6. String Interactions

So far, despite considerable effort, we've only discussed the free string. We now wish to consider interactions. If we take the analogy with quantum field theory as our guide, then we might be led to think that interactions require us to add various non-linear terms to the action. However, this isn't the case. Any attempt to add extra non-linear terms for the string won't be consistent with our precious gauge symmetries. Instead, rather remarkably, all the information about interacting strings is already contained in the free theory described by the Polyakov action. (Actually, this statement is almost true).

To see that this is at least feasible, try to draw a cartoon picture of two strings interacting. It looks something like the worldsheet shown in the figure. The worldsheet is smooth. In Feynman diagrams in quantum field theory, information about interactions is inserted at vertices, where different lines meet. Here there are no such points. Locally, every part of the diagram looks like a free propagating string. Only globally do we see that the diagram describes interactions.

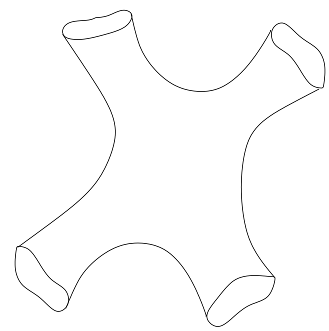


Figure 31:

6.1 What to Compute?

If the information about string interactions is already contained in the Polyakov action, let's go ahead and compute something! But what should we compute? One obvious thing to try is the probability for a particular configuration of strings at an early time to evolve into a new configuration at some later time. For example, we could try to compute the amplitude associated to the diagram above, stipulating fixed curves for the string ends.

No one knows how to do this. Moreover, there are words that we can drape around this failure that suggests this isn't really a sensible thing to compute. I'll now try to explain these words. Let's start by returning to the familiar framework of quantum field theory in a fixed background. There the basic objects that we can compute are correlation functions,

$$\langle \phi(x_1) \dots \phi(x_n) \rangle \tag{6.1}$$

After a Fourier transform, these describe Feynman diagrams in which the external legs carry arbitrary momenta. For this reason, they are referred to as *off-shell*. To get the scattering amplitudes, we simply need to put the external legs on-shell (and perform a few other little tricks captured in the LSZ reduction formula).

The discussion above needs amendment if we turn on gravity. Gravity is a gauge theory and the gauge symmetries are diffeomorphisms. In a gauge theory, only gauge invariant observables make sense. But the correlation function (6.1) is not gauge invariant because its value changes under a diffeomorphism which maps the points x_i to another point. This emphasizes an important fact: there are no local off-shell gauge invariant observables in a theory of gravity.

There is another way to say this. We know, by causality, that space-like separated operators should commute in a quantum field theory. But in gravity the question of whether operators are space-like separated becomes a dynamical issue and the causal structure can fluctuate due to quantum effects. This provides another reason why we are unable to define local gauge invariant observables in any theory of quantum gravity.

Let's now return to string theory. Computing the evolution of string configurations for a finite time is analogous to computing off-shell correlation functions in QFT. But string theory is a theory of gravity so such things probably don't make sense. For this reason, we retreat from attempting to compute correlation functions, back to the S-matrix.

The String S-Matrix

The object that we can compute in string theory is the S-matrix. This is obtained by taking the points in the correlation function to infinity: $x_i \to \infty$. This is acceptable because, just like in the case of QED, the redundancy of the system consists of those gauge transformations which die off asymptotically. Said another way, points on the boundary don't fluctuate in quantum gravity. (Such fluctuations would be over an infinite volume of space and are suppressed due to their infinite action).

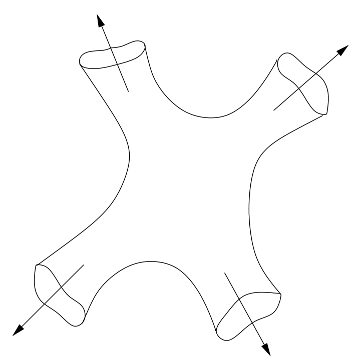


Figure 32:

So what we're really going to calculate is a diagram of the type shown in the figure, where all external legs are taken to infinity. Each of these legs can be placed in a different state of the free string and assigned some spacetime momentum p_i. The resulting expression is the string *S-matrix*.

Using the state-operator map, we know that each of these states at infinity is equivalent to the insertion of an appropriate vertex operator on the worldsheet. Therefore, to compute this S-matrix element we use a conformal transformation to bring each of these infinite legs to a finite distance. The end result is a worldsheet with the topology of the sphere, dotted with vertex operators where the legs used to be. However, we already saw in the previous section that the constraint of Weyl invariance meant that vertex operators are necessarily on-shell. Technically, this is the reason that we can only compute on-shell correlation functions in string theory.

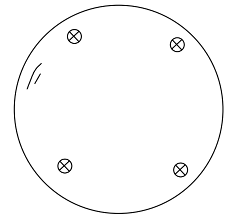
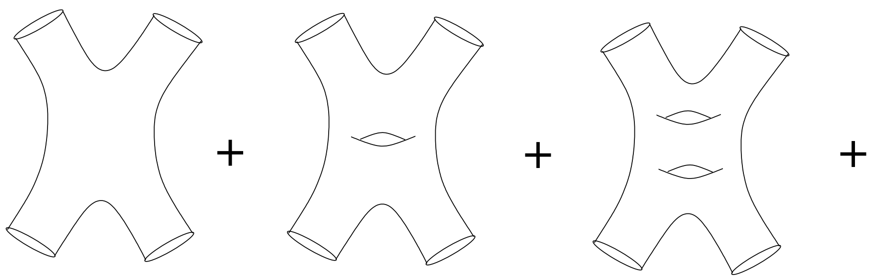


Figure 33:

6.1.1 Summing Over Topologies

The Polyakov path integral instructs us to sum over all metrics. But what about worldsheets of different topologies? In fact, we should also sum over these. It is this sum that gives the perturbative expansion of string theory. The scattering of two strings receives contributions from worldsheets of the form



(6.2)

The only thing that we need to know is how to weight these different worldsheets. Thankfully, there is a very natural coupling on the string that we have yet to consider and this will do the job. We augment the Polyakov action by

$$S_{\text{string}} = S_{\text{Poly}} + \lambda\chi \tag{6.3}$$

Here λ is simply a real number, while χ is given by an integral over the (Euclidean) worldsheet

$$\chi = \frac{1}{4\pi}\int d^2\sigma\ \sqrt{g}R \tag{6.4}$$

where R is the Ricci scalar of the worldsheet metric. This looks like the Einstein-Hilbert term for gravity on the worldsheet. It is simple to check that it is invariant under reparameterizations and Weyl transformations.

In four-dimensions, the Einstein-Hilbert term makes gravity dynamical. But life is very different in 2d. Indeed, we've already seen that all the components of the metric can be gauged away so there are no propagating degrees of freedom associated to $g_{\alpha\beta}$. So, in two-dimensions, the term (6.4) doesn't make gravity dynamical: in fact, classically, it doesn't do anything at all!

The reason for this is that χ is a topological invariant. This means that it doesn't actually depend on the metric $g_{\alpha\beta}$ at all – it depends only on the topology of the worldsheet. (More precisely, χ only depends on those global properties of the metric which themselves depend on the topology of the worldsheet). This is the content of the Gauss-Bonnet theorem: the integral of the Ricci scalar R over the worldsheet gives an integer, χ, known as the Euler number of the worldsheet. For a worldsheet without boundary (i.e. for the closed string) χ counts the number of handles h on the worldsheet. It is given by,

$$\chi = 2 - 2h = 2(1-g) \tag{6.5}$$

where g is called the *genus* of the surface. The simplest examples are shown in the figure. The sphere has $g=0$ and $\chi=2$; the torus has $g=1$ and $\chi=0$. For higher $g>1$, the Euler character χ is negative.

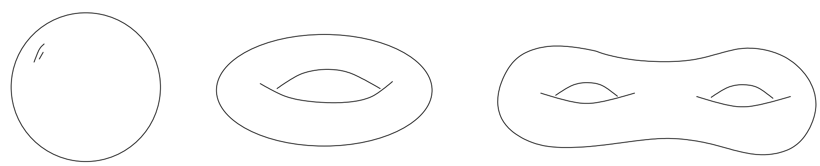


Figure 34: Examples of increasingly poorly drawn Riemann surfaces with $\chi = 2, 0$ and -2.

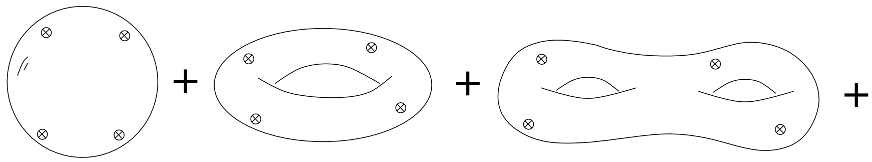
Now we see that the number λ — or, more precisely, e^λ — plays the role of the string coupling. The integral over worldsheets is weighted by,

$$\sum_{\substack{\text{topologies}\\ \text{metrics}}} e^{-S_{\text{string}}} \sim \sum_{\text{topologies}} e^{-2\lambda(1-g)} \int \mathcal{D}X\mathcal{D}g\; e^{-S_{\text{Poly}}}$$

For $e^\lambda \ll 1$, we have a good perturbative expansion in which we sum over all topologies. (In fact, it is an asymptotic expansion, just as in quantum field theory). It is standard to define the string coupling constant as

$$g_s = e^\lambda$$

After a conformal map, tree-level scattering corresponds to a worldsheet with the topology of a sphere: the amplitudes are proportional to $1/g_s^2$. One-loop scattering corresponds to toroidal worldsheets and, with our normalization, have no power of g_s. (Although, obviously, these are suppressed by g_s^2 relative to tree-level processes). The end result is that the sum over worldsheets in (6.2) becomes a sum over Riemann surfaces of increasing genus, with vertex operators inserted for the initial and final states,



The Riemann surface of genus g is weighted by

$$(g_s^2)^{g-1}$$

While it may look like we've introduced a new parameter g_s into the theory and added the coupling (6.3) by hand, we will later see why this coupling is a necessary part of the theory and provide an interpretation for g_s.

Scattering Amplitudes

We now have all the information that we need to explain how to compute string scattering amplitudes. Suppose that we want to compute the S-matrix for m states: we will label them as Λ_i and assign them spacetime momenta p_i. Each has a corresponding vertex operator $V_{\Lambda_i}(p_i)$. The S-matrix element is then computed by evaluating the correlation function in the 2d conformal field theory, with insertions of the vertex operators.

$$\mathcal{A}^{(m)}(\Lambda_i, p_i) = \sum_{\text{topologies}} g_s^{-\chi} \, \frac{1}{\text{Vol}} \int \mathcal{D}X \mathcal{D}g \; e^{-S_{\text{Poly}}} \; \prod_{i=1}^{m} \; V_{\Lambda_i}(p_i)$$

This is a rather peculiar equation. We are interpreting the correlation functions of a two-dimensional theory as the S-matrix for a theory in $D = 26$ dimensions!

To properly compute the correlation function, we should introduce the b and c ghosts that we saw in the last chapter and treat them carefully. However, if we're only interested in tree-level amplitudes, then we can proceed naively and ignore the ghosts. The reason can be seen in the ghost action (5.4) where we see that the ghosts couple only to the worldsheet metric, not to the other worldsheet fields. This means that if our gauge fixing procedure fixes the worldsheet metric completely — which it does for worldsheets with the topology of a sphere — then we can forget about the ghosts. (At least, we can forget about them as soon as we've made sure that the Weyl anomaly cancels). However, as we'll explain in 6.4, for higher genus worldsheets, the gauge fixing does not fix the metric completely and there are residual dynamical modes of the metric, known as moduli, which couple the ghosts and matter fields. This is analogous to the statement in field theory that we only need to worry about ghosts running in loops.

6.2 Closed String Amplitudes at Tree Level

The tree-level scattering amplitude is given by the correlation function of the 2d theory, evaluated on the sphere,

$$\mathcal{A}^{(m)} = \frac{1}{g_s^2} \frac{1}{\text{Vol}} \int \mathcal{D}X\mathcal{D}g \; e^{-S_{\text{Poly}}} \prod_{i=1}^{m} V_{\Lambda_i}(p_i)$$

where $V_{\Lambda_i}(p_i)$ are the vertex operators associated to the states.

We want to integrate over all metrics on the sphere. At first glance that sounds rather daunting but, of course, we have the gauge symmetries of diffeomorphisms and Weyl transformations at our disposal. Any metric on the sphere is conformally equivalent to the flat metric on the plane. For example, the round metric on the sphere of radius R can be written as

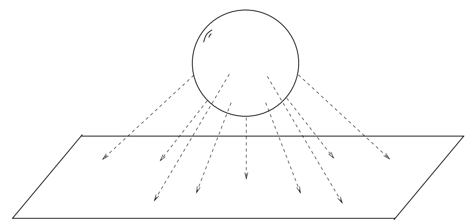


Figure 35:

$$ds^2 = \frac{4R^2}{(1+|z|^2)^2}\, dzd\bar{z}$$

which is manifestly conformally equivalent to the plane, supplemented by the point at infinity. The conformal map from the sphere to the plane is the stereographic projection depicted in the diagram. The south pole of the sphere is mapped to the origin; the north pole is mapped to the point at infinity. Therefore, instead of integrating over all metrics, we may gauge fix diffeomorphisms and Weyl transformations to leave ourselves with the seemingly easier task of computing correlation functions on the plane.

6.2.1 Remnant Gauge Symmetry: SL(2,C)

There's a subtlety. And it's a subtlety that we've seen before: there is a residual gauge symmetry. It is the conformal group, arising from diffeomorphisms which can be undone by Weyl transformations. As we saw in Section 4, there are an infinite number of such conformal transformations. It looks like we have a whole lot of gauge fixing still to do.

However, global issues actually mean that there's less remnant gauge symmetry than you might think. In Section 4, we only looked at infinitesimal conformal transformations, generated by the Virasoro operators L_n, $n \in \mathbf{Z}$. We did not examine whether these transformations are well-defined and invertible over all of space. Let's take a look at this. Recall that the coordinate changes associated to L_n are generated by the vector fields (4.44),

$$l_n = z^{n+1}\partial_z$$

which result in the shift $\delta z = \epsilon z^{n+1}$. This is non-singular at $z = 0$ only for $n \geq -1$. If we restrict to smooth maps, that gets rid of half the transformations right away. But, since we're ultimately interested in the sphere, we now also need to worry about the point at $z = \infty$ which, in stereographic projection, is just the north pole of the sphere. To do this, it's useful to work with the coordinate

$$u = \frac{1}{z}$$

The generators of coordinate transformations for the u coordinate are

$$l_n = z^{n+1}\partial_z = \frac{1}{u^{n+1}}\frac{\partial u}{\partial z}\partial_u = -u^{1-n}\partial_u$$

which is non-singular at $u = 0$ only for $n \leq 1$.

Combining these two results, the only generators of the conformal group that are non-singular over the whole Riemann sphere are l_{-1}, l_0 and l_1 which act infinitesimally as

$$\begin{aligned} l_{-1}: &\quad z \;\rightarrow\; z + \epsilon \\ l_0: &\quad z \;\rightarrow\; (1+\epsilon)z \\ l_1: &\quad z \;\rightarrow\; (1+\epsilon z)z \end{aligned}$$

The global version of these transformations is

$$\begin{aligned} l_{-1}: &\quad z \;\rightarrow\; z + \alpha \\ l_0: &\quad z \;\rightarrow\; \lambda z \\ l_1: &\quad z \;\rightarrow\; \frac{z}{1-\beta z} \end{aligned}$$

which can be combined to give the general transformation

$$z \;\rightarrow\; \frac{az+b}{cz+d} \tag{6.6}$$

with a, b, c and $d \in \mathbf{C}$. We have four complex parameters, but we've only got three transformations. What happened? Well, one transformation is fake because an overall scaling of the parameters doesn't change z. By such a rescaling, we can always insist that the parameters obey

$$ad - bc = 1$$

The transformations (6.6) subject to this constraint have the group structure $SL(2;\mathbf{C})$, which is the group of 2×2 complex matrices with unit determinant. In fact, since the transformation is blind to a flip in sign of all the parameters, the actual group of global conformal transformations is $SL(2;\mathbf{C})/\mathbf{Z}_2$, which is sometimes written as $PSL(2;\mathbf{C})$. (This $\mathbf{Z}_2$ subtlety won't be important for us in what follows).

The remnant global transformations on the sphere are known as *conformal Killing vectors* and the group $SL(2;\mathbf{C})/\mathbf{Z}_2$ is the *conformal Killing group*. This group allows us to take any three points on the plane and move them to three other points of our choosing. We will shortly make use of this fact to gauge fix, but for now we leave the $SL(2;\mathbf{C})$ symmetry intact.

6.2.2 The Virasoro-Shapiro Amplitude

We will now compute the S-matrix for closed string tachyons. You might think that this is the least interesting thing to compute: after all, we're ultimately interested in the superstring which doesn't have tachyons. This is true, but it turns out that tachyon scattering is much simpler than everything else, mainly because we don't have a plethora of extra indices on the states to worry about. Moreover, the lessons that we will learn from tachyon scattering hold for the scattering of other states as well.

The m-point tachyon scattering amplitude is given by the flat space correlation function

$$\mathcal{A}^{(m)}(p_1,\ldots,p_m) = \frac{1}{g_s^2}\frac{1}{\mathrm{Vol}(SL(2;\mathbf{C}))}\int \mathcal{D}X\ e^{-S_{\mathrm{Poly}}}\ \prod_{i=1}^{m} V(p_i)$$

where the tachyon vertex operator is given by,

$$V(p_i) = g_s\int d^2z\ e^{ip_i\cdot X} \equiv g_s\int d^2z\ \hat{V}(z,p_i) \tag{6.7}$$

Note that, in contrast to (5.7), we've added an appropriate normalization factor to the vertex operator. Heuristically, this reflects the fact that the operator is associated to the addition of a closed string mode. A rigorous derivation of this normalization can be found in Polchinski.

The amplitude can therefore be written as,

$$\mathcal{A}^{(m)}(p_1,\ldots,p_m) = \frac{g_s^{m-2}}{\mathrm{Vol}(SL(2;\mathbf{C}))}\int \prod_{i=1}^m d^2z_i\ \langle \hat{V}(z_1,p_1)\ldots\hat{V}(z_m,p_m)\rangle$$

where the expectation value $\langle\ldots\rangle$ is computed using the gauge fixed Polyakov action. But the gauge fixed Polyakov action is simply a free theory and our correlation function is something eminently computable: a Gaussian integral,

$$\langle \hat{V}(z_1,p_1)\ldots\hat{V}(z_m,p_m)\rangle = \int \mathcal{D}X \exp\left(-\frac{1}{2\pi\alpha'}\int d^2z\ \partial X\cdot\bar{\partial}X\right)\ \exp\left(i\sum_{i=1}^m p_i\cdot X(z_i,\bar{z}_i)\right)$$

The normalization in front of the Polyakov action is now $1/2\pi\alpha'$ instead of $1/4\pi\alpha'$ because we're working with complex coordinates and we need to remember that $\partial_\alpha\partial^\alpha = 4\partial\bar{\partial}$ and $d^2z = 2d^2\sigma$.

The Gaussian Integral

We certainly know how to compute Gaussian integrals. Let's go slow. Consider the following general integral,

$$\int \mathcal{D}X\ \exp\left(\int d^2z\ \frac{1}{2\pi\alpha'}X\cdot\partial\bar{\partial}X + iJ\cdot X\right) \sim \exp\left(\frac{\pi\alpha'}{2}\int d^2z d^2z'\ J(z,\bar{z})\,\frac{1}{\partial\bar{\partial}}\,J(z',\bar{z}')\right)$$

Here the $\sim$ symbol reflects the fact that we've dropped a whole lot of irrelevant normalization terms, including $\det^{-1/2}(-\partial\bar{\partial})$. The inverse operator $1/\partial\bar{\partial}$ on the right-hand-side of this equation is shorthand for the propagator $G(z,z')$ which solves

$$\partial\bar{\partial}G(z,\bar{z};z',\bar{z}') = \delta(z-z',\bar{z}-\bar{z}')$$

As we've seen several times before, in two dimensions this propagator is given by

$$G(z,\bar{z};z',\bar{z}') = \frac{1}{2\pi}\ln|z-z'|^2$$

Back to the Scattering Amplitude

Comparing our scattering amplitude with this general expression, we need to take the source J to be

$$J(z,\bar{z}) = \sum_{i=1}^m p_i\ \delta(z-z_i,\bar{z}-\bar{z}_i)$$

Inserting this into the Gaussian integral gives us an expression for the amplitude

$$\mathcal{A}^{(m)} \sim \frac{g_s^{m-2}}{\mathrm{Vol}(SL(2;\mathbf{C}))} \int \prod_{i=1}^{m} d^2 z_i \;\exp\left(\frac{\alpha'}{2}\sum_{j,l} p_j \cdot p_l \;\ln|z_j - z_l|\right)$$

The terms with $j = l$ seem to be problematic. In fact, they should just be left out. This follows from correctly implementing normal ordering and leaves us with

$$\mathcal{A}^{(m)} \sim \frac{g_s^{m-2}}{\mathrm{Vol}(SL(2;\mathbf{C}))} \int \prod_{i=1}^{m} d^2 z_i \prod_{j<l} |z_j - z_l|^{\alpha' p_j \cdot p_l} \tag{6.8}$$

Actually, there's something that we missed. (Isn't there always!). We certainly expect scattering in flat space to obey momentum conservation, so there should be a $\delta^{(26)}(\sum_{i=1}^{m} p_i)$ in the amplitude. But where is it? We missed it because we were a little too quick in computing the Gaussian integral. The operator $\partial\bar{\partial}$ annihilates the zero mode, x^μ, in the mode expansion. This means that its inverse, $1/\partial\bar{\partial}$, is not well-defined. But it's easy to deal with this by treating the zero mode separately. The derivatives ∂^2 don't see x^μ, but the source J does. Integrating over the zero mode in the path integral gives us our delta function

$$\int dx\;\exp(i\sum_{i=1}^{m} p_i \cdot x) \sim \delta^{26}(\sum_{i=1}^{m} p_i)$$

So, our final result for the amplitude is

$$\mathcal{A}^{(m)} \sim \frac{g_s^{m-2}}{\mathrm{Vol}(SL(2;\mathbf{C}))}\;\delta^{26}(\sum_i p_i)\;\int \prod_{i=1}^{m} d^2 z_i \prod_{j<l} |z_j - z_l|^{\alpha' p_j \cdot p_l} \tag{6.9}$$

The Four-Point Amplitude

We will compute only the four-point amplitude for two-to-two scattering of tachyons. The $\mathrm{Vol}(SL(2;\mathbf{C}))$ factor is there to remind us that we still have a remnant gauge symmetry floating around. Let's now fix this. As we mentioned before, it provides enough freedom for us to take any three points on the plane and move them to any other three points. We will make use of this to set

$$z_1 = \infty \quad , \quad z_2 = 0 \quad , \quad z_3 = z \quad , \quad z_4 = 1$$

Inserting this into the amplitude (6.9), we find ourselves with just a single integral to evaluate,

$$\mathcal{A}^{(4)} \sim g_s^2\;\delta^{26}(\sum_i p_i)\;\int d^2 z\;|z|^{\alpha' p_2 \cdot p_3}\,|1 - z|^{\alpha' p_3 \cdot p_4} \tag{6.10}$$

(There is also an overall factor of $|z_1|^4$, but this just gets absorbed into an overall normalization constant). We still need to do the integral. It can be evaluated exactly in terms of gamma functions. We relegate the proof to Appendix 6.5, where we show that

$$\int d^2z\ |z|^{2a-2}|1-z|^{2b-2} = \frac{2\pi\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)} \tag{6.11}$$

where $a+b+c=1$.

Four-point scattering amplitudes are typically expressed in terms of Mandelstam variables. We choose p_1 and p_2 to be incoming momenta and p_3 and p_4 to be outgoing momenta, as shown in the figure. We then define

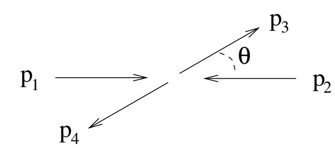


Figure 36:

$$s = -(p_1+p_2)^2 \quad , \quad t = -(p_1+p_3)^2 \quad , \quad u = -(p_1+p_4)^2$$

These obey

$$s+t+u = -\sum_i p_i^2 = \sum_i M_i^2 = -\frac{16}{\alpha'}$$

where, in the last equality, we've inserted the value of the tachyon mass (2.27). Writing the scattering amplitude (6.10) in terms of Mandelstam variables, we have our final answer

$$\mathcal{A}^{(4)} \sim g_s^2\ \delta^{26}(\sum_i p_i)\ \frac{\Gamma(-1-\alpha' s/4)\Gamma(-1-\alpha' t/4)\Gamma(-1-\alpha' u/4)}{\Gamma(2+\alpha' s/4)\Gamma(2+\alpha' t/4)\Gamma(2+\alpha' u/4)} \tag{6.12}$$

This is the *Virasoro-Shapiro amplitude* governing tachyon scattering in the closed bosonic string.

Remarkably, the Virasoro-Shapiro amplitude was almost the first equation of string theory! (That honour actually goes to the Veneziano amplitude which is the analogous expression for open string tachyons and will be derived in Section 6.3.1). These amplitudes were written down long before people knew that they had anything to do with strings: they simply exhibited some interesting and surprising properties. It took several years of work to realise that they actually describe the scattering of strings. We will now start to tease apart the Virasoro-Shapiro amplitude to see some of the properties that got people hooked many years ago.

6.2.3 Lessons to Learn

So what's the physics lying behind the scattering amplitude (6.12)? Obviously it is symmetric in s, t and u. That is already surprising and we'll return to it shortly. But we'll start by fixing t and looking at the properties of the amplitude as we vary s.

The first thing to notice is that $\mathcal{A}^{(4)}$ has poles. Lots of poles. They come from the factor of $\Gamma(-1-\alpha's/4)$ in the numerator. The first of these poles appears when

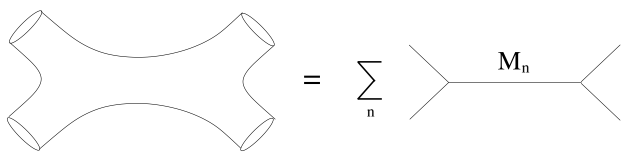
$$-1-\frac{\alpha's}{4}=0 \quad \Rightarrow \quad s=-\frac{4}{\alpha'}$$

But that's the mass of the tachyon! It means that, for s close to $-4/\alpha'$, the amplitude has the form of a familiar scattering amplitude in quantum field theory with a cubic vertex,

$$\sim \frac{1}{s-M^2}$$

where M is the mass of the exchanged particle, in this case the tachyon.

Other poles in the amplitude occur at $s=4(n-1)/\alpha'$ with $n \in \mathbf{Z}^+$. This is precisely the mass formula for the higher states of the closed string. What we're learning is that the string amplitude is summing up an infinite number of tree-level field theory diagrams,



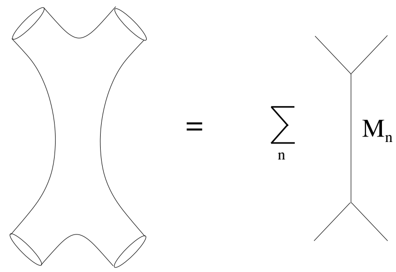
where the exchanged particles are all the different states of the free string.

In fact, there's more information about the spectrum of states hidden within these amplitudes. We can look at the residues of the poles at $s=4(n-1)/\alpha'$, for $n=0,1,\ldots$. These residues are rather complicated functions of t, but the highest power of momentum that appears for each pole is

$$\mathcal{A}^{(4)} \sim \sum_{n=0}^{\infty} \frac{t^{2n}}{s-M_n^2} \tag{6.13}$$

The power of the momentum is telling us the highest spin of the particle states at level n. To see why this is, consider a field corresponding to a spin J particle. It has a whole bunch of Lorentz indices, $\chi_{\mu_1 \ldots \mu_J}$. In a cubic interaction, each of these must be soaked up by derivatives. So we have J derivatives at each vertex, contributing powers of $(\text{momentum})^{2J}$ to the numerator of the Feynman diagram. Comparing with the string scattering amplitude, we see that the highest spin particle at level n has $J=2n$. This is indeed the result that we saw from the canonical quantization of the string in Section 2.

Finally, the amplitude (6.12) has a property that is very different from amplitudes in field theory. Above, we framed our discussion by keeping t fixed and expanding in s. We could just have well done the opposite: fix s and look at poles in t. Now the string amplitude has the interpretation of an infinite number of t-channel scattering amplitudes, one for each state of the string



Usually in field theory, we sum up both s-channel and t-channel scattering amplitudes. Not so in string theory. The sum over an infinite number of s-channel amplitudes can be reinterpreted as an infinite sum of t-channel amplitudes. We don't include both: that would be overcounting. (Similar statements hold for u). The fact that the same amplitude can be written as a sum over s-channel poles *or* a sum over t-channel poles is sometimes referred to as "duality". (A much overused word). In the early days, before it was known that string theory was a theory of strings, the subject inherited its name from this duality property of amplitudes: it was called the *dual resonance model*.

High Energy Scattering

Let's use this amplitude to see what happens when we collide strings at high energies. There are different regimes that we could look at. The most illuminating is s, $t \to \infty$, with s/t held fixed. In this limit, all the exchanged momenta become large. It corresponds to high-energy scattering with the angle θ between incoming and outgoing particles kept fixed. To see this consider, for example, massless particles (our amplitude is really for tachyons, but the same considerations hold). We take the incoming and outgoing momenta to be

$$p_1 = \frac{\sqrt{s}}{2}(1, 1, 0, \ldots) \quad , \quad p_2 = \frac{\sqrt{s}}{2}(1, -1, 0, \ldots)$$
$$p_3 = \frac{\sqrt{s}}{2}(1, \cos\theta, \sin\theta, \ldots) \quad , \quad p_4 = \frac{\sqrt{s}}{2}(1, -\cos\theta, -\sin\theta, \ldots)$$

Then we see explicitly that $s \to \infty$ and $t \to \infty$ with the ratio s/t fixed also keeps the scattering angle θ fixed.

We can evaluate the scattering amplitude $\mathcal{A}^{(4)}$ in this limit by using $\Gamma(x) \sim \exp(x \ln x)$. We send $s \to \infty$ avoiding the poles. (We can achieve this by sending $s \to \infty$ in a slightly imaginary direction. Ultimately this is valid because all the higher string states are actually unstable in the interacting theory which will shift their poles off the real axis once taken into account). It is simple to check that the amplitude drops off exponentially quickly at high energies,

$$\mathcal{A}^{(4)} \sim g_s^2\, \delta^{26}(\sum_i p_i)\, \exp\left(-\frac{\alpha'}{2}(s\ln s + t\ln t + u\ln u)\right) \quad \text{as } s \to \infty \tag{6.14}$$

The exponential fall-off seen in (6.14) is much faster than the amplitude of any field theory which, at best, fall off with power-law decay at high energies and, at worse, diverge. For example, consider the individual terms (6.13) corresponding to the amplitude for s-channel processes involving the exchange of particles with spin $2n$. We see that the exchange of a spin 2 particle results in a divergence in this limit. This is reflecting something you already know about gravity: the dimensionless coupling is $G_N E^2$ (in four-dimensions) which becomes large for large energies. The exchange of higher spin particles gives rise to even worse divergences. If we were to truncate the infinite sum (6.13) at any finite n, the whole thing would diverge. But infinite sums can do things that finite sums can't and the final behaviour of the amplitude (6.14) is much softer than any of the individual terms. The infinite number of particles in string theory conspire to render finite any divergence arising from an individual particle species.

Phrased in terms of the s-channel exchange of particles, the high-energy behaviour of string theory seems somewhat miraculous. But there is another viewpoint where it's all very obvious. The power-law behaviour of scattering amplitudes is characteristic of point-like charges. But, of course, the string isn't a point-like object. It is extended and fuzzy at length scales comparable to $\sqrt{\alpha'}$. This is the reason the amplitude has such soft high-energy behaviour. Indeed, this idea that smooth extended objects give rise to scattering amplitudes that decay exponentially at high energies is something that you've seen before in non-relativistic quantum mechanics. Consider, for example, the scattering of a particle off a Gaussian potential. In the Born approximation, the differential cross-section is just given by the Fourier transform which is again a Gaussian, now decaying exponentially for large momentum.

It's often said that theories of quantum gravity should have a "minimum length", sometimes taken to be the Planck scale. This is roughly true in string theory, although not in any crude simple manner. Rather, the minimum length reveals itself in different ways depending on which question is being asked. The above discussion highlights one example of this: strings can't probe distance scales shorter than $l_s = \sqrt{\alpha'}$ simply because they are themselves fuzzy at this scale. It turns out that D-branes are much better probes of sub-stringy physics and provide a different view on the short distance structure of spacetime. We will also see another manifestation of the minimal length scale of string theory in Section 8.3.

Graviton Scattering

Although we've derived the result (6.14) for tachyons, all tree-level amplitudes have this soft fall-off at high-energies. Most notably, this includes graviton scattering. As we noted above, this is in sharp contrast to general relativity for which tree-level scattering amplitudes diverge at high-energies. This is the first place to see that UV problems of general relativity might have a good chance of being cured in string theory.

Using the techniques described in this section, one can compute m-point tree-level amplitudes for graviton scattering. If we restrict attention to low-energies (i.e. much smaller than $1/\sqrt{\alpha'}$), one can show that these coincide with the amplitudes derived from the Einstein-Hilbert action in $D = 26$ dimensions

$$S = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-G}\;\mathcal{R}$$

where $\mathcal{R}$ is the $D = 26$ Ricci scalar (not to be confused with the worldsheet Ricci scalar which we call R). The gravitational coupling, κ^2 is related to Newton's constant in 26 dimensions. It plays no role for pure gravity, but is important when we couple to matter. We'll see shortly that it's given by

$$\kappa^2 \approx g_s^2(\alpha')^{12}$$

We won't explicitly compute graviton scattering amplitudes in this course, partly because they're fairly messy and partly because building up the Einstein-Hilbert action from m-particle scattering is hardly the best way to look at general relativity. Instead, we shall derive the Einstein-Hilbert action in a much better fashion in Section 7.

6.3 Open String Scattering

So far our discussion has been entirely about closed strings. There is a very similar story for open strings. We again compute S-matrix elements. Conformal symmetry now maps tree-level scattering to the disc, with vertex operators inserted on the boundary of the disc.

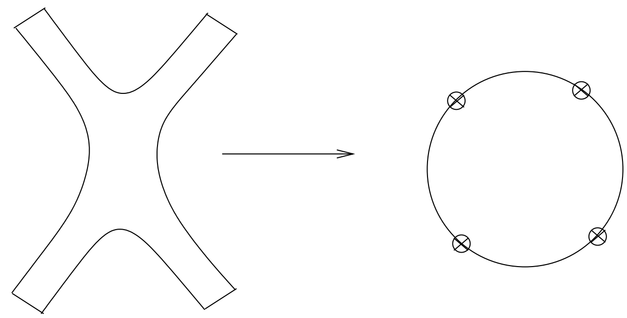


Figure 37: The conformal map from the open string worldsheet to the disc.

For the open string, the string coupling constant that we add to the Polyakov action requires the addition of a boundary term to make it well defined,

$$\chi = \frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma\ \sqrt{g}R + \frac{1}{2\pi}\int_{\partial\mathcal{M}} ds\, k \tag{6.15}$$

where k is the geodesic curvature of the boundary. To define it, we introduce two unit vectors on the worldsheet: t^α is tangential to the boundary, while n^α is normal and points outward from the boundary. The geodesic curvature is defined as

$$k = -t^\alpha n_\beta \nabla_\alpha t^\beta$$

Boundary terms of the type seen in (6.15) are also needed in general relativity for manifolds with boundaries: in that context, they are referred to as Gibbons-Hawking terms.

The Gauss-Bonnet theorem has an extension to surfaces with boundary. For surfaces with h handles and b boundaries, the Euler character is given by

$$\chi = 2 - 2h - b$$

Some examples are shown in Figure 38. The expansion for open-string scattering consists of adding consecutive boundaries to the worldsheet. The disc is weighted by $1/g_s$; the annulus has no factor of g_s and so on. We see that the open string coupling is related to the closed string coupling by

$$g^2_{\text{open}} = g_s \tag{6.16}$$

One of the key steps in computing closed string scattering amplitudes was the implementation of the conformal Killing group, which was defined as the surviving gauge

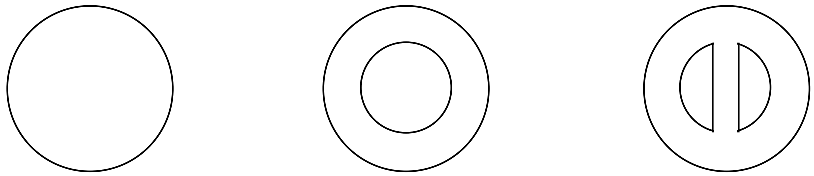


Figure 38: Riemann surfaces with boundary with $\chi = 1, 0$ and -1.

symmetry with a global action on the sphere. For the open string, there is again a residual gauge symmetry. If we think in terms of the upper-half plane, the boundary is $\text{Im}z = 0$. The conformal Killing group is composed of transformations

$$z \to \frac{az+b}{cz+d}$$

again with the requirement that $ad - bc = 1$. This time there is one further condition: the boundary $\text{Im}z = 0$ must be mapped onto itself. This requires $a, b, c, d \in \mathbf{R}$. The resulting conformal Killing group is $SL(2; \mathbf{R})/\mathbf{Z}_2$.

6.3.1 The Veneziano Amplitude

Since vertex operators now live on the boundary, they have a fixed ordering. In computing a scattering amplitude, we must sum over all orderings. Let's look again at the 4-point amplitude for tachyon scattering. The vertex operator is

$$V(p_i) = \sqrt{g_s} \int dx \; e^{ip_i \cdot X}$$

where the integral $\int dx$ is now over the boundary and $p^2 = 1/\alpha'$ is the on-shell condition for an open-string tachyon. The normalization $\sqrt{g_s}$ is that appropriate for the insertion of an open-string mode, reflecting (6.16).

Going through the same steps as for the closed string, we find that the amplitude is given by

$$\mathcal{A}^{(4)} \sim \frac{g_s}{\text{Vol}(SL(2; \mathbf{R}))} \, \delta^{26}(\sum_i p_i) \int \prod_{i=1}^{4} dx_i \prod_{j<l} |\, x_j - x_l|^{2\alpha' p_j \cdot p_l} \tag{6.17}$$

Note that there's a factor of 2 in the exponent, differing from the closed string expression (6.8). This comes about because the boundary propagator (4.52) has an extra factor of 2 due to the image charge.

We now use the $SL(2;\mathbf{R})$ residual gauge symmetry to fix three points on the boundary. We choose a particular ordering and set $x_1 = 0$, $x_2 = x$, $x_3 = 1$ and $x_4 \to \infty$. The only free insertion point is $x_2 = x$ but, because of the restriction of operator ordering, this must lie in the interval $x \in [0,1]$. The interesting part of the integral is then given by

$$\mathcal{A}^{(4)} \sim g_s \int_0^1 dx \; |x|^{2\alpha' p_1 \cdot p_2} \, |1-x|^{2\alpha' p_2 \cdot p_3}$$

This integral is well known: as shown in Appendix 6.5, it is the Euler beta function

$$B(a,b) = \int_0^1 dx \; x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

After summing over the different orderings of vertex operators, the end result for the amplitude for open string tachyon scattering is,

$$\mathcal{A}^{(4)} \sim g_s \left[B(-\alpha' s - 1, -\alpha' t - 1) + B(-\alpha' s - 1, -\alpha' u - 1) + B(-\alpha' t - 1, -\alpha' u - 1) \right]$$

This is the famous *Veneziano Amplitude*, first postulated in 1968 to capture some observed features of the strong interactions. This was before the advent of QCD and before it was realised that the amplitude arises from a string.

The open string scattering amplitude contains the same features that we saw for the closed string. For example, it has poles at

$$s = \frac{n-1}{\alpha'} \qquad n = 0, 1, 2, \ldots$$

which we recognize as the spectrum of the open string.

6.3.2 The Tension of D-Branes

Recall that we introduced D-branes as surfaces in space on which strings can end. At the time, I promised that we would eventually discover that these D-branes are dynamical objects in their own right. We'll look at this more closely in the next section, but for now we can do a simple computation to determine the tension of D-branes.

The tension T_p of a Dp-brane is defined as the energy per spatial volume. It has dimension $[T_p] = p+1$. The tension is telling us the magnitude of the coupling between the brane and gravity. Or, in our new language, the strength of the interaction between a closed string state and an open string. The simplest such diagram is shown in the figure, with a graviton vertex operator inserted. Although we won't compute this diagram completely, we can figure out its most important property just by looking at it: it has the topology of a disc, so is proportional to $1/g_s$. Adding powers of α' to get the dimension right, the tension of a Dp-brane must scale as

$$T_p \sim \frac{1}{l_s^{p+1}} \frac{1}{g_s} \tag{6.18}$$

where the string length is defined as $l_s = \sqrt{\alpha'}$. The $1/g_s$ scaling of the tension is one of the key characteristic features of a D-brane.

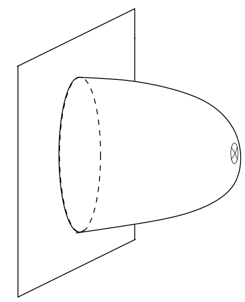


Figure 39:

I should confess that there's a lot swept under the carpet in the above discussion, not least the question of the correct normalization of the vertex operators and the difference between the string frame and the Einstein frame (which we will discuss shortly). Nonetheless, the end result (6.18) is correct. For a fuller discussion, see Section 8.7 of Polchinski.

6.4 One-Loop Amplitudes

We now return to the closed string to discuss one-loop effects. As we saw above, this corresponds to a worldsheet with the topology of a torus. We need to integrate over all metrics on the torus.

For tree-level processes, we used diffeomorphisms and Weyl transformations to map an arbitrary metric on the sphere to the flat metric on the plane. This time, we use these transformations to map an arbitrary metric on the torus to the flat metric on the torus. But there's a new subtlety that arises: not all flat metrics on the torus are equivalent.

6.4.1 The Moduli Space of the Torus

Let's spell out what we mean by this. We can construct a torus by identifying a region in the complex z-plane as shown in the figure. In general, this identification depends on a single complex parameter, $\tau \in \mathbf{C}$.

$$z \equiv z + 2\pi \quad \text{and} \quad z \equiv z + 2\pi\tau$$

Do not confuse τ with the Minkowski worldsheet time: we left that behind way back in Section 3. Everything here is Euclidean worldsheet and τ is just a parameter telling us how skewed the torus is. The flat metric on the torus is now simply

$$ds^2 = dz d\bar{z}$$

subject to the identifications above.

A general metric on a torus can always be transformed to a flat metric for some value of τ. But the question that interests us is whether two tori, parameterized by different τ, are conformally equivalent. In general, the answer is no. The space of conformally inequivalent tori, parameterized by τ, is called the *moduli space* $\mathcal{M}$.

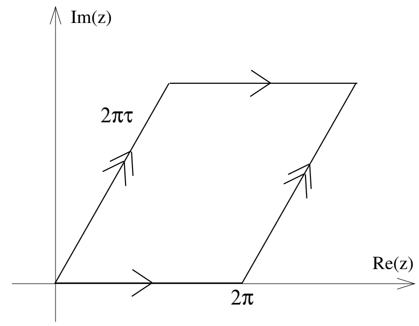


Figure 40:

However, there are some values of τ that do correspond to the same torus. In particular, there are a couple of obvious ways in which we can change τ without changing the torus. They go by the names of the S and T transformations:

- $T : \tau \rightarrow \tau + 1$: This clearly gives rise to the same torus, because the identification is now

$$z \equiv z + 2\pi \quad \text{and} \quad z \equiv z + 2\pi(\tau + 1) \equiv z + 2\pi\tau$$

- $S : \tau \rightarrow -1/\tau$: This simply flips the sides of the torus. For example, if $\tau = ia$ is purely imaginary, then this transformation maps $\tau \rightarrow i/a$, which can then be undone by a scaling.

It turns out that these two changes S and T are the only ones that keep the torus intact. They are sometimes called *modular transformations*. A general modular transformations is constructed from combinations of S and T and takes the form,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \qquad \text{with } ad - bc = 1 \tag{6.19}$$

where a, b, c and $d \in \mathbf{Z}$. This is the group $SL(2, \mathbf{Z})$. (In fact, we have our usual $\mathbf{Z}_2$ identification and the group is actually $PSL(2, \mathbf{Z}) = SL(2; \mathbf{Z})/\mathbf{Z}_2$). The moduli space $\mathcal{M}$ of the torus is given by

$$\mathcal{M} \cong \mathbf{C}/SL(2; \mathbf{Z})$$

What does this space look like? Using $T : \tau \rightarrow \tau + 1$, we can always shift τ until it lies within the interval

$$\mathrm{Re}\,\tau \;\in\; [-\tfrac{1}{2}, +\tfrac{1}{2}\,]$$

where the edges of the interval are identified. Meanwhile, $S : \tau \rightarrow -1/\tau$ inverts the

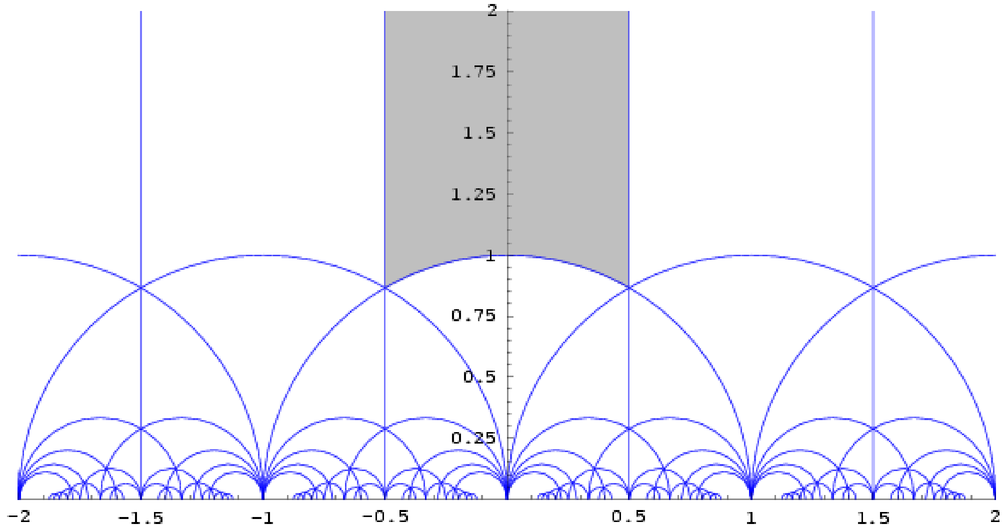


Figure 41: The fundamental domain.

modulus $|\tau|$, so we can use this to map a point inside the circle $|\tau| < 1$ to a point outside $|\tau| > 1$. One can show that by successive combinations of S and T, it is possible to map any point to lie within the shaded region shown in the figure, defined by

$$|\tau| \geq 1 \quad \text{and} \quad \mathrm{Re}\,\tau \in [-\tfrac{1}{2}, +\tfrac{1}{2}\,]$$

This is referred to as the *fundamental domain* of $SL(2;\mathbf{Z})$.

We could have just as easily chosen one of the other fundamental domains shown in the figure. But the shaded region is the standard one.

Integrating over the Moduli Space

In string theory we're invited to sum over all metrics. After gauge fixing diffeomorphisms and Weyl invariance, we still need to integrate over all inequivalent tori. In other words, we integrate over the fundamental domain. The $SL(2;\mathbf{Z})$ invariant measure over the fundamental domain is

$$\int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}$$

To see that this is $SL(2;\mathbf{Z})$ invariant, note that under a general transformation of the form (6.19) we have

$$d^2\tau \;\to\; \frac{d^2\tau}{|c\tau + d|^4} \quad \text{and} \quad \mathrm{Im}\,\tau \;\to\; \frac{\mathrm{Im}\,\tau}{|c\tau + d|^2}$$

There's some physics lurking within these rather mathematical statements. The integration over the fundamental domain in string theory is analogous to the loop integral over momentum in quantum field theory. Consider the square tori defined by $\mathrm{Re}\,\tau = 0$. The tori with $\mathrm{Im}\,\tau \to \infty$ are squashed and chubby. They correspond to the infra-red region of loop momenta in a Feynman diagram. Those with $\mathrm{Im}\,\tau \to 0$ are long and thin. Those correspond to the ultra-violet limit of loop momenta in a Feynman diagram. Yet, as we have seen, we should not integrate over these UV regions of the loop since the fundamental domain does not stretch down that far. Or, more precisely, the thin tori are mapped to chubby tori. This corresponds to the fact that any putative UV divergence of string theory can always be reinterpreted as an IR divergence. This is the second manifestation of the well-behaved UV nature of string theory. We will see this more explicitly in the example of Section 6.4.2.

Finally, when computing a loop amplitude in string theory, we still need to worry about the residual gauge symmetry that is left unfixed after the map to the flat torus. In the case of tree-level amplitudes on the sphere, this residual gauge symmetry was due to the conformal Killing group $SL(2;\mathbf{C})$. For the torus, the conformal Killing group is generated by the obvious generators ∂_z and $\bar{\partial}_{\bar{z}}$. It is $U(1) \times U(1)$.

Higher Genus Surfaces

The moduli space $\mathcal{M}_g$ of the Riemann surface of genus $g > 1$ can be shown to have dimension,

$$\dim \mathcal{M}_g = 3g - 2$$

There are no conformal Killing vectors when $g > 1$. These facts can be demonstrated as an application of the Riemann-Roch theorem. For more details, see section 5.2 of Polchinski, or sections 3.3 and 8.2 of Green, Schwarz and Witten.

6.4.2 The One-Loop Partition Function

We won't compute any one-loop scattering amplitudes in string theory. Instead, we will look at something a little simpler: the one-loop vacuum to vacuum amplitude. A Euclidean worldsheet with periodic time has the interpretation of a finite temperature partition function for the theory defined on a cylinder. In $D = 26$ dimensional spacetime, it is related to the cosmological constant in bosonic string theory.

Consider firstly the partition function of a theory on a square torus, with $\mathrm{Re}\,\tau = 0$. Compactifying Euclidean time, with period $(\mathrm{Im}\,\tau)$ is equivalent to putting the theory at temperature $T = 1/(\mathrm{Im}\,\tau)$,

$$Z[\tau] = \mathrm{Tr}\; e^{-2\pi(\mathrm{Im}\,\tau)H}$$

where the Tr is over all states in the theory. For any CFT defined on a cylinder, the Hamiltonian given by

$$H = L_0 + \tilde{L}_0 - \frac{c + \tilde{c}}{24}$$

where the final term is the Casimir energy computed in Section 4.4.1.

What then is the interpretation of the vacuum amplitude computed on a torus with $\mathrm{Re}\,\tau \neq 0$? From the diagram, we see that the effect of such a skewed torus is to translate a given point around the cylinder by $\mathrm{Re}\,\tau$. But we know which operator implements such a translation: it is $\exp(2\pi i(\mathrm{Re}\,\tau)P)$, where P is the momentum operator on the cylinder. After the map to the plane, this becomes the rotation operator

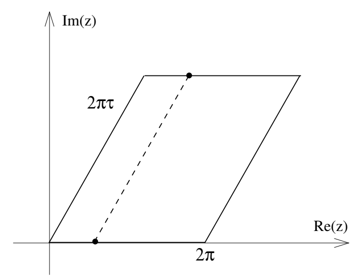


Figure 42:

$$P = L_0 - \tilde{L}_0$$

So the vacuum amplitude on the torus has the interpretation of the sum over all states in the theory, weighted by

$$Z[\tau] = \mathrm{Tr}\; e^{-2\pi(\mathrm{Im}\,\tau)(L_0+\tilde{L}_0)}\; e^{-2\pi i(\mathrm{Re}\,\tau)(L_0-\tilde{L}_0)}\; e^{2\pi(\mathrm{Im}\,\tau)(c+\tilde{c})/24}$$

We define

$$q = e^{2\pi i\tau} \quad , \quad \bar{q} = e^{-2\pi i\bar{\tau}}$$

The partition function can then be written in slick notation as

$$Z[\tau] = \mathrm{Tr}\; q^{L_0 - c/24}\; \bar{q}^{\tilde{L}_0 - \tilde{c}/24}$$

Let's compute this for the free string. We know that each scalar field X decomposes into a zero mode and an infinite number harmonic oscillator modes α_{-n} which create states of energy n. We'll deal with the zero mode shortly but, for now, we focus on the oscillators. Acting d times with the operator α_{-n} creates states with energy dn. This gives a contribution to $\mathrm{Tr} q^{L_0}$ of the form

$$\sum_{d=0}^{\infty} q^{nd} = \frac{1}{1-q^n}$$

But the Fock space of a single scalar field is built by acting with oscillator modes $n \in \mathbf{Z}^+$. Including the central charge, $c = 1$, the contribution from the oscillator modes of a single scalar field is therefore

$$\text{Tr } q^{L_0 - c/24} = \frac{1}{q^{1/24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

There is a similar expression from the $\bar{q}^{\tilde{L}_0 - \tilde{c}/24}$ sector. We're still left with the contribution from the zero mode p of the scalar field. The contribution to the energy H of the state on the worldsheet is

$$\frac{1}{4\pi\alpha'} \int d\sigma \ (\alpha' p)^2 = \frac{1}{2}\alpha' p^2$$

The trace in the partition function requires us to sum over all states, which gives

$$\int \frac{dp}{2\pi} \ e^{-\pi\alpha' \, (\text{Im}\, \tau) p^2} \sim \frac{1}{\sqrt{\alpha' \text{Im}\, \tau}}$$

So, including both the zero mode and oscillators, we get the partition function for a single free scalar field,

$$Z_{\text{scalar}}[\tau] \sim \frac{1}{\sqrt{\alpha' \text{Im}\, \tau}} \frac{1}{(q\bar{q})^{1/24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \prod_{n=1}^{\infty} \frac{1}{1-\bar{q}^n} \tag{6.20}$$

where I haven't been careful to keep track of constant factors.

To build the string partition function, we should really work in covariant quantization and include the ghost fields. Here we'll cheat and work in lightcone gauge. This is dodgy because, if we do it honestly, much of the physics gets pushed to the $p^+ = 0$ limit of the lightcone momentum where the gauge choice breaks down. So instead we'll do it dishonestly.

In lightcone gauge, we have 24 oscillator modes. But we have 26 zero modes. (You may worry that we still have to impose level matching...this is the dishonest part of the calculation. We'll see partly where it comes from shortly). Finally, there's a couple of extra steps. We need to divide by the volume of the conformal Killing group. This is just $U(1) \times U(1)$, acting by translations along the cycles of the torus. The volume is just $\text{Vol} = 4\pi^2 \, \text{Im}\, \tau$. Finally, we also need to integrate over the moduli space of the torus. Our final result, neglecting all constant factors, is

$$Z_{\text{string}} = \int d^2\tau \ \frac{1}{(\text{Im}\, \tau)} \frac{1}{(\alpha' \text{Im}\, \tau)^{13}} \ \frac{1}{q\bar{q}} \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n} \right)^{24} \left(\prod_{n=1}^{\infty} \frac{1}{1-\bar{q}^n} \right)^{24} \tag{6.21}$$

Modular Invariance

The function appearing in the partition function for the scalar field has a name: it is the inverse of the Dedekind eta function

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$$

It was studied in the 1800s by mathematicians interested in the properties of functions under modular transformations $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$. The eta-function satisfies the identities

$$\eta(\tau+1) = e^{2\pi i/24}\eta(\tau) \quad \text{and} \quad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$

These two statements ensure that the scalar partition function (6.20) is a modular invariant function. Of course, that kinda had to be true: it follows from the underlying physics.

Written in terms of η, the string partition function (6.21) takes the form

$$Z_{\text{string}} = \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \left(\frac{1}{\sqrt{\mathrm{Im}\,\tau}} \frac{1}{\eta(q)} \frac{1}{\bar{\eta}(\bar{q})} \right)^{24}$$

Both the measure and the integrand, are individually modular invariant.

6.4.3 Interpreting the String Partition Function

It's probably not immediately obvious what the string partition function (6.21) is telling us. Let's spend some time trying to understand it in terms of some simpler concepts.

We know that the free string describes an infinite number of particles with mass $m_n^2 = 4(n-1)/\alpha'$, $n = 0, 1, \ldots$. The string partition function should just be a sum over vacuum loops of each of these particles. We'll now show that it almost has this interpretation.

Firstly, let's figure out what the contribution from a single particle would be? We'll consider a free massive scalar field ϕ in D dimensions. The partition function is given by,

$$\begin{aligned} Z &= \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int d^Dx\; \phi(-\partial^2 + m^2)\phi\right) \\ &\sim \det{}^{-1/2}(-\partial^2 + m^2) \\ &= \exp\left(\frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\; \ln(p^2 + m^2)\right) \end{aligned}$$

This is the partition function of a field theory. It contains vacuum loops for all numbers of particles. To compare to the string partition function, we want the vacuum amplitude for just a single particle. But that's easy to extract. We write the field theory partition function as,

$$Z = \exp(Z_1) = \sum_{n=0}^{\infty} \frac{Z_1^n}{n!}$$

Each term in the sum corresponds to n particles propagating in a vacuum loop, with the $n!$ factor taking care of Bosonic statistics. So the vacuum amplitude for a single, free massive particle is simply

$$Z_1 = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \; \ln(p^2 + m^2)$$

Clearly this diverges in the UV range of the integral, $p \to \infty$. There's a nice way to rewrite this integral using something known as Schwinger parameterization. We make use of the identity

$$\int_0^\infty dl \; e^{-xl} = \frac{1}{x} \quad \Rightarrow \quad \int_0^\infty dl \; \frac{e^{-xl}}{l} = -\ln x$$

We then write the single particle partition function as

$$Z_1 = \int \frac{d^D p}{(2\pi)^D} \int_0^\infty \frac{dl}{2l} \; e^{-(p^2+m^2)l} \tag{6.22}$$

It's worth mentioning that there's another way to see that this is the single particle partition function that is a little closer in spirit to the method we used in string theory. We could start with the einbein form of the relativistic particle action (1.8). After fixing the gauge to $e = 1$, the exponent in (6.22) is the energy of the particle traversing a loop of length l. The integration measure dl/l sums over all possible sizes of loops.

We can happily perform the $\int d^D p$ integral in (6.22). Ignoring numerical factors, we have

$$Z_1 = \int_0^\infty dl \; \frac{1}{l^{1+D/2}} \, e^{-m^2 l} \tag{6.23}$$

Note that the UV divergence as $p \to \infty$ has metamorphosised into a divergence associated to small loops as $l \to 0$.

Equation (6.23) gives the answer for a single particle of mass m. In string theory, we expect contributions from an infinite number of species of particles of mass m_n. Specializing to $D = 26$, we expect the partition function to be

$$Z = \int_0^\infty dl \, \frac{1}{l^{14}} \sum_{n=0}^\infty e^{-m_n^2 l}$$

But we know that the mass spectrum of the free string: it is given in terms of the L_0 and $\tilde{L}_0$ operators by

$$m^2 = \frac{4}{\alpha'}(L_0 - 1) = \frac{4}{\alpha'}(\tilde{L}_0 - 1) = \frac{2}{\alpha'}(L_0 + \tilde{L}_0 - 2)$$

subject to the constraint of level matching, $L_0 = \tilde{L}_0$. It's easy to impose level matching: we simply throw in a Kronecker delta in its integral representation,

$$\frac{1}{2\pi} \int_{-1/2}^{+1/2} ds \; e^{2\pi i s (L_0 - \tilde{L}_0)} = \delta_{L_0, \tilde{L}_0} \tag{6.24}$$

Replacing the sum over species, with the trace over the spectrum of states subject to level matching, the partition function becomes,

$$Z = \int_0^\infty dl \, \frac{1}{l^{14}} \int_{-1/2}^{+1/2} ds \; \mathrm{Tr}\, e^{2\pi i s (L_0 - \tilde{L}_0)} \, e^{-2(L_0 + \tilde{L}_0 - 2) l / \alpha'} \tag{6.25}$$

We again use the definition $q = \exp(2\pi i \tau)$, but this time the complex parameter τ is a combination of the length of the loop l and the auxiliary variable that we introduced to impose level matching,

$$\tau = s + \frac{2li}{\alpha'}$$

The trace over the spectrum of the string once gives the eta-functions, just as it did before. We're left with the result for the partition function,

$$Z_{\text{string}} = \int \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \left(\frac{1}{\sqrt{\mathrm{Im}\,\tau}} \frac{1}{\eta(q)} \frac{1}{\bar{\eta}(\bar{q})} \right)^{24}$$

But this is exactly the same expression that we saw before. With a difference! In fact, the difference is hidden in the notation: it is the range of integration for $d^2\tau$ which can be found in the original expressions (6.23) and (6.24). $\mathrm{Re}\,\tau$ runs over the same interval $[-\frac{1}{2}, +\frac{1}{2}]$ that we saw in string theory. As is clear from this discussion, it is this integral which implements level matching. The difference comes in the range of $\mathrm{Im}\,\tau$ which, in this naive analysis, runs over $[0, \infty)$. This is in stark contrast to string theory where we only integrate over the fundamental domain.

This highlights our previous statement: the potential UV divergences in field theory are encountered in the region $\mathrm{Im}\,\tau \sim l \to 0$. In the above analysis, this corresponds to particles traversing small loops. But this region is simply absent in the correct string theory computation. It is mapped, by modular invariance, to the infra-red region of large loops.

It is often said that in the $g_s \to 0$ limit string theory becomes a theory of an infinite number of free particles. This is true of the spectrum. But this calculation shows that it's not really true when we compute loops because the modular invariance means that we integrate over a different range of momenta in string theory than in a naive field theory approach.

So what happens in the infra-red region of our partition function? The easiest place to see it is in the $l \to \infty$ limit of the integral (6.25). We see that the integral is dominated by the lightest state which, for the bosonic string is the tachyon. This has $m^2 = -4/\alpha'$, or $(L_0 + \tilde{L}_0 - 2) = -2$. This gives a contribution to the partition function of,

$$\int^{\infty} \frac{dl}{l^{14}}\; e^{+4l/\alpha'}$$

which clearly diverges. This IR divergence of the one-loop partition function is another manifestation of tachyonic trouble. In the superstring, there is no tachyon and the IR region is well-behaved.

6.4.4 So is String Theory Finite?

The honest answer is that we don't know. The UV finiteness that we saw above holds for all one-loop amplitudes. This means, in particular, that we have a one-loop finite theory of gravity interacting with matter in higher dimensions. This is already remarkable.

There is more good news: One can show that UV finiteness continues to hold at the two-loops. And, for the superstring, state-of-the-art techniques using the "pure-spinor" formalism show that certain objects remain finite up to five-loops. Moreover, the exponential suppression (6.14) that we saw when all momentum exchanges are large continues to hold for all amplitudes.

However, no general statement of finiteness has been proven. The danger lurks in the singular points in the integration over Riemann surfaces of genus 3 and higher.

6.4.5 Beyond Perturbation Theory?

From the discussion in this section, it should be clear that string perturbation theory is entirely analogous to the Feynman diagram expansion in field theory. Just as in field theory, one can show that the expansion in g_s is asymptotic. This means that the series does not converge, but we can nonetheless make sense of it.

However, we know that there are many phenomena in quantum field theory that aren't captured by Feynman diagrams. These include confinement in the strongly coupled regime and instantons and solitons in the weakly coupled regime. Does this mean that we are missing similarly interesting phenomena in string theory? The answer is almost certainly yes! In this section, I'll very briefly allude to a couple of more advanced topics which allow us to go beyond the perturbative expansion in string theory. The goal is not really to teach you these things, but merely to familiarize you with some words.

One way to proceed is to keep quantum field theory as our guide and try to build a non-perturbative definition of string theory in terms of a path integral. We've already seen that the Polyakov path integral over worldsheets is equivalent to Feynman diagrams. So we need to go one step further. What does this mean? Recall that in QFT, a field creates a particle. In string theory, we are now looking for a field which creates a loop of string. We should have a different field for each configuration of the string. In other words, our field should itself be a function of a function: $\Phi(X^\mu(\sigma))$. Needless to say, this is quite a complicated object. If we were brave, we could then consider the path integral for this field,

$$Z = \int \mathcal{D}\Phi \; e^{iS[\Phi(X(\sigma))]}$$

for some suitable action $S[\Phi]$. The idea is that this path integral should reproduce the perturbative string expansion and, furthermore, defines a non-perturbative completion of the theory. This line of ideas is known as *string field theory*. It should be clear that this is one step further in the development: particles $\rightarrow$ fields $\rightarrow$ string fields. Or, in more historical language, if field theory is "second quantization", then string field theory is "third quantization".

String field theory has been fairly successful for the open string and some interesting non-perturbative results have been obtained in this manner. However, for the closed string this approach has been much less useful. It is usually thought that there are deep reasons behind the failure of closed string field theory, related to issues that we mentioned at the beginning of this section: there are no off-shell quantities in a theory of gravity. Moreover, we mentioned in Section 4 that a theory of interacting open strings necessarily includes closed strings, so somehow the open string field theory should already contain gravity and closed strings. Quite how this comes about is still poorly understood.

There are other ways to get a handle on non-perturbative aspects of string theory using the low-energy effective action (we will describe what the "low-energy effective action" is in the next section). Typically these techniques rely on supersymmetry to provide a window into the strongly coupled regime and so work only for the superstring. These methods have been extremely successful and any course on superstring theory would be devoted to explaining various aspects of such as dualities and M-theory.

Finally, in asymptotically AdS spacetimes, the AdS/CFT correspondence gives a non-perturbative definition of string theory and quantum gravity in the bulk in terms of Yang-Mills theory, or something similar, on the boundary. In some sense, the boundary field theory is a "string field theory".

6.5 Appendix: Games with Integrals and Gamma Functions

The gamma function is defined by the integral representation

$$\Gamma(z) = \int_0^\infty dt\ t^{z-1} e^{-t} \qquad (6.26)$$

which converges if Re$z > 0$. It has a unique analytic expression to the whole z-plane. The absolute value of the gamma function over the z-plane is shown in the figure.

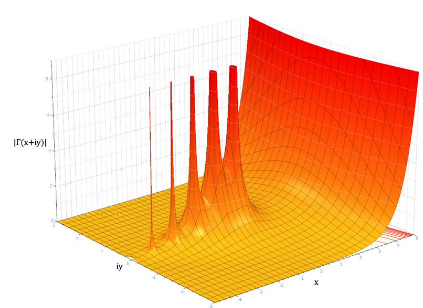


Figure 43:

The gamma function has a couple of important properties. Firstly, it can be thought of as the analytic continuation of the factorial function for positive integers, meaning

$$\Gamma(n) = (n-1)! \qquad n \in \mathbf{Z}^+$$

Secondly, $\Gamma(z)$ has poles at non-positive integers. More precisely when $z \approx -n$, with $n = 0, 1, \ldots$, there is the expansion

$$\Gamma(z) \approx \frac{1}{z+n} \frac{(-1)^n}{n!}$$

The Euler Beta Function

The Euler beta function is defined for $x,\, y \in \mathbf{C}$ by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

It has the integral representation

$$B(x,y) = \int_0^1 dt\; t^{x-1}(1-t)^{y-1} \tag{6.27}$$

Let's prove this statement. We start by looking at

$$\Gamma(x)\Gamma(y) = \int_0^\infty du \int_0^\infty dv\; e^{-u}u^{x-1}e^{-v}v^{y-1}$$

We write $u = a^2$ and $v = b^2$ so the integral becomes

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 4\int_0^\infty da \int_0^\infty db\; e^{-(a^2+b^2)}a^{2x-1}b^{2y-1} \\ &= \int_{-\infty}^\infty da \int_{-\infty}^\infty db\; e^{-(a^2+b^2)}|a|^{2x-1}|b|^{2y-1}\end{aligned}$$

We now change coordinates once more, this time to polar $a = r\cos\theta$ and $b = r\sin\theta$. We get

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty rdr\; e^{-r^2}r^{2x+2y-2}\; \int_0^{2\pi} d\theta\; |\cos\theta|^{2x-1}|\sin\theta|^{2y-1} \\ &= \frac{1}{2}\Gamma(x+y)\times 4\int_0^{\pi/2} d\theta\; (\cos\theta)^{2x-1}(\sin\theta)^{2y-1} \\ &= \Gamma(x+y)\int_0^1 dt\; (1-t)^{y-1}t^{x-1}\end{aligned}$$

where, in the final line, we made the substitution $t = \cos^2\theta$. This completes the proof.

The Virasoro-Shapiro Amplitude

In the closed string computation, we came across the integral

$$C(a,b) = \int d^2z\; |z|^{2a-2}|1-z|^{2b-2}$$

We will now evaluate this and show that it is given by (6.11). We start by using a trick. We can write

$$|z|^{2a-2} = \frac{1}{\Gamma(1-a)}\int_0^\infty dt\; t^{-a}e^{-|z|^2t}$$

which follows from the definition (6.26) of the gamma function. Similarly, we can write

$$|1-z|^{2b-2} = \frac{1}{\Gamma(1-b)} \int_0^\infty du\ u^{-b} e^{-|1-z|^2 u}$$

We decompose the complex coordinate $z = x + iy$, so that the measure of the integral is $d^2z = 2dxdy$. We can then write the integral $C(a,b)$ as

$$\begin{aligned}
C(a,b) &= \int \frac{d^2z\, du\, dt}{\Gamma(1-a)\Gamma(1-b)}\ t^{-a} u^{-b} e^{-|z|^2 t} e^{-|1-z|^2 u} \\
&= 2\int \frac{dx\, dy\, du\, dt}{\Gamma(1-a)\Gamma(1-b)}\ t^{-a} u^{-b}\ e^{-(t+u)(x^2+y^2)+2xu-u} \\
&= 2\int \frac{dx\, dy\, du\, dt}{\Gamma(1-a)\Gamma(1-b)}\ t^{-a} u^{-b}\ \exp\left(-(t+u)\left[\left(x - \frac{u}{t+u}\right)^2 + y^2\right] - u + \frac{u^2}{t+u}\right)
\end{aligned}$$

Now we do the $dxdy$ integral which is simply Gaussian. We find

$$C(a,b) = \frac{2\pi}{\Gamma(1-a)\Gamma(1-b)} \int_0^\infty du\, dt\ \frac{t^{-a}u^{-b}}{t+u}\ e^{-tu/(t+u)}$$

Finally, we make a change of variables. We write $t = \alpha\beta$ and $u = (1-\beta)\alpha$. In order for t and u to take values in the range $[0,\infty)$, we require $\alpha \in [0,\infty)$ and $\beta \in [0,1]$. Taking into account the Jacobian arising from this transformation, which is simply α, the integral becomes

$$C(a,b) = \frac{2\pi}{\Gamma(1-a)\Gamma(1-b)} \int d\alpha\, d\beta\ \frac{\alpha^{1-a-b}}{\alpha}\ \beta^{-a}(1-\beta)^{-b} e^{-\alpha\beta(1-\beta)}$$

But we recognize the integral over $d\alpha$: it is simply

$$\int_0^\infty d\alpha\ \alpha^{-a-b} e^{-\beta\alpha(1-\beta)} = [\beta(1-\beta)]^{a+b-1}\Gamma(1-a-b)$$

We write $c = 1 - a - b$. Finally, we're left with

$$C(a,b) = \frac{2\pi\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)} \int_0^1 d\beta\ (1-\beta)^{a-1}\beta^{b-1}$$

But the final integral is the Euler beta function (6.27). This gives us our promised result,

$$C(a,b) = \frac{2\pi\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)}$$