SIGMA MODELS AND STRING THEORY

C. Callan[†⋆] and L. Thorlacius[†⋆]

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94309

ABSTRACT

We present a pedagogical discussion of conformally invariant two-dimensional nonlinear sigma models and their relation to string theory. Our main goal is show how to calculate the Weyl anomaly coefficients in the bosonic, supersymmetric and heterotic sigma models and to explain their interpretation as string theory equations of motion for spacetime background fields.

† Supported in part by DOE grant DE-AC02-84-15534
⋆ On leave from Princeton University

1. INTRODUCTION

The subject of this review is the connection between conformally invariant, two-dimensional, non-linear sigma models and string theory. Two-dimensional conformal field theories are of course of interest in their own right as models of critical phenomena, but we will focus on those aspects of the subject which are of use in understanding string theory.

We will cover the following topics: In Chapter 2 we ask the question why is two-dimensional, conformal field theory relevant to string theory? In Chapter 3 we do a simple, back of the envelope, one-loop calculation of the Weyl anomaly (roughly, but not identically, equivalent to the renormalization group beta function) for the bosonic non-linear sigma model. If one is interested in using beta functions to discuss string theory, there are a number of issues (not arising in the statistical mechanics application) which must be resolved. We use the one-loop calculation as a guide to what we can expect in general. In Chapter 4 we turn to systematics and the problem of higher loop calculations. There are some subtleties here and we will give a guide to what is known about them. In Chapter 5 we will discuss the new issues which arise when we consider the supersymmetric and heterotic sigma models appropriate to the study of the physics of superstrings.

These notes represent a modest expansion of material presented in four lectures at the 1988 TASI School and follow the format of those lectures quite closely. A vast amount of work has been done in this field over the last few years and we do not claim, or aim, to cover it all. We do hope to have explained the basic ideas and shown, explicitly where possible, how to do some of the basic calculations in this important area of string theory. Given the limitations on time and energy available to the authors, much of importance has inevitably been left out. In particular, the references are not meant to be comprehensive, but simply to provide the reader with useful entry points into the literature. In various places we have made heavy use of existing reviews or comprehensive articles on restricted aspects of this subject (referred to at appropriate places in the text) and we heartily recommend them to the attention of the serious student.

2. WHY TWO-DIMENSIONAL CONFORMAL FIELD THEORY?

2.1. The Nambu-Goto and Polyakov actions

When we discuss string theory it is natural to write down a two-dimensional field theory. One starts with the Nambu-Goto action[1)], describing a string propagating in some D dimensional spacetime. The simplest action one can write down is the invariant area of the worldsheet that is swept out by that string as it moves through spacetime. The string is moving in a spacetime that has some D-dimensional metric which is not necessarily flat. (The original Nambu-Goto action described strings in flat spacetime but some critical problems in string theory, such as compactification of extra dimensions, can only be discussed in non-flat spaces, so we might as well let our original classical string propagate in curved spacetime.) To calculate the invariant area, put a coordinate system down on the worldsheet and calculate the induced metric. The Nambu-Goto action density is the square root of the determinant of that induced metric and can be shown to have the explicit form

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{(\dot{X} \cdot \dot{X})(\acute{X} \cdot \acute{X}) - (\dot{X} \cdot \acute{X})^2}. \tag{2.1}$$

The notation is

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \quad \text{and} \quad \acute{X}^\mu = \frac{\partial X^\mu}{\partial \sigma} \quad \text{for} \quad \mu = 1, \ldots, D$$

and

$$(A \cdot B) = G_{\mu\nu}(X) A^\mu B^\nu.$$

It is easy to see that this is indeed the invariant area of the worldsheet swept out by some embedding $X^\mu(\sigma, \tau)$ of a string into a spacetime that has a metric

$G_{\mu\nu}(X)$. The spacetime coordinate functions, $X^\mu(\sigma, \tau)$, have units of length, so a dimensional parameter is needed in front of the action. The α' in (2.1) has units of $(\text{length})^2$. The length scale it sets depends on the physics the string theory is designed to describe. In its most convincing form, string theory is a quantum theory of gravity and the natural length scale is the Planck length,

$$L_{Planck} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-34}\,\text{m}$$

Whatever its value, α' plays the role of a coupling constant in the two-dimensional quantum field theory. People often adopt "natural" units in which α' is something simple like 2 or $\frac{1}{2}$, but we find it convenient to keep it visible in the notation to keep track of loop orders in perturbation theory.

This is all very nice except that the equations of motion that follow from the Nambu-Goto action are hideously non-linear. A more civilized, easier to quantize, and classically equivalent choice is the Polyakov action[2,3)]

$$S_P = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X). \tag{2.2}$$

This action has a new element that was in a way implicitly present in S_{NG}: the worldsheet metric, γ_{ab}, now appears as an independent object, whereas before we identified it with the induced metric. The Polyakov action has a number of virtues over the Nambu-Goto action. Taken as an abstract object it is something which is quadratic in derivatives of the X's. Eventually the X's are going to be dynamical variables. The string is wiggling along in spacetime and we want to quantize this motion. Something which is quadratic in derivatives is relatively simple to quantize. The Nambu-Goto action is, on the other hand, non-polynomial in derivatives and does not yield to quantization easily. String physics should of course not depend on the coordinates one chooses on the worldsheet, and both actions are invariant to worldsheet reparametrizations (the Polyakov action is invariant only if one transforms the metric along with the coordinates).

A crucial point is that, at the classical level, the physics of S_P reduces to that of S_{NG}. Define the worldsheet energy momentum tensor in the standard action principle way as the variation of S_P with respect to the worldsheet metric, γ^{ab}:

$$\begin{aligned} T_{ab} &= \frac{4\pi}{\sqrt{\gamma}} \frac{\delta S_P}{\delta \gamma^{ab}} \\ &= \frac{1}{\alpha'} G_{\mu\nu}(X)\left(\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\gamma_{ab}\partial^c X^\mu \partial_c X^\nu\right). \end{aligned} \tag{2.3}$$

If one imposes the constraint $T_{ab} = 0$, that is to say, if one focuses on those solutions of the equations of motion which correspond to vanishing T_{ab}, then something nice happens. Namely, (2.3) fixes γ_{ab} and if you insert it into the Polyakov action, that action becomes identical to the Nambu-Goto action. Such an insertion is in order because S_P does not depend on derivatives of the worldsheet metric, so setting its variation with respect to γ_{ab} to zero is an appropriate constraint equation. This eliminates the extra degrees of freedom, associated with the variables γ_{ab}, which have no obvious physical interpretation anyway. All the physics should be contained in the embedding of the string into spacetime, that is, in the X's. The Nambu-Goto action has only the variables one wants but it is not a good starting point for a quantum theory. It is straightforward to quantize the Polyakov action but on the other hand it has more degrees of freedom than one cares to have. The constraint that the two-dimensional energy momentum tensor vanish remedies that defect, at least at the classical level. It therefore seems reasonable that the Polyakov action is the place to start learning about the physics of strings.

There is another way in which to arrive at S_P. Given that we want to study two-dimensional field theory of scalar variables like the X's, the available field is considerably narrowed down if we insist on both two-dimensional reparametrization invariance and renormalizability. A two-dimensional action is renormalizable if the Lagrangian is of scaling dimension two or less. Scalars are of dimension zero and each derivative adds one to the dimension. So if one wants an action which is just exactly renormalizable, neither super-renormalizable nor non-renormalizable, one can only write down something which is quadratic in derivatives of X, but it can

have an arbitrary function of X going along with it. The Polyakov action is not the only action of this kind. There are several other possible terms, which we will eventually write down, when we discuss the full sigma model approach to bosonic string theory.

We first want to study the system described by the Polyakov action alone. We will treat it as a two-dimensional quantum field theory. For our purposes here, we take the two-dimensional γ_{ab} to be some fixed metric and eventually we shall see that the physics we are interested in is by and large quite independent of the choice of this metric. So, we have some crumpled two-dimensional surface and on it live scalar variables, X^μ. There are D of them and at the start we do not necessarily know what D is. The Lagrangian for the X^μ fields has a kinetic term and there are also potential terms. In fact, for particular choices of the function $G_{\mu\nu}(X)$ this class of actions is something one is familiar with in a variety of contexts. For example when $G_{\mu\nu}$ is the metric on the surface of a sphere, this becomes the $SO(D)$ non-linear sigma model, which is commonly studied in statistical mechanics. We are not going to make any *a priori* specifications about the D-dimensional world in which the X's take values and we want to consider a perfectly general $G_{\mu\nu}(X)$. Of course, we shall see later on that one must make restrictions in order for this two-dimensional field theory to correspond to a satisfactory string theory.

2.2. Weyl invariance versus conformal invariance

Let us turn back to the constraint equations. We would like all components of T_{ab} to vanish. One would therefore expect three independent equations. However, in two dimensions, the trace of the classical expression (2.3) is identically zero, as is easily checked. This is an important point, which we will have more to say about later, because while this identity holds in the classical theory it is not necessarily true after we have quantized it. That is, in fact, where the question of the precise form of the spacetime metric, $G_{\mu\nu}(X)$, comes in. The classical vanishing trace identity is anomalous and the requirement that the anomaly vanish leads to equations for the spacetime metric. As we shall see later, making the trace of the energy momentum tensor vanish is a symmetry condition which corresponds to maintaining conformal invariance in the two-dimensional quantum theory. This should be taken care of first, after which it is possible to impose the conditions that the remaining two independent components of the energy momentum tensor vanish also.

Now we turn to a general discussion of conformal invariance in two-dimensional field theories to provide us with a framework for our subsequent discussion. Ref. [4] gives an excellent detailed account of this subject. Consider some two-dimensional theory with an action $A(X,\gamma)$, which is perhaps more general than S_P. From the point of view of string theory, there are invariances which the action $A(X,\gamma)$ should have. First of all there is reparametrization invariance in the two dimensional sense. If you change the worldsheet coordinates by some function v^a,

$$\xi^a \longrightarrow \xi^a + v^a(\xi) \tag{2.4}$$

the variation of the metric is

$$\delta\gamma_{ab} = \nabla_a v_b + \nabla_b v_a \tag{2.5}$$

and that of the scalar fields is

$$\delta X^\mu = v^a \nabla_a X^\mu. \tag{2.6}$$

A reparametrization invariant action does not change under such a variation

$$A(X',\gamma') = A(X,\gamma). \tag{2.7}$$

That, as you know, leads to the conservation of the two-dimensional energy momentum tensor. In fact it is easy to see, using the equations of motion for the X's

(*i.e.*, invariance of $A(X,\gamma)$ under small changes of the X's alone), that

$$\nabla^a T_{ab} = 0, \tag{2.8}$$

where, as before, the energy momentum tensor is the response of the action to a variation in the metric

$$T_{ab} = \frac{4\pi}{\sqrt{\gamma(\xi)}} \frac{\delta A}{\delta\gamma^{ab}(\xi)}. \tag{2.9}$$

We would be ill-advised to write down an action that does not have reparametrization invariance and S_P, for example, certainly has that. S_P is also invariant to another class of transformations, the Weyl transformations, and our more general action may or may not have that invariance. A Weyl transformation only acts on the two-dimensional metric while leaving the X's unchanged,

$$\begin{aligned} \delta\gamma_{ab} &= -\,\delta\phi(\xi)\,\gamma_{ab}, \\ \delta X^\mu &= 0. \end{aligned} \tag{2.10}$$

This is an infinitesimal local rescaling of the metric. It changes the length scale on the worldsheet in a position dependent way. S_P is invariant to the finite version of this transformation

$$\gamma_{ab} \longrightarrow e^{\phi(\xi)}\gamma_{ab} \tag{2.11}$$

because, in two dimensions, the product of the square root of the determinant of the covariant components of the metric and any contravariant component of the metric is completely independent of such a rescaling. If our general action has this invariance it immediately follows that the trace of the classical energy momentum tensor vanishes,

$$\frac{\delta A}{\delta\phi} = 0 \Rightarrow T_a^a = 0. \tag{2.12}$$

Thus the trace of the energy momentum tensor obtained from S_P is identically zero because S_P has Weyl invariance. One can discover, in a number of ways, that this invariance cannot in general be maintained in the quantum theory. To characterize that failure (the anomaly) in a convenient way we want to introduce some more notation.

First of all, it is very convenient to make a coordinate transformation to a conformal gauge. Reparametrization invariance allows us to choose the two functions in (2.4) in any way that suits us. Since the two-dimensional metric has three independent components to begin with one can in general expect to be able to choose worldsheet coordinates such that only one degree of freedom remains. A convenient choice is the conformal gauge metric

$$\gamma_{ab} = e^{\phi(\xi)}\,\delta_{ab}, \tag{2.13}$$

i.e., a Weyl transform on a flat metric. We can represent the flat metric with complex coordinates and write the conformal gauge line element as

$$ds^2 = e^{\phi}\,dz d\overline{z} \tag{2.14}$$

so that the components of the metric are

$$\gamma_{zz} = \gamma_{\overline{z}\,\overline{z}} = 0, \qquad \gamma_{z\overline{z}} = \gamma_{\overline{z}z} = \frac{e^{\phi}}{2}. \tag{2.15}$$

A nice feature of this choice of parametrization is that the Christoffel connection turns out to have no mixed components. The only non-zero ones are

$$\Gamma^{z}_{zz} = \partial_z\phi \quad \text{and} \quad \Gamma^{\overline{z}}_{\overline{z}\,\overline{z}} = \partial_{\overline{z}}\phi. \tag{2.16}$$

The two-dimensional curvature is also particularly simple in this coordinate system:

$$\sqrt{\gamma}\,{}^{(2)}R = -2\partial_z\partial_{\overline{z}}\phi. \tag{2.17}$$

Covariant derivatives are simple in conformal gauge because the connection only

has a few non-vanishing components:

$$\begin{aligned} \nabla_{\overline{z}} v^z &= \partial_{\overline{z}} v^z \\ \nabla_z v^z &= \partial_z v^z + \partial_z\phi\, v^z. \end{aligned} \tag{2.18}$$

These formulas will be of use to us later on.

Once we are in conformal gauge, a generic coordinate transformation will take us out of it, but there is a special subset, called conformal reparametrizations, which still leaves us in a conformal gauge. A conformal reparametrization is a coordinate transformation characterized by a $v^z(z)$ which is a function of z alone (and likewise $v^{\overline{z}}(\overline{z})$ is a function of $\overline{z}$ only), whereas in general v^z can be a function of both z and $\overline{z}$. A quick look at (2.5) should convince you that such a reparametrization leaves γ_{zz} and $\gamma_{\overline{z}\,\overline{z}}$ unchanged and that the metric remains in the form (2.13). The variation $\delta\gamma_{z\overline{z}}$ does not vanish and indeed this transformation just leads to a change in the scale factor ϕ:

$$\delta\phi = \nabla_z v^z + \nabla_{\overline{z}} v^{\overline{z}}. \tag{2.19}$$

A conformal reparametrization is simply a coordinate change using a transformation function which is appropriately analytic. It causes a change both in the two-dimensional metric and the fields. Let us also define a conformal transformation:

$$\begin{aligned} \delta X^\mu &= v^z\partial_z X^\mu + v^{\overline{z}}\partial_{\overline{z}} X^\mu, \\ \delta\gamma_{ab} &= 0\,, \end{aligned} \tag{2.20}$$

where $v^z(z)$ $\left(v^{\overline{z}}(\overline{z})\right)$ is analytic (antianalytic). A conformal transformation is a dynamical transformation which acts only on the fields (the X's) with no compensating action on the metric and it is a good idea to distinguish it from a conformal reparametrization which changes both the fields and the metric. In point of fact the variation of the metric under a conformal reparametrization is just a change in the scale factor, ϕ, and that is precisely what a Weyl transformation would cause.

Therefore if one has a conformal transformation, which acts only on the X's, accompanied by an appropriate Weyl transform, which acts only on the scale factor of the metric, the two taken together are a conformal reparametrization. Since we insist on reparametrization invariance this means that the effect of a conformal transformation on the fields in the action can be achieved by an astutely chosen Weyl transformation of the two-dimensional metric. Therefore it is actually safe to be careless and not distinguish between the two: invariance under one is equivalent to invariance under the other.

2.3. Quantum theory and the anomaly

Now we finally turn to the quantum mechanical path integral and discuss the effect of the above transformations on it. Our treatment of these matters follows closely that of S. Jain [5]. The result of doing a path integral over the X's with our classical action defines the partition function of the system and that can, as usual, be written as the exponential of a connected generating functional

$$\begin{aligned} Z[\gamma] &= \int [DX]_\gamma \, e^{-A[X,\gamma]} \\ &= e^{-W[\gamma]}. \end{aligned} \tag{2.21}$$

We only integrate over the X's and take the worldsheet metric to be a fixed (but arbitrary) background. Elsewhere in this volume, Giddings talks about further integrations over the two-dimensional metric but we will anticipate that, at least for the applications we discuss, such integrations will not be needed. On the other hand the effective action, W, certainly depends on the worldsheet metric we choose. The classical action is explicitly a functional of γ_{ab} but the functional measure of the X's also implicitly depends on it. We need to introduce some inner product in order to measure volume elements on the function space of the X's. Doing that in a reparametrization invariant way, from the worldsheet point of view, inevitably brings in the two-dimensional metric[4]. When we work with the Polyakov action

a natural choice of inner product is

$$||\delta X||^2 = \int d^2\xi \, \sqrt{\gamma(\xi)} \, \delta X^\mu \delta X^\nu G_{\mu\nu}(X) \tag{2.22}$$

which in fact involves both γ_{ab} and $G_{\mu\nu}(X)$, the spacetime metric.

Doing the path integral is of course no trivial matter and the only general method we know is to calculate in some perturbative fashion. Eventually we will do explicit perturbation theory calculations but for the moment let us pretend someone has given us some non-perturbative results and proceed to discuss general properties of the quantum effective action. Now W had better be invariant to worldsheet reparametrizations (2.4). Since it depends only on the two-dimensional metric, we must insist that

$$0 = \int d^2\xi \, \frac{\delta W}{\delta\gamma^{ab}} (\nabla^a v^b + \nabla^b v^a). \tag{2.23}$$

In a conformal gauge this reads

$$0 = \int d^2\xi \, [\frac{\delta W}{\delta\gamma^{zz}} \nabla^z v^z + \frac{\delta W}{\delta\gamma^{\bar z \bar z}} \nabla^{\bar z} v^{\bar z} - \frac{\delta W}{\delta\phi} (\nabla_z v^z + \nabla_{\bar z} v^{\bar z})] \tag{2.24}$$

and by integrating the various terms by parts we can lift the covariant derivatives from v^z and $v^{\bar z}$ to obtain

$$\begin{aligned} 0 = \int d^2\xi \, \sqrt{\gamma} \Big[&\Big(\nabla_z (\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta\phi}) - \nabla^z (\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta\gamma^{zz}}) \Big) v^z \\ + &\Big(\nabla_{\bar z} (\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta\phi}) - \nabla^{\bar z} (\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta\gamma^{\bar z \bar z}}) \Big) v^{\bar z} \Big]. \end{aligned} \tag{2.25}$$

The functions v^z and $v^{\bar z}$ are arbitrary so we can read off two equations, i.e.

$$\nabla_z (\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta\phi}) = \nabla^z (\frac{1}{\sqrt{\gamma}} \frac{\delta W}{\delta\gamma^{zz}}) \tag{2.26}$$

and the corresponding one with z replaced by $\bar z$. This identity is the quantum analog of the classical conservation equation (2.8), if we think of the variation of the effective action, W, with respect to the two-dimensional metric as playing the role of the energy momentum tensor. We can legitimately argue that the right hand side of (2.26) is ∇^z of the quantum expectation value of the zz component of the energy momentum tensor. The reasoning goes as follows[5]: When we vary W with respect to γ^{zz} we get

$$\frac{\delta W}{\delta \gamma^{zz}} = -\frac{1}{Z}\frac{\delta Z}{\delta \gamma^{zz}}. \tag{2.27}$$

The partition function depends on the two-dimensional metric both through the classical action and the path integral measure. The variation of the classical action brings down a factor of the classical energy momentum tensor inside the path integral, which gives us an expectation value when we divide by Z. The question is whether any unwanted terms arise from the variation of the path integral measure. A reparametrization invariant measure is defined using the inner product (2.22), which in turn only depends on the determinant of the worldsheet metric. In a conformal gauge the form of the metric plus its variation is

$$\begin{pmatrix} \delta\gamma_{zz} & \gamma_{z\bar{z}} + \delta\gamma_{z\bar{z}} \\ \gamma_{\bar{z}z} + \delta\gamma_{\bar{z}z} & \delta\gamma_{\bar{z}\bar{z}} \end{pmatrix}$$

From this it is apparent that a first order variation of the determinant of the metric with respect to γ^{zz} vanishes, and therefore the path integral measure is invariant to that variation. This means that we can write

$$\nabla^z\left(\frac{4\pi}{\sqrt{\gamma}}\frac{\delta W}{\delta\gamma^{zz}}\right) = \nabla^z \langle T_{zz}\rangle . \tag{2.28}$$

It is tempting to similarly identify the left hand side of (2.26) with ∇_z of the expectation value of the trace of the energy momentum tensor. In a conformally invariant theory the trace of the energy momentum tensor vanishes, and (2.26) would be telling us that $\partial_{\bar{z}}\langle T_{zz}\rangle = 0$, a powerful and useful statement. In reality it does not quite work like that. Unfortunately the left hand side of (2.26) is not in general zero because due to a quantum anomaly the variation of the path integral

measure with respect to ϕ does not vanish. However, we do know that the left hand side of (2.26) is a z derivative of something. If we make the assumption that this something is local on the worldsheet we can pretty much pin down its structure. A more general assumption is conceivable but in explicit calculations in free field theory (i.e. with a flat spacetime metric), or even interacting theories evaluated perturbatively, it has always turned out to be a local object and we will assume that is true in general. The right hand side of (2.26) is a tensor of type t_z under conformal reparametrization and is of scaling dimension one on the worldsheet. In the absence of any extraneous dimensional parameters, the only local function of γ_{ab} which has those conformal transformation properties and scaling dimension is a z derivative of the scalar curvature ${}^{(2)}R$. The assumption of locality combined with dimensional analysis therefore tells us what the form of the left hand side of (2.26) is:

$$\nabla_z\left(\frac{1}{\sqrt{\gamma}}\frac{\delta W}{\delta\phi}\right) = \frac{\lambda}{48\pi}\nabla_z\, {}^{(2)}R. \tag{2.29}$$

We cannot determine the constant of proportionality, λ, by general arguments, as it is characteristic of the theory in question. The $\frac{1}{48\pi}$ normalization is of historical origin. The fact that we can have anything else than zero on the right in (2.29) signals that conformal symmetry is in general anomalous in two-dimensional field theory, but we have parametrized that potential anomaly in a particularly simple way. From (2.29) we can get the form of the anomalous part of the effective action itself. First we integrate both sides with respect to z to get

$$\frac{\delta W}{\delta\phi} = \frac{\lambda}{48\pi}\sqrt{\gamma}\left({}^{(2)}R + \mu^2\right). \tag{2.30}$$

Now recall that the conformal scale factor always refers the metric to some given reference metric,

$$\gamma_{ab} = e^{\phi}\,\hat{\gamma}_{ab}. \tag{2.31}$$

(In a conformal gauge $\hat{\gamma}$ is chosen to be the flat metric.) Keeping this in mind it is easily checked that (2.30) is satisfied by

$$W = \frac{\lambda}{48\pi}\int d^2\xi\,\sqrt{\hat{\gamma}}\,\big(\frac{1}{2}\,\hat{\gamma}^{ab}\,\partial_a\phi\,\partial_b\phi + \mu^2\,e^{\phi}\big) \\ + \quad \text{conformally invariant terms.} \tag{2.32}$$

The assumption that the anomaly is local has enabled us to characterize the conformally non-invariant part of the quantum effective action by one dimensionless parameter, λ, and a dimensional parameter μ (which turns out not to play any role in our considerations). The terms in W that depend on the two-dimensional scale factor, ϕ, are called the Liouville action. It is awkward, from the point of view of string theory, that W depends on our choice of worldsheet metric at all, and a lot of work has been concerned with getting rid of this dependence in various ways. As we shall see later on from an explicit computation, the coefficient λ is not zero even in free field theory, much less in interacting systems. The fact that the path integral is not Weyl invariant is generic to conformal field theory in two dimensions and is something one has to deal with.

2.4. Operator product expansion and the Virasoro algebra

One can make some further observations. Combining (2.26) and (2.29) one sees that the vacuum expectation value of T_{zz} is not analytic because of the Weyl (or conformal) anomaly:

$$\gamma^{z\,\bar{z}}\,\partial_{\bar{z}}\,\langle T_{zz}\rangle = \nabla^z\,\langle T_{zz}\rangle = \frac{\lambda}{48\pi}\nabla_z\,{}^{(2)}R. \tag{2.33}$$

However it is possible to 'improve' the energy momentum tensor so as to make the expectation value analytic. We can use the conformal gauge identities (2.15)-(2.18) to show that

$$\nabla_z^{(2)}R = \nabla^z(-2\partial_z\partial_z\phi + \partial_z\phi\,\partial_z\phi). \tag{2.34}$$

This means that if we define the zz component of an improved energy momentum tensor by

$$T_{zz}^{(0)} = T_{zz} + \frac{\lambda}{48\pi}(2\partial_z^2\phi - (\partial_z\phi)^2), \tag{2.35}$$

then its expectation value is indeed analytic,

$$\gamma^{z\,\bar{z}}\,\partial_{\bar{z}}\left\langle T_{zz}^{(0)}\right\rangle = \nabla^z\left\langle T_{zz}^{(0)}\right\rangle = 0. \tag{2.36}$$

Because of the special role of analytic reparametrizations, and the power of analytic function theory, it is very useful to have operators with analytic expectation values.

In the remainder of this section we will outline how one obtains an operator product expansion for this improved energy momentum tensor and indicate how it can be converted into the Virasoro algebra. First note that the expectation value in (2.36) refers to a specific worldsheet metric. The equation of course holds for any choice of metric and we can, if we like, do a variation with respect to the metric. The variation of the classical action inside the path integral brings down a second factor of the energy momentum tensor in the expectation value. The variation of course also acts on the covariant derivatives and the scalar curvature and that generates other terms. Say the variation is with respect to γ^{zz} at a point, w, different than at which the original $T_{zz}(z)$ is evaluated. Then we get an equation involving a differentiated correlation function of two T's. After performing an integration and going through the steps of replacing the energy momentum tensor with the improved one we arrive at (see the work of S. Jain[5]) for a detailed derivation)

$$\left\langle T_{zz}^{(0)}T_{ww}^{(0)}\right\rangle = \frac{\lambda}{2}\frac{1}{(z-w)^4} + \frac{\left\langle T_{ww}^{(0)}\right\rangle}{(z-w)^2} + \frac{\partial_w\left\langle T_{ww}^{(0)}\right\rangle}{(z-w)} + \text{ regular terms.} \tag{2.37}$$

The remaining terms are regular in the sense that they have no singularities as z approaches w. The coefficient of the leading singularity is precisely the λ which characterizes the failure of naive conformal invariance. Here that failure manifests itself as an anomalous term in a short distance expansion of a product of analytic energy momentum tensors.

(2.37) has the same content as the energy momentum tensor operator product expansion that comes up in discussions of conformal field theories. In fact it can be used to generate the Virasoro algebra. Expand $T^{(0)}_{zz}$ in a Laurent series

$$T^{(0)}_{zz}(z) = \sum_{n=-\infty}^{\infty} z^{-n-2} L_n. \tag{2.38}$$

The operator coefficients, L_n, are the famous Virasoro generators. By integrating along astutely chosen contours and making use of the singularity structure of the various terms one can turn (2.37) into a statement about commutators of these L_n's [6]. They turn out to satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{\lambda}{12} n(n^2-1)\, \delta_{m,-n}. \tag{2.39}$$

All along there was the anti-analytic $T_{\bar{z}\,\bar{z}}$ component, which is an independent object and defines for you another set of operators, $\tilde{L}_n$'s. They also satisfy the Virasoro algebra among themselves but commute with the L_n's.

At this point we can finally state how the classical constraint that the components of the energy momentum tensor vanish is implemented in the quantum theory. Because of Weyl anomalies we can in general expect complications as the trace of the energy momentum tensor fails to be zero. In spite of that, if the anomaly has a simple local form characterized by a single c-number, we still find that expectation values of the other independent components of the energy momentum tensor behave like analytic objects. In order to enforce the remaining constraint equations we would like to put T_{zz} and $T_{\bar{z}\,\bar{z}}$ to zero, that is have the L_n and $\tilde{L}_n$ operators annihilate all states. However, that is not consistent with the algebra these operators satisfy. The algebra does allow us to set half of them equal to zero. One defines physical states to be those that are annihilated by all L_n and $\tilde{L}_n$ with $n > 0$ and not by the negative n operators. This means that expectation values of all the Virasoro generators, except maybe L_0 and $\tilde{L}_0$, vanish for physical states. In a healthy theory it is only the physical states that contribute to S-matrix elements.

The bottom line is the following. If you start with a two-dimensional theory which is conformally invariant at the classical level you have to be aware that the conformal symmetry will be anomalous. That means basically an anomaly in the trace of the energy momentum tensor. In the next chapter we will compute it explicitly for a specific form of the action, $A[X,\gamma]$, with non-trivial interactions built into it. What we wanted to show in this section was that even when there is an anomaly, most of the physics one wants to extract from the theory is maintained. There is still an improved energy momentum tensor, whose components have nice analytic or anti-analytic short distance expansions, from which you can get an algebra for the Virasoro generators and that algebra allows you to define physical states which give zero expectation values of the analytic and anti-analytic components of the energy momentum tensor. So even if there are anomalies, they are not catastrophic and in all the cases we will come across they leave the interesting physics of two-dimensional conformal field theories intact.

3. GENERAL RENORMALIZABLE, WEYL INVARIANT, TWO-DIMENSIONAL SCALAR FIELD THEORY

3.1. Reparametrization invariant, renormalizable sigma models

The Polyakov action (2.2) is both reparametrization invariant and power counting renormalizable. Furthermore, it is Weyl invariant. We can only add one other term with all these properties and that is one where the spacetime coupling function is antisymmetric in spacetime indices:

$$S_{AS} = \frac{1}{4\pi\alpha'} \int d^2\xi\, \epsilon^{ab}\, \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}. \tag{3.1}$$

Here ϵ^{ab} is the two-dimensional, antisymmetric Levi-Civita symbol. It is a tensor density rather than a tensor, so we do not need a factor of $\sqrt{\gamma}$ for a reparametrization invariant two-dimensional measure.

In general we can expect renormalization to bring in mixing with all possible terms of the same (or lower) dimension as S_P and S_{AS}. On a curved worldsheet we can write down one more term of dimension two[7],

$$S_D = \frac{1}{8\pi}\int d^2\xi\,\sqrt{\gamma}\,{}^{(2)}\!R\,\Phi(X). \tag{3.2}$$

This is reparametrization invariant and the coupling, $\Phi(X)$, is a scalar function in spacetime. On the other hand this term is not Weyl invariant. Since we want our classical theory to be Weyl invariant the plan is to view S_D as entering at higher loop order than the other two terms, and cancel its tree level Weyl variation against one-loop Weyl anomalies arising from the others, and so on. This view of S_D is supported by dimensional analysis in spacetime. The coupling function, $\Phi(X)$, carries no units so we do not need a factor of α' in the normalization of S_D as we do in front of S_P and S_{AS}. Since α' is the loop counting parameter this means that S_D first contributes at one-loop rather than the classical level. It is perhaps not obvious that this procedure is correct but at the end of the day we will see that it leads to consistent results describing reasonable spacetime physics.

There are no other reparametrization invariant terms, of dimension two, we can add to our action. The coupling functions $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$ correspond to condensates of the massless states of the bosonic string theory: gravitons, antisymmetric tensors and dilatons[7,8,9] . This sigma model approach is therefore often referred to as the study of strings in background fields and we will soon see why this identification of coupling constant functions with spacetime fields makes sense.

Renormalization will not only mix the terms of dimension two, but also brings in mixing to any terms of lower dimension. There is one possible dimension zero term that can be added to the action

$$S_T = \frac{1}{4\pi}\int d^2\xi\,\sqrt{\gamma}\,T(X). \tag{3.3}$$

It describes a coupling to a background of tachyon field. It is reparametrization invariant but not Weyl invariant. Counterterms of this form are actually needed to

eat quadratic divergences in vacuum diagrams of the theory. They are not needed in superstring theory and, it turns out, play no significant role in the following discussion of purely bosonic theories.

We have not said anything yet about the topology of the string worldsheet. First of all we have to decide whether the game is to be played with open strings or not. An open string worldsheet has boundaries where boundary conditions for the X's need to be specified. In fact there can be a coupling on the boundary to a condensate of open string massless states, *i.e.*, a Wilson line coupling to a gauge field,

$$S_A = i\oint ds\,A_\mu(X)\,\frac{dX^\mu}{ds}, \tag{3.4}$$

where $X^\mu(s)$ is the mapping of the boundary into spacetime. An open string can only have gauge charges on its ends, so any interaction with a background gauge field necessarily takes place there. Maintaining conformal invariance of the full quantum theory, with interactions both in the interior of the worldsheet and on the boundary, places restrictions on all the spacetime couplings [10,11]. However, one can consistently study closed strings on their own, and in fact the most promising string theories do not have any open strings. We will, for simplicity, work only with closed string backgrounds and assume that the worldsheet has no boundaries. In Chapter 5, when we talk about heterotic string theory, we will see how coupling to gauge fields can arise even in the absence of open strings.

A closed string worldsheet can have any number of handles. The Polyakov approach to quantum string theory in fact instructs us to sum over amplitudes on surfaces of every genus. We will only be concerned with 'tree level' string theory, in that our calculations assume that the worldsheet is conformally equivalent to the Riemann sphere, *i.e.*, has no handles. On the other hand, we will be computing an anomaly, which describes ultra-violet, or short distance, physics of the non-linear sigma model. The result does not depend on the global properties of the worldsheet. From the two-dimensional field theory point of view the structure of a local anomaly is indifferent to handles far away on the worldsheet. Since the requirement that the Weyl anomaly vanish is what places restrictions on spacetime fields, this would seem to say that the spacetime physics is completely determined at string tree level without any corrections from string loop amplitudes (*i.e.*, from quantum string theory). That is an unlikely state of affairs and recent years have seen an effort to generate such string loop corrections. A promising approach involves canceling divergences in so called modular integrations, associated with higher genus surfaces, against field theory divergences of the kind we are discussing in this review. We refer the interested reader to the literature[12,11] and turn back to standard sigma model considerations.

3.2. Background field expansion and normal coordinates

In the previous section we wrote down the three possible reparametrization invariant action terms of scaling dimension two. Consider a two-dimensional field theory with a classical action which includes these three terms,

$$A[X,\gamma] = S_P + S_{AS} + S_D. \tag{3.5}$$

It defines a general, bosonic non-linear sigma model and is usually taken as the starting point of any systematic study of string theory in non-trivial background fields. All three coupling functions $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$ transform covariantly under spacetime general coordinate transformations and in addition S_{AS} is invariant to spacetime 'gauge transformations' of the antisymmetric tensor

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \tag{3.6}$$

where $\Lambda_\mu(X)$ is some vector function. It is desirable to arrange the perturbation expansion so that these spacetime symmetries are manifest. That means we want to calculate diagrams using covariant vertices and propagators and use a regularization procedure for divergent loop integrals which is compatible with general covariance in spacetime. Then any counterterms we come across will also be covariant and the conformal anomaly will be expressed in a spacetime coordinate

invariant manner. The way to achieve this is through a trick, called the covariant background field expansion[13]. The basic idea is to separate the two-dimensional fields into a 'background' part and a 'quantum' part

$$X^\mu(\xi) = X_0^\mu(\xi) + \pi^\mu(\xi) \tag{3.7}$$

and then shift the path integration to be over the π^μ's only (which is why they are termed quantum fields). The background fields here live in the two-dimensional theory and should not be confused with the spacetime backgrounds described by the coupling functions. Following ref. [13] we define the background field partition function as

$$\Omega[X_0,\gamma] = \int [D\pi]\, e^{-\{A[X_0+\pi]-A[X_0]-\int d^2\xi \, \frac{\delta A}{\delta X_0^\mu(\xi)}\pi^\mu(\xi)\}}. \tag{3.8}$$

The next step is to expand the classical action in powers of the quantum field, π^μ, and derive Feynman rules for diagrams. As usual the propagator is obtained from the quadratic term. It depends on the background X_0^μ's, which are considered as classical functions here. The cubic and higher terms in the expansion give rise to background field dependent interaction vertices with ever more legs. $\Omega[X_0,\gamma]$ can be viewed as a generating functional for loop diagrams with all external trees amputated. At one-loop order, cancelling the divergent part of the quantity $-\log\Omega[X_0,\gamma]$ gives the counterterms needed for the effective action. At two loops it is the usual story: we have to compute all one-particle-irreducible two-loop diagrams and also one-loop diagrams involving insertions of the counterterms obtained at one loop.

We actually have to do a little more work before we can proceed. While this diagrammatic expansion leads to a well defined perturbation theory, and would doubtless give us correct results, it is not manifestly covariant from the spacetime point of view. The reason is that the quantum field, $\pi^\mu(\xi)$, is defined as a coordinate difference in spacetime (between the value of the full field $X^\mu(\xi)$ and the background field $X_0^\mu(\xi)$) and therefore does not transform as a vector under general coordinate transformations. We need to replace it with an integration variable, in the path integral, which is a spacetime vector. A natural choice is a tangent vector to the spacetime geodesic which connects the points X_0^μ and $X_0^\mu + \pi^\mu$. We assume that there exists a unique such geodesic, $\lambda^\mu(t)$, and we will choose the affine parameter t so that $\lambda^\mu(0) = X_0^\mu(\xi)$ and $\lambda^\mu(1) = X_0^\mu(\xi) + \pi^\mu(\xi)$. Now let η^μ be the tangent to $\lambda^\mu(t)$ at X_0^μ, that is define $\eta^\mu = \dot\lambda^\mu(0)$, where the dot stands for $\frac{d}{dt}$. The geodesic equation for $\lambda^\mu(t)$ is

$$\ddot\lambda^\mu(t) + \Gamma^\mu_{\nu\sigma}\dot\lambda^\nu(t)\dot\lambda^\sigma(t) = 0. \tag{3.9}$$

Repeated use of this equation allows us to write the Taylor expansion of $\lambda^\mu(t)$, around $t = 0$, in terms of η^μ and spacetime Christoffel symbols:

$$\lambda^\mu(t) = X_0^\mu + \eta^\mu t - \frac{1}{2}\Gamma^\mu_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2}\,t^2 - \frac{1}{3}\Gamma^\mu_{\sigma_1\sigma_2\sigma_3}\eta^{\sigma_1}\eta^{\sigma_2}\eta^{\sigma_3}\,t^3 + \ldots \tag{3.10}$$

The higher order Γ symbols stand for differentiated Christoffel symbols, $\Gamma^\mu_{\sigma_1\ldots\sigma_n} \equiv \nabla'_{\sigma_1}\ldots\nabla'_{\sigma_{n-2}}\Gamma^\mu_{\sigma_{n-1}\sigma_n}$ where ∇'_σ means a formal covariant derivative acting only on the lower indices. At t=1 (3.10) defines a coordinate transformation in the neighborhood of X_0^μ to a set of coordinates, η^μ, which are called Riemann normal coordinates. The old quantum field, π^μ, is expressed as a power series in the normal coordinates

$$\pi^\mu = \eta^\mu - \frac{1}{2}\Gamma^\mu_{\sigma_1\sigma_2}\eta^{\sigma_1}\eta^{\sigma_2} + \ldots \tag{3.11}$$

If we had started out in the normal coordinate system we could still have gone through the arguments leading to (3.11) but this time of course only the first term on the right hand side would be there. This allows us to deduce some nice properties of the normal coordinate system. In it all components of the Christoffel symbol vanish and furthermore all the higher order $\Gamma^\mu_{\sigma_1\ldots\sigma_n}$ symbols also vanish

when you symmetrize their lower indices,

$$\begin{aligned} \overline{\Gamma}^\mu_{\sigma\rho} &= 0 \\ \overline{\Gamma}^\mu_{(\sigma_1\ldots\sigma_n)} &= 0. \end{aligned} \tag{3.12}$$

The bars indicate that these relations only hold in the normal coordinate system. The expression for the curvature tensor also simplifies in normal coordinates,

$$\overline{R}^\mu{}_{\nu\sigma\rho} = \partial_\sigma\overline{\Gamma}^\mu_{\nu\rho} - \partial_\rho\overline{\Gamma}^\mu_{\nu\sigma}, \tag{3.13}$$

and we can in fact combine (3.13) with (3.12), for $n = 3$, to express derivatives of the Christoffel symbols in terms of the Riemann tensor,

$$\partial_\nu\overline{\Gamma}^\mu_{\sigma\rho} = \frac{1}{3}(\overline{R}^\mu{}_{\sigma\nu\rho} + \overline{R}^\mu{}_{\rho\nu\sigma}). \tag{3.14}$$

Using (3.12) for $n > 3$, one can derive normal coordinate formulas relating symmetrized higher derivatives of Christoffel symbols to covariant derivatives of the curvature tensor, but we do not work them out here.

Once these normal coordinate relations have been established we can 'covariantize' the Taylor expansion of an arbitrary tensor. In the normal coordinate system itself, the Taylor expansion reads

$$\overline{T}_{\mu_1\ldots\mu_n}(X_0 + \eta) = \sum_{m=0}^{\infty}\frac{1}{m!}(\partial_{\nu_1}\ldots\partial_{\nu_m}\overline{T}_{\mu_1\ldots\mu_n}(X_0))\eta^{\nu_1}\ldots\eta^{\nu_m}. \tag{3.15}$$

Replacing the derivatives in the Taylor coefficients by covariant derivatives generates terms involving (symmetrized) derivatives of the connection. We can then use (3.14), and its higher order counterparts, to rewrite these in terms of the curvature tensor and its covariant derivatives. For example in the case of a second rank tensor we get

$$
\begin{aligned}
\overline{T}_{\mu\nu}(X_0+\eta) &= \overline{T}_{\mu\nu}(X_0) + \nabla_\lambda \overline{T}_{\mu\nu}(X_0)\,\eta^\lambda \\
&+ \frac{1}{2}\{\nabla_\lambda\nabla_\sigma \overline{T}_{\mu\nu}(X_0) - \frac{1}{3}\overline{R}^{\rho}_{\lambda\mu\sigma}\,\overline{T}_{\rho\nu}(X_0) - \frac{1}{3}\overline{R}^{\rho}_{\lambda\nu\sigma}\,\overline{T}_{\mu\rho}(X_0)\}\,\eta^\lambda\eta^\sigma \\
&+ \ldots
\end{aligned}
\tag{3.16}
$$

This expansion only involves spacetime tensors and covariant derivatives so it holds in any coordinate system (provided η is a vector) even though we used normal coordinates to derive it. It is a general covariant expansion of $T_{\mu\nu}(X_0+\pi)$ and we can drop the bars from the notation.

Now let us find the covariant expansion of the various sigma model terms. First consider the Polyakov action. The spacetime metric has a particularly simple expansion because it is symmetric and its covariant derivative vanishes. In that case (3.16) reduces to

$$G_{\mu\nu}(X_0+\pi) = G_{\mu\nu}(X_0) + \frac{1}{3}R_{\mu\lambda\sigma\nu}(X_0)\,\eta^\lambda\eta^\sigma + \ldots \tag{3.17}$$

In order to expand $\partial_a(X_0^\mu+\pi^\mu)$ we take a ∂_a derivative of both sides of (3.11) and then apply (3.14) to get

$$\partial_a(X_0^\mu+\pi^\mu) = \partial_a X_0^\mu + \nabla_a\eta^\mu + \frac{1}{3}R^{\mu}_{\lambda\sigma\nu}(X_0)\,\partial_a X_0^\nu\,\eta^\lambda\eta^\sigma + \ldots \tag{3.18}$$

where $\nabla_a\eta^\mu \equiv \partial_a\eta^\mu + \Gamma^\mu_{\lambda\sigma}(X_0)\partial_a X_0^\lambda\,\eta^\sigma$. Combining (3.17) and (3.18) gives

$$
\begin{aligned}
S_P[X_0+\pi] &= S_P[X_0] + \frac{1}{2\pi\alpha'}\int d^2\xi\,\sqrt{\gamma}\gamma^{ab}\,G_{\mu\nu}(X_0)\partial_a X_0^\mu\,\nabla_b\eta^\nu \\
&+ \frac{1}{4\pi\alpha'}\int d^2\xi\,\sqrt{\gamma}\gamma^{ab}\{G_{\mu\nu}(X_0)\nabla_a\eta^\mu\nabla_b\eta^\nu + R_{\mu\lambda\sigma\nu}(X_0)\partial_a X_0^\mu\partial_b X_0^\nu\eta^\lambda\eta^\sigma\} \\
&+ \frac{1}{3\pi\alpha'}\int d^2\xi\,\sqrt{\gamma}\gamma^{ab}\,R_{\mu\lambda\sigma\nu}(X_0)\partial_a X_0^\mu\,\eta^\lambda\eta^\sigma\nabla_b\eta^\nu \\
&+ \frac{1}{12\pi\alpha'}\int d^2\xi\,\sqrt{\gamma}\gamma^{ab}\,R_{\mu\lambda\sigma\nu}(X_0)\eta^\lambda\eta^\sigma\nabla_a\eta^\mu\nabla_b\eta^\nu \\
&+ \ldots
\end{aligned}
\tag{3.19}
$$

The term linear in η^μ is not of interest. We are free to choose any background X_0^μ

we like and if we arrange it to satisfy the classical equations of motion that follow from the Polyakov action the linear term in fact vanishes. The first of the quadratic terms in (3.19) involves two derivatives on the quantum fields, and is therefore the kinetic term of the theory. On the other hand it involves the spacetime metric, which is a function of the background field X_0^μ, so a propagator derived from it would be non-trivial. The way around this obstacle is to introduce a vielbein, $e^i_\mu(X_0), i=1,\ldots,D$, which refers the vectors η^μ to a local Lorentz frame,

$$\eta^i = e^i_\mu(X_0)\,\eta^\mu. \tag{3.20}$$

The vielbein satisfies

$$e^i_\mu(X_0)\,e^j_\nu(X_0)\,\delta_{ij} = G_{\mu\nu}(X_0) \tag{3.21}$$

where δ_{ij} is a flat, D-dimensional metric. The kinetic term is diagonal in the η^i coordinate system,

$$G_{\mu\nu}(X_0)\nabla_a\eta^\mu\nabla_b\eta^\nu = (\nabla_a\eta)^i(\nabla_b\eta)^i. \tag{3.22}$$

Here $(\nabla_a\eta)^i = \partial_a\eta^i + \omega^{ij}_\mu\partial_a X_0^\mu\eta^j$ and ω^{ij}_μ is the spin connection on spacetime. General coordinate invariance in spacetime is an SO(D-1,1) internal symmetry from the two-dimensional sigma model point of view and the field $A^{ij}_a(X_0) = \omega^{ij}_\mu(X_0)\partial_a X_0^\mu$ transforms like a Yang-Mills gauge potential under local Lorentz transformations. We of course have to break the gauge invariance to define a propagator for the η^i, but the point of the covariant background field expansion is that we maintain gauge covariance in terms of the background X_0^μ fields. It simplifies our work considerably to know that diagrams involving insertions of the gauge potential $A_a(X_0)$ have to combine to give gauge covariant objects (such as the spacetime curvature tensor) in order to give a non-vanishing contribution.

These considerations along with the fact that we get a particularly simple propagator from the $\partial_a\eta^i\partial^a\eta^i$ piece of (3.22) are good reasons to change variables in the path integral and integrate over the η^i local Lorentz frame fields. The path integral measure is defined in a spacetime coordinate invariant manner so the change of variables from π^μ to η^μ and then to η^i does not affect it.

Let us turn back to the covariant expansion of the action. (3.19) includes the terms in the expansion up to first order in the spacetime curvature. For one-loop diagrams we only use the second-order terms in η^i but we will need the quartic term for a two-loop calculation later on. Using the method we outlined above it is straightforward to generate further terms involving higher derivatives of the spacetime metric but it quickly gets tedious. We will not have use for such higher-order terms in the calculations we do here but if you embark on any computation beyond leading order in the curvature they will of course be needed. Fortunately there is an easier way to get them. In ref. [14] a simple recursive algorithm is developed which allows you to derive each successive order in the covariant background field expansion of S_P from the previous one, and it is used there to display the terms to sixth order in η^μ.

The covariant expansion of the antisymmetric tensor action out to second order in η^i reads:

$$
\begin{aligned}
&S_{AS}[X_0+\pi] = S_{AS}[X_0] \\
&+\frac{1}{2\pi\alpha'}\int d^2\xi\,\epsilon^{ab}\Big\{B_{\mu\nu}(X_0)\partial_a X_0^\mu\,\nabla_b\eta^\nu+\frac{1}{2}\nabla_\lambda B_{\mu\nu}(X_0)\,\partial_a X_0^\mu\partial_b X_0^\nu\,\eta^\lambda\Big\} \\
&+\frac{1}{4\pi\alpha'}\int d^2\xi\,\epsilon^{ab}\Big\{B_{\mu\nu}(X_0)\,\nabla_a\eta^\mu\nabla_b\eta^\nu+2\nabla_\lambda B_{\mu\nu}(X_0)\partial_a X_0^\mu\,\nabla_b\eta^\nu\eta^\lambda \\
&\quad+\frac{1}{2}[\nabla_\lambda\nabla_\sigma B_{\mu\nu}(X_0)+B_{\mu\rho}(X_0)\,R^\rho_{\ \lambda\sigma\nu}+B_{\rho\nu}(X_0)\,R^\rho_{\ \lambda\sigma\mu}]\partial_a X_0^\mu\partial_b X_0^\nu\,\eta^\lambda\eta^\sigma\Big\} \\
&+\dots
\end{aligned}
\tag{3.23}
$$

Again we assume that the background field satisfies the classical equation of motion and drop the linear terms. It is desirable to write the quadratic piece in terms of the antisymmetric tensor field strength $H_{\mu\nu\lambda}\equiv\nabla_\mu B_{\nu\lambda}+\nabla_\nu B_{\lambda\mu}+\nabla_\lambda B_{\mu\nu}$. Because of the gauge invariance (3.6) the spacetime physics of the antisymmetric tensor field only depends on its field strength, which is gauge invariant. After some integrating

by parts and rearranging terms the quadratic part of (3.23) is seen to take the form

$$
\frac{1}{4\pi\alpha'}\int d^2\xi\,\epsilon^{ab}\{H_{\mu ij}(X_0)\partial_a X_0^\mu\,\nabla_b\eta^i\eta^j+\frac{1}{2}\nabla_i H_{\mu\nu j}(X_0)\partial_a X_0^\mu\partial_b X_0^\nu\,\eta^i\eta^j\}. \tag{3.24}
$$

The algorithm of ref. [14] can also be applied here to generate higher order terms from this one. In particular the existence of such a recursive procedure ensures that at every order in the covariant expansion only the gauge invariant combination $H_{\mu\nu\lambda}$ appears. The only higher term we will find use for in what follows is cubic in η^i and involves the field strength with no derivatives,

$$
\frac{1}{12\pi\alpha'}\int d^2\xi\,\epsilon^{ab}\,H_{ijk}(X_0)\eta^i\nabla_a\eta^j\nabla_b\eta^k. \tag{3.25}
$$

The dilaton coupling function is a scalar in spacetime so the expansion of S_D is very simple,

$$
\begin{aligned}
S_D[X_0+\pi] =&S_D[X_0]+\frac{1}{4\pi}\int d^2\xi\,\sqrt{\gamma}\,^{(2)}\!R\,\nabla_i\Phi(X_0)\,\eta^i \\
&+\frac{1}{8\pi}\int d^2\xi\,\sqrt{\gamma}\,^{(2)}\!R\,\nabla_i\nabla_j\Phi(X_0)\,\eta^i\eta^j \\
&+\cdots
\end{aligned}
\tag{3.26}
$$

Armed with these covariant expansions of the terms in the classical action and with a diagonal kinetic term for the quantum fields we can now read off simple Feynman rules for diagrams and start doing perturbation theory. But first we should ask ourselves whether perturbative calculations can make any sense in this theory. A necessary requirement is that there be a small, dimensionless parameter to expand in. All the interaction terms in our covariant expansions involve one or more derivatives of the spacetime coupling functions. The theory is therefore weakly coupled only if we make the assumption that the spacetime background fields are slowly varying on the length scale defined by α'.

3.3. Lowest order calculation of the Weyl anomaly

In this section we will do a simple calculation to obtain the one-loop Weyl anomaly of the sigma model using the covariant background field expansion. We will employ a trick to make things easy. What we are after is the variation of the effective action, with respect to the scale factor of the worldsheet metric. Rather than attempt a direct computation we will obtain it *via* the conservation equation (2.26). Let us identify the energy momentum tensor in the quantum theory with the variation of the effective action with respect to the two-dimensional metric:

$$\langle T_{ab}\rangle = \frac{4\pi}{\sqrt{\gamma}}\frac{\delta W}{\delta\gamma^{ab}}. \tag{3.27}$$

The arguments leading up to (2.28) tell us that this is valid for the zz and $\bar{z}\bar{z}$ components. General renormalization theory arguments (see Chapter 4) guarantee that (3.27) is also a good definition of $\langle T_{\bar{z}z}\rangle$ even when that quantity vanishes classically. With this identification the conservation equation (2.26) takes a more familiar form,

$$\nabla^{\bar{z}}\langle T_{\bar{z}z}\rangle + \nabla^{z}\langle T_{zz}\rangle = 0. \tag{3.28}$$

The trick is the observation that $\langle T_{zz}\rangle$ is finite and therefore well-defined so that (3.28) can be used to infer a value for $\langle T_{\bar{z}z}\rangle$ [15]. We will find that the one-loop value of $\langle T_{zz}\rangle$ is such that the one-loop value of $\langle T_{\bar{z}z}\rangle$ cannot be zero if we insist on conservation of two-dimensional energy-momentum. It is a familiar story when dealing with anomalies that one has to give up some symmetry in the quantum system, but one generally has a choice. Here the choice is between Weyl invariance, which would imply $\langle T_{\bar{z}z}\rangle = 0$, on the one hand, and two-dimensional reparametrization invariance, which is associated with the conservation of the energy momentum tensor, on the other. We have no hesitation to maintain conservation of energy and momentum over Weyl invariance and the price is a non-zero $\langle T_{\bar{z}z}\rangle$.

Now let us get down to the business of calculation. For simplicity we will first obtain the anomaly using a flat worldsheet metric, and then later consider the effect of worldsheet curvature. For the purpose of evaluating the Feynman diagrams it is convenient to let the worldsheet have Minkowski signature and go to two-dimensional momentum space. Then the conservation equation reads:

$$q_+\langle T_{-+}\rangle + q_-\langle T_{++}\rangle = 0. \tag{3.29}$$

Let us first compute the contribution to $\langle T_{++}\rangle$ that comes from S_P. To leading order in the spacetime curvature only one diagram needs to be calculated.

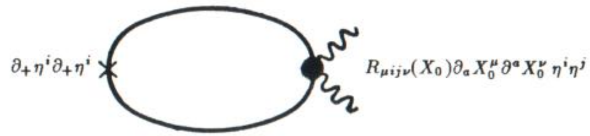

FIGURE 3.1.

The $\partial_+\eta^i\partial_+\eta^i$ comes from an insertion of T_{++} and the vertex involving the spacetime curvature comes from the expansion of S_P. The T_{++} insertion of course also has pieces involving the SO(D-1,1) gauge potential (because we have covariant derivatives, $\nabla_+\eta^i\nabla_+\eta^i$, in (3.19)) and a piece involving the spacetime curvature. The diagrams with insertions of the gauge potential cannot combine to give a covariant result (because one also needs a derivative of the connection to form a curvature tensor) so their contribution must vanish and we ignore them. The diagram with two insertions of the curvature tensor (one from T_{++} and one from S_P) can be neglected if are only working to first order in $R_{\mu\lambda\sigma\nu}$.

The contribution of the diagram above is

$$\int\frac{d^2l}{2\pi}\frac{l_+(l_++q_+)}{l^2(l+q)^2}\{R_{\mu\nu}\partial_a X_0^\mu\partial^a X_0^\nu\}(q). \tag{3.30}$$

Here l is the loop momentum and q is the momentum with which we insert $\partial_+\eta^i\partial_+\eta^i$. Momentum conservation tells us that the momentum q is carried away by the background $\partial_a X_0^\mu$'s on the right in the diagram. $R_{\mu\nu}$ is the Ricci tensor of the spacetime metric and since we are working in momentum space on the worldsheet we have a Fourier transform of the product of functions in the curly bracket.

The momentum integral in (3.30) is superficially logarithmically divergent, but the ++ tensor character of the diagram indicates that the result must have what amounts to two factors of q_+, and the diagram is therefore in fact finite. This is the whole point of defining T_{-+} from T_{++} via conservation. The integral is easily performed, for example by using standard dimensional regularization formulas[16] and one obtains

$$\int \frac{d^2 l}{2\pi} \frac{l_+(l_+ + q_+)}{l^2(l+q)^2} = -\frac{1}{4}\frac{q_+}{q_-}. \tag{3.31}$$

Having obtained the one-loop contribution to $\langle T_{++}\rangle$ we can now use the conservation equation (3.29) to get

$$\langle T_{-+}(\xi)\rangle = \frac{1}{4} R_{\mu\nu}(X_0)\partial_a X_0^\mu(\xi)\partial^a X_0^\nu(\xi). \tag{3.32}$$

This is the conformal anomaly. The trace of the energy momentum tensor is non-zero even if we start out with a conformally invariant classical action and we have discovered that at the one-loop level the anomalous $\langle T_{-+}\rangle$ depends on the curvature of spacetime. If we restrict our spacetime coupling function, $G_{\mu\nu}(X_0)$, to be such that $R_{\mu\nu}(X_0) = 0$ this anomaly goes away, and we will eventually do something like that.

The anomaly (3.32) does not have any power of α' in front of it. The energy momentum tensor insertion and the interaction term in the Lagrangian each have a $\frac{1}{\alpha'}$ and each propagator an α' so the factors of α' cancel in the diagram. This is as expected if α' is the parameter that counts loops, since at tree level the energy momentum tensor has a $\frac{1}{\alpha'}$ in front.

We do not have the full anomaly yet. Two diagrams derived from S_{AS} also contribute. One has two vertices involving the field strength $H_{\mu\nu\lambda}$:

FIGURE 3.2.

By power counting the superficial degree of divergence is the same as in the diagram derived from S_P, but again the ++ character of the diagram tells us that it is finite. A calculation quite analogous to the one we did already, shows that this diagram produces an anomaly also,

$$\langle T_{-+}\rangle = -\frac{1}{16} H_{\mu\lambda\sigma}(X_0)H_\nu^{\ \lambda\sigma}(X_0)\partial_a X_0^\mu(\xi)\partial^a X_0^\nu(\xi). \tag{3.33}$$

This has the same form as (3.32) with the Ricci tensor replaced by the square of the antisymmetric field strength.

The other diagram derived from S_{AS} that contributes at this order is

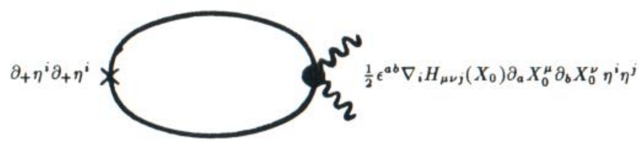

FIGURE 3.3.

This time the loop momentum integral is identical to (3.31) and we immediately find that the anomaly receives a contribution,

$$\langle T_{-+}\rangle = \frac{1}{8}\nabla^\lambda H_{\lambda\mu\nu}(X_0)\partial_a X_0^\mu\partial_b X_0^\nu\,\epsilon^{ab}. \tag{3.34}$$

It involves an antisymmetric, rather than symmetric, combination of worldsheet derivatives on the background fields, X_0^μ.

We will see later that these terms we have just computed, which form the Weyl anomaly, can be arrived at in a quite different way, which reveals their physics content from another point of view. It turns out that they are the same as the renormalization group beta functions for S_P and S_{AS}. The symmetric function $G_{\mu\nu}(X)$ and the antisymmetric function $B_{\mu\nu}(X)$ play the role of coupling constants in the non-linear sigma model and one can do a renormalization group calculation to obtain the associated beta functions. What one finds is that $\beta^G_{\mu\nu}$ is obtained from the sum of (3.32) and (3.33) while $\beta^B_{\mu\nu}$ is given by (3.34). A very nice picture is emerging. The Weyl anomaly can be expressed in terms of spacetime tensors built out of the curvature and antisymmetric tensor field strength, and can be identified with the beta functions of the couplings in the theory. But a key element is still missing. There was a third coupling, arising from the dilaton. It appeared in the classical action as a non-standard term involving the worldsheet curvature, and its influence is not easy to see from the standpoint of the renormalization group. However the calculations which gave us the one-loop Weyl anomaly in the simple setting of this section can easily be generalized to include the effect of the dilaton term. We will find that its presence adds further pieces to the terms in the anomaly which correspond to $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ plus a new kind of anomaly, which can be viewed as a generalization of the central charge of the Virasoro algebra, and can also be interpreted as the beta function of the dilaton coupling itself.

If we are only considering field theory on flat two-dimensional space, $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ are the complete one-loop anomaly. In general they are two independent tensor structures and if one wants the anomaly to vanish the two have to be set equal to zero separately. There are exceptions to this. If the target manifold (spacetime) happens to be the manifold of some group the Ricci tensor does not vanish but the Weyl anomaly from S_P can be cancelled by an astute choice of antisymmetric tensor field in S_{AS}. In the sigma model literature an antisymmetric tensor coupling on a group manifold is referred to as a Wess-Zumino term. We will not discuss further the special case of group manifolds here since we want to study string theory in a generic spacetime.

3.4. Including the dilaton coupling

In the previous section we established the contributions from S_P and S_{AS} to the one-loop Weyl anomaly. We computed the expectation value of the trace of the two-dimensional energy momentum tensor, in the presence of spacetime background fields, on a flat worldsheet. We would now like to obtain the contribution of the dilaton term (3.2) in the classical action to the Weyl anomaly. In S_D the spacetime dilaton is coupled to the two-dimensional curvature but we can still compute the anomaly without having to do all our calculations on a curved worldsheet. Although the dilaton coupling itself vanishes in the limit of a flat worldsheet its variation with respect to the worldsheet metric does not. In other words having included such a term in the original action on a curved worldsheet we have changed the two-dimensional energy momentum tensor even on a flat worldsheet. The energy momentum tensor is always the response of the action to an infinitesimal variation of the metric and the dilaton coupling affects this response even when the term itself in the Lagrangian vanishes. With a little algebra we find that this addition to the energy momentum tensor is

$$T^{dil}_{ab} = \left(\partial_a\partial_b - \delta_{ab}\Box\right)\Phi(X). \tag{3.35}$$

It clearly had to be of this form. It must be linear in Φ, have vanishing divergence (*i.e.*, be conserved), and be of dimension two. Notice that this part of the energy momentum tensor has a non-vanishing trace,

$$T^{dil}_{-+} = \Box_\xi\, \Phi\big(X(\xi)\big). \tag{3.36}$$

The ξ subscript is to remind you that this is the D'Alembertian on the worldsheet, and not in spacetime, which acts on Φ. This non-zero trace was to be expected because S_D is not Weyl invariant, even at the classical level. As we discussed earlier the idea is to cancel this tree level contribution to the trace of the energy momentum tensor against the one-loop anomalies coming from the terms in the action that were Weyl invariant classically. The powers of the loop counting parameter α' in front of the various terms in the action are so arranged that the tree level trace of the energy momentum tensor coming from S_D enters at the same order as the one-loop pieces we computed in the previous section (that is order$(\alpha')^0$). What we need to calculate is simply the classical trace in (3.36) in the background field X_0^μ. There are two terms,

$$\Box_\xi \Phi(X_0) = \Box X_0^\mu \, \partial_\mu \Phi(X_0) + \partial_a X_0^\mu \partial^a X_0^\nu \, \partial_\mu \partial_\nu \Phi(X_0). \tag{3.37}$$

These terms are not covariant from the spacetime point of view and we have been careful to write partial derivatives rather than attempting to covariantize the expression at this point. Now we use the classical equation of motion for X_0^μ to rewrite this. Since we are still working on a flat worldsheet the relevant equation of motion is the one derived from S_P and S_{AS} together, not including S_D:

$$\Box X_0^\mu = \Gamma^\mu_{\lambda\sigma} \, \partial_a X_0^\lambda \partial^a X_0^\sigma - \frac{1}{2} H^\mu{}_{\lambda\sigma} \, \partial_a X_0^\lambda \partial_b X_0^\sigma \epsilon^{ab}. \tag{3.38}$$

If we insert this into (3.37) things combine in such a way as to give a spacetime covariant result,

$$\Box_\xi \Phi = \nabla_\mu \nabla_\nu \Phi(X_0) \partial_a X_0^\mu \partial^a X_0^\nu - \frac{1}{2} \nabla^\lambda \Phi(X_0) \, H_{\lambda\mu\nu}(X_0) \partial_a X_0^\mu \partial_b X_0^\nu \epsilon^{ab}. \tag{3.39}$$

These terms have the same structure as the one-loop anomaly coming from S_P and S_{AS}, which we calculated before, and we can conclude that the trace of the full energy momentum tensor on a flat world sheet is obtained by adding these two terms to what we have called $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$.

$$\begin{aligned} \langle T_{-+} \rangle = \; & \frac{1}{4} \left\{ R_{\mu\nu} - \frac{1}{4} H^2_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \Phi \right\} \partial_a X_0^\mu \partial_b X_0^\nu \sqrt{\gamma} \gamma^{ab} \\ & + \frac{1}{8} \left\{ \nabla^\lambda H_{\lambda\mu\nu} - 2 \nabla^\lambda \Phi \, H_{\lambda\mu\nu} \right\} \partial_a X_0^\mu \partial_b X_0^\nu \epsilon^{ab}. \end{aligned} \tag{3.40}$$

The objects in the curly brackets are $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ with the effect of the dilaton coupling taken into account,

$$\begin{aligned} \beta^G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{4} H^2_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \Phi, \\ \beta^B_{\mu\nu} &= \frac{1}{2} \nabla^\lambda H_{\lambda\mu\nu} - \nabla^\lambda \Phi \, H_{\lambda\mu\nu}. \end{aligned} \tag{3.41}$$

These are not quite the renormalization group beta functions of the sigma model but rather Weyl anomaly coefficients. The formal connection between the two will be explained in the next chapter when we discuss the renormalization group approach to the sigma model.

This is still not the whole story. We have included the contribution of the dilaton coupling to $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ but the trace of the energy momentum tensor in (3.40) is nevertheless evaluated on a flat worldsheet. If the general non-linear sigma model is to be Weyl invariant at the quantum level we must impose that the trace of the energy momentum tensor vanish on any worldsheet. However we would like to avoid having to deal with the complications that accompany calculations on an arbitrary curved two-dimensional space. It turns out that we can get at the piece of the anomaly on a curved worldsheet, which we are still missing, by calculating two point functions of the energy momentum tensor on a flat worldsheet. First we recall that locally on the worldsheet we can always write a curved metric as a conformal scale factor times a flat metric, $\gamma_{ab} = e^\phi \, \delta_{ab}$. Weyl invariance means that the trace of the energy momentum tensor, evaluated using the metric γ_{ab}, vanishes regardless of what the scale factor ϕ happens to be. A minimum requirement is that its first variation with respect to ϕ be equal to zero. If we evaluate that variation on a flat worldsheet we formally have a two point function of the trace of the energy momentum tensor.

$$\frac{\delta}{\delta \phi(\xi)} \langle T_{-+}(0) \rangle_{e^\phi \delta_{ab}} \bigg|_{\phi = o} = -\frac{1}{4\pi} \langle T_{-+}(\xi) T_{-+}(0) \rangle_{\delta_{ab}}. \tag{3.42}$$

Consider for the moment the part of the theory which is classically Weyl invariant and then we will include the dilaton coupling later on. In that case the two point function in (3.42) is something which vanishes at the classical level and is only non-zero because the theory is anomalous. We will use conservation of the energy momentum tensor to define its value in much the same way as we defined the expectation value of the trace of the energy momentum tensor in Section 3.3. At the classical level the only non-vanishing two-point functions of the energy momentum tensor are $\langle T_{++}T_{++}\rangle$ and the one with all minus indices. It is straightforward to calculate $\langle T_{++}T_{++}\rangle$. To lowest order in α' there is only one diagram that needs to be evaluated.

FIGURE 3.4.

It is the same story as before. The diagram is superficially divergent but the answer must have ++++ tensor character and that means that it is actually finite. It is convenient to do a Fourier transform and calculate the diagram in momentum space. Conservation of momentum tells us that the two $\partial_+\eta^i\partial_+\eta^i$ insertions must have opposite momentum. The loop momentum integral is relatively simple:

$$\langle T_{++}(q)T_{++}(-q)\rangle = 2D\int d^2l\,\frac{l_+^2(l_+ + q_+)^2}{l^2(l+q)^2} = -\frac{\pi D}{6}\frac{q_+^3}{q_-}. \tag{3.43}$$

The factor of D is there because there is an independent loop contribution from each component field η^i. As in the other one-loop graphs the factors of α' cancel between the $\partial_+\eta^i\partial_+\eta^i$ insertions and the propagators.

Now we obtain one-loop values for those two-point functions that vanish in the classical theory by implementing conservation of the energy momentum tensor[15],

$$q_+\,T_{-+} + q_-\,T_{++} = 0. \tag{3.44}$$

Applying the conservation equation once to (3.43) gives

$$\langle T_{-+}(q)T_{++}(-q)\rangle = \frac{\pi D}{6}\,q_+^2. \tag{3.45}$$

This is a polynomial in q, a contact term with no singularity. It can be viewed as coming from a finite part of a renormalization subtraction. A two-point function of the energy momentum tensor with the tensor character of (3.45) is logarithmically divergent by power counting and we can add any finite number times q_+^2 to it. One usually invokes symmetry principles to fix finite parts of counterterms. Weyl invariance would have that this two-point function be equal to zero but we are finding out that conservation of the energy momentum tensor (which we take to be more fundamental than Weyl invariance) does not allow that. The point is that $\langle T_{++}T_{++}\rangle$ is finite so we have no freedom to add anything to it and the conservation equation then fixes a unique value for $\langle T_{-+}T_{++}\rangle$. Proceeding on, we use (3.44) again to obtain the two-point function of the trace of the energy momentum tensor with itself,

$$\langle T_{-+}(q)T_{-+}(-q)\rangle = -\frac{\pi D}{6}\,q_+q_-. \tag{3.46}$$

It is clearly non-zero so there is an anomaly. The product q_+q_- is a Lorentz scalar on the worldsheet. It simply turns into the D'Alembertian upon returning to coordinate space and since the right hand side of (3.46) is otherwise a constant, it will act on a δ-function of location,

$$\langle T_{-+}(\xi)T_{-+}(0)\rangle = \frac{\pi D}{12}\,\Box\,\delta^{(2)}(\xi). \tag{3.47}$$

This two point function is the response of T_{-+} to a variation of the metric scale factor, evaluated on a flat worldsheet. Using (3.47) we can integrate (3.42) to get the trace of the energy momentum tensor

$$\langle T_{-+}(\xi)\rangle_{e^\phi\delta_{ab}} = -\frac{D}{48}\,\Box\phi. \tag{3.48}$$

The D'Alembertian of the scale factor is the two-dimensional curvature of the worldsheet and if we use (2.17) we can rewrite the above equation as

$$\langle T_{-+}(\xi)\rangle_{e^\phi\delta_{ab}} = \frac{D}{24}\sqrt{\gamma}\,{}^{(2)}R. \tag{3.49}$$

Once again we have found an anomaly but this contribution to the trace of the energy momentum tensor is of a different form than the pieces we obtained in the previous section. Like the dilaton coupling, it is proportional to the two-dimensional curvature and we call the coefficient of proportionality β^{Φ}. This anomaly only depends on D, the dimension of spacetime, but does not seem to involve the spacetime fields in any way. Of course we have only obtained it to zeroth order in α' so far. In order to see a dependence on the sigma model coupling functions we have to go to order α' and calculate two-loop diagrams. At this stage we are considering the contribution to the anomaly of the classically Weyl invariant part of the theory, so the interaction vertices in the two-loop diagrams are those derived from S_P and S_{AS}. There are a number of two-loop diagrams we can write down but only three of them are relevant to the calculation of β^{Φ}.

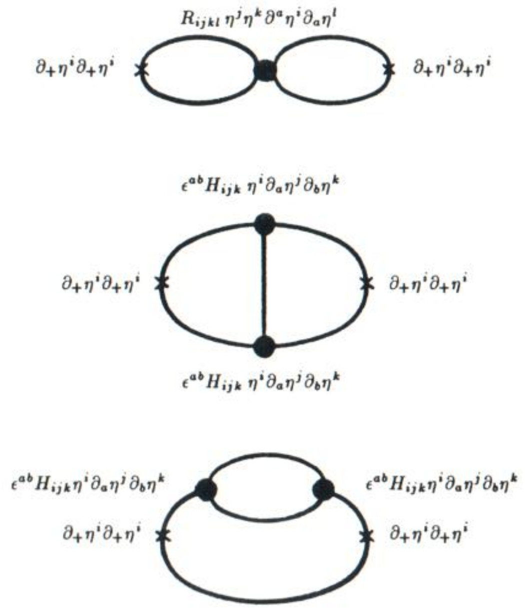

FIGURE 3.5.

The vertices in these diagrams come from the interaction terms, in the background field expansion, which we displayed in (3.19) and (3.25). All other two-loop diagrams either contribute only to $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ or can be shown to be unimportant by symmetry arguments. The actual calculation we are proposing is quite involved. These are two-loop diagrams, which contribute to $\langle T_{++}T_{++}\rangle$. They can have subdivergences which need to be properly renormalized, so there are counterterm diagrams that need to be considered and so on. We will not go into the details here but if one is careful, and subtracts divergences in a way which respects energy-momentum conservation on the worldsheet, each diagram in the end gives a unique finite answer. Conservation of the energy momentum tensor is the guiding principle which allows us to define the diagrams unambiguously. The momentum structure of all three diagrams turns out to be identical to what we had in the one-loop case, $\frac{q_+^3}{q_-}$, but now there are coefficients in front that depend on the spacetime backgrounds. The vertex indices in the graphs are contracted in such a way as to give spacetime scalars. Doing the actual loop calculations therefore has to give a term involving the Ricci scalar and another involving the scalar square of $H_{\mu\nu\lambda}$. The conservation argument which we used to obtain $\langle T_{-+}T_{-+}\rangle$ from $\langle T_{++}T_{++}\rangle$ only depends on the momentum structure of a given diagram, but not the coefficient that goes with it. If we follow the argument through we find a new set of terms, involving R and H^2, to be added to β^{Φ}. It is easy to check by counting vertices and propagators that the two-loop diagrams enter at order α' and it is a matter of detail to get the precise numbers in front.

Now we have the order α' contribution to β^{Φ} coming from the spacetime metric and antisymmetric tensor field but we are still missing the piece coming from the dilaton coupling itself. That is of course because we only calculated two point functions of the classically Weyl invariant part of the energy momentum tensor. There is also an explicit dilaton contribution (3.36) to the trace of the energy momentum tensor which we have called T^{dil}_{-+}. Recall that it has an extra power of α' compared with the classical energy momentum tensor derived from the Weyl invariant terms in the action, so there are two ways it can give a contribution at the two-loop level. One is a tree-level two-point function of T^{dil}_{-+} with itself and the other comes from a one-loop diagram with one insertion of T^{dil}_{-+} and one insertion of $\partial_+\eta^i\partial_+\eta^i$.

The only tree diagram, which is relevant to the calculation of β^Φ, connects two $\nabla_\mu\Phi\,\Box X_0^\mu$ vertices with a propagator. (In a tree diagram we consider X_0^μ as a quantum field.)

FIGURE 3.6.

The propagator cancels one of the D'Alembertians and contributes a factor of α' to give:

$$\left\langle T^{dil}_{-+}(\xi)T^{dil}_{-+}(0)\right\rangle = \pi\alpha'\,(\nabla\Phi)^2\,\Box\delta^{(2)}(\xi). \tag{3.50}$$

The only one-loop diagram that gives a dilaton contribution to β^Φ has a $\partial_+\eta^i\partial_+\eta^i$ insertion and a vertex $\nabla_i\nabla_j\Phi\,\partial_+\eta^i\partial_+\eta^j$, which comes from the background field expansion of T^{dil}_{++}.

FIGURE 3.7.

Then we use the by now familiar conservation procedure to define $\langle T_{-+}T^{dil}_{-+}\rangle$ from $\langle T_{++}T^{dil}_{++}\rangle$, and we get

$$\left\langle T_{-+}(\xi)T^{dil}_{-+}(0)\right\rangle = -\pi\alpha'\nabla^2\Phi\,\Box\delta^{(2)}(\xi). \tag{3.51}$$

We see that β^Φ receives two terms from the dilaton coupling, one involving $(\nabla\Phi)^2$ and the other proportional to $\nabla^2\Phi$. Gathering together the various pieces

of β^Φ that we have found gives:

$$\beta^\Phi = \frac{D}{6} + \frac{\alpha'}{2}\Big\{-R + \frac{H^2}{12} + 4(\nabla\Phi)^2 - 4\nabla^2\Phi\Big\}. \tag{3.52}$$

There can be no further contributions to the Weyl anomaly at this order in the perturbation expansion. The trace of the energy momentum tensor is of scaling dimension two and the only things that can give you an anomaly are operators of dimension two. In Section 3.1 we argued that there are only three independent structures of dimension two in the two-dimensional sigma model and the coefficients $\beta^G_{\mu\nu}$, $\beta^B_{\mu\nu}$ and β^Φ multiply precisely those objects. We have obtained β^Φ to order α' but $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ only to order $(\alpha')^0$. Nevertheless all the coefficients have been calculated to the same physical order as they all involve two derivatives of the spacetime coupling functions.

3.5. Consistency of the Weyl anomaly conditions

In string theory we want the non-linear sigma model to be Weyl invariant so we must impose the condition that all the Weyl anomaly coefficients vanish: $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$. The leading order part of β^Φ does not depend on the spacetime backgrounds at all and is present even in flat, empty spacetime. It had better be gotten rid of somehow because we know string theory can be defined in (26 dimensional) flat space. It is in fact cancelled by a contribution from the conformal ghost fields which are needed for fixing the two-dimensional metric in the original string path integral. We have not discussed the ghosts here because they form a free system, completely decoupled from the degrees of freedom we have been looking at. From our point of view their only significance is that they contribute a constant piece to β^Φ which is $-\frac{26}{6}$ in our units. The order $(\alpha')^0$ part of β^Φ therefore goes away if we choose the number of scalar fields in the sigma model, that is the dimension of spacetime, to be 26.

Choosing $D = 26$ is all we have to do in order to get rid of the Weyl anomaly in a free two-dimensional theory but in the general non-linear sigma model there are further terms. The question is whether there exists a configuration of the spacetime metric, antisymmetric tensor field and dilaton field such that all three Weyl anomaly coefficients vanish. It should be kept in mind that we have only found these coefficients to leading order in a power series in α'. It is really the full power series, or rather the function it approximates, that is to be set equal to zero. We will show explicitly how this question is answered in the context of these first order calculations. A similar procedure can be carried through for higher order corrections but we will not go into that in any detail in this section.

First we want to settle a potentially troubling issue. In the general non-linear sigma model β^Φ plays the role of the central charge of the Virasoro algebra. It is essentially the parameter λ we discussed in Chapter 2. We found that in a generic two-dimensional conformal field theory the effective action depends on the choice of two-dimensional metric. Its variation with respect to the conformal factor of the metric is of the form: a number, λ, times the two-dimensional curvature, and that coefficient of ${}^{(2)}R$ is precisely what we have called β^Φ in the sigma model. The cause for concern is that while λ is simply a number which characterizes the theory, β^Φ in (3.52) looks like an operator that depends on position in spacetime! The spacetime metric, antisymmetric tensor field and dilaton field are in general complicated functions in spacetime so how are we guaranteed that β^Φ is only a c-number?

The theory defeats this problem in an interesting way. If we want the non-linear sigma model to be a conformal quantum field theory we certainly must impose that $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ be equal to zero. Otherwise there is a conformal anomaly even on a flat worldsheet. The remarkable fact is that this condition actually implies that β^Φ is a constant. If $\beta^G_{\mu\nu}(X_0)$ is identically zero then its spacetime divergence also vanishes. On the other hand if $\beta^B_{\mu\nu}$ is equal to zero one can use that to show that the divergence of $\beta^G_{\mu\nu}$ is actually equal to the gradient of β^Φ;

$$0 = \nabla^\mu \beta^G_{\mu\nu}(X_0) = \nabla_\nu \beta^\Phi(X_0). \tag{3.53}$$

We feed in the vanishing of $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ and get out that β^Φ is constant[8,17].

Showing that (3.53) holds is a recommended exercise in using the Bianchi identities that the tensors $R_{\mu\lambda\sigma\nu}$ and $H_{\mu\nu\lambda}$ satisfy. Some work is required and at the end one might be inclined to think that the argument only goes through because of the specific form of the various leading order β-coefficients, and that it might break down when we add higher loop corrections. However general arguments have been given[18] that vanishing $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ together imply a constant β^Φ at all orders in perturbation theory. We will come back to that in the next chapter.

It still remains to demonstrate that all three Weyl anomaly coefficients can be consistently set equal to zero. If we write down an arbitrary set of equations for the spacetime coupling functions we cannot expect them to be simultaneously satisfied, in particular since $R_{\mu\lambda\sigma\nu}$ and $H_{\mu\nu\lambda}$ are constrained objects derived from potentials. Nevertheless the Weyl anomaly conditions are mutually consistent. One way to see that is to observe that they can all be derived from a single spacetime action. In the process of showing that we will see more explicitly how conformal invariance on the worldsheet leads to Einstein gravity in spacetime. First we rearrange the Weyl anomaly equations into a more suggestive form:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}\, R &= \frac{1}{4}\big[H^2_{\mu\nu} - \frac{1}{6}G_{\mu\nu}\, H^2\big] - 2\nabla_\mu\nabla_\nu\Phi + 2G_{\mu\nu}\,\nabla^2\Phi, \\ \nabla^\lambda H_{\lambda\mu\nu} &= 2\nabla^\lambda\Phi\, H_{\lambda\mu\nu}, \\ \nabla^2\Phi - 2\big(\nabla\Phi\big)^2 &= -\frac{1}{2}H^2. \end{aligned} \tag{3.54}$$

We have used the trace of the top equation to eliminate R from the bottom or $\beta^\Phi = 0$ equation. The three equations in (3.54) are completely equivalent to imposing $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$ but now the left hand side of the top equation is the spacetime Einstein tensor and the bottom two can be viewed as equations of motion for the spacetime antisymmetric tensor and dilaton fields. We can define the spacetime energy momentum tensor to be the right hand side of the top equation in (3.54);

$$\Theta_{\mu\nu} = \frac{1}{4}\big[H^2_{\mu\nu} - \frac{1}{6}G_{\mu\nu}\, H^2\big] - 2\nabla_\mu\nabla_\nu\Phi + 2G_{\mu\nu}\,\nabla^2\Phi. \tag{3.55}$$

It is a symmetric tensor and it had better be conserved, because the Einstein tensor is always conserved as a result of a Bianchi identity for $R_{\mu\lambda\sigma\nu}$. For general $H_{\mu\nu\lambda}$ and Φ (3.55) has no reason to be a conserved tensor but it is easy to see that when the spacetime fields satisfy their equations of motion (3.54) it follows that $\nabla^\mu \Theta_{\mu\nu} = 0$. In other words the vanishing of the conformal anomaly is equivalent to the Einstein equations in spacetime with a covariantly conserved source term. The system of equations is therefore certainly consistent and must in fact follow from an action principle. The D dimensional covariant action whose variational equations reproduce (3.54) is:

$$S_D = \int d^D X \sqrt{G}\, e^{-2\Phi} \Big\{ R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 \Big\}. \tag{3.56}$$

This can be recast into a more standard form by performing a Weyl rescaling of the spacetime metric,

$$G_{\mu\nu} = \tilde{G}_{\mu\nu}\, e^{\frac{4}{D-2}\Phi}. \tag{3.57}$$

If we redefine in this way what we mean by the spacetime metric tensor the spacetime action (3.56) becomes:

$$S_D = \int d^D X \sqrt{\tilde{G}} \Big\{ \tilde{R} - \frac{4}{D-2} (\widetilde{\nabla\Phi})^2 - \frac{1}{12} e^{-\frac{8\Phi}{D-2}} \tilde{H}^2 \Big\}. \tag{3.58}$$

The twiddles on the $(\nabla\Phi)^2$ and H^2 terms are to remind you that these scalars involve contractions of spacetime indices which are carried out with the new metric. Now the spacetime action looks more familiar. It is simply the Einstein action along with a kinetic term for the dilaton field and a Maxwell type kinetic term for the antisymmetric tensor field. There is a complication in that the analog of the coupling constant strength for the antisymmetric tensor field now depends upon the dilaton field.

The bottom line is that when we study the general non-linear sigma model we find lurking behind the scenes a spacetime action functional from which one can derive the Weyl anomaly conditions. This is a reflection of the deep connection

between the sigma model and string theory for something like this is certainly not true for renormalizable quantum field theories in general. Again it can be asked whether this is true to all orders in perturbation theory or is just a special feature of the leading order approximation. The complete answer to that question is not known, but it has been checked up to three loops that the connection between the vanishing of the Weyl anomaly and a spacetime action survives the calculation of higher order corrections[19,20]. Based on that it is generally believed that the full expansion, in a power series in α', of the Weyl anomaly coefficients can be derived from a master spacetime action, which is itself given by a power series in α'. The higher terms in α' involve more spacetime derivatives of the coupling functions and give rise to short distance (*i.e.*, Planck scale) corrections to the Einstein equations and the equations of motion for the matter fields.

There is a direct link of all this to string theory. If we take the spacetime action (3.58) as an effective action for the spacetime gravitational, antisymmetric tensor and dilaton fields then it precisely generates the string theory S-matrix (at string tree level). By that we mean the following. If you take the spacetime action, expand it in powers of the linearized graviton, antisymmetric tensor and dilaton fields to generate spacetime propagators and interaction vertices, and compute all the resulting tree level diagrams (one normally only computes tree diagrams from an effective action as the effective interactions are usually non-renormalizable), then the resulting S-matrix elements are exactly those one obtains, for the massless states of the bosonic string, from the standard string theory operator formalism. This connection should convince you that the graviton described by the spacetime metric tensor of the sigma model is indeed the same as the graviton of string theory. It has been checked to high enough order in sigma model perturbation theory for one to have faith in it (and may even have been proven[21]) to all orders) that the spacetime effective action is the generating functional of the string S-matrix.

We have seen how the two-dimensional non-linear sigma model has encoded in it all of the spacetime physics of the massless modes of string theory at string tree level but the picture is incomplete in a number of ways. As we mentioned at the end of Chapter 2 it is not clear at present how string loop corrections affect these considerations. Also one would like to be able to generate the full string S-matrix in a sigma model approach, not just those elements which involve massless external states. The trouble is that the sigma model interaction terms, which correspond to backgrounds of the massive modes of string theory, are non-renormalizable in two dimensions and the perturbation theory is not well defined.

4. HIGHER-LOOP METHODS: RELATION TO RENORMALIZATION GROUP

In the previous chapter we presented a quick and dirty calculation of the one-loop Weyl anomaly of a bosonic nonlinear sigma model. The one-loop results had some surprising features (all anomalies derivable from a spacetime action function, for instance) and it is clearly important to see whether these features remain true at higher loop orders. The method used previously is not very systematic and does not generalize very easily to higher orders. In this chapter we therefore outline a systematic method for calculating higher-loop Weyl anomalies and present selected concrete results. This whole subject is very complicated and limitations of space prevent us from doing much more than giving the reader some notion of what the essential issues are.

The most systematic way to proceed is to exploit the fact that the Weyl anomaly is almost, but not quite, identical to the renormalization group beta functions which can in turn be computed by standard methods to any loop order. Analyses of this connection have been given by several authors[22,23,24]. The work of Metsaev and Tseytlin[24] is the most systematic, but we will find it convenient follow a subsequent treatment[18] which is particularly concise and transparent and lends itself to useful generalizations. The starting point is the dimensionally-regulated sigma model defined in $2+\epsilon$ dimensions by the action

$$S = \frac{1}{4\pi\alpha'}\int dv[\gamma^{ab}G^{B}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \frac{\alpha'}{2}R^{(2)}\Phi^{B}], \tag{4.1}$$

where dv stands for the dimensionally continued reparametrization invariant measure $d^{2+\epsilon}\xi\sqrt{\gamma}$. We have dropped the antisymmetric tensor field for the time being. Amplitudes generated from this action are singular as $\epsilon \to 0$ but they are rendered finite by using minimal subtraction to express bare couplings as power series in $1/\epsilon$ with coefficients built out of renormalized couplings $G_{\mu\nu}$ and Φ :

$$\begin{aligned}
G^{B}_{\mu\nu} &= \mu^{\epsilon}(G_{\mu\nu} + T_{\mu\nu}) \\
\Phi^{B} &= \mu^{\epsilon}(Z\cdot\Phi + \Psi) \\
T_{\mu\nu}(G) &= \sum_{L=1}^{\infty}(\alpha')^L \sum_{i=1}^{L}\frac{1}{\epsilon^i}T^{(L,i)}_{\mu\nu}(G)) \\
Z\cdot\Phi &= \Phi + \sum_{L=1}^{\infty}(\alpha')^L\sum_{i=1}^{L}\frac{1}{\epsilon^i}Z^{(L,i)}(G)\,\Phi \\
\Psi &= \sum_{i=1}^{\infty}\frac{1}{\epsilon^i}\Psi^{(i)}(G)
\end{aligned} \tag{4.2}$$

In these expressions, $T^{(L,i)}_{\mu\nu}$ and $Z^{(L,i)}$ are generally covariant functions of the renormalized metric $G_{\mu\nu}$ and μ is an arbitrary mass scale. Calculations up to two-loop order yield the results[25]

$$\begin{aligned}
T^{(1,1)}_{\mu\nu} &= -R_{\mu\nu} \qquad & T^{(2,1)}_{\mu\nu} &= -\frac{1}{4}R_{\mu\lambda\sigma\rho}R_{\nu}{}^{\lambda\sigma\rho} \\
Z^{(1,1)} &= \frac{1}{2}\nabla^2 & Z^{(2,1)} &= 0\ .
\end{aligned}$$

A major novelty here is that along with a multiplicative renormalization by Z, Φ also suffers an additive renormalization Ψ. This occurs because on a curved worldsheet the dimension-two interaction based on $G_{\mu\nu}$ requires new counterterms of the form of a dimension zero operator (scalar field) times worldsheet curvature scalar. This is in accord with general lore about renormalization in curved space[26] and a major nuisance insofar as determining Ψ requires explicit curved worldsheet computations. A major point of ref. [18] is that such calculations can be avoided by astute use of consistency conditions on flat worldsheet results.

At this stage one defines the renormalization group beta functions as the variation of the renormalized couplings under a variation of the arbitrary finite scale μ which leaves the bare couplings (and therefore meaningful physical quantities) fixed. It is a basic requirement of renormalizability that a finite renormalization group flow can be found. By expanding the equations $\mu\frac{\partial}{\partial\mu} G^B_{\mu\nu} = 0$ and $\mu\frac{\partial}{\partial\mu}\Phi^B = 0$ in powers of ϵ^{-1} one obtains from the nonsingular parts the flow equations

$$\begin{aligned} \mu\frac{\partial}{\partial\mu} G_{\mu\nu} &= -\epsilon G_{\mu\nu} + \beta_{\mu\nu} \\ \mu\frac{\partial}{\partial\mu}\Phi &= -\epsilon\Phi + \gamma(\Phi) + \theta(G) \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \beta_{\mu\nu} &= -\sum_L (\alpha')^L L\, T^{(L,1)}_{\mu\nu}\,, \\ \gamma(\Phi) &= -\sum_L (\alpha')^L L\, Z^{(L,1)}\cdot\Phi \\ \theta(G) &= (G\frac{\partial}{\partial G} - 1)\Psi^{(1)}\,. \end{aligned} \tag{4.4}$$

The terms in the equation that are singular as inverse powers of ϵ yield a series of consistency conditions

$$\begin{gathered} \sum_L (\alpha')^L L\, T^{(L,i+1)}_{\mu\nu} + \beta\frac{\partial}{\partial G}\sum_L (\alpha')^L\, T^{(L,i)}_{\mu\nu} = 0\,, \\ (1 - G\frac{\partial}{\partial G})\Psi^{(i+1)} + \beta\frac{\partial}{\partial G}\Psi^i + \sum_L (\alpha')^L Z^{(L,i)}\cdot(G\frac{\partial}{\partial G} - 1)\Psi^{(1)} = 0\,. \end{gathered} \tag{4.5}$$

In deriving the above results essential use is made of a simple scaling law which follows from dimensional analysis on the perturbation expansion:

$$\begin{aligned} G\frac{\partial}{\partial G}\, T^{(L,i)}_{\mu\nu} &= -(L-1)\, T^{(L,i)}_{\mu\nu}, \\ G\frac{\partial}{\partial G}\, Z^{(L,i)} &= -L Z^{(L,i)}. \end{aligned} \tag{4.6}$$

Physical quantities, like the partition function $\mathcal{Z}$, can be expressed solely in terms of bare quantities. When they are reexpressed in terms of μ and the renormalized couplings, they must satisfy the renormalization group equation $\mathcal{D}\mathcal{Z} = 0$

where

$$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + (-\epsilon G_{\mu\nu} + \beta_{\mu\nu})\frac{\partial}{\partial G_{\mu\nu}} + (-\epsilon\Phi + \gamma(\Phi) + \theta(G))\frac{\partial}{\partial\Phi}. \tag{4.7}$$

This is because $\mathcal{D}$ generates a joint variation of μ and the renormalized couplings which keeps the bare couplings fixed. Apart from the special features arising from the curved worldsheet, this is all standard dimensional regularization lore[13)].

To discuss the Weyl anomaly (trace of the energy-momentum tensor) we need to be able to construct renormalized composite operators of dimension two or less. The simplest way to define the insertion of a dimension-two operator of type $F_{\mu\nu}\partial X^\mu\partial X^\nu$ is to parametrically differentiate with respect to the renormalized metric. Because the theory is renormalized the result is finite. Because the metric appears only in the action, the result is equivalent to the insertion of the following parametric derivative of the action (4.1):

$$\begin{aligned} &\int dv\mu^\epsilon N\left[F_{\mu\nu}\partial X^\mu\partial X^\nu\right] \\ &\quad = 4\pi\alpha' F_{\mu\nu}\frac{\partial S}{\partial G_{\mu\nu}} \\ &\quad = \int dv\mu^\epsilon[(F_{\mu\nu} + F\cdot\frac{\partial}{\partial G}T_{\mu\nu})\partial X^\mu\partial X^\nu - \frac{\alpha'}{2}R^{(2)}(F_{\mu\nu}\frac{\partial Z}{\partial G_{\mu\nu}}\cdot\Phi + F_{\mu\nu}\frac{\partial\Psi}{\partial G_{\mu\nu}})]\,. \end{aligned} \tag{4.8}$$

(We denote the renormalized version of an operator O, or its normal product, by $N[O]$.) The quantities T, Z and Ψ are the same power series in ϵ^{-1} as appear in the relation between bare and renormalized coupling constants. They specify the counterterms needed to renormalize the original dimension two operator (note that, as promised, a subtraction operator proportional to worldsheet curvature is needed). Note that because the action S is an integral over the worldsheet, this procedure only gives a renormalized version of integrated operators: the subtractions specified above may not be enough to renormalize the local, unintegrated operator $F_{\mu\nu}\partial X^\mu\partial X^\nu$. We also need to be able to define the insertion of a dimension zero operator of type $H(X)$. It is relatively easy to see that

$$N[H(X)] = Z\,H(X) \tag{4.9}$$

where Z is defined in (4.2).

The main operator of interest to us is the trace of the energy-momentum tensor. It is defined by a parametric derivative with respect to the worldsheet metric

$$T^a{}_a = \frac{4\pi}{\sqrt{\gamma}}\gamma^{ab}\frac{\delta S}{\delta\gamma^{ab}}$$

and, by the sort of argument given above, is guaranteed finite. When the parametric differentiation is carried out on the bare action one gets, after using standard expressions for dimensionally continued versions of γ and $R^{(2)}$,

$$T^a{}_a = \frac{1}{2\alpha'}(-\epsilon G^B_{\mu\nu}\partial X^\mu\partial X^\nu + \frac{\epsilon}{2}\alpha' R^{(2)}\Phi^B - \alpha'\Box\Phi^B) \tag{4.10}$$

Using the results of the previous paragraph, the integral of $T^a{}_a$ over the worldsheet can be expressed directly in terms of renormalized quantities. By performing straightforward algebra on the definition (4.8) of the integrated normal product operator, the definition (4.4) of the beta functions and the various consistency conditions (4.5) on higher pole counterterms, we can derive the important result

$$\begin{aligned}2\alpha'\int dv\,T^a{}_a = \int dv\,\{&N[\beta_{\mu\nu}\partial X^\mu\partial X^\nu] - \epsilon N[G_{\mu\nu}\partial X^\mu\partial X^\nu] \\ &- \frac{\alpha'}{4}R^{(2)}N[\gamma(\Phi)+\theta(G)] + \frac{\alpha'}{2}\epsilon\,R^{(2)}N[\Phi]\}\ .\end{aligned} \tag{4.11}$$

In other words, the vanishing of the beta functions and anomalous dimensions in the renormalization group flow equation implies the vanishing of the integrated trace of the energy-momentum tensor. This is sufficient to impose rigid scale invariance, but not full conformal invariance, which requires the vanishing of the local energy-momentum tensor trace.

To discuss conformal invariance, we need a normal product definition which applies to local operators. The essential point is that singular pieces in the form of a total derivative may have to be added to (4.11) to render the unintegrated operator finite[27]. By the usual rules of renormalizable field theory, since the operator $F_{\mu\nu}\partial X^\mu\partial X^\nu$ is of dimension two, it only needs renormalization subtractions of dimension two or less. The only possible such object which is a total derivative is

$$\partial_a(\partial^a X^\mu A_\mu(F)) \quad , \quad A_\mu(F) = A_\mu^{\lambda\sigma}(G)F_{\lambda\sigma} \tag{4.12}$$

(with $A_\mu = \sum_L(\alpha')^L\sum_{i=1}^L \epsilon^{-i}A_\mu^{(L,i)}$ as usual). A remarkable and useful feature of the version of dimensional regulation we are using is that it is legitimate to relate operators by using the classical equations of motion inside the normal product (for this and other general features of the dimensional regulation normal product formalism, see ref. [28]). This allows us to write

$$\partial_a(\partial^a X^\mu A_\mu(F)) = \nabla_\mu A_\nu(F)\partial X^\mu\partial X^\nu + \alpha' R^{(2)}\nabla^\mu\Phi A_\mu(F)$$

and to express the unintegrated normal product in the form

$$\begin{aligned}N[F_{\mu\nu}\partial X^\mu\partial X^\nu] = \{&(F_{\mu\nu} + F\frac{\partial}{\partial G}T_{\mu\nu} + \nabla_\mu A_\nu(F))\partial X^\mu\partial X^\nu \\ &- \frac{\alpha'}{2}R^{(2)}(F\frac{\partial Z}{\partial G}\cdot\Phi + F\frac{\partial\Psi}{\partial G} + \nabla^\mu\Phi\,A_\mu(F))\}\ .\end{aligned} \tag{4.13}$$

The A_μ are not known *a priori* but reasonably simple calculations suffice to determine them to low loop order. Representative results for the ϵ^{-1} pole terms are[27,29]

$$\begin{aligned}A_\mu^{(1,1)} &= \nabla^\lambda F_{\lambda\mu} - \frac{1}{2}\nabla_\mu F^\lambda{}_\lambda\ , \\ A_\mu^{(2,1)} &= \frac{1}{4}R_\mu{}^{\lambda\sigma\rho}\nabla_\rho F_{\lambda\sigma}\ .\end{aligned} \tag{4.14}$$

It will be important to know the behavior of insertions of renormalized operators under renormalization group flows of the couplings. Since the general operator is not expressed in terms of the bare couplings only, there is no reason for it to be annihilated by $\mathcal{D}$. Indeed, general renormalization lore tells us that $\mathcal{D}N[O]$ will be a linear combination of other renormalized operators $N[\tilde{O}]$ of the same dimension. For the dimension two operator we have been considering, the expected relation is of the form

$$\mathcal{D}N\left[F_{\mu\nu}\partial X^{\mu}\partial X^{\nu}\right] + N\left[\gamma^{F}_{\mu\nu}\partial X^{\mu}\partial X^{\nu}\right] - \frac{\alpha'}{2}R^{(2)}N\left[\gamma^{F\Phi}\right] = 0 . \tag{4.15}$$

For the general dimension zero operator it is

$$\mathcal{D}N\left[H(X)\right] + N\left[\gamma^{H}\right] = 0 .$$

It is a straightforward exercise to read off the γ operators (anomalous dimensions) from our explicit expressions for $\mathcal{D}$ and $N[O]$. As usual, the essential information is contained in ϵ^{-1} pole terms while the higher poles express consistency conditions whose satisfaction is, in principle, automatic. Specifically,

$$\begin{aligned}\gamma^{F}_{\mu\nu} =& F\frac{\partial}{\partial G}\beta_{\mu\nu} - \sum_{L}(\alpha')^{L}L\left(\nabla_{\mu}A^{(L,1)}_{\nu}(F) + \nabla_{\nu}A^{(L,1)}_{\mu}(F)\right)\\ \gamma^{F\Phi} =& F\frac{\partial}{\partial G}\left(\gamma(\Phi) + \theta(G)\right) - \sum_{L}(\alpha')^{L}L\, A^{(L,1)}_{\mu}(F)\,\nabla^{\mu}\Phi\\ \gamma^{H} =& -\sum_{L}(\alpha')^{L}L\, Z^{(l,1)}\cdot H .\end{aligned} \tag{4.16}$$

Once again, the standard dimension two operator (based on $F_{\mu\nu}$) mixes with operators of the same type (through $\gamma^{F}_{\mu\nu}$) and operators of dimension zero times worldsheet curvature (through $\gamma^{F\Phi}$).

We can now discuss the renormalization of the unintegrated trace of the energy-momentum tensor. Applying the results expressed in (4.13) to (4.10) we obtain

$$\begin{aligned}2\alpha' T^{a}{}_{a} =& \mu^{\epsilon}\Big(\left\{N[\beta^{\star}_{\mu\nu}\partial X^{\mu}\partial X^{\nu}] - \epsilon\, N[G_{\mu\nu}\partial X^{\mu}\partial X^{\nu}]\right\}\\ & - \frac{\alpha'}{4}R^{(2)}N[\beta^{\star}_{\Phi}] + \frac{\alpha'}{2}\epsilon\, R^{(2)}N[\Phi]\Big) .\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}\beta^{\star}_{\mu\nu} =& \beta_{\mu\nu} - \alpha'\nabla_{\mu}\nabla_{\nu}\Phi + \nabla_{\mu}S_{\nu} + \nabla_{\nu}S_{\mu} ,\\ \beta^{\star}_{\Phi} =& -2\gamma(\Phi) - 2\theta(\Phi) + \alpha'(\nabla_{\mu}\Phi\nabla^{\mu}\Phi - 2\nabla^{2}\Phi) ,\end{aligned} \tag{4.18}$$

To simplify subsequent equations, we have elected to write the quantity A_{μ} appearing in (4.12) as $S_{\mu} - \frac{\alpha'}{2}\nabla_{\mu}\Phi$.

Both S_{μ} and $\theta(G)$ have somehow to be evaluated in terms of the 'coupling constants' $g_{\mu\nu}$ and Φ. There are various tricks which have been employed to accomplish essentially this goal but the most elegant is to make use of a renormalization group consistency condition[18)]. The crucial point is that since $T^{a}{}_{a}$ is expressible directly in terms of bare quantities, it is a physical object and must satisfy $\mathcal{D}T^{a}{}_{a} = 0$ (the anomalous dimension of the energy-momentum tensor must vanish). The explicit action of $\mathcal{D}$ on the various normal product terms in (4.17) is known from (4.15) and the net result is a set of differential equations which, as it turns out, can be solved to determine S_{μ} and θ. We will omit the details of this demonstration and simply record the results. The first useful result is that, while $S_{\mu} \neq 0$, it begins to receive contributions only at three-loop order: $S_{\mu} = O((\alpha')^{3})$. θ likewise begins at three-loop order and its leading term can easily be evaluated:

$$\theta = -\frac{(\alpha')^{2}}{8}R^{\mu\lambda\sigma\nu}R_{\mu\lambda\sigma\nu} + O((\alpha')^{3}). \tag{4.19}$$

Putting all of this together, we get the following explicit results for the $\beta^{\star}$ functions which determine $T^{a}{}_{a}$:

$$\begin{aligned}\beta^{\star}_{\mu\nu} =& \alpha' R_{\mu\nu} + \frac{(\alpha')^{2}}{2}R_{\mu}{}^{\lambda\sigma\rho}R_{\nu\lambda\sigma\rho} + 2\alpha'\nabla_{\mu}\nabla_{\nu}\Phi + O((\alpha')^{3})) ,\\ \beta^{\star}_{\Phi} =& \alpha'(-\nabla^{2}\Phi + (\nabla\Phi)^{2}) + \frac{(\alpha')^{2}}{4}R^{\mu\lambda\sigma\nu}R_{\mu\lambda\sigma\nu} + O((\alpha')^{3}) .\end{aligned} \tag{4.20}$$

(Note that there is a factor of α' difference in the normalization of the Weyl anomaly coefficients (3.41) and the renormalization group beta function $\beta_{\mu\nu}$.) The condition for conformal invariance is that both of these functions vanish. If the dilaton could be set equal to a constant, the conformal invariance condition would reduce to the vanishing of the standard renormalization group beta function for the metric, but it is obvious from the $\beta^\star_\Phi$ equation that this is not possible (except in the rather trivial case that the Riemann tensor vanishes). The important point is that, while the Weyl invariance conditions are not simply that the renormalization group beta functions vanish, there is a systematic procedure for generating the Weyl anomaly coefficients from the beta functions.

Finally, we should say something about the consistency of these equations. In the previous lecture, we argued that, at the one-loop level, Bianchi identities imply that if $\beta^\star_{\mu\nu}$ vanishes, then $\beta^\star_\Phi$ is a constant, consistent with zero. It is not obvious how this argument generalizes to higher loop order, to say the least. A remarkable feature of the arguments of ref. [18] is that they allow one to reach the same conclusion to all orders in α' by a systematic use of the dimensional regulation pole consistency conditions. We have no space to explain this argument, but it is of the utmost importance for the sigma model approach to string theory (it is roughly equivalent to the statement that the various conformal invariance conditions are variational derivatives of a single master spacetime action, even though we do not have a direct construction of that action).

5. SUPERSYMMETRIC NON-LINEAR SIGMA MODELS

5.1. Superstrings and worldsheet supersymmetry

In previous chapters we discussed two-dimensional non-linear sigma models whose couplings correspond to backgrounds of massless states of the bosonic string theory. We have seen some interesting spacetime physics, involving gravitation and the antisymmetric tensor and dilaton fields, arise in this framework. Such a description nevertheless has serious limitations that ultimately trace back to shortcomings of the bosonic string theory itself. Bosonic strings have tachyon modes and their presence means that the theory is not consistent. The tachyon

states may simply reflect that we have not identified the true vacuum of the theory but people have yet to find a stable vacuum without tachyons. In our calculations we managed to sidestep the problems that tachyons cause but that does not mean they can be ignored. Another issue is that so far we have only come across bosons, but any theory which attempts to describe nature had better include fermions.

A more realistic and successful approach is to study superstring theory, where the string is not only described by its embedding into spacetime but also has internal degrees of freedom, which can be anticommuting. There are many ways in which one might envisage generalizing the bosonic string theory to include fermionic modes but consistency conditions on the quantum theory place severe restrictions on the possible choices. We will not go into those matters here but the list of viable superstring theories is actually quite short. The generalizations of the bosonic string theory can all be described by two-dimensional field theories with fermion fields living on the worldsheet in addition to the bosonic coordinate fields. For reasons, which will become clear in a moment, the two-dimensional theories are all supersymmetric but the extent of the supersymmetry algebra in each case depends on the superstring theory in question. In this chapter we write down the various supersymmetric non-linear sigma models that correspond to superstrings in non-trivial backgrounds and give an outline of the analysis of anomalies and the resulting spacetime equations. The actual computations are analogous to what we have already seen for the bosonic theory so we will not delve into the details but rather try to give an overview of the general structure and review the salient results. We recommend ref. [30] to the reader interested in supersymmetric sigma models.

Consider a two-dimensional field theory with D scalar fields, which describe the embedding of a string worldsheet into D-dimensional spacetime, but also with two-dimensional fermion fields living on the worldsheet, which enable us to build fermionic states in the string spectrum. When we only had the bosonic coordinates we found that the non-linear sigma model actually describes string theory only if the coupling functions are chosen so that the two-dimensional theory is conformally invariant. One way to see this is to observe that there are spurious states in the spectrum of the bosonic string and the theory must have an infinite dimensional symmetry to effect the decoupling of those spurious states from physical processes. Conformal invariance is precisely that symmetry; it is expressed in terms of the Virasoro algebra and, as we mentioned in Chapter 2, the Virasoro generators are used to determine the physical states of the theory. Now that we also have fermions on the worldsheet there will be further spurious states associated with them and an enlarged symmetry algebra is needed to decouple them also. This can be accomplished by requiring supersymmetry on top of conformal invariance. In a supersymmetric theory every bosonic object will have a fermion partner so corresponding to each Virasoro generator there will be a fermionic generator. The Virasoro generators are moments of the worldsheet energy momentum tensor and their fermionic counterparts are moments of the worldsheet supercurrent (the conserved current associated with the worldsheet supersymmetry). These new generators along with the L_n's satisfy a supersymmetric extension of the Virasoro algebra and can be used to define physical state conditions for the full superstring spectrum. All this is explained in detail in standard string theory texts (for example ref. [31]) but the point we wish to emphasize here is that supersymmetry leads to a natural extension of the Virasoro algebra and it is necessary to impose it in addition to conformal invariance in order for a non-linear sigma model with fermions to describe superstring theory. Our strategy will therefore be to write down supersymmetric, renormalizable, non-linear sigma model actions and then proceed to calculate Weyl anomalies to derive conditions on the spacetime physics, in much the same way as in the purely bosonic case.

In Chapter 3 we set out assembling all possible terms in a general, reparametrization invariant, renormalizable scalar field theory on a curved two-dimensional space. The resulting action described a scalar field with non-linear interactions along with gravity in two dimensions. There is no Einstein term to describe the dynamics of the metric in two dimensions because the Einstein action, *i.e.*, the integral of the curvature scalar, is a topological invariant (proportional to the Euler

character of the worldsheet). The two-dimensional metric is therefore not dynamical. Reparametrization invariance and classical Weyl invariance along with some tricks enabled us to eliminate it to the extent that we could do all our computations on a flat worldsheet.

A corresponding approach to the study of supersymmetric non-linear sigma models would be to write down the most general two-dimensional supergravity action with scalars and spinors. We would then be doing physics on a curved worldsheet with both a metric and a gravitino field, which is a vector-spinor (spin $\frac{3}{2}$ field). The question then is whether we can eliminate not only the metric degrees of freedom but also the gravitino. That turns out to be possible if the action has a supersymmetry and is conformally invariant (the combination of the two is called superconformal invariance). There is a horde of possible renormalizable and superconformal terms in two-dimensional supergravity. However, we will not write down the most general classical action of that form because at the end of the day, when the superconformal invariance has been used to fix the metric and to gauge away the gravitino field, things simplify considerably. In fact the physics we are interested in is all contained in an action with simply a rigid supersymmetry plus ordinary reparametrization invariance and conformal invariance at the classical level. We will arrange things so that the quantum theory is supersymmetric and at the conformal fixed point the supersymmetry will be promoted to superconformal invariance. So rather than deal with supergravity, which is complicated and not really necessary in order to get the results we are after, we can keep things simple and calculate ordinary Weyl anomalies in a globally supersymmetric theory. We will actually take the worldsheet to be flat and compute renormalization group beta functions for the various supersymmetric sigma models. This is simpler than obtaining the Weyl anomaly coefficients. The difference between the two has to do with a dilaton field that varies in spacetime and can be sorted out at the end.

There are different ways to introduce fermions into the two-dimensional non-linear sigma model and a number of supersymmetric extensions of the bosonic theory exist. Only a few choices however, lead to interesting theories and we restrict our attention to those that correspond to consistent superstring theories. The N=1 or (1,1) supersymmetric sigma model is a good starting point. The bosonic coordinates are the same as before: a set of D scalar fields $X^\mu(\xi), \mu = 1, \cdots, D$, on the worldsheet. The fermions are taken to be Majorana spinors $\psi^\mu(\xi)$. In a supersymmetric theory they have to be equal in number to the bosons so the index μ runs from 1 to D. The worldsheet fermions transform as a vector under spacetime coordinate changes, which are internal symmetry transformations from the two-dimensional point of view. The number of fields is chosen to eliminate the leading order Weyl anomaly (the part that is present even in flat spacetime) and one finds that $D = 10$ is the critical dimension in the supersymmetric theory. In two dimensions a Majorana spinor has two components. We can choose a basis for the two-dimensional Dirac matrices in which the two components are a pair of Weyl spinors, each with a definite chirality:

$$\psi^\mu(\xi) = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}. \tag{5.1}$$

A good way to express supersymmetry and construct supersymmetric Lagrangians is to use superspace techniques and superfields (which are described in a more general setting by Dine[32] elsewhere in this volume). Our two dimensional superspace will consist of the commuting worldsheet coordinates ξ_+ and ξ_- (we are using Minkowski light-cone coordinates) and a pair of anticommuting (Grassmann) variables θ_+ and θ_-. The notation is very simple because we need not worry about conventions for raising and lowering spinor indices in the basis we have chosen. A superfield is a function on superspace. It always has a finite expansion in powers of the nilpotent Grassmann coordinates. The coefficients of the different powers

of θ_+ and θ_- are ordinary fields (called the components of the superfield) which transform into each other under supersymmetry transformations. Our basic sigma model fields can be assembled into a set of superfields:

$$\Phi^\mu(\xi, \theta_+, \theta_-) = X^\mu(\xi) - i\theta_-\psi^\mu_+(\xi) + i\theta_+\psi^\mu_-(\xi) + \theta_+\theta_- F^\mu(\xi). \tag{5.2}$$

We need the $F^\mu(\xi)$ to complete the superfield expansion but they turn out to be auxiliary fields which have no dynamics in the theory. Supersymmetry is realized on superspace through the action of differential operators. The N=1 supersymmetry has two generators, one associated with each chirality of spinor components (hence the term $(1,1)$ supersymmetry):

$$Q_\pm = i\frac{\partial}{\partial\theta_\mp} - \theta_\mp\,\partial_\pm\,. \tag{5.3}$$

It is easy to check that these Q's satisfy the usual supersymmetry algebra and give the correct transformation rules for the component fields. There are also supercovariant derivatives on superspace,

$$D_\pm = i\frac{\partial}{\partial\theta_\mp} + \theta_\mp\,\partial_\pm\,, \tag{5.4}$$

which anticommute with the supersymmetry generators. Finally we adopt the usual Berezin definitions for integrals over Grassman variables:

$$\int d\theta_+ = \int d\theta_- = 0, \qquad \int d\theta_+\,\theta_+ = \int d\theta_-\,\theta_- = 1\,. \tag{5.5}$$

It is easy to see using these formal rules for integration that the Berezin integral of a total derivative vanishes, *i.e.*, for any superfield Ψ we have

$$\int d\theta_+\,\frac{\partial\Psi}{\partial\theta_+} = \int d\theta_-\,\frac{\partial\Psi}{\partial\theta_-} = 0. \tag{5.6}$$

This superspace formalism is very useful because it gives a compact way to write supersymmetric Lagrangians. In fact the integral of any function of a superfield and its supercovariant derivatives is invariant under supersymmetry transformations.

$$I = \int d^2\xi d\theta_+ d\theta_-\, V(\Phi, D_\pm\Phi, \ldots). \tag{5.7}$$

A superfield transforms as a scalar under a supersymmetry transformation and since supercovariant derivatives anticommute with the supersymmetry generators the function V in (5.7) also transforms as a scalar, $\delta_\pm V = Q_\pm V$. The variation of the integral is therefore

$$\begin{aligned}\delta_\pm I &= \int d^2\xi d\theta_+ d\theta_-\, (i\frac{\partial}{\partial\theta_\mp} - \theta_\mp \partial_\pm)\, V(\Phi, D_+\Phi, D_-\Phi, \ldots) \\ &= 0.\end{aligned} \tag{5.8}$$

It vanishes because we can integrate by parts either in the Berezin or in the conventional integral.

We want the theory to be renormalizable so terms in the action must be of the right dimension. If the superfields in a given term are expanded in component fields and the Berezin integrals over θ_+ and θ_- performed any term in the resulting expression involving only the bosonic fields $X^\mu(\xi)$ should be of scaling dimension two. This tells us that the original superfield term has to have precisely two supercovariant derivatives. The supersymmetric extension of the Polyakov action is an example:

$$S_1^{(1,1)}[\Phi] = \frac{1}{4\pi\alpha'} \int d^2\xi d\theta_+ d\theta_-\, G_{\mu\nu}(\Phi)\, D_+\Phi^\mu D_-\Phi^\nu\,. \tag{5.9}$$

It has a pair of supercovariant derivatives of opposite chirality acting on the superfield and the spacetime indices are contracted on an arbitrary symmetric tensor function $G_{\mu\nu}(\Phi)$, which will be interpreted as the metric tensor of spacetime. It is instructive to obtain this action in component form. After inserting (5.2) for the superfield in (5.9) and performing the Berezin integrals we have a number of terms involving the X^μ, $\psi^\mu_\pm$ and F^μ fields. An important thing to note is that the F^μ are

auxiliary fields. They have no derivatives acting on them and can be eliminated by their equations of motion. After that has been taken care of the resulting terms can be gathered into the following spacetime coordinate invariant expression:

$$\begin{aligned}S_1^{(1,1)}[X,\psi] = \frac{1}{4\pi\alpha'} \int d^2\xi \Big\{ & G_{\mu\nu}(X)\big(\partial_+ X^\mu \partial_- X^\nu + i\psi^\mu_+ \nabla_- \psi^\nu_+ + i\psi^\mu_- \nabla_+ \psi^\nu_-\big) \\ & + \frac{1}{2} R_{\mu\nu\lambda\sigma}(X)\, \psi^\mu_+ \psi^\nu_+ \psi^\lambda_- \psi^\sigma_- \Big\},\end{aligned} \tag{5.10}$$

The purely bosonic term in (5.10) is precisely S_P. In addition we have quadratic kinetic terms for fermions of both chiralities and a four fermion interaction with a Riemann tensor coupling.

$S_1^{(1,1)}$ is not the most general (1,1) supersymmetric renormalizable action on a flat worldsheet. There is another term we can write down and it involves an antisymmetric coupling:

$$I_2^{(1,1)}[\Phi] = \frac{1}{4\pi\alpha'} \int d^2\xi d\theta_+ d\theta_-\, B_{\mu\nu}(\Phi)\, D_+\Phi^\mu D_-\Phi^\nu\,. \tag{5.11}$$

This is of course the supersymmetric extension of S_{AS}. In component form it reads

$$\begin{aligned}S_2^{(1,1)}[X,\psi] = \frac{1}{4\pi\alpha'} \int d^2\xi \Big\{ & B_{\mu\nu}(X)\, \partial_+ X^\mu \partial_- X^\nu - \frac{i}{2} H_{\mu\nu\lambda}(X)\, \psi^\mu_+ \psi^\nu_+ \partial_- X^\lambda \\ & - \frac{i}{2} H_{\mu\nu\lambda}(X)\, \psi^\mu_- \psi^\nu_- \partial_+ X^\lambda + \frac{1}{2} \nabla_\mu H_{\nu\lambda\sigma}(X) \psi^\mu_- \psi^\nu_+ \psi^\lambda_+ \psi^\sigma_- \\ & - \frac{1}{4} H^\rho{}_{\mu\nu}(X) H_{\rho\lambda\sigma}(X) \psi^\mu_+ \psi^\nu_- \psi^\lambda_+ \psi^\sigma_- \Big\}.\end{aligned} \tag{5.12}$$

The action of the $(1,1)$ supersymmetric non-linear sigma model on a flat worldsheet is the sum of $S_1^{(1,1)}$ and $S_2^{(1,1)}$. In a two-dimensional supergravity theory it is also possible to write down a supersymmetric extension of the dilaton term, S_D, of the bosonic sigma model, but we do not need that here. We want to calculate renormalization group beta functions on a flat worldsheet with the gravitino field gauged to zero in which case the dilaton term is absent. The effects of the dilaton can be obtained from the beta functions by the methods described in the previous chapter.

The spacetime coupling functions we have written down for the N=1 supersymmetric sigma model are of the same type as the ones we had in the bosonic theory. They correspond to background condensates of massless bosonic states of the closed superstring in spacetime. One of the motivations to study superstrings in the first place was to get spacetime fermions into the game, so why not add terms to the sigma model action which describe fermion backgrounds? This would be an interesting thing to do but there are technical obstacles which have to be overcome first. Spacetime fields appear as coupling functions in the two-dimensional theory and a fermion in spacetime is therefore described by a sigma model with anticommuting couplings. Such models can be written down using techniques from two-dimensional superconformal field theory[33]. One problem which makes those theories unwieldy is that the worldsheet fields are non-trivially coupled to the superconformal ghost system in the fermion coupling function, which gives rise to notorious technical problems. This subject has not been fully developed and we will not go into it.

When we extend the two-dimensional non-linear sigma model to an N=1 supersymmetric theory we go from describing bosonic string theory in 26-dimensional spacetime to ten-dimensional superstring theory which contains much more interesting physics. It is natural to ask what happens if we start with a larger supersymmetry algebra on the worldsheet. There is a conformally invariant N=2 supersymmetric sigma model which has two supersymmetry generators of each chirality. It turns out that the bosonic fields are complex and the critical dimension is only two so this theory does not describe realistic spacetime physics. It is possible to extend the worldsheet supersymmetry even further to N=4 but that leads to a negative critical dimension. It thus appears one should start with no more than N=1 supersymmetry on the worldsheet in order to describe the sort of physics we are interested in. We will see later on that it can be advantageous to restrict the N=1 supersymmetry to a smaller algebra. That will lead us to the study of heterotic sigma models.

Extended worldsheet supersymmetry nevertheless does play an important role

in string theory. When the coupling functions of the N=1 sigma model satisfy certain conditions a new pair of supersymmetry generators appears and the (1,1) supersymmetry is enhanced to (2,2). The resulting N=2 theory is of course different from the one we mentioned above since restricting the form of the sigma model couplings does not change the critical dimension. We do not show it here but the simplest way to realize this enhanced supersymmetry is to set the antisymmetric tensor field $B_{\mu\nu}$ equal to zero and choose a $G_{\mu\nu}$ which is Kähler, which means one can define complex coordinates on spacetime in a consistent way (a Kähler manifold by definition has a covariantly constant complex structure). The point is that the enhanced worldsheet supersymmetry is needed to get a theory with spacetime supersymmetry and therefore plays a central role when one studies compactification of extra dimensions and string phenomenology from the worldsheet point of view.

5.3. Beta functions of the N=1 supersymmetric sigma model

The calculation of the renormalization group beta functions of the supersymmetric non-linear sigma model is very similar to that of the bosonic theory. The spacetime tensor structure of the couplings is identical in the two models (a symmetric metric and an antisymmetric tensor). In the bosonic theory the beta functions got a contribution from every object of dimension two that could be built using those couplings. This continues to be true in the supersymmetric theory so the beta functions will have the same sorts of terms but there are now more Feynman graphs to calculate and the coefficients of the terms are different. In a supersymmetric theory we expect cancellations between boson and fermion loops which can lead to interesting effects.

In order to compute the beta functions we need to derive Feynman rules and identify relevant diagrams. There are two ways to go in setting up the perturbation theory. One is to use superspace techniques to generate a manifestly supersymmetric diagrammatic expansion. In that case one derives a superfield propagator and works out a background field expansion for superfields. The notation is very compact and leads to a powerful formalism which is especially useful for higher loop calculations. It has been used to obtain $\beta^G_{\mu\nu}$ to four loops in the N=1 supersymmetric sigma model with only a metric coupling[34] and to five loops for a model with extended N=2 supersymmetry[35]. We will not develop superfield perturbation theory here but rather take the other approach which is to use the component form of the action and calculate conventional diagrams with both boson and fermion legs. One has more graphs to consider than in superfield calculations but for low loop orders we find this formalism more transparent to the physics and the comparison to the bosonic theory more straightforward.

For simplicity we will drop the antisymmetric tensor from the discussion for the moment and consider an action with only a symmetric spacetime coupling function. The normal coordinate expansion of (5.10) is straightforward. We want to compute $\beta^G_{\mu\nu}$ which is a commuting object. We therefore only introduce background X^μ's and consider the fermion ψ^μ's as quantum fields from the start. We already have the expansion of the bosonic part of the action in (3.19). The following are those terms coming from the fermion pieces of the action which are needed for calculations to two-loop order:

$$\begin{aligned}\frac{1}{4\pi\alpha'}\int d^2\xi\Big\{ & i\psi^i_+\nabla_-\psi^i_+ + i\psi^i_-\nabla_+\psi^i_+ - \frac{1}{2}R_{\mu ijk}(X_0)\partial_- X^\mu_0\,\eta^i\psi^j_+\psi^k_+ \\ & - \frac{1}{2}R_{\mu ijk}(X_0)\partial_+ X^\mu_0\,\eta^i\psi^j_-\psi^k_- + \frac{1}{2}R_{ijkl}(X_0)\,\psi^i_+\psi^j_+\psi^k_-\psi^l_- \\ & + \frac{1}{3}R_{iklj}(X_0)\,\eta^k\eta^l\,\psi^i_+\nabla_-\psi^j_+ + \frac{1}{3}R_{iklj}(X_0)\,\eta^k\eta^l\,\psi^i_-\nabla_+\psi^j_-\Big\}.\end{aligned} \tag{5.13}$$

The spacetime indices of the quantum fields are referred to a local Lorentz frame as before.

We can immediately see without any calculation that the fermions do not contribute to $\beta^G_{\mu\nu}$ at the one-loop level. In the background field approach we are instructed to compute vacuum diagrams so at one-loop order it is only vertices with two quantum fields that contribute. Thus the only one-loop graph with fermion legs which could contribute to $\beta^G_{\mu\nu}$ is

FIGURE 5.1.

where the insertions of the spacetime spin connection come from the covariant derivatives in the fermion kinetic terms in (5.13). However, as we argued in Chapter 3, this diagram cannot give a contribution to $\beta^G_{\mu\nu}$ because it is not covariant in spacetime. So there simply is no diagram with only a single fermion loop that contributes to $\beta^G_{\mu\nu}$ and the one-loop result of the bosonic theory, $\beta^G_{\mu\nu} = R_{\mu\nu}$, remains unchanged in the supersymmetric theory.

At the two-loop level there was an order α' correction to $\beta^G_{\mu\nu}$ of the form $R_{\mu\lambda\sigma\rho}R_\nu{}^{\lambda\sigma\rho}$ in the bosonic sigma model. In the $(1,1)$ supersymmetric theory it vanishes. There is only one new divergent diagram with fermions that contributes to $\beta^G_{\mu\nu}$ at this order:

FIGURE 5.2.

The value of this diagram is precisely equal to the contribution of the purely bosonic graphs and because of the minus sign associated with a fermion loop there is exact cancellation. This cancellation generated some excitement when it was discovered[13], which only grew when further calculations showed that the three-loop contribution to $\beta^G_{\mu\nu}$ also vanishes in the N=1 supersymmetric sigma model[36]. This was naturally taken as evidence that $\beta^G_{\mu\nu}$ vanished to all higher orders and effort was put into trying to prove it. The idea that a Ricci flat spacetime manifold was the solution to the beta function equations to all orders in perturbation theory was especially appealing for sigma models with enhanced supersymmetry because Ricci flat Kähler manifolds have been extensively studied by mathematicians and have many nice properties. An explicit four-loop computation[34] however, established a non-vanishing contribution to $\beta^G_{\mu\nu}$ at order $(\alpha')^3$. This caused a certain amount of consternation. On closer examination it turned out that the higher order corrections to $\beta^G_{\mu\nu}$ could be accounted for by certain field redefinitions[37]. Therefore the essential physical results which were derived thinking the one-loop result was the final answer remain valid. Having a superfield formalism proved crucial for carrying out the four-loop calculation. It would hardly be feasible using component fields because of the number of diagrams. The result is a complicated expression involving terms with eight derivatives of the spacetime metric.

The perturbation theory is more complicated if the sigma model includes an antisymmetric tensor coupling function. There are more diagrams to calculate and subtleties involved in defining the ϵ^{ab} symbol in higher loop dimensional regularization caused some confusion for a while. The controversy has been resolved by now and the beta functions computed to two-loop order[38]. In the (1,1) supersymmetric theory the two-loop correction to both $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ vanishes.

5.4. Spacetime gauge fields and heterotic sigma models

So far we have written down a supersymmetric non-linear sigma model with coupling functions that correspond to a background metric tensor and an antisymmetric tensor field in spacetime. The action of the theory is invariant under general coordinate transformations and does indeed describe gravity in spacetime. In a theory of nature it is also crucial to have non-abelian gauge symmetry to account for the other interactions of matter. Two very different ways of introducing spacetime gauge fields into the theory are known. One is to allow open strings with gauge charges (which are called Chan-Paton factors in string theory) on their ends. The worldsheet can then have edges and the corresponding sigma model will include one more renormalizable term which lives only on the worldsheet boundary. It is a supersymmetric generalization of the Wilson line coupling to a spacetime gauge field that we mentioned in Chapter 3. The renormalization group analysis can of course be carried out for the theory with interactions both on the boundary and in the interior of the worldsheet. The beta functions of all the couplings, including

the Wilson line, can be identified and there is a spacetime action from which the vanishing beta function conditions can be derived. There is a lot of interesting physics in the study of open strings but we will not pursue it here.

The other method of incorporating gauge symmetry into string theory involves only closed strings and uses the fact that the right moving and left moving modes on the worldsheet are decoupled. It is possible to build a consistent theory of closed strings in which the right moving modes are those of the superstring but some of the left movers carry gauge degrees of freedom. Such a construction, with a mismatch between the left and right moving modes on the same string worldsheet, is called a heterotic theory[39]. There is a corresponding non-linear sigma model which describes the physics of the heterotic string in background fields. We have as before D scalar fields $X^\mu(\xi)$, which describe the embedding of the string worldsheet into spacetime, and the same number of right moving fermions $\psi^\mu_+(\xi)$. Worldsheet supersymmetry is the basis for all the nice features of the N=1 superstring theory and we want to retain it in the right moving sector of the heterotic theory. Therefore, we look for an action with a supersymmetry which relates the right moving bosons and fermions. It is convenient to use superspace techniques to construct such an action. Since the supersymmetry is only among right movers the appropriate superspace only has one anticommuting coordinate, θ_-. There is one supersymmetry generator

$$Q_+ = i\frac{\partial}{\partial\theta_-} - \theta_-\,\partial_+\,, \tag{5.14}$$

and one supercovariant derivative which anticommutes with it

$$D_+ = i\frac{\partial}{\partial\theta_-} + \theta_-\,\partial_+\,. \tag{5.15}$$

Because there is only the one right moving supersymmetry generator the heterotic sigma model is also called the $(1,0)$ supersymmetric sigma model. (The (1,0) supersymmetry is also sometimes referred to as N=$\frac{1}{2}$ supersymmetry.) The bosonic fields and right moving fermions combine into a set of superfields:

$$\Phi^\mu(\xi,\theta_-) = X^\mu(\xi) - i\theta_-\psi_+^\mu(\xi). \tag{5.16}$$

There is only one Grassman variable so the component expansion is simple. The general, renormalizable action that can be built out of the superfield Φ^μ and its derivatives has two terms:

$$S_1^{(1,0)}[\Phi] = \frac{1}{4\pi\alpha'}\int d^2\xi d\theta_-\, G_{\mu\nu}(\Phi)\, D_+\Phi^\mu\partial_-\Phi^\nu \tag{5.17}$$

and

$$S_2^{(1,0)}[\Phi] = \frac{1}{4\pi\alpha'}\int d^2\xi d\theta_-\, B_{\mu\nu}(\Phi)\, D_+\Phi^\mu\partial_-\Phi^\nu, \tag{5.18}$$

where as usual $G_{\mu\nu}(\Phi)$ is a symmetric coupling function and $B_{\mu\nu}(\Phi)$ is antisymmetric. There is no left moving supersymmetry generator or supercovariant derivative so we simply use the partial derivative ∂_-.

The heterotic string also has left moving fermions, which we will call λ_-^A, and in the sigma model they appear in a natural way as components of a set of left moving, anticommuting superfields:

$$\Lambda_-^A(\xi,\theta_-) = \lambda_-^A + \theta_- f^A(\xi). \tag{5.19}$$

The $f^A(\xi)$ are scalar fields needed to complete the superfield expansion but they turn out to be auxiliary fields like the F^μ's in the $(1,1)$ model. Of course Φ^μ and Λ_-^A are totally unrelated superfields at this stage so the A index has nothing to do with the spacetime index μ. We can construct another renormalizable interaction term using the anticommuting superfield. Its general form is:

$$S_3^{(1,0)}[\Phi,\Lambda_-] = -\frac{i}{4\pi\alpha'}\int d^2\xi d\theta_-\, g_{AB}(\Phi)\,\Lambda_-^A\left(D_+ + A_+(\Phi)\right)^B{}_C\,\Lambda_-^C. \tag{5.20}$$

The notation is chosen to suggest an interpretation for $g_{AB}(\Phi)$ and $A_{+C}^{\;B}$. We can view the λ_-^A as taking value in some fiber bundle over spacetime and then $g_{AB}(\Phi)$ is a metric on fibres while $A_{+C}^{\;B} = \left(A_\mu(\Phi)\right)^B{}_C D_+\Phi^\mu$ is a connection. In order to implement the usual spacetime non-abelian gauge symmetry in the sigma model we take $g_{AB}(\Phi) = \delta_{AB}$ and identify A_μ with a gauge connection for some Lie algebra. From the two-dimensional point of view the gauge coupling $A_{\mu C}^{\;B}(\Phi(\xi))$ is a function on the worldsheet like the spacetime metric or antisymmetric tensor field. The gauge charge is thus distributed along the closed string.

The action of the heterotic non-linear sigma model is the sum of $S_1^{(1,0)}$, $S_2^{(1,0)}$ and $S_3^{(1,0)}$. It is easier to see its physical content in component form which is obtained by expanding the superfields, integrating over θ_- and then eliminating the auxiliary fields f^A by their equations of motion. We get a number of terms which combine neatly into the following spacetime coordinate invariant and gauge invariant form:

$$\begin{aligned} A^{(1,0)}[X,\psi_+,\lambda_-] = \frac{1}{4\pi\alpha'}\int d^2\xi \Big\{ & G_{\mu\nu}(X)\,\partial_+X^\mu\partial_-X^\nu + B_{\mu\nu}(X)\,\partial_+X^\mu\partial_-X^\nu \\ & + iG_{\mu\nu}(X)\,\psi_+^\mu\nabla_-\psi_+^\nu + ig_{AB}(X)\,\lambda_-^A\hat\nabla_+\lambda_-^B \\ & + \frac{1}{2}\left(\hat F_{\mu\nu}\right)_{AB}(X)\,\psi_+^\mu\psi_+^\nu\lambda_-^A\lambda_-^B\Big\}. \end{aligned} \tag{5.21}$$

The spacetime covariant derivative, which acts on right moving fermions, involves both the Christoffel connection and a torsion piece,

$$\nabla_-\psi_+^\mu = \partial_-\psi_+^\mu + \Gamma^\mu_{\lambda\sigma}\partial_-X^\sigma\,\psi_+^\lambda - \frac{1}{2}H^\mu{}_{\lambda\sigma}\partial_-X^\sigma\,\psi_+^\lambda\,. \tag{5.22}$$

The gauge covariant derivative that acts on the left moving fermions is:

$$\hat\nabla_+\lambda_-^A = \partial_+\lambda_-^A + \hat A_{\mu B}^{\;A}\,\partial_+X^\mu\,\lambda_-^B\,, \tag{5.23}$$

where $\hat A_{\mu B}^{\;A} = A_{\mu B}^{\;A} + \frac{1}{2}g^{AC}\partial_\mu g_{BC}$. A gauge transformation acts in the usual way on the fields with gauge degrees of freedom:

$$\begin{aligned} \lambda_-^A &\to U(X)^{AB}\,\lambda_-^B \\ A_{\mu B}^{\;A} &\to U^A_{\;C}\,A_{\mu D}^{\;C}\,U^{\dagger D}{}_B - \partial_\mu U^A_{\;C}\,U^{\dagger C}{}_B\,. \end{aligned} \tag{5.24}$$

The action (5.21) has all the ingredients to describe the physics of heterotic strings in background fields. The worldsheet fields are coupled to a spacetime metric, an antisymmetric tensor field and a non-abelian gauge potential which are precisely the massless bosons of the heterotic string theory (there is also a dilaton field but the techniques of the previous chapter can presumably be used to calculate its effect without having to introduce an explicit dilaton term in the action). This theory has less supersymmetry but a much richer structure than the model we presented in the previous section. The restricted supersymmetry in the right moving sector is enough to keep tachyons out of the theory while leaving the left moving fermions free to carry gauge degrees of freedom.

The heterotic theory is in a sense more general than the N=1 supersymmetric sigma model because for a special choice of heterotic coupling functions it reproduces the N=1 theory. If we take the antisymmetric tensor field strength to be equal to zero and choose the gauge group to be the same as the holonomy group of spacetime then the spin connection and the gauge connection can be identified. The index A on the left moving fermions then runs over the same values as the spacetime index μ on the right movers, the non-abelian field strength $(F_{\mu\nu})_{AB}$ becomes the curvature tensor $R_{\mu\nu\lambda\sigma}$, and it is not hard to see that $A^{(1,0)}$ in (5.21) is then precisely the same as the action (5.10) of the N=1 supersymmetric sigma model. Of course this is a very strong condition on the gauge group (and the gauge fields) but the point we wish to illustrate is that for a certain choice of the spacetime coupling functions the heterotic sigma model reduces to the N=1 supersymmetric model and the (1,0) supersymmetry is promoted to the larger (1,1) supersymmetry.

The (1,0) worldsheet supersymmetry of the heterotic theory can be enhanced to (2,0) by restricting the spacetime metric to be Kähler and setting the antisymmetric tensor field equal to zero. Enhanced worldsheet supersymmetry is necessary to have spacetime supersymmetry in the heterotic theory [40,32].

5.5. Gauge covariant background field expansion and heterotic beta functions

The heterotic string is the most promising string theory from the point of view of model building and phenomenology. It is therefore of fundamental importance to obtain the beta functions of the corresponding sigma model since they carry information about the low energy spacetime physics of the theory. The action of the heterotic sigma model looks a bit odd with the mismatch between right moving and left moving fermions but it is straightforward to derive its Feynman rules and the diagrams are not hard to evaluate. The chiral nature of the theory does complicate things in that there are chiral as well as the usual Weyl anomalies at the quantum level. The chiral anomalies of the sigma model in general break both gauge invariance and local Lorentz invariance in spacetime. It turns out that these symmetries can be recovered by modifying the spacetime physics in a way that is familiar from the study of supergravity theories.

We will not attempt to compute the beta functions in the most general heterotic sigma model. Such a calculation is quite complicated due to the many diagrams and interaction terms. Instead we will focus on those features which are special to the heterotic theory and particularly relevant to spacetime considerations. First of all there is a new coupling function $A_\mu(X)$ in the sigma model and a corresponding beta function. Let us for simplicity concentrate on the gauge field and assume a flat spacetime metric and vanishing antisymmetric tensor field for the moment. We do a background field expansion as usual. In flat spacetime we do not have to worry about choosing normal coordinates but it is very convenient to arrange the expansion of the action so that the interaction vertices are gauge covariant with respect to the background field (*i.e.*, covariant under the gauge transformation (5.24) with $U^A_{\ B}$ a function of the background X_0^μ's only). This is accomplished by imposing a 'radial gauge' condition on the gauge connection[41] :

$$\pi^\mu A^{\ B}_{\mu C}(X_0+\pi) = 0. \tag{5.25}$$

It may sound strange that a gauge covariant perturbation expansion is generated by fixing a gauge for the quantum field but that is how background field expansions work. The situation here can be compared to that of Section 3.2. There we obtained a spacetime generally covariant expansion by choosing a special coordinate system (the normal coordinates) for the quantum field.

The spacetime gauge field coupling in the heterotic sigma model action is the connection piece of the kinetic term of the left moving fermions:

$$\frac{i}{4\pi\alpha'}\int d^2\xi\, \lambda_-^B\, A_\mu(X)_{BC}\partial_+ X^\mu\, \lambda_-^C. \tag{5.26}$$

The gauge covariant background field expansion of this term reads:

$$\frac{i}{4\pi\alpha'}\int d^2\xi\Big\{A_\mu(X_0)_{BC}\partial_+ X_0^\mu\, \lambda_-^B\lambda_-^C + \big(F_{\mu\nu}(X_0)\big)_{BC}\pi^\mu\partial_+ X_0^\nu\, \lambda_-^B\lambda_-^C + \frac{1}{2}\big(F_{\mu\nu}(X_0)\big)_{BC}\pi^\mu\partial_+\pi^\nu\, \lambda_-^B\lambda_-^C + \frac{1}{2}\big(D_\lambda F_{\mu\nu}(X_0)\big)_{BC}\pi^\lambda\pi^\mu\partial_+ X_0^\nu\, \lambda_-^B\lambda_-^C\Big\}. \tag{5.27}$$

The D_λ denotes a gauge covariant spacetime derivative. Only the last term in this expansion gives rise to a diagram which has the right background field content to contribute to β_μ^A at one-loop order.

$(D_\lambda F_{\mu\nu})_{BC}\, \partial_+ X_0^\nu\, \lambda_-^B\lambda_-^C\, \pi^\lambda\pi^\mu$

FIGURE 5.3.

This diagram is logarithmically divergent and the resulting one-loop beta function is

$$\beta_\mu^A(X_0) = \frac{1}{2}\, D^\lambda F_{\lambda\mu}(X_0)\,. \tag{5.28}$$

We do not have to worry about the chiral anomaly at this stage. It is associated with fermion loops and the diagram above has a boson propagating in the loop.

This result supports our interpretation of A_μ as a spacetime gauge potential. The condition that its beta function be equal to zero is the Yang-Mills field equation. Higher loop calculations give short distance corrections to the Yang-Mills equation. In ref. [41], β_μ^A is computed to three-loop order. It can be derived from a spacetime action which is found to agree with that derived from the tree-level S-matrix of the heterotic string[41,42].

Life gets more interesting and calculations get quite tedious when the heterotic string is in a non-trivial background metric and antisymmetric tensor field. There are more diagrams that contribute to the one-loop value of β_μ^A and the end result is that the gauge covariant derivative on the field strength in (5.28) gets completed to a spacetime covariant (including torsion piece) and gauge covariant derivative[43]. One can argue without doing calculations that the one-loop results for $\beta_{\mu\nu}^G$ and $\beta_{\mu\nu}^B$ receive no contribution from the spacetime gauge coupling. There are no logarithmically divergent one-loop diagrams involving gauge field vertices which have the appropriate background field content. The right moving fermions do not contribute either, so at one loop $\beta_{\mu\nu}^G$ and $\beta_{\mu\nu}^B$ of the heterotic sigma model are given by their values in the bosonic theory. The same was true in the N=1 supersymmetric model.

5.6. Chiral anomalies and higher-loop results

At higher loop order, sigma model chiral anomalies[44] play a crucial role in the computation of heterotic beta functions. The nature of the problem can easily be understood by looking for possible contributions to β^G which depend on spacetime gauge fields (in particular one should be able to find a term of order F^2 representing the gauge field contribution to the energy-momentum tensor of matter). The vertices out of which the relevant diagrams must be constructed are displayed in (5.27). Since they involve only $\partial_+ X_0$ there would appear to be no way to construct an amplitude proportional to $\partial_+ X_0 \partial_- X_0$, the background field structure which accompanies β^G, and therefore no F-dependent correction to β^G! The origin of this problem is that, since the spacetime gauge fields couple to purely left-moving worldsheet fermions λ_-^A, Lorentz-invariance considerations dictate that only $\partial_+ X_0$ can appear in interaction vertices. Fortunately, this argument is wrong because anomalies cause loop diagrams in a theory with only left-moving dynamical fermions to behave in a certain sense 'as if' fermions of the other handedness were present as well. The issues involved here are the same as those arising in the study of gauge anomalies in two-dimensional chiral gauge theories. A simple example will suffice to show how standard anomaly arguments resolve the sigma model beta function paradox described above.

At this stage it is convenient to note that the bilinear $\lambda_-^A \lambda_-^B$ is the left-moving component J_-^{AB} of a worldsheet vector current J_a^{AB} (whose right-moving component J_+^{AB} vanishes, at least formally). We can then rewrite (5.26) in the less manifestly chiral form

$$\frac{i}{4\pi\alpha'} \int d^2\xi \, A_\mu(X)_{BC} \partial^a X^\mu \, J_a^{BC}. \tag{5.29}$$

and rewrite its background field expansion (5.27) in a parallel fashion. A general diagram will have some number of closed λ_- loops each one of which, by virtue of the above interaction structure, is equivalent to some vacuum n-point function of the chiral current J_a. It is in evaluating these current n-point functions that the anomaly comes in.

Consider first the two-point functions $\langle J_a(q) J_b(-q) \rangle$ (we suppress gauge indices for the moment). The quantity $\langle J_- J_- \rangle$ is easy to find. The diagram is

FIGURE 5.4.

where the dashed lines stand for the λ_- propagators. This graph is superficially logarithmically divergent, but the $--$ tensor structure guarantees that it is actually

finite and unambiguous. A straightforward calculation gives the result

$$\langle J_-(q) J_-(-q) \rangle = \frac{q_-}{q_+} \, . \tag{5.30}$$

By similar reasoning, the $\langle J_+ J_+ \rangle$ amplitude is unambiguously zero. The two point function $\langle J_+ J_- \rangle$, on the other hand, is a worldsheet scalar and therefore truly logarithmically divergent. By the usual rules of renormalization theory, it is defined only up to an arbitrary momentum space constant contact term. Since there is no diagram for this amplitude, the contact term is the entire story and we have

$$\langle J_+(q) J_-(-q) \rangle = c \, . \tag{5.31}$$

Note that the ambiguity discussed here does not exist for three- and higher-point functions since they are all power-counting convergent.

The simplest way to fix a value for c is to consider the Ward identity obtained by taking the divergence of one of the currents in $\langle J_a J_b \rangle$. Using (5.30) and (5.31) we find, after a little algebra, that

$$q^a \, \langle J_a(q) J_b(-q) \rangle = (c + \tfrac{1}{2}\,) \, q_b + \tfrac{1}{2} \, \epsilon_{bc} \, q^c. \tag{5.32}$$

Although the current J_a is formally conserved, it is apparent that no choice of c can make the right-hand side of (5.32) vanish. In fact, it is the sum of a parity-conserving and a parity-violating piece (the term proportional to ϵ_{ab}) and the only apparent distinguished choice is $c = -\frac{1}{2}$ which makes the right-hand side of (5.32) pure parity-violating. General discrete spacetime symmetry arguments in fact require that the anomalous nonconservation of chiral gauge currents be pure parity violating[15], and it is the choice we will make. This is precisely the strategy one adopts in trying to define chiral gauge theories in two dimensions.

The upshot of all this is that a non-zero $\langle J_+ J_- \rangle$ is forced upon us at the one-loop level. This means for example that $\beta_{\mu\nu}^G$ receives a contribution from the following diagram:

FIGURE 5.5.

The contribution to $\beta^G_{\mu\nu}$ comes from the part of the diagram which is proportional to $\partial_+ X_0^\mu \partial_- X_0^\nu$ so it is entirely due to the anomaly. Using (5.31) it is not hard to do the calculation and find the following addition to $\beta^G_{\mu\nu}$:

$$-\frac{\alpha'}{2}\,\mathrm{tr}\,(F_{\mu\lambda}F^\lambda{}_\nu)\,. \tag{5.33}$$

The trace is over the gauge group. Recall that the vanishing of $\beta^G_{\mu\nu}$ is the string theory generalization of Einstein's equation (ignoring dilaton effects). This F^2 term in $\beta^G_{\mu\nu}$ supplies a gauge field piece to the spacetime energy momentum tensor on the right hand side of the Einstein equation. The chiral nature of the heterotic sigma model was what enabled us to introduce a gauge field into the theory in the first place and we have now seen the first example of how the subsequent chiral anomaly provides essential ingredients of the spacetime physics.

There is also a chiral anomaly involving the right moving fermions ψ^μ_+. Exactly the same arguments as we used in the left moving sector tell us that there is a non-vanishing one-loop contribution to $\left\langle \tilde{J}_+ \tilde{J}_- \right\rangle$ where $\tilde{J}^{\mu\nu}_+ = \psi^\mu_+\psi^\nu_+$. A diagram like the one in Figure 5.3 contributes an R^2 term to $\beta^G_{\mu\nu}$ but the coefficient in front is different from the one in the N=1 sigma model and it does not cancel the bosonic diagrams. The (1,0) worldsheet supersymmetry of the heterotic theory is not enough to remove the two-loop contribution to the beta function. Up to terms coming from the antisymmetric tensor the two-loop beta function of the metric is

$$\beta^G_{\mu\nu} = R_{\mu\nu} + \frac{\alpha'}{2}\left(R_\mu{}^{\lambda\rho\sigma}R_{\nu\lambda\rho\sigma} - \mathrm{tr}\,(F_\mu{}^\lambda F_{\lambda\nu})\right)\,. \tag{5.34}$$

The chiral anomalies of the heterotic sigma model couple to the antisymmetric

tensor field in an interesting way. Focusing on the left moving sector we can draw the following diagram:

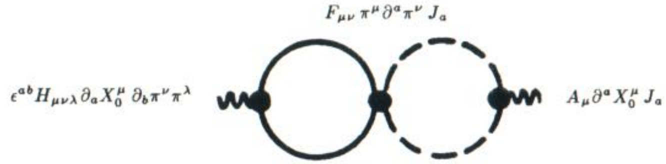

FIGURE 5.6

It contributes to $\beta^G_{\mu\nu}$ because of the chiral anomaly. We can use (5.31) to evaluate the graph and find that it adds to $\beta^G_{\mu\nu}$ the term

$$\frac{\alpha'}{16\pi}\,H_{\mu\lambda\sigma}\,\mathrm{tr}\,(F\wedge A)_\nu{}^{\lambda\sigma}\,. \tag{5.35}$$

The $\wedge$ denotes the usual wedge product of the spacetime forms $F_{\mu\nu}$ and A_μ and the trace is over the gauge group. The expression in (5.35) is certainly not invariant under spacetime gauge transformations which should cause some concern since the vanishing of the beta function is an equation of motion and has direct physical meaning in spacetime. Furthermore there is a corresponding diagram involving the right moving fermions which also gives a contribution to $\beta^G_{\mu\nu}$. This time the term is

$$-\frac{\alpha'}{16\pi}\,H_{\mu\lambda\sigma}\,\mathrm{tr}\,(R\wedge\omega)_\nu{}^{\lambda\sigma}\,. \tag{5.36}$$

Here $R_{\mu\nu ij}$ is the curvature tensor with mixed spacetime and local Lorentz indices and ω^{ij}_μ is the spin connection. The wedge product is in spacetime while the trace is over the local Lorentz indices. The expression in (5.36) is not invariant under local Lorentz transformations which indicates a spacetime gravitational anomaly.

This failure of spacetime symmetries looks disastrous but there is a way out. We define a new antisymmetric field strength

$$\hat{H} = H + \frac{\alpha'}{8\pi}\,\mathrm{tr}\,(F\wedge A) - \frac{\alpha'}{8\pi}\,\mathrm{tr}\,(R\wedge\omega)\,. \tag{5.37}$$

The additional terms (5.35) and (5.36) in $\beta^G_{\mu\nu}$, which come from the anomalous two-loop diagrams, can then be absorbed into the one-loop result:

$$\beta^G_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}\hat{H}^2_{\mu\nu} + \frac{\alpha'}{2}(R_\mu{}^{\lambda\sigma\rho}R_{\nu\lambda\sigma\rho} - \mathrm{tr}\,(F_\mu{}^\lambda F_{\nu\lambda}))\,. \tag{5.38}$$

This expression contains non-gauge-invariant terms of order $(\alpha')^2$ which arise from anomalous three-loop graphs we have not discussed. Following this line of argument to arbitrary orders[41)], one finds that all the gauge and Lorentz non-invariance can be absorbed by a slight generalization of (5.37):

$$\hat{H} = H + \frac{\alpha'}{4\pi}\,\omega_3(A) - \frac{\alpha'}{4\pi}\,\omega_3(\omega)\,, \tag{5.39}$$

where ω_3 are the Chern-Simons forms defined by $\omega_3(A) = \mathrm{tr}\,(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$ in the gauge case and with the replacement of A^{AB}_μ by the spin connection ω^{ij}_μ in the Lorentz case.

This method of collecting the gauge non-invariant terms finally suggests how the spacetime gauge symmetries can be restored[45,8]. Under gauge transformations, with parameter Λ, the variation of a Chern-Simons three-form is the curl of a specific two-form:

$$\delta_\Lambda\omega_3(A) = d\wedge\omega^1_2(\Lambda, A) \tag{5.40}$$

Since H is itself the curl of a two-form potential, $H = d\wedge B$, one can make $\hat{H}$, and therefore beta functions such as (5.38), gauge invariant by assigning to B the simple gauge variation

$$\delta_\Lambda B = -\frac{\alpha'}{4\pi}\omega^1_2(\Lambda, A) \tag{5.41}$$

(and a similar assignment for local Lorentz transformations). This is not something one would normally do since $B_{\mu\nu}$ is gauge neutral and a tangent space scalar, but it is known that precisely such a gauge variation of $B_{\mu\nu}$ is needed in a consistent ten-dimensional supergravity theory[46)]. The chiral anomalies of the two-dimensional theory thus play a key role in reproducing a consistent ten dimensional supergravity theory.

The entire two-loop correction to β^A_μ comes from the chiral anomalies and it can be accounted for by replacing the torsion piece of the connection in the curved space Yang-Mills equation with its Chern-Simons completion (5.39) [43,41)].

Finally we point out a non-trivial check of these higher-loop results in the heterotic sigma model. If the gauge connection and the spin connection are identified the beta functions should reduce to those of the N=1 supersymmetric theory. In particular chiral anomalies should then be absent since the interactions of the right and left moving fermions are exactly the same and the two-loop contribution to $\beta^G_{\mu\nu}$ should vanish. This is easily checked keeping in mind that the curvature tensor and gauge field strength are now the same. The order α' piece in (5.34) goes away and the gauge and Lorentz Chern-Simons forms in (5.39) cancel so that $\hat{H}$ reduces to H.

6. Conclusion

Finally, it is time to take stock of what we have done. Almost without exception, our considerations have been exercises in the renormalized perturbation expansion of two-dimensional field theories on curved worldsheets. The principles involved are, to say the least, not new and have been part of the arsenal of field theory and statistical mechanics since the mid-seventies. String theory has simply raised a number of new questions (such as the role of the dilaton coupling, heterotic theories and the role of conformal as opposed to scale invariance) which would not have arisen, or would not have been considered particularly interesting, in older contexts. The exploration of these matters has given useful employment to old physicists and old ideas but hasn't really added to our intellectual capital! It is probably fair to say that a clear and comprehensive answer to all the questions we have touched on in these lectures either exists or could be obtained with a finite amount of effort. A definitive review of perturbative conformal field theory (i.e. perturbative classical string field theory) could, and should, be written in the near future.

String theory does nevertheless raise genuinely new questions which connect organically to the subjects we have discussed, and which should at least be mentioned, if only to let the reader know where the skeletons are buried. The first concerns nonperturbative treatments of conformal field theory. QCD taught us that essential qualitative aspects of field theory physics are not visible in perturbation expansions, and the same is true for the sigma models of interest to us. It is possible to get a glimpse of this physics by studying worldsheet instantons and this is an activity that has received a certain amount of attention. It is in direct competition with, and probably less powerful than, the method of direct solution of extended Virasoro algebras, which encompasses the bulk of string theory activity these days. The other issue that we have neglected is that of string loops. There is a growing literature on the problem of computing string loop corrections to the Weyl anomaly coefficients so as to capture quantum corrections to spacetime physics. This is a matter of learning how to sum two-dimensional field theories over worldsheets of different topologies and to renormalize new divergences which arise in that sum. The problems which arise in carrying out this program parallel those of the old quantum field theory perturbative renormalization program, but are not yet understood in any systematic way. The problem of perturbative versus nonperturbative physics arises here also, and virtually nothing is known about it. This all reflects the unsatisfactory state of our understanding of string field theory. Here are worthy topics for a future TASI session.

We would like to thank the organizers of the 1988 TASI School for putting together a well-run and informative session, and for giving us the opportunity to lecture. We thank the students for their active interest and intelligent questions. We would also like to thank D. Hadden and J. McCown for reading various versions of these lectures and helping us to find embarrassing errors.

REFERENCES

1. Y. Nambu, Lectures at the Copenhagen symposium, 1970;
   T. Goto, *Prog. Theor. Phys.* **46** (1970), 1560.
2. L. Brink, P. Di Vecchia and P. Howe, *Phys. Lett.* **65B** (1976), 471.
3. A. M. Polyakov, *Phys. Lett.* **103B** (1981), 207.
4. D. Friedan, 'Introduction to Polyakov's string theory', in *Recent Advances in Field Theory and Statistical Mechanics*, eds. J. B. Zuber and R. Stora. Proc. of 1982 Les Houches Summer School (Elsevier), p. 839.
5. S. Jain, *Int. Jour. Mod. Phys.* **A3** (1988), 1759.
6. M. Peskin, 'Introduction to String and Superstring Theory II', in *From the Planck Scale to the Weak Scale: Toward a Theory of the Universe*, ed. H. E. Haber, Proc. of the 1986 Theoretical Advanced Study Institute in Elementary Particle Physics, UC Santa Cruz (World Scientific, 1987) p. 277.
7. E. S. Fradkin and A. A. Tseytlin, *Nucl. Phys.* **B261** (1985), 1.
8. C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, *Nucl.Phys.* **B262** (1985), 593.
9. A. Sen, *Phys. Rev. Lett.* **55** (1985), 1846.
10. A. Abouelsaood, C. G. Callan, C. R. Nappi and S. A. Yost, *Nucl. Phys.* **B280** (1987), 599.
11. C. G. Callan, C. Lovelace, C.R. Nappi and S. A. Yost, *Nucl. Phys.* **B288** (1987), 525.
12. W. Fischler and L. Susskind, *Phys. Lett.* **B171** (1986), 383.
13. L. Alvarez-Gaume, D. Z. Freedman and S. Mukhi, *Ann. Phys.* **134** (1981), 85.
14. S. Mukhi, *Nucl. Phys.* **B264** (1986), 640.
15. L. Alvarez-Gaume and E. Witten, *Nucl. Phys.* **B234** (1984), 269.

16. P. Ramond, *Field Theory: A Modern Primer*, Benjamin-Cummings, 1981.

17. C. G. Callan, *Phys. Scr.* **T15** (1987), 114.

18. G. Curci and G. Paffuti, *Nucl. Phys.* **B286** (1987), 399.

19. A. P. Foakes and N. Mohammedi, *Nucl. Phys.* **B306** (1988), 343.

20. S. J. Graham, *Phys. Lett.* **197B** (1987), 543.

21. R. Brustein, D. Nemeschansky and S. Yankielowicz, *Nucl. Phys.* **B301** (1988), 224.

22. C. G. Callan, I. R. Klebanov and M. J. Perry, *Nucl. Phys.* **B278** (1986), 78.

23. C. M. Hull and P. K. Townsend, *Nucl. Phys.* **B274** (1986), 349.

24. R. R. Metsaev and A. A. Tseytlin, *Nucl. Phys.* **B293** (1987), 385.

25. D. H. Friedan, *Ann. Phys.* **163** (1985), 318.

26. S. J. Hathrell, *Ann. Phys.* **142** (1982), 34, *Ann. Phys.* **139** (1982), 136.

27. A. A. Tseytlin, *Nucl. Phys.* **B276** (1986), 391, *erratum-ibid.* **B291** (1987), 876.

28. P. Breitenlohner and D. Maison, *Comm. Math. Phys.* **52** (1977), 11, 39, 55.

29. C. G. Callan and Z. Gan, *Nucl. Phys.* **B272** (1986), 647.

30. S. Mukhi, 'Nonlinear Sigma Models, Scale Invariance and String Theories: A Pedagogical Review', Tata Institute preprint TIFR/TH/86-24, Jan 1986.

31. M. Green, J. Schwarz and E. Witten, *Superstring Theory*, Vols. I and II, Cambridge University Press, 1987.

32. M. Dine, 'Supersymmetry: Microphysics and Macrophysics', lectures in this volume.

33. D. Friedan, E. Martinec and S. Shenker, *Nucl. Phys.* **B271** (1986), 93.

34. M. Grisaru, A. van de Ven and D. Zanon, *Nucl. Phys.* **B277** (1986), 409.

35. M. Grisaru, D. Kazakov and D. Zanon, *Nucl. Phys.* **B287** (1987), 189.

36. L. Alvarez-Gaume, *Nucl. Phys.* **B184** (1981), 180.

37. D. Nemeschansky and A. Sen, *Phys. Lett.* **B178** (1986), 365.

38. R. Allen and D. Jones, *Nucl. Phys.* **B303** (1988), 271.

39. D. Gross, J. Harvey, E. Martinec and R. Rohm, *Nucl. Phys.* **B256** (1985), 253.

40. P. Candelas, G. Horowitz, A. Strominger and E. Witten, *Nucl. Phys.* **B258** (1985), 46.

41. U. Ellwanger, J. Fuchs and M. Schmidt, 'The Heterotic Sigma Model with Background Gauge Fields', Heidelberg University preprint HD-THEP-88-2a, Apr 1988.

42. D. Gross and J. Sloan, *Nucl. Phys.* **B291** (1987), 41.

43. K. Hamada, J. Kodaira and J. Saito, 'Chern-Simons Terms and the Sigma Model Approach to the Heterotic String', Hiroshima University preprint HUPD-8701, Feb 1987.

44. G. Moore and P. Nelson, *Phys. Rev. Lett.* **53** (1984), 1510.

45. C. Hull and E. Witten, *Phys. Lett.* **B160** (1985), 398.

46. G. Chapline and N. Manton, *Phys. Lett.* **120B** (1983), 105.