5. The Polyakov Path Integral and Ghosts

At the beginning of the last chapter, we stressed that there are two very different interpretations of conformal symmetry depending on whether we're thinking of a fixed 2d background or a dynamical 2d background. In applications to statistical physics, the background is fixed and conformal symmetry is a global symmetry. In contrast, in string theory the background is dynamical. Conformal symmetry is a gauge symmetry, a remnant of diffeomorphism invariance and Weyl invariance.

But gauge symmetries are not symmetries at all. They are redundancies in our description of the system. As such, we can't afford to lose them and it is imperative that they don't suffer an anomaly in the quantum theory. At worst, theories with gauge anomalies make no sense. (For example, Yang-Mills theory coupled to only left-handed fundamental fermions is a nonsensical theory for this reason). At best, it may be possible to recover the quantum theory, but it almost certainly has nothing to do with the theory that you started with.

Piecing together some results from the previous chapter, it looks like we're in trouble. We saw that the Weyl symmetry is anomalous since the expectation value of the stress-energy tensor takes different values on backgrounds related by a Weyl symmetry:

$$\langle T^{\alpha}_{\ \alpha}\rangle = -\frac{c}{12}\,R$$

On fixed backgrounds, that's merely interesting. On dynamical backgrounds, it's fatal. What can we do? It seems that the only way out is to ensure that our theory has $c = 0$. But we've already seen that $c > 0$ for all non-trivial, unitary CFTs. We seem to have reached an impasse. In this section we will discover the loophole. It turns out that we do indeed require $c = 0$, but there's a way to achieve this that makes sense.

5.1 The Path Integral

In Euclidean space the Polyakov action is given by,

$$S_{\rm Poly} = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\; g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\;\delta_{\mu\nu}$$

From now on, our analysis of the string will be in terms of the path integral[5]. We integrate over all embedding coordinates X^μ and all worldsheet metrics $g_{\alpha\beta}$. Schematically,

[5]The analysis of the string path integral was first performed by Polyakov in "*Quantum geometry of bosonic strings*,", Phys. Lett. B **103**, 207 (1981). The paper weighs in at a whopping 4 pages. As a follow-up, he took another 2.5 pages to analyze the superstring in "*Quantum geometry of fermionic strings*," Phys. Lett. B **103**, 211 (1981).

the path integral is given by,

$$Z = \frac{1}{\text{Vol}} \int \mathcal{D}g\mathcal{D}X \; e^{-S_{\text{Poly}}[X,g]}$$

The "Vol" term is all-important. It refers to the fact that we shouldn't be integrating over all field configurations, but only those physically distinct configurations not related by diffeomorphisms and Weyl symmetries. Since the path integral, as written, sums over all fields, the "Vol" term means that we need to divide out by the volume of the gauge action on field space.

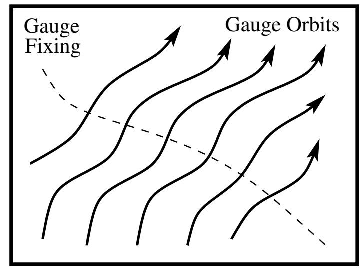

Figure 30:

To make the situation more explicit, we need to split the integration over all field configurations into two pieces: those corresponding to physically distinct configurations — schematically depicted as the dotted line in the figure — and those corresponding to gauge transformations — which are shown as solid lines. Dividing by "Vol" simply removes the piece of the partition function which comes from integrating along the solid-line gauge orbits.

In an ordinary integral, if we change coordinates then we pick up a Jacobian factor for our troubles. The path integral is no different. We want to decompose our integration variables into physical fields and gauge orbits. The tricky part is to figure out what Jacobian we get. Thankfully, there is a standard method to determine the Jacobian, first introduced by Faddeev and Popov. This method works for all gauge symmetries, including Yang-Mills and you will also learn about it in the "Advanced Quantum Field Theory" course.

5.1.1 The Faddeev-Popov Method

We have two gauge symmetries: diffeomorphisms and Weyl transformations. We will schematically denote both of these by ζ. The change of the metric under a general gauge transformation is $g \to g^{\zeta}$. This is shorthand for,

$$g_{\alpha\beta}(\sigma) \;\longrightarrow\; g^{\zeta}_{\alpha\beta}(\sigma') = e^{2\omega(\sigma)} \frac{\partial\sigma^{\gamma}}{\partial\sigma'^{\alpha}} \frac{\partial\sigma^{\delta}}{\partial\sigma'^{\beta}} \, g_{\gamma\delta}(\sigma)$$

In two dimensions these gauge symmetries allow us to put the metric into any form that we like — say, $\hat{g}$. This is called the fiducial metric and will represent our choice of gauge fixing. Two caveats:

- Firstly, it's not true that we can put any 2d metric into the form $\hat{g}$ of our choosing. This is only true locally. Globally, it remains true if the worldsheet has the topology of a cylinder or a sphere, but not for higher genus surfaces. We'll revisit this issue in Section 6.

- Secondly, fixing the metric locally to $\hat{g}$ does not fix all the gauge symmetries. We still have the conformal symmetries to deal with. We'll revisit this in the Section 6 as well.

Our goal is to only integrate over physically inequivalent configurations. To achieve this, first consider the integral over the gauge orbit of $\hat{g}$. For some value of the gauge transformation ζ, the configuration g^ζ will coincide with our original metric g. We can put a delta-function in the integral to get

$$\int \mathcal{D}\zeta \; \delta(g - \hat{g}^\zeta) = \Delta_{FP}^{-1}[g] \tag{5.1}$$

This integral isn't equal to one because we need to take into account the Jacobian factor. This is analogous to the statement that $\int dx\, \delta(f(x)) = 1/|f'|$, evaluated at points where $f(x) = 0$. In the above equation, we have written this Jacobian factor as Δ_{FP}^{-1}. The inverse of this, namely Δ_{FP}, is called the *Faddeev-Popov determinant*. We will evaluate it explicitly shortly. Some comments:

- This whole procedure is rather formal and runs into the usual difficulties with trying to define the path integral. Just as for Yang-Mills theory, we will find that it results in sensible answers.
- We will assume that our gauge fixing is good, meaning that the dotted line in the previous figure cuts through each physically distinct configuration exactly once. Equivalently, the integral over gauge transformations $\mathcal{D}\zeta$ clicks exactly once with the delta-function and we don't have to worry about discrete ambiguities (known as Gribov copies in QCD).
- The measure is taken to be the analogue of the Haar measure for Lie groups, invariant under left and right actions

$$\mathcal{D}\zeta = \mathcal{D}(\zeta'\zeta) = \mathcal{D}(\zeta\zeta')$$

Before proceeding, it will be useful to prove a quick lemma:

Lemma: $\Delta_{FP}[g]$ is gauge invariant. This means that

$$\Delta_{FP}[g] = \Delta_{FP}[g^\zeta]$$

Proof:

$$\begin{aligned}\Delta_{FP}^{-1}[g^\zeta] &= \int \mathcal{D}\zeta' \; \delta(g^\zeta - \hat{g}^{\zeta'}) \\ &= \int \mathcal{D}\zeta' \; \delta(g - \hat{g}^{\zeta^{-1}\zeta'}) \\ &= \int \mathcal{D}\zeta'' \; \delta(g - \hat{g}^{\zeta''}) = \Delta_{FP}^{-1}[g]\end{aligned}$$

where, in the second line, we have used the fact that the measure is invariant. □

Now we can employ the Faddeev-Popov procedure. We start by inserting a factor of unity into the path integral, in the guise of

$$1 = \Delta_{FP}[g] \int \mathcal{D}\zeta \; \delta(g - \hat{g}^\zeta)$$

We'll call the resulting path integral expression $Z[\hat{g}]$ since it depends on the choice of fiducial metric $\hat{g}$. The first thing we do is use the $\delta(g - \hat{g}^\zeta)$ delta-function to do the integral over metrics,

$$\begin{aligned}Z[\hat{g}] &= \frac{1}{\text{Vol}} \int \mathcal{D}\zeta \mathcal{D}X \mathcal{D}g \; \Delta_{FP}[g] \, \delta(g - \hat{g}^\zeta) \, e^{-S_{\text{Poly}}[X,g]} \\ &= \frac{1}{\text{Vol}} \int \mathcal{D}\zeta \mathcal{D}X \; \Delta_{FP}[\hat{g}^\zeta] \, e^{-S_{\text{Poly}}[X,\hat{g}^\zeta]} \\ &= \frac{1}{\text{Vol}} \int \mathcal{D}\zeta \mathcal{D}X \; \Delta_{FP}[\hat{g}] \, e^{-S_{\text{Poly}}[X,\hat{g}]}\end{aligned}$$

where to go to the last line, we've used the fact that the action is gauge invariant, so $S[X,g] = S[X,g^\zeta]$ and, as we proved in the lemma above, the Faddeev-Popov determinant is also gauge invariant.

But now, nothing depends on the gauge transformation ζ. Indeed, this is precisely the integration over the gauge orbits that we wanted to isolate and it cancels the "Vol" factor sitting outside. We're left with

$$Z[\hat{g}] = \int \mathcal{D}X \; \Delta_{FP}[\hat{g}] \, e^{-S_{\text{Poly}}[X,\hat{g}]} \tag{5.2}$$

This is the integral over physically distinct configurations — the dotted line in the previous figure. We see that the Faddeev-Popov determinant is precisely the Jacobian factor that we need.

5.1.2 The Faddeev-Popov Determinant

We still need to compute $\Delta_{FP}[\hat{g}]$. It's defined in (5.1). Let's look at gauge transformations ζ which are close to the identity. In this case, the delta-function $\delta(g - \hat{g}^\zeta)$ is going to be non-zero when the metric g is close to the fiducial metric $\hat{g}$. In fact, it will be sufficient to look at the delta-function $\delta(\hat{g} - \hat{g}^\zeta)$, which is only non-zero when $\zeta = 0$. We take an infinitesimal Weyl transformation parameterized by $\omega(\sigma)$ and an infinitesimal diffeomorphism $\delta\sigma^\alpha = v^\alpha(\sigma)$. The change in the metric is

$$\delta\hat{g}_{\alpha\beta} = 2\omega\hat{g}_{\alpha\beta} + \nabla_\alpha v_\beta + \nabla_\beta v_\alpha$$

Plugging this into the delta-function, the expression for the Faddeev-Popov determinant becomes

$$\Delta_{FP}^{-1}[\hat{g}] = \int \mathcal{D}\omega\mathcal{D}v\ \delta(2\omega\hat{g}_{\alpha\beta} + \nabla_\alpha v_\beta + \nabla_\beta v_\alpha) \tag{5.3}$$

where we've replaced the integral $\mathcal{D}\zeta$ over the gauge group with the integral $\mathcal{D}\omega\mathcal{D}v$ over the Lie algebra of group since we're near the identity. (We also suppress the subscript on v_α in the measure factor to keep things looking tidy).

At this stage it's useful to represent the delta-function in its integral, Fourier form. For a single delta-function, this is $\delta(x) = \int dp\ \exp(2\pi ip\,x)$. But the delta-function in (5.3) is actually a delta-functional: it restricts a whole function. Correspondingly, the integral representation is in terms of a functional integral,

$$\Delta_{FP}^{-1}[\hat{g}] = \int \mathcal{D}\omega\mathcal{D}v\mathcal{D}\beta\ \exp\left(2\pi i\int d^2\sigma\ \sqrt{\hat{g}}\ \beta^{\alpha\beta}[2\omega\hat{g}_{\alpha\beta} + \nabla_\alpha v_\beta + \nabla_\beta v_\alpha]\right)$$

where $\beta^{\alpha\beta}$ is a symmetric 2-tensor on the worldsheet.

We now simply do the $\int\mathcal{D}\omega$ integral. It doesn't come with any derivatives, so it merely acts as a Lagrange multiplier, setting

$$\beta^{\alpha\beta}\hat{g}_{\alpha\beta} = 0$$

In other words, after performing the ω integral, $\beta^{\alpha\beta}$ is symmetric and traceless. We'll take this to be the definition of $\beta^{\alpha\beta}$ from now on. So, finally we have

$$\Delta_{FP}^{-1}[\hat{g}] = \int \mathcal{D}v\mathcal{D}\beta\ \exp\left(4\pi i\int d^2\sigma\ \sqrt{\hat{g}}\ \beta^{\alpha\beta}\nabla_\alpha v_\beta\right)$$

5.1.3 Ghosts

The previous manipulations give us an expression for Δ_{FP}^{-1}. But we want to invert it to get Δ_{FP}. Thankfully, there's a simple way to achieve this. Because the integrand is quadratic in v and β, we know that the integral computes the inverse determinant of the operator ∇_α. (Strictly speaking, it computes the inverse determinant of the projection of ∇_α onto symmetric, traceless tensors. This observation is important because it means the relevant operator is a square matrix which is necessary to talk about a determinant). But we also know how to write down an expression for the determinant Δ_{FP}, instead of its inverse, in terms of path integrals: we simply need to replace the commuting integration variables with anti-commuting fields,

$$\begin{aligned} \beta_{\alpha\beta} &\longrightarrow b_{\alpha\beta} \\ v^\alpha &\longrightarrow c^\alpha \end{aligned}$$

where b and c are both Grassmann-valued fields (i.e. anti-commuting). They are known as *ghost fields*. This gives us our final expression for the Faddeev-Popov determinant,

$$\Delta_{FP}[g] = \int \mathcal{D}b\mathcal{D}c\ \exp[iS_{\text{ghost}}]$$

where the ghost action is defined to be

$$S_{\text{ghost}} = \frac{1}{2\pi}\int d^2\sigma\sqrt{g}\ b_{\alpha\beta}\nabla^\alpha c^\beta \tag{5.4}$$

and we have chosen to rescale the b and c fields at this last step to get a factor of $1/2\pi$ sitting in front of the action. (This only changes the normalization of the partition function which doesn't matter). Rotating back to Euclidean space, the factor of i disappears. The expression for the full partition function (5.2) is

$$Z[\hat{g}] = \int \mathcal{D}X\mathcal{D}b\mathcal{D}c\ \exp\left(-S_{\text{Poly}}[X,\hat{g}] - S_{\text{ghost}}[b,c,\hat{g}]\right)$$

Something lovely has happened. Although the ghost fields were introduced as some auxiliary constructs, they now appear on the same footing as the dynamical fields X. We learn that gauge fixing comes with a price: our theory has extra ghost fields.

The role of these ghost fields is to cancel the unphysical gauge degrees of freedom, leaving only the $D-2$ transverse modes of X^μ. Unlike lightcone quantization, they achieve this in a way which preserves Lorentz invariance.

Simplifying the Ghost Action

The ghost action (5.4) looks fairly simple. But it looks even simpler if we work in conformal gauge,

$$\hat{g}_{\alpha\beta} = e^{2\omega}\delta_{\alpha\beta}$$

The determinant is $\sqrt{\hat{g}} = e^{2\omega}$. Recall that in complex coordinates, the measure is $d^2\sigma = \frac{1}{2}d^2z$, while we can lower the index on the covariant derivative using $\nabla^z = g^{z\bar{z}}\nabla_{\bar{z}} = 2e^{-2\omega}\nabla_{\bar{z}}$. We have

$$S_{\text{ghost}} = \frac{1}{2\pi}\int d^2z\;\left(b_{zz}\nabla_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}}\right)$$

In deriving this, remember that there is no field $b_{z\bar{z}}$ because $b_{\alpha\beta}$ is traceless. Now comes the nice part: the covariant derivatives are actually just ordinary derivatives. To see why this is the case, look at

$$\nabla_{\bar{z}}c^z = \partial_{\bar{z}}c^z + \Gamma^z_{\bar{z}\alpha}c^\alpha$$

But the Christoffel symbols are given by

$$\Gamma^z_{\bar{z}\alpha} = \frac{1}{2}g^{z\bar{z}}\left(\partial_{\bar{z}}g_{\alpha\bar{z}} + \partial_\alpha g_{\bar{z}\bar{z}} - \partial_{\bar{z}}g_{\bar{z}\alpha}\right) = 0 \qquad \text{for } \alpha = z, \bar{z}$$

So in conformal gauge, the ghost action factorizes into two free theories,

$$S_{\text{ghost}} = \frac{1}{2\pi}\int d^2z\; b_{zz}\,\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\,\partial_z c^{\bar{z}}$$

The action doesn't depend on the conformal factor ω. In other words, it is Weyl invariant without any need to change b and c: these are therefore both neutral under Weyl transformations.

(It's worth pointing out that $b_{\alpha\beta}$ and c^α are neutral under Weyl transformations. But if we raise or lower these indices, then the fields pick up factors of the metric. So $b^{\alpha\beta}$ and c_α would not be neutral under Weyl transformations).

5.2 The Ghost CFT

Fixing the Weyl and diffeomorphism gauge symmetries has left us with two new dynamical ghost fields, b and c. Both are Grassmann (i.e. anti-commuting) variables. Their dynamics is governed by a CFT. Define

$$b = b_{zz} \quad , \quad \bar{b} = b_{\bar{z}\bar{z}}$$
$$c = c^z \quad , \quad \bar{c} = c^{\bar{z}}$$

The ghost action is given by

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2z \;\left(b\,\bar{\partial}c + \bar{b}\,\partial\bar{c}\right)$$

Which gives the equations of motion

$$\bar{\partial}b = \partial\bar{b} = \bar{\partial}c = \partial\bar{c} = 0$$

So we see that b and c are holomorphic fields, while $\bar{b}$ and $\bar{c}$ are anti-holomorphic.

Before moving onto quantization, there's one last bit of information we need from the classical theory: the stress tensor for the bc ghosts. The calculation is a little bit fiddly. We use the general definition of the stress tensor (4.4), which requires us to return to the theory (5.4) on a general background and vary the metric $g^{\alpha\beta}$. The complications are twofold. Firstly, we pick up a contribution from the Christoffel symbol that is lurking inside the covariant derivative ∇^α. Secondly, we must also remember that $b_{\alpha\beta}$ is traceless. But this is a condition which itself depends on the metric: $b_{\alpha\beta}g^{\alpha\beta} = 0$. To account for this we should add a Lagrange multiplier to the action imposing tracelessness. After correctly varying the metric, we may safely retreat back to flat space where the end result is rather simple. We have $T_{z\bar{z}} = 0$, as we must for any conformal theory. Meanwhile, the holomorphic and anti-holomorphic parts of the stress tensor are given by,

$$T = 2(\partial c)\,b + c\,\partial b \quad , \quad \bar{T} = 2(\bar{\partial}\bar{c})\,\bar{b} + \bar{c}\,\bar{\partial}\bar{b}. \tag{5.5}$$

Operator Product Expansions

We can compute the OPEs of these fields using the standard path integral techniques that we employed in the last chapter. In what follows, we'll just focus on the holomorphic piece of the CFT. We have, for example,

$$0 = \int \mathcal{D}b\mathcal{D}c\;\frac{\delta}{\delta b(\sigma)}\;\left[e^{-S_{\text{ghost}}}\,b(\sigma')\right] = \int \mathcal{D}b\mathcal{D}c\;e^{-S_{\text{ghost}}}\left[-\frac{1}{2\pi}\,\bar{\partial}c(\sigma)\,b(\sigma') + \delta(\sigma-\sigma')\right]$$

which tells us that

$$\bar{\partial}c(\sigma)\,b(\sigma') = 2\pi\,\delta(\sigma-\sigma')$$

Similarly, looking at $\delta/\delta c(\sigma)$ gives

$$\bar{\partial}b(\sigma)\,c(\sigma') = 2\pi\,\delta(\sigma-\sigma')$$

We can integrate both of these equations using our favorite formula $\bar{\partial}(1/z) = 2\pi\delta(z,\bar{z})$. We learn that the OPEs between fields are given by

$$b(z)\,c(w) = \frac{1}{z-w} + \ldots$$
$$c(w)\,b(z) = \frac{1}{w-z} + \ldots$$

In fact the second equation follows from the first equation and Fermi statistics. The OPEs of $b(z)\,b(w)$ and $c(z)\,c(w)$ have no singular parts. They vanish as $z \to w$.

Finally, we need the stress tensor of the theory. After normal ordering, it is given by

$$T(z) = 2 :\partial c(z)\,b(z): + :c(z)\partial b(z):$$

We will shortly see that with this choice, b and c carry appropriate weights for tensor fields which are neutral under Weyl rescaling.

Primary Fields

We will now show that both b and c are primary fields, with weights $h = 2$ and $h = -1$ respectively. Let's start by looking at c. The OPE with the stress tensor is

$$\begin{aligned} T(z)\,c(w) &= 2 :\partial c(z)\,b(z): c(w) + :c(z)\,\partial b(z): c(w) \\ &= \frac{2\partial c(z)}{z-w} - \frac{c(z)}{(z-w)^2} + \ldots = -\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w} + \ldots \end{aligned}$$

confirming that c has weight -1. When taking the OPE with b, we need to be a little more careful with minus signs. We get

$$\begin{aligned} T(z)\,b(w) &= 2 :\partial c(z)\,b(z): b(w) + :c(z)\,\partial b(z): b(w) \\ &= -2b(z)\left(\frac{-1}{(z-w)^2}\right) - \frac{\partial b(z)}{z-w} = \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} + \ldots \end{aligned}$$

showing that b has weight 2. As we've pointed out a number of times, conformal = diffeo + Weyl. We mentioned earlier that the fields b and c are neutral under Weyl transformations. This is reflected in their weights, which are due solely to diffeomorphisms as dictated by their index structure: b_{zz} and c^z.

The Central Charge

Finally, we can compute the TT OPE to determine the central charge of the bc ghost system.

$$\begin{aligned} T(z)\,T(w) = 4 &:\partial c(z)b(z)::\partial c(w)b(w): + 2 :\partial c(z)b(z)::c(w)\partial b(w): \\ &+ 2 :c(z)\partial b(z)::\partial c(w)b(w): + :c(z)\partial b(z)::c(w)\partial b(w): \end{aligned}$$

For each of these terms, making two contractions gives a $(z-w)^{-4}$ contribution to the OPE. There are also two ways to make a single contraction. These give $(z-w)^{-1}$ or $(z-w)^{-2}$ or $(z-w)^{-3}$ contributions depending on what the derivatives hit. The end result is

$$\begin{aligned} T(z)\,T(w) = & \frac{-4}{(z-w)^4} + \frac{4 :\partial c(z) b(w):}{(z-w)^2} - \frac{4 : b(z)\partial c(w):}{(z-w)^2} \\ & - \frac{4}{(z-w)^4} + \frac{2 :\partial c(z)\,\partial b(w):}{z-w} - \frac{4 : b(z) c(w):}{(z-w)^3} \\ & - \frac{4}{(z-w)^4} - \frac{4 : c(z) b(w):}{(z-w)^3} + \frac{2 :\partial b(z)\partial c(w):}{z-w} \\ & - \frac{1}{(z-w)^4} - \frac{: c(z)\partial b(w):}{(z-w)^2} + \frac{\partial b(z) c(w):}{(z-w)^2} + \ldots \end{aligned}$$

After some Taylor expansions to turn $f(z)$ functions into $f(w)$ functions, together with a little collecting of terms, this can be written as,

$$T(z)\,T(w) = \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

The first thing to notice is that it indeed has the form expected of TT OPE. The second, and most important, thing to notice is the central charge of the bc ghost system: it is

$$c = -26$$

5.3 The Critical "Dimension" of String Theory

Let's put the pieces together. We've learnt that gauge fixing the diffeomorphisms and Weyl gauge symmetries results in the introduction of ghosts which contribute central charge $c = -26$. We've also learnt that the Weyl symmetry is anomalous unless $c = 0$. Since the Weyl symmetry is a gauge symmetry, it's crucial that we keep it. We're forced to add exactly the right degrees of freedom to the string to cancel the contribution from the ghosts.

The simplest possibility is to add D free scalar fields. Each of these contributes $c = 1$ to the central charge, so the whole procedure is only consistent if we pick

$$D = 26$$

This agrees with the result we found in Chapter 2: it is the critical dimension of string theory.

However, there's no reason that we have to work with free scalar fields. The consistency requirement is merely that the degrees of freedom of the string are described by a CFT with $c = 26$. Any CFT will do. Each such CFT describes a different background in which a string can propagate. If you like, the space of CFTs with $c = 26$ can be thought of as the space of classical solutions of string theory.

We learn that the "critical dimension" of string theory is something of a misnomer: it is really a "critical central charge". Only for rather special CFTs can this central charge be thought of as a spacetime dimension.

For example, if we wish to describe strings moving in 4d Minkowski space, we can take $D = 4$ free scalars (one of which will be timelike) together with some other $c = 22$ CFT. This CFT may have a geometrical interpretation, or it may be something more abstract. The CFT with $c = 22$ is sometimes called the "internal sector" of the theory. It is what we really mean when we talk about the "extra hidden dimensions of string theory". We'll see some examples of CFTs describing curved spaces in Section 7.

There's one final subtlety: we need to be careful with the transition back to Minkowski space. After all, we want one of the directions of the CFT, X^0, to have the wrong sign kinetic term. One safe way to do this is to keep X^0 as a free scalar field, with the remaining degrees of freedom described by some $c = 25$ CFT. This doesn't seem quite satisfactory though since it doesn't allow for spacetimes which evolve in time — and, of course, these are certainly necessary if we wish to understand early universe cosmology. There are still some technical obstacles to understanding the worldsheet of the string in time-dependent backgrounds. To make progress, and discuss string cosmology, we usually bi-pass this issue by working with the low-energy effective action which we will derive in Section 7.

5.3.1 The Usual Nod to the Superstring

The superstring has another gauge symmetry on the worldsheet: supersymmetry. This gives rise to more ghosts, the so-called $\beta\gamma$ system, which turns out to have central charge $+11$. Consistency then requires that the degrees of freedom of the string have central charge $c = 26 - 11 = 15$.

However, now the CFTs must themselves be invariant under supersymmetry, which means that bosons come matched with fermions. If we add D bosons, then we also need to add D fermions. A free boson has $c = 1$, while a free fermion has $c = 1/2$. So, the total number of free bosons that we should add is $D(1 + 1/2) = 15$, giving us the critical dimension of the superstring:

$$D = 10$$

5.3.2 An Aside: Non-Critical Strings

Although it's a slight departure from the our main narrative, it's worth pausing to mention what Polyakov actually did in his four page paper. His main focus was not critical strings, with $D = 26$, but rather *non-critical* strings with $D \neq 26$. From the discussion above, we know that these suffer from a Weyl anomaly. But it turns out that there is a way to make sense of the situation.

The starting point is to abandon Weyl invariance from the beginning. We start with D free scalar fields coupled to a dynamical worldsheet metric $g_{\alpha\beta}$. (More generally, we could have any CFT). We still want to keep reparameterization invariance, but now we ignore the constraints of Weyl invariance. Of course, it seems likely that this isn't going to have too much to do with the Nambu-Goto string, but let's proceed anyway. Without Weyl invariance, there is one extra term that it is natural to add to the 2d theory: a worldsheet cosmological constant μ,

$$S_{\rm non-critical} = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\,\left(g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + \mu\right)$$

Our goal will be to understand how the partition function changes under a Weyl rescaling. There will be two contributions: one from the explicit μ dependence and one from the Weyl anomaly. Consider two metrics related by a Weyl transformation

$$\hat{g}_{\alpha\beta} = e^{2\omega}g_{\alpha\beta}$$

As we vary ω, the partition function $Z[\hat{g}]$ changes as

$$\begin{aligned}\frac{1}{Z}\frac{\partial Z}{\partial\omega} &= \frac{1}{Z}\int\mathcal{D}X\ e^{-S}\left(-\frac{\partial S}{\partial\hat{g}_{\alpha\beta}}\frac{\partial\hat{g}_{\alpha\beta}}{\partial\omega}\right)\\ &= \frac{1}{Z}\int\mathcal{D}X\ e^{-S}\left(-\frac{1}{2\pi}\sqrt{\hat{g}}\,T^\alpha_{\ \alpha}\right)\\ &= \frac{c}{24\pi}\sqrt{\hat{g}}\,\hat{R} - \frac{1}{2\pi\alpha'}\,\mu e^{2\omega}\\ &= \frac{c}{24\pi}\sqrt{g}(R - 2\nabla^2\omega) - \frac{1}{2\pi\alpha'}\,\mu e^{2\omega}\end{aligned}$$

where, in the last two lines, we used the Weyl anomaly (4.34) and the relationship between Ricci curvatures (1.29). The central charge appearing in these formulae includes the contribution from the ghosts,

$$c = D - 26$$

We can now just treat this as a differential equation for the partition function Z and solve. This allows us to express the partition function $Z[\hat{g}]$, defined on one worldsheet metric, in terms of $Z[g]$, defined on another. The relationship is,

$$Z[\hat{g}] = Z[g] \exp\left[-\frac{1}{4\pi\alpha'}\int d^2\sigma\ \sqrt{g}\left(2\mu e^{2\omega} - \frac{c\alpha'}{6}\left(g^{\alpha\beta}\,\partial_\alpha\omega\,\partial^\beta\omega + R\omega\right)\right)\right]$$

We see that the scaling mode ω inherits a kinetic term. It now appears as a new dynamical scalar field in the theory. It is often called the Liouville field on account of the exponential potential term multiplying μ. Solving this theory is quite hard[6]. Notice also that our new scalar field ω appears in the final term multiplying the Ricci scalar R. We will describe the significance of this in Section 7.2.1. We'll also see another derivation of this kind of Lagrangian in Section 7.4.4.

5.4 States and Vertex Operators

In Chapter 2 we determined the spectrum of the string in flat space. What is the spectrum for a general string background? The theory consists of the b and c ghosts, together with a $c = 26$ CFT. At first glance, it seems that we have a greatly enlarged Hilbert space since we can act with creation operators from all fields, including the ghosts. However, as you might expect, not all of these states will be physical. After correctly accounting for the gauge symmetry, only some subset survives.

The elegant method to determine the physical Hilbert space in a gauge fixed action with ghosts is known as *BRST quantization*. You will learn about it in the "Advanced Quantum Field Theory" course where you will apply it to Yang-Mills theory. Although a correct construction of the string spectrum employs the BRST method, we won't describe it here for lack of time. A very clear description of the general method and its application to the string can be found in Section 4.2 of Polchinski's book.

Instead, we will make do with a poor man's attempt to determine the spectrum of the string. Our strategy is to simply pretend that the ghosts aren't there and focus on the states created by the fields of the matter CFT (i.e. the X^μ fields if we're talking about flat space). As we'll explain in the next section, if we're only interested in tree-level scattering amplitudes then this will suffice.

To illustrate how to compute the spectrum of the string, let's go back to flat $D = 26$ dimensional Minkowski space and the discussion of covariant quantization in Section

[6] A good review can be found Seiberg's article "*Notes on Quantum Liouville Theory and Quantum Gravity*", Prog. Theor. Phys. Supl. 102 (1990) 319.

2.1. We found that physical states $|\Psi\rangle$ are subject to the Virasoro constraints (2.6) and (2.7) which read

$$\begin{aligned} L_n\,|\Psi\rangle &= 0 \qquad\qquad \text{for } n>0 \\ L_0\,|\Psi\rangle &= a\,|\Psi\rangle \end{aligned}$$

and similar for $\tilde{L}_n$,

$$\begin{aligned} \tilde{L}_n\,|\Psi\rangle &= 0 \qquad\qquad \text{for } n>0 \\ \tilde{L}_0\,|\Psi\rangle &= \tilde{a}\,|\Psi\rangle \end{aligned}$$

where we have, just briefly, allowed for the possibility of different normal ordering coefficients a and $\tilde{a}$ for the left- and right-moving sectors. But there's a name for states in a conformal field theory obeying these requirements: they are primary states of weight $(a, \tilde{a})$.

So how do we fix the normal ordering ambiguities a and $\tilde{a}$? A simple way is to first replace the states with operator insertions on the worldsheet using the state-operator map: $|\Psi\rangle \to \mathcal{O}$. But we have a further requirement on the operators $\mathcal{O}$: gauge invariance. There are two gauge symmetries: reparameterization invariance and Weyl symmetry. Both restrict the possible states.

Let's start by considering reparameterization invariance. In the last section, we happily placed operators at specific points on the worldsheet. But in a theory with a dynamical metric, this doesn't give rise to a diffeomorphism invariant operator. To make an object that is invariant under reparameterizations of the worldsheet coordinates, we should integrate over the whole worldsheet. Our operator insertions (in conformal gauge) are therefore of the form,

$$V \sim \int d^2z\; \mathcal{O} \tag{5.6}$$

Here the $\sim$ sign reflects the fact that we've dropped an overall normalization constant which we'll return to in the next section.

Integrating over the worldsheet takes care of diffeomorphisms. But what about Weyl symmetries? The measure d^2z has weight $(-1,-1)$ under rescaling. To compensate, the operator $\mathcal{O}$ must have weight $(+1,+1)$. This is how we fix the normal ordering ambiguity: we require $a = \tilde{a} = 1$. Note that this agrees with the normal ordering coefficient $a = 1$ that we derived in lightcone quantization in Chapter 2.

This, then, is the rather rough derivation of the string spectrum. The physical states are the primary states of the CFT with weight $(+1,+1)$. The operators (5.6) associated to these states are called *vertex operators*.

5.4.1 An Example: Closed Strings in Flat Space

Let's use this new language to rederive the spectrum of the closed string in flat space. We start with the ground state of the string, which was previously identified as a tachyon. As we saw in Section 4, the vacuum of a CFT is associated to the identity operator. But we also have the zero modes. We can give the string momentum p^μ by acting with the operator $e^{ip\cdot X}$. The vertex operator associated to the ground state of the string is therefore

$$V_{\text{tachyon}} \sim \int d^2z \;\, : e^{ip\cdot X} : \tag{5.7}$$

In Section 4.3.3, we showed that the operator $e^{ip\cdot X}$ is primary with weight $h = \tilde{h} = \alpha' p^2/4$. But Weyl invariance requires that the operator has weight $(+1,+1)$. This is only true if the mass of the state is

$$M^2 \equiv -p^2 = -\frac{4}{\alpha'}$$

This is precisely the mass of the tachyon that we saw in Section 2.

Let's now look at the first excited states. In covariant quantization, these are of the form $\zeta_{\mu\nu}\,\alpha^\mu_{-1}\,\tilde{\alpha}^\nu_{-1}\,|0;p\rangle$, where $\zeta_{\mu\nu}$ is a constant tensor that determines the type of state, together with its polarization. (Recall: traceless symmetric $\zeta_{\mu\nu}$ corresponds to the graviton, anti-symmetric $\zeta_{\mu\nu}$ corresponds to the $B_{\mu\nu}$ field and the trace of $\zeta_{\mu\nu}$ corresponds to the scalar known as the dilaton). From (4.51), the vertex operator associated to this state is,

$$V_{\text{excited}} \sim \int d^2z \;\, : e^{ip\cdot X}\,\partial X^\mu \bar{\partial} X^\nu :\, \zeta_{\mu\nu} \tag{5.8}$$

where ∂X^μ gives us a α^μ_{-1} excitation, while $\bar{\partial} X^\mu$ gives a $\tilde{\alpha}^\mu_{-1}$ excitation. It's easy to check that the weight of this operator is $h = \tilde{h} = 1 + \alpha' p^2/4$. Weyl invariance therefore requires that

$$p^2 = 0$$

confirming that the first excited states of the string are indeed massless. However, we still need to check that the operator in (5.8) is actually primary. We know that ∂X is primary and we know that $e^{ip\cdot X}$ is primary, but now we want to consider them both sitting together inside the normal ordering. This means that there are extra terms in the Wick contraction which give rise to $1/(z-w)^3$ terms in the OPE, potentially ruining the primacy of our operator. One such term arises from a double contraction, one of which includes the $e^{ip\cdot X}$ operator. This gives rise to an offending term proportional to $p^\mu \zeta_{\mu\nu}$. The same kind of contraction with $\bar{T}$ gives rise to a term proportional to $p^\nu \zeta_{\nu\mu}$. In order for these terms to vanish, the polarization tensor must satisfy

$$p^\mu \zeta_{\mu\nu} = p^\nu \zeta_{\mu\nu} = 0$$

which is precisely the transverse polarization condition expected for a massless particle.

5.4.2 An Example: Open Strings in Flat Space

As explained in Section 4.7, vertex operators for the open-string are inserted on the boundary $\partial\mathcal{M}$ of the worldsheet. We still need to ensure that these operators are diffeomorphism invariant which is achieved by integrating over $\partial\mathcal{M}$. The vertex operator for the open string tachyon is

$$V_{\text{tachyon}} \sim \int_{\partial\mathcal{M}} ds \ : e^{ip\cdot X} :$$

We need to figure out the dimension of the boundary operator $: e^{ip\cdot X} :$. It's not the same as for the closed string. The reason is due to presence of the image charge in the propagator (4.52) for a free scalar field on a space with boundary. This propagator appears in the Wick contractions in the OPEs and affects the weights. Let's see why this is the case. Firstly, we look at a single scalar field X,

$$\begin{aligned}\partial X(z) \ : e^{ipX(w,\bar{w})} : &= \sum_{n=1}^{\infty} \frac{(ip)^n}{(n-1)!} : X(w,\bar{w})^{n-1} : \left(-\frac{\alpha'}{2}\frac{1}{z-w} - \frac{\alpha'}{2}\frac{1}{z-\bar{w}} \right) + \dots \\ &= -\frac{i\alpha' p}{2} \ : e^{ipX(w,\bar{w})} : \left(\frac{1}{z-w} + \frac{1}{z-\bar{w}} \right) + \dots\end{aligned}$$

With this result, we can now compute the OPE with T,

$$T(z) \ : e^{ipX(w,\bar{w})} : = \frac{\alpha' p^2}{4} \ : e^{ipX} : \left(\frac{1}{z-w} + \frac{1}{z-\bar{w}} \right)^2 + \dots$$

When the operator $: e^{ipX(w,\bar{w})} :$ is placed on the boundary $w = \bar{w}$, this becomes

$$T(z) : e^{ipX(w,\bar{w})} := \frac{\alpha' p^2 \ : e^{ipX(w,\bar{w})} :}{(z-w)^2} + \dots$$

This tells us that the boundary operator $: e^{ip\cdot X} :$ is indeed primary, with weight $\alpha' p^2$.

For the open string, Weyl invariance requires that operators have weight +1 in order to cancel the scaling dimension of −1 coming from the boundary integral $\int ds$. So the mass of the open string ground state is

$$M^2 \equiv -p^2 = -\frac{1}{\alpha'}$$

in agreement with the mass of the open string tachyon computed in Section 3.

The vertex operator for the photon is

$$V_{\text{photon}} \sim \int_{\partial\mathcal{M}} ds\ \zeta_a\ :\partial X^a\, e^{ip\cdot X}: \tag{5.9}$$

where the index $a = 0, \ldots, p$ now runs only over those directions with Neumann boundary conditions that lie parallel to the brane worldvolume. The requirement that this is a primary operator gives $p^a\zeta_a = 0$, while Weyl invariance tells us that $p^2 = 0$. This is the expected behaviour for the momentum and polarization of a photon.

5.4.3 More General CFTs

Let's now consider a string propagating in four-dimensional Minkowski space $\mathcal{M}_4$, together with some internal CFT with $c = 22$. Then any primary operator of the internal CFT with weight (h, h) can be assigned momentum p^μ, for $\mu = 0, 1, 2, 3$ by dressing the operator with $e^{ip\cdot X}$. In order to get a primary operator of weight $(+1, +1)$ as required, we must have

$$\frac{\alpha' p^2}{4} = 1 - h$$

We see that the mass spectrum of closed string states is given by

$$M^2 = \frac{4}{\alpha'}(h - 1)$$

where h runs over the spectrum of primary operators of the internal CFT. Some comments:

- Relevant operators in the internal CFT have $h < 1$ and give rise to tachyons in the spectrum. Marginal operators, with $h = 1$, give massless particles. And irrelevant operators result in massive states.

- Notice that requiring the vertex operators to be Weyl invariant determines the mass formula for the state. We say that the vertex operators are "on-shell", in the same sense that external legs of Feynman diagrams are on-shell. We will have more to say about this in the next section.