String Theory: Example Sheet 3

Dr David Tong, February 2010

1. The "Polyakov-style" action for a massive relativistic point particle, involving the Minkowski space coordinate $X^\mu(\tau)$ and the einbein $e(\tau)$, is given by

$$S = \frac{1}{2}\int d\tau \, \left(e^{-1}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} - em^2\right)$$

a. Show that this action has the reparameterization invariance $\tau \to \tilde{\tau} - \eta(\tau)$, to linear order in η, with

$$\delta e \;=\; \frac{d}{d\tau}(\eta(\tau)e) \quad , \quad \delta X^\mu = \frac{dX^\mu}{d\tau}\eta(\tau)$$

b. Consider a path starting at $X^\mu(\tau_1) = X_1^\mu$ and finishing at $X^\mu(\tau_2) = X_2^\mu$. Show that reparameterization invariance allows the choice of gauge

$$e(\tau) = \frac{l}{\tau_2 - \tau_1}$$

where l is the invariant length of the worldline.

c. The Feynman propagator for a massive scalar particle defined by the path integral,

$$G(X_1 - X_2) = \mathcal{N}\int De\, DX\, e^{iS[X,e]}$$

with $\mathcal{N}$ the usual normalization constant. By transforming to the gauge $e = l/(\tau_2 - \tau_1)$, together with an appropriate redefinition of the parameter τ, show that

$$G(X_1 - X_2) = \mathcal{N}'\int_0^\infty dl\, DX\, \exp\left(\frac{i}{2}\int_0^l d\tau\, (\dot{X}^2 - m^2)\right)$$

Why does the integral over the length of path l remain? (You may find the discussion in Polchinski chapter 5.1 useful to answer this).

d. Compute the functional integral over X to deduce that the propagator in momentum space is given by

$$\tilde{G}(p) = \frac{1}{p^2 + m^2}$$

2. The scattering amplitude for m closed string tachyons is given by,

$$\mathcal{A}^{(m)}(p_1,\ldots,p_m) = \frac{g_s^{m-2}}{\mathrm{Vol}(SL(2;\mathbf{C}))} \int \prod_{i=1}^{m} d^2 z_i \ \langle \hat{V}(z_1,p_1)\ldots\hat{V}(z_m,p_m)\rangle$$

where $\hat{V}(z,p) = e^{ip\cdot X(z,\bar{z})}$ and the correlation function is computed using the gauge fixed free Polyakov action

$$S_{\mathrm{Poly}} = \frac{1}{2\pi\alpha'}\int d^2 z\ \partial X\cdot\bar{\partial}X$$

a. By expressing the correlation function as a Gaussian integral, show that the amplitude is given by

$$\mathcal{A}^{(m)} \sim \frac{g_s^{m-2}}{\mathrm{Vol}(SL(2;\mathbf{C}))}\ \delta^{26}(\sum_i p_i)\ \int \prod_{i=1}^{m} d^2 z_i\ \prod_{j<l}|z_j - z_l|^{\alpha' p_j\cdot p_l}$$

b. Show that this integral is invariant under the $SL(2;\mathbf{C})$ transformation

$$z_i \longrightarrow \frac{az_i + b}{cz_i + d}$$

only when the momenta are on-shell, i.e. $p_i^2 = 4/\alpha'$.

c. Explain why this means that the 4-point amplitude can be reduced to the integral

$$\mathcal{A}^{(4)} \sim g_s^2\ \delta^{26}(\sum_i p_i)\ \int d^2 z\ |z|^{\alpha' p_2\cdot p_3}|1-z|^{\alpha' p_3\cdot p_4}$$

d. Evaluate this integral in terms of gamma functions. Show that, when written in Mandelstam variables, it is given by the Virasoro-Shapiro amplitude

$$\mathcal{A}^{(4)} \sim g_s^2\ \frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' t/4)}$$

3. Explain why the limit $s\to\infty$, with t fixed corresponds to small angle scattering at high energy. Show that in this limit the Virasoro-Shapiro amplitude exhibits so-called Regge behaviour,

$$\mathcal{A}^{(4)} \to\ g_s^2\ \delta^{26}(\sum_i p_i)\ \frac{\Gamma(-1-\alpha' t/4)}{\Gamma(2+\alpha' t/4)}\ s^{2+\alpha' t/2}$$

4a. Write down the vertex operator for a massless closed string state with polarization $\zeta_{\mu\nu}$ and momentum p^μ. What are the restrictions on p^μ and $\zeta_{\mu\nu}$?

b. Consider the scattering of a massless closed string mode with momentum p_1 and two tachyons with momentum p_2 and p_3. Show that $p_1 \cdot p_2 = p_1 \cdot p_3 = 0$ and $p_2 \cdot p_3 = -4/\alpha'$.

c. Show that the 3-point scattering amplitude for these particles is given by

$$\mathcal{A}^{(3)} \sim \frac{g_s}{\mathrm{Vol}(SL(2;\mathbf{C}))}\,\delta^{26}(\sum_i p_i)\,\int \prod_{i=1}^{3} d^2 z_i\,\frac{1}{|z_{23}|^4}\,\zeta_{\mu\nu}\left(\frac{p_2^\mu}{z_{12}} + \frac{p_3^\mu}{z_{13}}\right)\left(\frac{p_2^\nu}{\bar{z}_{12}} + \frac{p_3^\nu}{\bar{z}_{13}}\right)$$

where $z_{ij} = z_i - z_j$.

d. Explain why the $SL(2;\mathbf{C})$ gauge symmetry allows us to simplify this to

$$\mathcal{A}^{(3)} \sim g_s \delta^{(26)}(\sum_i p_i)\ \zeta_{\mu\nu}(p_2^\mu - p_3^\mu)(p_2^\nu - p_3^\nu)$$

5a. After using $SL(2;\mathbf{C})$ to fix the positions of 3 vertex operators, the tree-level m-point amplitude for tachyon scattering reduces to an integral over the positions of the remaining $m-3$ vertex operator insertions,

$$\mathcal{A}^{(4)} \sim g_s^{m-2}\,\delta^{26}(\sum_i p_i)\,\int \prod_{i=4}^{m} d^2 z_i\,\prod_{j<l} |z_{jl}|^{\alpha' p_j \cdot p_l}$$

where $z_{jl} = z_j - z_l$. The variables describing the exchange of momentum are $s_{ij} = -(p_i + p_j)^2$. The hard scattering limit is defined by $s_{ij} \to \infty$. Explain why the integral can be evaluated using a saddle-point approximation in this limit.

b. For the 4-point amplitude, use the saddle point approximation to show that

$$\mathcal{A}^{(4)} \sim g_s^2\,\delta^{26}(\sum_i p_i)\,\exp\left(-\frac{\alpha'}{2}(s\ln s + t\ln t + u\ln u)\right)$$