higgs vev). As we will see in Section 5.3, particles that decay through the weak force do so by the emission of an intermediate W or Z boson. The large mass of these bosons translates to a small decay rate.

It's also fruitful to compare the couplings for the weak and strong force. Measured at the weak scale M_Z, we have

$$\alpha_s(M_Z) \approx 0.12 \quad \text{and} \quad \alpha_w(M_Z) \approx 0.034 \; . \tag{5.69}$$

So the weak force is indeed weaker than the strong force.

Asymptotic freedom ensures that both g_s and g_w get smaller as we look at higher energies. But they do so at different speeds. The running of the strong coupling (assuming six massless generations) is dictated by

$$b_{\text{strong}} = \frac{11}{3} \times 3 - \frac{2}{3} \times 6 = 7 \; . \tag{5.70}$$

Because we have $b_{\text{strong}} > b_{\text{weak}}$, the two couplings will converge as we go to higher energies. And it's natural to ask: where does this convergence take place?

You have to be a little bold to do this calculation. We will take $\Lambda_{UV} = M_W$ in (5.66) and then extrapolate the equation to energy scales $\mu \gg M_W$ and, moreover, to energy scales beyond those that we've probed experimentally. There's nothing wrong with this per se, since the equation is invertible: if you know the coupling at one scale, then we can always determine it at any other scale, whether lower or higher. But we are assuming that there's no additional matter to discover which would change the coefficient b_0 as we go to higher energies. That seems like a rather big assumption.

With these health warnings in place, the two couplings meet at a scale μ given by

$$\frac{1}{g_s^2} - \frac{b_{\text{strong}}}{(4\pi)^2}\log\frac{M_W^2}{\mu^2} = \frac{1}{g_w^2} - \frac{b_{\text{weak}}}{(4\pi)^2}\log\frac{M_W^2}{\mu^2}\ . \tag{5.71}$$

Solving, we find

$$\mu = M_W \exp\left(\frac{2\pi}{b_{\text{strong}} - b_{\text{weak}}}\left(\frac{1}{\alpha_w} - \frac{1}{\alpha_s}\right)\right) \approx 2\times 10^{16}\,\text{GeV}\ . \tag{5.72}$$

So the two couplings do indeed meet, although it takes them a long time because the running is only logarithmic.

Nonetheless, the couplings meet in an intriguing place. The Planck scale sits sits at about $M_{\text{pl}} \sim 10^{19}$ GeV (or a bit less depending on where you put factors of 8π.) Had the two couplings converged at a scale $\mu \gg M_{\text{pl}}$ then we could have simply discarded this computation. We did it assuming that there was nothing new to find as we went to higher energies but as soon as quantum gravity effects kick in there's certainly no reason to trust the formula (5.66). The fact that the two lines meet at a scale just below M_{pl} is, if nothing else, telling us that we don't have an immediate reason to discard it. It also suggests that perhaps something more interesting is going on.

That something is the idea of *unification*. Is it perhaps possible that the two coupling constants are meeting because the $SU(2)$ and $SU(3)$ forces sit within a larger gauge group? The answer is: we don't know. But it is a compelling idea. Proposals for this larger gauge group include $SU(5)$ and $SO(10)$ (strictly $Spin(10)$).

There is, of course, a third coupling constant in the Standard Model. This is the hypercharge coupling g'. This is the smallest of the three couplings and it too runs, now getting bigger as we go to higher energies. This means that it must also meet the other two. But where? A similar calculation shows that $\alpha_Y = g'^2/4\pi$ meets the strong and weak couplings at

$$\begin{aligned} \alpha_Y = \alpha_s \quad &\text{at} \quad \mu \approx 5\times 10^{19}\ \text{GeV} \\ \alpha_Y = \alpha_w \quad &\text{at} \quad \mu \approx 10^{21}\ \text{GeV}\ . \end{aligned} \tag{5.73}$$

We see that the three lines don't meet. Things aren't as clean as that. Moreover, the unification of the hypercharge coupling seems to be in the regime where quantum gravity comes into play. Nonetheless, it's still in the same ballpark. So, while not perfect, this also lends credence to the idea of unification. Needless to say, we don't know if unification does indeed take place. But if we're searching the Standard Model for clues for what lies beyond, this is certainly one of the most striking.

5.2.3 A First Look at Fermion Masses

The Higgs gives mass to the W and Z boson. But it also gives masses to all the fundamental fermions in the Standard Model. These arise through the Yukawa interactions.

First, a repeat of a comment that we made previously: it's not possible to write down straightforward mass terms for the fermions in the Standard Model. This is because it is a chiral theory, with left- and right-handed fermions transforming differently under the gauge group. This means that any mass term necessarily violates gauge symmetry. The Yukawa terms are the gauge invariant interaction terms and give a mass only once the Higgs field gets an expectation value.

To kick things off, let's ignore the fact that we have three generations of fermions and focus only on the first. This will allow us to see how the basic structure of particles arises. We will then see the complications that arise from having multiple generations in Section 6.

The Yukawa couplings for a single generation were given in (5.14) and (5.15),

$$\mathcal{L}_{\text{Yuk}} = -y^d\,\bar{Q}_L H d_R - y^u\,\bar{Q}_L \tilde{H} u_R - y^e\,\bar{L}_L H e_R - y^\nu\,\bar{L}_L \tilde{H} \nu_R + \text{h.c.}\ . \tag{5.74}$$

Here H is the Higgs doublet that transforms in the **2** of the $SU(2)$ gauge group, and $\tilde{H}$ is the conjugated Higgs doublet, contracted with an ϵ so that it too transforms in the **2**,

$$\tilde{H}^a = \epsilon^{ab} H_b^\dagger \quad \text{with} \quad a,b = 1,2\ . \tag{5.75}$$

Meanwhile, y^d, y^u, y^e and y^ν are dimensionless Yukawa couplings. We'll give their values in Section 6. (This is one place where we really should include all three generations to appreciate the values.) Recall, also, that we're not sure if there is a right-handed neutrino field ν_R, so we might have to dispense with the final term in (5.74).

Our immediate interest is to understand the implications of the Higgs vev (5.42)

$$\langle H\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \quad \Longrightarrow \quad \langle \tilde{H}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} \ . \tag{5.76}$$

This will distinguish the two components of the $SU(2)$ doublets Q_L and L_L, giving them different masses and, as we will see, different charges under the unbroken symmetry of electromagnetism. For this reason, it's useful to introduce different names for the two components of these doublets. We write

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \text{and} \quad L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \ . \tag{5.77}$$

(We already introduced these names in (5.18) and (5.20) although, as we noted at the time, it was premature before we discussed electroweak symmetry breaking.)

Now we can look at the Yukawa couplings (5.74), focussing only on the role of the vev v and ignoring the interactions with the fluctuations of the Higgs boson h. We have

$$\mathcal{L}_{\text{Yuk}} = -\frac{v}{\sqrt{2}}\Big(y^d\,\bar{d}_L d_R + y^u\,\bar{u}_L u_R + y^e\,\bar{e}_L e_R + y^\nu\,\bar{\nu}_L \nu_R\Big) \ . \tag{5.78}$$

We see that each of the fermions gets a mass, given by

$$m^X = \frac{1}{\sqrt{2}} y^X v \tag{5.79}$$

where $X = d, u, e, \nu$ labels the appropriate Yukawa coupling y^X. The scale of all these masses is, like all particles in the Standard Model, set by Higgs vev. If the Higgs did not condense, all fermions would be massless.

This is the source of the oft-repeated claim that the Higgs boson is responsible for all mass in the Standard Model. It is, as we stressed in Section 3, a lie. It is true that the Higgs vev v is the only dimensionful scale in the Standard Model Lagrangian and that all fundamental particles would be massless if it were to vanish. But there is another, more subtle, scale in the Standard Model itself which is Λ_{QCD}, the scale at which the strong force lives up to its name. And this scale would exist even in the absence of the Higgs vev and would continue to give a mass to the proton and neutron. Of course, that's not to say that the Higgs is unimportant: in this hypothetical world in which $v = 0$, electrons would be massless so physics, atoms, and life would be vastly different.

We can also determine the electric charges of each of the fermions using the formula (5.59)

$$Q = T^3 + Y \ . \tag{5.80}$$

We listed the hypercharges Y of all particles previously. They are

$$\begin{array}{c|ccccc} & Q_L & L_L & u_R & d_R & e_R \\ \hline Y & \frac{1}{6} & -\frac{1}{2} & \frac{2}{3} & -\frac{1}{3} & -1 \end{array} \tag{5.81}$$

Each of the right-handed fermions is uncharged under the $SU(2)$ gauge group and so we have simply $Q = Y$. Indeed, we recognise the hypercharge as the usually advertised electric charge of these particles.

For the $SU(2)$ doublets Q_L and L_L, we have a small calculation to do. The T^3 eigenvalues are $\pm\frac{1}{2}$, with $+$ for the upper component and $-$ for the lower component. This means that the electric charges $Q = T^3 + Y$ are:

$$\begin{aligned} u_L: \quad Q = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \quad &\text{and} \quad d_L: \quad Q = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3} \\ \nu_L: \quad Q = \frac{1}{2} - \frac{1}{2} = 0 \quad &\text{and} \quad e_L: \quad Q = -\frac{1}{2} - \frac{1}{2} = -1 \ . \end{aligned} \tag{5.82}$$

We see that the electric charges of the left-handed fermions coincide with those of the right-handed fermions in (5.81), as indeed they must so that the mass terms (5.78) are invariant under the surviving $U(1)_{\rm EM} \subset SU(2) \times U(1)_Y$.

The upshot of symmetry breaking is that we are left with four *Dirac* fermions. These are the up quark u with charge $+2/3$, the down quark d with charge $-1/3$, the electron e with charge -1, and the neutral neutrino ν. If the right-handed neutrino ν_R doesn't exist then the neutrino is a Weyl fermion and cannot get a mass through the simple mechanism described above. We will discuss the issue of neutrino masses further in Section **??**.

The collection of electric charges of fermions in the Standard Model look kind of random. And, viewed as a low-energy vector-like theory, they are! But, as we have seen, there is a deeper reason underlying this choice that only becomes apparent when you realise that the Standard Model is a chiral theory, subject to the stringent constraints of anomaly cancellation.

5.3 Weak Decays

Since the time of Newton, we've tended to think of forces as things that push and pull. That's an intuition that holds well for QED and the Coulomb force, and also for QCD which binds quarks together into hadrons. But it's not the best way to think about the weak force. Instead, the weak force is an instrument of decay.

One of the consequences of the weak force is that it rents asunder what the strong force so carefully put together. We saw in Section 3 that quarks are bound into baryons and mesons. In a world of just QCD, the baryon octet that contains, among other things, the proton and neutron would be stable. So too would the octet of pseudoscalar mesons that includes the pions and kaons. But in our world, only the proton is stable. (Admittedly, we can also have stable nuclei consisting of bound states of protons and neutrons.) Everything else decays through the weak force.

In this section, we will start to understand how these decay processes take place. We will start by better understanding what fermions the W and Z bosons couple to and constructing the relevant Feynman diagrams.

5.3.1 Electroweak Currents

To start, we understand how the various gauge bosons couple to the fermions. For now, we will again stick with just a single generation. (There is an interesting twist to the story when we introduce multiple generations that we describe in Section 6.)

The fermion kinetic terms are

$$\mathcal{L}_{\text{fermi}} = -i\Big(\bar{Q}_L\bar{\sigma}^\mu\mathcal{D}_\mu Q_L + \bar{L}_L\bar{\sigma}^\mu\mathcal{D}_\mu L_L + \bar{u}_R\sigma^\mu\mathcal{D}_\mu u_R + \bar{d}_R\sigma^\mu\mathcal{D}_\mu d_R + \bar{e}_R\sigma^\mu\mathcal{D}_\mu e_R\Big)\ . \tag{5.83}$$

We haven't included the right-handed neutrino ν_R because it is neutral under all gauge symmetries. We'll ignore the gluon fields for now, and just focus on the terms that involve interactions with the electroweak gauge bosons. These are

$$\begin{aligned}\mathcal{L}_{\text{kin}}\Big|_{\text{weak}} = &-\frac{g}{2}W^3_\mu\left(\bar{u}_L\bar{\sigma}^\mu u_L - \bar{d}_L\bar{\sigma}^\mu d_L + \bar{\nu}_L\bar{\sigma}^\mu\nu_L - \bar{e}_L\bar{\sigma}^\mu e_L\right)\\ &-\frac{g}{\sqrt{2}}W^+_\mu(\bar{u}_L\bar{\sigma}^\mu d_L + \bar{\nu}_L\bar{\sigma}^\mu e_L) + \frac{g}{\sqrt{2}}W^-_\mu(\bar{d}_L\bar{\sigma}^\mu u_L + \bar{e}_L\bar{\sigma}^\mu\nu_L)\\ &-g'B_\mu\Big(\frac{1}{6}\bar{u}_L\bar{\sigma}^\mu u_L + \frac{1}{6}\bar{d}_L\bar{\sigma}^\mu d_L - \frac{1}{2}\bar{\nu}_L\bar{\sigma}^\mu\nu_L - \frac{1}{2}\bar{e}_L\bar{\sigma}^\mu e_L\\ &\qquad\qquad -\frac{2}{3}\bar{u}_R\sigma^\mu u_R - \frac{1}{3}\bar{d}_R\sigma^\mu d_R - \bar{e}_R\sigma^\mu e_R\Big)\ .\end{aligned} \tag{5.84}$$

If we replace W^3_μ and B_μ with the Z boson and photon fields, as in (5.50), these terms can be written as

$$\mathcal{L}_{\text{kin}}\Big|_{\text{weak}} = -\frac{e}{\sqrt{2}\sin\theta_W}(W^+_\mu J^\mu_+ + W^-_\mu J^\mu_-) - \frac{e}{\sin\theta_W\,\cos\theta_W} Z^\mu J^Z_\mu - eA^\mu J^{\text{EM}}_\mu \ . \quad (5.85)$$

Here we've replaced the two coupling constants g and g' with the Weinberg angle $\tan\theta_W = g'/g$ and the electromagnetic coupling $e = g\sin\theta_W = g'\cos\theta_W$ and we've introduced various currents that interact with the gauge fields. The electromagnetic current that couples to the photon is given by

$$\begin{aligned} J^{\text{EM}}_\mu &= \frac{2}{3}(\bar{u}_L\bar{\sigma}_\mu u_L + \bar{u}_R\sigma^\mu u_R) - \frac{1}{3}(\bar{d}_L\bar{\sigma}_\mu d_L + \bar{d}_R\sigma^\mu d_R) - (\bar{e}_L\bar{\sigma}_\mu e_L + \bar{e}_R\sigma^\mu e_R) \\ &= \left(\frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \bar{e}\gamma_\mu e\right) \ . \end{aligned} \quad (5.86)$$

This takes the expected form, with each fermion multiplied by its electric charge. In the second line, we've written this in terms of Dirac spinors u, d, and e and the gamma matrices γ^μ to emphasise that, despite its chiral origins, this is the kind of vector-like current that we're used to in QED.

For the Z boson, we have a little more work to do. Some algebra reveals that the current takes the form

$$J^Z_\mu = \frac{1}{2}(\bar{u}_L\bar{\sigma}_\mu u_L - \bar{d}_L\bar{\sigma}_\mu d_L + \bar{\nu}_L\bar{\sigma}_\mu\nu_L - \bar{e}_L\bar{\sigma}_\mu e_L) - \sin^2\theta_W\, J^{\text{EM}}_\mu \ . \quad (5.87)$$

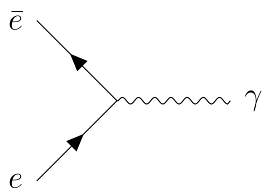
Finally, the currents for the W bosons can be read off immediately from (5.85); they are

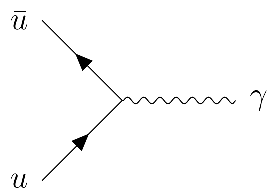
$$J^+_\mu = \bar{u}_L\bar{\sigma}_\mu d_L + \bar{\nu}_L\bar{\sigma}_\mu e_L \quad \text{and} \quad J^-_\mu = \bar{d}_L\bar{\sigma}_\mu u_L + \bar{e}_L\bar{\sigma}_\mu\nu_L \ . \quad (5.88)$$

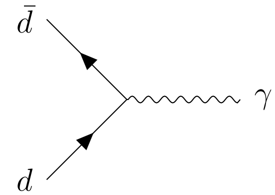
The currents for both the W and Z bosons are chiral, treating left-handed fermions differently from their right-handed counterparts.

5.3.2 Feynman Diagrams

From the interaction terms (5.85), we can read off the Feynman rules for the electroweak sector. We see from (5.86) that the photon couples in the usual way to the up and down quarks and to the electron, with coupling constant given by eq with q the charge. This gives rise to the kind of Feynman diagram that we met in our first course on Quantum Field Theory.

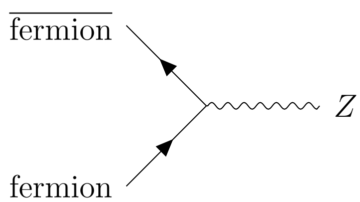






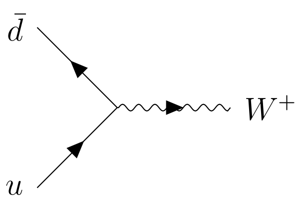
The photon couples to the up and down quarks and the electron. It doesn't couple to the neutrino because it's neutral.

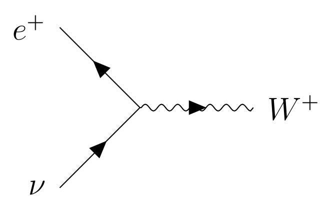
From (5.87), we see that there are similar diagrams involving the Z boson. But, in contrast to the photon, this couples to all low energy particles, including the neutrino. So we have diagrams of the form



where the fermion could be u, d, e or ν. This time, the coupling is more complicated: there is an overall factor of $e/\sin\theta_W\,\cos\theta_W$, with different coefficients depending on the fermion species. And more care is needed with the spinor indices because of the chiral nature of the coupling.

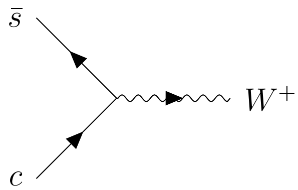
Finally, the W boson relates two different fermions. We have the Feynman diagrams:

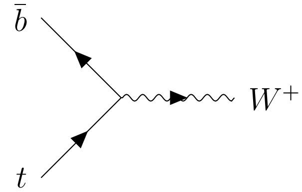




The two fermions in these diagrams have electric charges that differ by ± 1 to ensure that the overall electric charge is conserved at the vertex. We've included an arrow on the gauge boson propagator because it is now a complex spin 1 field. The arrow going the other way corresponds to the anti-particle W^-.

Here, we've only focussed on a single generation. There are similar diagrams where u, d, e and ν_e are replaced by their higher generational cousins. So, for example, there are additional W boson diagrams that connect the strange and charm quark, and the bottom and top quark:





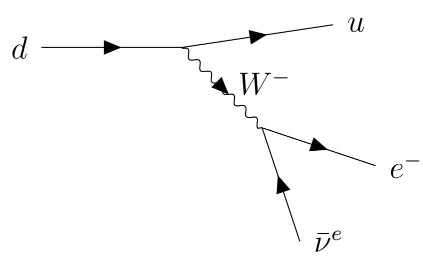
There are also diagrams with muons and taus replacing electrons. In fact, it turns out that there is an additional subtlety when considering these higher generations that we will turn to in Section 6.

5.3.3 A First Look at Weak Processes

Historically, the weak force was first observed in beta decay of nuclei. We can view this as a neutron decaying to a proton, electron and anti-neutrino

$$n \to p + e^- + \bar{\nu}^e \ . \tag{5.89}$$

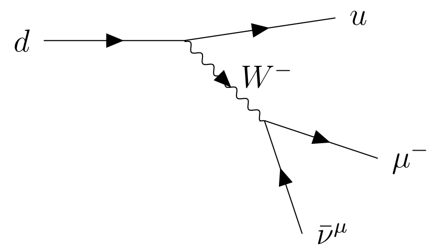
The possibility of such a process follows immediately from our discussion above. As we saw in Section 3, a neutron is a baryon with quark content *udd*. This decays to a proton with quark content *uud* through the tree level Feynman diagram



The lifetime of the neutron is about 10 minutes.

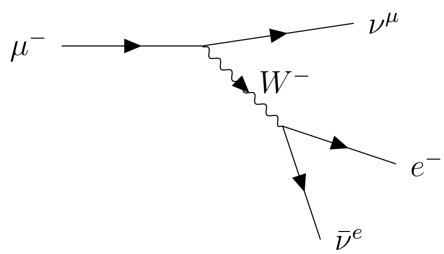
An obvious comment: the reason that down quarks decay into up quarks, rather than the other way around, is because the mass of the down quark is heavier than the masses of the decay products, $m_d > m_u + m_e + m_{\nu_e}$. As we've mentioned previously, we have no understanding of why the masses of fundamental particles are ordered in this way.

Neutrons are not the only victim of the weak force. A world without the weak force would be awash with pions which, as we saw in Section 3, are the lightest of the hadrons. The vast majority of the time (something like 99.99%) charged pion $\pi^- = d\bar{u}$ decays through the weak force to a muon and anti-neutrino. This occurs through a similar Feynman diagram to that responsible for beta decay, but with muons replacing electrons as the end products,



The resulting up quark then combines with the anti-up quark in the pion, and the two rapidly decay into photons. The lifetime of the charged pion is about 10^{-8} seconds.

The resulting muons don't live too long either. Their demise is also due to the weak force and they decay to electrons and neutrinos through the process



The lifetime of the muon is around 2×10^{-6} seconds. All other particles involving quarks and leptons from the second and third generation have the same fate, decaying through the weak force to the more familiar particles from the first generation.

5.3.4 4-Fermi Theory

Although the weak force is mediated by W and Z bosons, if we focus on processes that take place at low energies, $E \ll M_W,\ M_Z$, then it's possible to ignore these gauge bosons and write down interaction terms that describe the relevant physics directly.

There are a couple of (essentially equivalent) ways to remove the W and Z bosons while leaving behind the processes that they induce. The first, and most direct, way to see this is to start with the terms linear and quadratic in W bosons. (We'll ignore the Higgs field h in what follows but, crucially, keep its vev v.) We have

$$\begin{aligned}\mathcal{L}_{\text{weak}} = &-\frac{1}{2}(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial^\mu W^{-\,\nu} - \partial^\nu W^{-\,\mu}) \\ &- \frac{g^2v^2}{4}W^+_\mu W^{-\,\mu} - \frac{g}{\sqrt{2}}(W^+_\mu J^\mu_+ + W^-_\mu J^\mu_-)\ . \end{aligned} \tag{5.90}$$

At low energies, we can neglect the kinetic terms for the W bosons. We then proceed by completing the square in the remaining terms,

$$\mathcal{L}_{\text{weak}} \approx -\frac{g^2v^2}{4}\left(W^+_\mu + \frac{2\sqrt{2}}{gv^2}J_{-\,\mu}\right)\left(W^{-\,\mu} + \frac{2\sqrt{2}}{gv^2}J^\mu_+\right) + \frac{2}{v^2}J_{+\,\mu}J^\mu_- \ . \tag{5.91}$$

Performing the path integral over the W bosons effectively sets the first term to zero, leaving us just with the current-current interaction. We write this, for historic reasons, as

$$\mathcal{L}_{\text{weak}} = \frac{4G_F}{\sqrt{2}}\, J_{+\,\mu}\, J^\mu_- \tag{5.92}$$

with

$$G_F = \frac{1}{\sqrt{2}v^2} \approx 1.16 \times 10^{-5}\ \text{GeV}^{-2} \ . \tag{5.93}$$

Our final result (5.92) is a 4-fermion interaction. The coupling constant G_F is called the *Fermi coupling* and provides a direct measurement of the Higgs vev. It has dimensions $[G_F] = -2$ (because the fermion has dimension $3/2$ so the $J_\mu J^\mu$ term has dimension 6). This means that the four fermi term is irrelevant in the renormalisation group sense. It is, however, very relevant in the cosmic sense. For example, it is what makes the Sun shine.

There is a second way to arrive at the same result (5.92) using Feynman diagrams. In this approach, we start by examining the propagator for a massive vector field. In momentum space, it takes the form

$$D_{\mu\nu}(p) = \frac{i}{p^2 - M^2}\left(-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{M}\right) \ . \tag{5.94}$$

In the limit $E \ll M$, we ignore the momentum terms and get

$$D_{\mu\nu}(p) \approx \frac{i}{M^2}\eta_{\mu\nu} \quad \Longrightarrow \quad D_{\mu\nu}(x-y) = \frac{i}{M^2}\eta_{\mu\nu}\delta^4(x-y) \ . \tag{5.95}$$

In this limit, the propagator in position space becomes a delta-function, as shown, and the kind of couplings induced by the massive gauge boson, which are generally of the form $J^\mu(x)D_{\mu\nu}(x,y)J^\nu(y)$ collapse to the direct current-current interaction that we saw in (5.92).

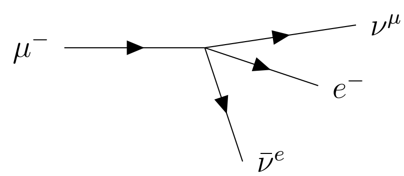
We can see what this means for, say, muon decay. If we ignore the quarks for now, but include both electron and muon contributions, then the W boson current (5.88) includes the term

$$J_\mu^+ = \bar{\nu}_L^e \bar{\sigma}_\mu e_L + \bar{\nu}_L^\mu \bar{\sigma}_\mu \mu_L \ . \tag{5.96}$$

The 4-fermi terms then include

$$\mathcal{L}_{\text{weak}} \sim \frac{4G_F}{\sqrt{2}} \, (\bar{\nu}_L^e \bar{\sigma}_\mu e_L)(\bar{\mu}_L \bar{\sigma}^\mu \nu_L^\mu) \ . \tag{5.97}$$

This gives rise directly to muon decay through the Feynman diagram



It's as if we've squinted and ignored the W boson that mediates the weak force.

These kinds of 4-fermion interactions were first written down by Fermi in 1933. His purpose was to describe beta decay, with the neutron coupled to the proton, electron and neutrino fields (the latter later realised to be an anti-neutrino). This was an important breakthrough in our understanding of particle physics because it changed the way we think about particles. In beta decay, a neutron decays into a proton and electron. But that doesn't mean that the neutron is *made* of a proton and electron! They're not sitting there inside the neutron all along, waiting to escape. Instead, the key idea of quantum field theory is that the four-fermion couplings allow one type of field to transmute into the others.

Second, there's some spin structure going on in (5.97) that Fermi was unaware of. This arises because the W boson couples only to left-handed fermions, not their right-handed counterparts. We can also write the resulting coupling in terms of Dirac spinors where we need a projection operator onto the left-handed part. The coupling (5.97) can then be written as

$$\mathcal{L}_{\text{weak}} \sim \frac{G_F}{\sqrt{2}} \left(\bar{\nu}^e \gamma_\mu (1+\gamma^5) e\right) \left(\bar{\mu} \gamma^\mu (1+\gamma^5) \nu^\mu\right) \ . \tag{5.98}$$

This is referred to as the "V-A" theory, because the coupling involves the difference between the vector current $\bar{\psi}\gamma^\mu\psi$ and the axial current $\bar{\psi}\gamma^5\gamma^\mu\psi$. (Admittedly, the term V-A would probably have made more sense if I'd defined my γ^5 matrix with a different sign so that it appeared as $(1-\gamma^5)$ rather than $(1+\gamma^5)$ in the expressions above. Oh well.