7. Low Energy Effective Actions

So far, we've only discussed strings propagating in flat spacetime. In this section we will consider strings propagating in different backgrounds. This is equivalent to having different CFTs on the worldsheet of the string.

There is an obvious generalization of the Polyakov action to describe a string moving in curved spacetime,

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\; g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,G_{\mu\nu}(X) \tag{7.1}$$

Here $g_{\alpha\beta}$ is again the worldsheet metric. This action describes a map from the worldsheet of the string into a spacetime with metric $G_{\mu\nu}(X)$. (Despite its name, this metric is not to be confused with the Einstein tensor which we won't have need for in this lecture notes).

Actions of the form (7.1) are known as *non-linear sigma models*. (This strange name has its roots in the history of pions). In this context, the D-dimensional spacetime is sometimes called the *target space*. Theories of this type are important in many aspects of physics, from QCD to condensed matter.

Although it's obvious that (7.1) describes strings moving in curved spacetime, there's something a little fishy about just writing it down. The problem is that the quantization of the closed string already gave us a graviton. If we want to build up some background metric $G_{\mu\nu}(X)$, it should be constructed from these gravitons, in much the same manner that a laser beam is made from the underlying photons. How do we see that the metric in (7.1) has anything to do with the gravitons that arise from the quantization of the string?

The answer lies in the use of vertex operators. Let's expand the metric as a small fluctuation around flat space

$$G_{\mu\nu}(X) = \delta_{\mu\nu} + h_{\mu\nu}(X)$$

Then the partition function that we build from the action (7.1) is related to the partition function for a string in flat space by

$$Z = \int \mathcal{D}X\mathcal{D}g\; e^{-S_{\rm Poly}-V} = \int \mathcal{D}X\mathcal{D}g\; e^{-S_{\rm Poly}}(1 - V + \frac{1}{2}V^2 + \ldots)$$

where $S_{\rm Poly}$ is the action for the string in flat space given in (1.22) and V is the expression

$$V = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\; g^{\alpha\beta}\partial_\alpha X^\mu\,\partial_\beta X^\nu\, h_{\mu\nu}(X) \tag{7.2}$$

But we've seen this before: it's the vertex operator associated to the graviton state of the string! For a plane wave, corresponding to a graviton with polarization given by the symmetric, traceless tensor $\zeta_{\mu\nu}$ and momentum p^μ, the fluctuation is given by

$$h_{\mu\nu}(X) = \zeta_{\mu\nu}\; e^{ip\cdot X}$$

With this choice, the expression (7.2) agrees with the vertex operator (5.9). But in general, we could take any linear superposition of plane waves to build up a general fluctuation $h_{\mu\nu}(X)$.

We know that inserting a single copy of V in the path integral corresponds to the introduction of a single graviton state. Inserting e^V in the path integral corresponds to a coherent state of gravitons, changing the metric from $\delta_{\mu\nu}$ to $\delta_{\mu\nu} + h_{\mu\nu}$. In this way we see that the background curved metric of (7.1) is indeed built of the quantized gravitons that we first met back in Section 2.

7.1 Einstein's Equations

In conformal gauge, the Polyakov action in flat space reduces to a free theory. This fact was extremely useful, allowing us to compute the spectrum of the theory. But on a curved background, it is no longer the case. In conformal gauge, the worldsheet theory is described by an interacting two-dimensional field theory,

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\; G_{\mu\nu}(X)\,\partial_\alpha X^\mu\,\partial^\alpha X^\nu \tag{7.3}$$

To understand these interactions in more detail, let's expand around a classical solution which we take to simply be a string sitting at a point $\bar{x}^\mu$.

$$X^\mu(\sigma) = \bar{x}^\mu + \sqrt{\alpha'}\,Y^\mu(\sigma)$$

Here Y^μ are the dynamical fluctuations about the point which we assume to be small. The factor of $\sqrt{\alpha'}$ is there for dimensional reasons: since $[X] = -1$, we have $[Y] = 0$ and statements like $Y \ll 1$ make sense. Expanding the Lagrangian gives

$$G_{\mu\nu}(X)\,\partial X^\mu\partial X^\nu = \alpha'\left[G_{\mu\nu}(\bar{x}) + \sqrt{\alpha'}G_{\mu\nu,\omega}(\bar{x})\,Y^\omega + \frac{\alpha'}{2}\,G_{\mu\nu,\omega\rho}(\bar{x})\,Y^\omega Y^\rho + \ldots\right]\partial Y^\mu\,\partial Y^\nu$$

Each of the coefficients $G_{\mu\nu,\ldots}$ in the Taylor expansion are coupling constants for the interactions of the fluctuations Y^μ. The theory has an infinite number of coupling constants and they are nicely packaged into the function $G_{\mu\nu}(X)$.

We want to know when this field theory is weakly coupled. Obviously this requires the whole infinite set of coupling constants to be small. Let's try to characterize this in a crude manner. Suppose that the target space has characteristic radius of curvature r_c, meaning schematically that

$$\frac{\partial G}{\partial X} \sim \frac{1}{r_c}$$

The radius of curvature is a length scale, so $[r_c] = -1$. From the expansion of the metric, we see that the effective dimensionless coupling is given by

$$\frac{\sqrt{\alpha'}}{r_c} \tag{7.4}$$

This means that we can use perturbation theory to study the CFT (7.3) if the spacetime metric only varies on scales much greater than $\sqrt{\alpha'}$. The perturbation series in $\sqrt{\alpha'}/r_c$ is usually called the α'-expansion to distinguish it from the g_s expansion that we saw in the previous section. Typically a quantity computed in string theory is given by a double perturbation expansion: one in α' and one in g_s.

If there are regions of spacetime where the radius of curvature becomes comparable to the string length scale, $r_c \sim \sqrt{\alpha'}$, then the worldsheet CFT is strongly coupled and we will need to develop new methods to solve it. Notice that strong coupling in α' is hard, but the problem is at least well-defined in terms of the worldsheet path integral. This is qualitatively different to the question of strong coupling in g_s for which, as discussed in Section 6.4.5, we're really lacking a good definition of what the problem even means.

7.1.1 The Beta Function

Classically, the theory defined by (7.3) is conformally invariant. But this is not necessarily true in the quantum theory. To regulate divergences we will have to introduce a UV cut-off and, typically, after renormalization, physical quantities depend on the scale of a given process μ. If this is the case, the theory is no longer conformally invariant. There are plenty of theories which classically possess scale invariance which is broken quantum mechanically. The most famous of these is Yang-Mills.

As we've discussed several times, in string theory conformal invariance is a gauge symmetry and we can't afford to lose it. Our goal in this section is to understand the circumstances under which (7.3) retains conformal invariance at the quantum level.

The object which describes how couplings depend on a scale μ is called the β-function. Since we have a functions worth of couplings, we should really be talking about a β-functional, schematically of the form

$$\beta_{\mu\nu}(G) \sim \mu \frac{\partial G_{\mu\nu}(X;\mu)}{\partial\mu}$$

The quantum theory will be conformally invariant only if

$$\beta_{\mu\nu}(G) = 0$$

We now compute this for the non-linear sigma model at one-loop. Our strategy will be to isolate the UV divergence of the theory and figure out what kind of counterterm we should add. The beta-function will vanish if this counterterm vanishes.

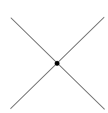
The analysis is greatly simplified by a cunning choice of coordinates. Around any point $\bar{x}$, we can always pick Riemann normal coordinates such that the expansion in $X^\mu = \bar{x}^\mu + \sqrt{\alpha'}\,Y^\mu$ gives

$$G_{\mu\nu}(X) = \delta_{\mu\nu} - \frac{\alpha'}{3}\mathcal{R}_{\mu\lambda\nu\kappa}(\bar{x})Y^\lambda Y^\kappa + \mathcal{O}(Y^3)$$

To quartic order in the fluctuations, the action becomes

$$S = \frac{1}{4\pi}\int d^2\sigma\ \partial Y^\mu\,\partial Y^\nu\delta_{\mu\nu} - \frac{\alpha'}{3}\mathcal{R}_{\mu\lambda\nu\kappa}\,Y^\lambda Y^\kappa\partial Y^\mu\partial Y^\nu$$

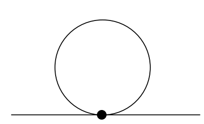
We can now treat this as an interacting quantum field theory in two dimensions. The quartic interaction gives a vertex with the Feynman rule,



$$\sim \mathcal{R}_{\mu\lambda\nu\kappa}\,(k^\mu\cdot k^\nu)$$

where k^μ_α is the 2d momentum ($\alpha = 1, 2$ is a worldsheet index) for the scalar field Y^μ. It sits in the Feynman rules because we are talking about derivative interactions.

Now we've reduced the problem to a simple interacting quantum field theory, we can compute the β-function using whatever method we like. The divergence in the theory comes from the one-loop diagram



It's actually simplest to think about this diagram in position space. The propagator for a scalar particle is

$$\langle Y^\lambda(\sigma) Y^\kappa(\sigma')\rangle = -\frac{1}{2}\,\delta^{\lambda\kappa}\,\ln|\sigma-\sigma'|^2$$

For the scalar field running in the loop, the beginning and end point coincide. The propagator diverges as $\sigma \to \sigma'$, which is simply reflecting the UV divergence that we would see in the momentum integral around the loop.

To isolate this divergence, we choose to work with dimensional regularization, with $d = 2+\epsilon$. The propagator then becomes,

$$\begin{aligned}\langle Y^\lambda(\sigma) Y^\kappa(\sigma')\rangle &= 2\pi\delta^{\lambda\kappa}\int \frac{d^{2+\epsilon}k}{(2\pi)^{2+\epsilon}}\,\frac{e^{ik\cdot(\sigma-\sigma')}}{k^2} \\ &\longrightarrow \frac{\delta^{\lambda\kappa}}{\epsilon} \qquad \text{as } \sigma\to\sigma'\end{aligned}$$

The necessary counterterm for this divergence can be determined simply by replacing $Y^\lambda Y^\kappa$ in the action with $\langle Y^\lambda Y^\kappa\rangle$. To subtract the $1/\epsilon$ term, we add the counterterm

$$\mathcal{R}_{\mu\lambda\nu\kappa}\,Y^\lambda Y^\kappa \partial Y^\mu \partial Y^\nu \;\to\; \mathcal{R}_{\mu\lambda\nu\kappa}\,Y^\lambda Y^\kappa \partial Y^\mu \partial Y^\nu - \frac{1}{\epsilon}\mathcal{R}_{\mu\nu}\,\partial Y^\mu \partial Y^\nu$$

One can check that this can be absorbed by a wavefunction renormalization $Y^\mu \to Y^\mu + (\alpha'/6\epsilon)\,\mathcal{R}^\mu{}_\nu Y^\nu$, together with the renormalization of the coupling constant which, in our theory, is the metric $G_{\mu\nu}$. We require,

$$G_{\mu\nu} \;\to\; G_{\mu\nu} + \frac{\alpha'}{\epsilon}\,\mathcal{R}_{\mu\nu} \qquad (7.5)$$

From this we learn the beta function of the theory and the condition for conformal invariance. It is

$$\beta_{\mu\nu}(G) = \alpha'\mathcal{R}_{\mu\nu} = 0 \qquad (7.6)$$

This is a magical result! The requirement for the sigma-model to be conformally invariant is that the target space must be Ricci flat: $\mathcal{R}_{\mu\nu} = 0$. Or, in other words, the background spacetime in which the string moves must obey the vacuum Einstein equations! We see that the equations of general relativity also describe the renormalization group flow of 2d sigma models.

There are several more magical things just around the corner, but it's worth pausing to make a few diverse comments.

Beta Functions and Weyl Invariance

The above calculation effectively studies the breakdown of conformal invariance in the CFT (7.3) on a flat worldsheet. We know that this should be the same thing as the breakdown of Weyl invariance on a curved worldsheet. Since this is such an important result, let's see how it works from this other perspective. We can consider the worldsheet metric

$$g_{\alpha\beta} = e^{2\phi}\delta_{\alpha\beta}$$

Then, in dimensional regularization, the theory is not Weyl invariant in $d = 2 + \epsilon$ dimensions because the contribution from $\sqrt{g}$ does not quite cancel that from the inverse metric $g^{\alpha\beta}$. The action is

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int d^{2+\epsilon}\sigma\; e^{\phi\epsilon}\partial_\alpha X^\mu\,\partial^\alpha X^\nu\,G_{\mu\nu}(X) \\ &\approx \frac{1}{4\pi\alpha'}\int d^{2+\epsilon}\sigma\;(1+\phi\epsilon)\,\partial_\alpha X^\mu\,\partial^\alpha X^\nu\,G_{\mu\nu}(X) \end{aligned}$$

where, in this expression, the $\alpha = 1, 2$ index is now raised and lowered with $\delta_{\alpha\beta}$. If we replace $G_{\mu\nu}$ in this expression with the renormalized metric (7.5), we see that there's a term involving ϕ which remains even as $\epsilon \to 0$,

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\;\partial_\alpha X^\mu\partial^\alpha X^\nu\;[G_{\mu\nu}(X) + \alpha'\phi\,\mathcal{R}_{\mu\nu}(X)]$$

This indicates a breakdown of Weyl invariance. Indeed, we can look at our usual diagnostic for Weyl invariance, namely the vanishing of $T^\alpha_{\ \alpha}$. In conformal gauge, this is given by

$$T_{\alpha\beta} = +\frac{4\pi}{\sqrt{g}}\frac{\partial S}{\partial g^{\alpha\beta}} = -2\pi\frac{\partial S}{\partial\phi}\delta_{\alpha\beta} \quad\Rightarrow\quad T^\alpha_{\ \alpha} = -\frac{1}{2}\mathcal{R}_{\mu\nu}\,\partial X^\mu\,\partial X^\nu$$

In this way of looking at things, we define the β-function to be the coefficient in front of $\partial X\partial X$, namely

$$T^\alpha_{\ \alpha} = -\frac{1}{2\alpha'}\,\beta_{\mu\nu}\,\partial X^\mu\partial X^\nu$$

Again, we have the result

$$\beta_{\mu\nu} = \alpha'\mathcal{R}_{\mu\nu}$$

7.1.2 Ricci Flow

In string theory we only care about conformal theories with Ricci flat metrics. (And generalizations of this result that we will discuss shortly). However, in other areas of physics and mathematics, the RG flow itself is important. It is usually called Ricci flow,

$$\mu \frac{\partial G_{\mu\nu}}{\partial \mu} = \alpha' \mathcal{R}_{\mu\nu} \tag{7.7}$$

which dictates how the metric changes with scale μ.

As an illustrative and simple example, consider the target space $\mathbf{S}^2$ with radius r. This is an important model in condensed matter physics where it describes the low-energy limit of a one-dimensional Heisenberg spin chain. It is sometimes called the $O(3)$ sigma-model. Because the sphere is a symmetric space, the only effect of the RG flow is to make the radius scale dependent: $r = r(\mu)$. The beta function is given by

$$\mu \frac{\partial r^2}{\partial \mu} = \frac{\alpha'}{2\pi}$$

Hence r gets large as we go towards the UV and small towards the IR. Since the coupling is $1/r$, this means that the non-linear sigma model with $\mathbf{S}^2$ target space is asymptotically free. At low energies, the theory is strongly coupled and perturbative calculations — such as this one-loop beta function — are no longer trusted. In particular, one can show that the $\mathbf{S}^2$ sigma-model develops a mass gap in the IR.

The idea of Ricci flow (7.7) was recently used by Perelman to prove the Poincaré conjecture. In fact, Perelman used a slightly generalized version of Ricci flow which we will see shortly. In the language of string theory, he introduced the dilaton field.

7.2 Other Couplings

We've understood how strings couple to a background spacetime metric. But what about the other modes of the string? In Section 2, we saw that a closed string has further massless states which are associated to the anti-symmetric tensor $B_{\mu\nu}$ and the dilaton Φ. We will now see how the string reacts if these fields are turned on in spacetime.

7.2.1 Charged Strings and the B field

Let's start by looking at how strings couple to the anti-symmetric field $B_{\mu\nu}$. We discussed the vertex operator associated to this state in Section 5.4.1. It is given in

(5.9) and takes the same form as the graviton vertex operator, but with $\zeta_{\mu\nu}$ anti-symmetric. It is a simple matter to exponentiate this, to get an expression for how strings propagate in background $B_{\mu\nu}$ field. We'll keep the curved metric $G_{\mu\nu}$ as well to get the general action,

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g}\ \left(G_{\mu\nu}(X)\, \partial_\alpha X^\mu\, \partial_\beta X^\nu g^{\alpha\beta} + iB_{\mu\nu}(X)\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, \epsilon^{\alpha\beta} \right) \tag{7.8}$$

Where $\epsilon^{\alpha\beta}$ is the anti-symmetric 2-tensor, normalized such that $\sqrt{g}\epsilon^{12} = +1$. (The factor of i is there in the action because we're in Euclidean space and this new term has a single "time" derivative). The action retains invariance under worldsheet reparameterizations and Weyl rescaling.

So what is the interpretation of this new term? We will now show that we should think of the field $B_{\mu\nu}$ as analogous to the gauge potential A_μ in electromagnetism. The action (7.8) is telling us that the string is "electrically charged" under $B_{\mu\nu}$.

Gauge Potentials

We'll take a short detour to remind ourselves about some pertinent facts in electromagnetism. Let's start by returning to a point particle. We know that a charged point particle couples to a background gauge potential A_μ through the addition of a worldline term to the action,

$$\int d\tau\ A_\mu(X)\, \dot{X}^\mu\ . \tag{7.9}$$

If this relativistic form looks a little unfamiliar, we can deconstruct it by working in static gauge with $X^0 \equiv t = \tau$, where it reads

$$\int dt\ A_0(X) + A_i(X)\, \dot{X}^i\ ,$$

which should now be recognizable as the Lagrangian that gives rise to the Coulomb and Lorentz force laws for a charged particle.

So what is the generalization of this kind of coupling for a string? First note that (7.9) has an interesting geometrical structure. It is the pull-back of the one-form $A = A_\mu dX^\mu$ in spacetime onto the worldline of the particle. This works because A is a one-form and the worldline is one-dimensional. Since the worldsheet of the string is two-dimensional, the analogous coupling should be to a two-form in spacetime. This is an anti-symmetric tensor field with two indices, $B_{\mu\nu}$. The pull-back of $B_{\mu\nu}$ onto the worldsheet gives the interaction,

$$\int d^2\sigma \; B_{\mu\nu}(X)\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\epsilon^{\alpha\beta}\;. \tag{7.10}$$

This is precisely the form of the interaction we found in (7.8).

The point particle coupling (7.9) is invariant under gauge transformations of the background field $A_\mu \to A_\mu + \partial_\mu \alpha$. This follows because the Lagrangian changes by a total derivative. There is a similar statement for the two-form $B_{\mu\nu}$. The spacetime gauge symmetry is,

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu \tag{7.11}$$

under which the Lagrangian (7.10) changes by a total derivative.

In electromagnetism, one can construct the gauge invariant electric and magnetic fields which are packaged in the two-form field strength $F = dA$. Similarly, for $B_{\mu\nu}$, the gauge invariant field strength $H = dB$ is a three-form,

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\;.$$

This 3-form H is sometimes known as the *torsion*. It plays the same role as torsion in general relativity, providing an anti-symmetric component to the affine connection.

7.2.2 The Dilaton

Let's now figure out how the string couples to a background dilaton field $\Phi(X)$. This is more subtle. A naive construction of the vertex operator is not primary and one must work a little harder. The correct derivation of the vertex operators can be found in Polchinski. Here I will simply give the coupling and explain some important features.

The action of a string moving in a background involving profiles for the massless fields $G_{\mu\nu}$, $B_{\mu\nu}$ and $\Phi(X)$ is given by

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}\;\Big(G_{\mu\nu}(X)\,\partial_\alpha X^\mu\,\partial_\beta X^\nu g^{\alpha\beta} + iB_{\mu\nu}(X)\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\epsilon^{\alpha\beta} + \alpha'\,\Phi(X)\,R^{(2)}\Big) \tag{7.12}$$

where $R^{(2)}$ is the two-dimensional Ricci scalar of the worldsheet. (Up until now, we've always denoted this simply as R but we'll introduce the superscript from hereon to distinguish the worldsheet Ricci scalar from the spacetime Ricci scalar).

The coupling to the dilaton is surprising for several reasons. Firstly, we see that the term in the action vanishes on a flat worldsheet, $R^{(2)} = 0$. This is one of the reasons that it's a little trickier to determine this coupling using vertex operators.

However, the most surprising thing about the coupling to the dilaton is that it *does not* respect Weyl invariance! Since a large part of this course has been about understanding the implications of Weyl invariance, why on earth are we willing to throw it away now?! The answer, of course, is that we're not. Although the dilaton coupling does violate Weyl invariance, there is a way to restore it. We will explain this shortly. But firstly, let's discuss one crucially important implication of the dilaton coupling (7.12).

The Dilaton and the String Coupling

There is an exception to the statement that the classical coupling to the dilaton violates Weyl invariance. This arises when the dilaton is constant. For example, suppose

$$\Phi(X) = \lambda \quad , \text{ a constant}$$

Then the dilaton coupling reduces to something that we've seen before: it is

$$S_{\text{dilaton}} = \lambda\chi$$

where χ is the Euler character of the worldsheet that we introduced in (6.4). This tells us something important: the constant mode of the dilaton, $\langle\Phi\rangle$ determines the string coupling constant. This constant mode is usually taken to be the asymptotic value of the dilaton,

$$\Phi_0 = \underset{X\to\infty}{\text{limit}}\ \Phi(X) \tag{7.13}$$

The string coupling is then given by

$$g_s = e^{\Phi_0} \tag{7.14}$$

So the string coupling is not an independent parameter of string theory: it is the expectation value of a field. This means that, just like the spacetime metric $G_{\mu\nu}$ (or, indeed, like the Higgs vev) it can be determined dynamically.

We've already seen that our perturbative expansion around flat space is valid as long as $g_s \ll 1$. But now we have a stronger requirement: we can only trust perturbation theory if the string is localized in regions of space where $e^{\Phi(X)} \ll 1$ for all X. If the string ventures into regions where $e^{\Phi(X)}$ is of order 1, then we will need to use techniques that don't rely on string perturbation theory as described in Section 6.4.5.

7.2.3 Beta Functions

We now return to understanding how we can get away with the violation of Weyl invariance in the dilaton coupling (7.12). The key to this is to notice the presence of α' in front of the dilaton coupling. It's there simply on dimensional grounds. (The other two terms in the action both come with derivatives $[\partial X] = -1$, so don't need any powers of α').

However, recall that α' also plays the role of the loop-expansion parameter (7.4) in the non-linear sigma model. This means that the classical lack of Weyl invariance in the dilaton coupling can be compensated by a one-loop contribution arising from the couplings to $G_{\mu\nu}$ and $B_{\mu\nu}$.

To see this explicitly, one can compute the beta-functions for the two-dimensional field theory (7.12). In the presence of the dilaton coupling, it's best to look at the breakdown of Weyl invariance as seen by $\langle T^{\alpha}_{\ \alpha}\rangle$. There are three different kinds of contribution that the stress-tensor can receive, related to the three different spacetime fields. Correspondingly, we define three different beta functions,

$$\langle T^{\alpha}_{\ \alpha}\rangle = -\frac{1}{2\alpha'}\beta_{\mu\nu}(G)\, g^{\alpha\beta}\partial_\alpha X^\mu\, \partial_\beta X^\nu - \frac{i}{2\alpha'}\beta_{\mu\nu}(B)\, \epsilon^{\alpha\beta}\partial_\alpha X^\mu\, \partial_\beta X^\nu - \frac{1}{2}\beta(\Phi) R^{(2)} \qquad (7.15)$$

We will not provide the details of the one-loop beta function computations. We merely state the results[8],

$$\begin{aligned}
\beta_{\mu\nu}(G) &= \alpha' \mathcal{R}_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H_\nu^{\ \lambda\kappa} \\
\beta_{\mu\nu}(B) &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi\, H_{\lambda\mu\nu} \\
\beta(\Phi) &= -\frac{\alpha'}{2}\nabla^2\Phi + \alpha' \nabla_\mu \Phi\, \nabla^\mu \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}
\end{aligned}$$

A consistent background of string theory must preserve Weyl invariance, which now requires $\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$.

7.3 The Low-Energy Effective Action

The equations $\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$ can be viewed as the equations of motion for the background in which the string propagates. We now change our perspective: we

[8] The relationship between the beta function and Einstein's equations was first shown by Friedan in his 1980 PhD thesis. A readable account of the full beta functions can be found in the paper by Callan, Friedan, Martinec and Perry "*Strings in Background Fields*", Nucl. Phys. B262 (1985) 593. The full calculational details can be found in TASI lecture notes by Callan and Thorlacius which can be downloaded from the course webpage.

look for a $D=26$ dimensional spacetime action which reproduces these beta-function equations as the equations of motion. This is the *low-energy effective action* of the bosonic string,

$$S = \frac{1}{2\kappa_0^2}\int d^{26}X\sqrt{-G}\,e^{-2\Phi}\,\left(\mathcal{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\,\partial^\mu\Phi\right) \tag{7.16}$$

where we have taken the liberty of Wick rotating back to Minkowski space for this expression. Here the overall constant involving κ_0 is not fixed by the field equations but can be determined by coupling these equations to a suitable source as described, for example, in 7.4.2. On dimensional grounds alone, it scales as $\kappa_0^2 \sim l_s^{24}$ where $\alpha' = l_s^2$.

Varying the action with respect to the three fields can be shown to yield the beta functions thus,

$$\delta S = \frac{1}{2\kappa_0^2\alpha'}\int d^{26}X\sqrt{-G}\,e^{-2\Phi}\,\Big(\delta G_{\mu\nu}\,\beta^{\mu\nu}(G) - \delta B_{\mu\nu}\,\beta^{\mu\nu}(B) - (2\delta\Phi + \frac{1}{2}G^{\mu\nu}\,\delta G_{\mu\nu})(\beta^\lambda{}_\lambda(G) - 4\beta(\Phi))\Big)$$

Equation (7.16) governs the low-energy dynamics of the spacetime fields. The caveat "low-energy" refers to the fact that we only worked with the one-loop beta functions which requires large spacetime curvature.

Something rather remarkable has happened here. We started, long ago, by looking at how a single string moves in flat space. Yet, on grounds of consistency alone, we're led to the action (7.16) governing how spacetime and other fields fluctuate in $D=26$ dimensions. It feels like the tail just wagged the dog. That tiny string is seriously high-maintenance: its requirements are so stringent that they govern the way the whole universe moves.

You may also have noticed that we now have two different methods to compute the scattering of gravitons in string theory. The first is in terms of scattering amplitudes that we discussed in Section 6. The second is by looking at the dynamics encoded in the low-energy effective action (7.16). Consistency requires that these two approaches agree. They do.

7.3.1 String Frame and Einstein Frame

The action (7.16) isn't quite of the familiar Einstein-Hilbert form because of that strange factor of $e^{-2\Phi}$ that's sitting out front. This factor simply reflects the fact that the action has been computed at tree level in string perturbation theory and, as we saw in Section 6, such terms typically scale as $1/g_s^2$.

It's also worth pointing out that the kinetic terms for Φ in (7.16) seem to have the wrong sign. However, it's not clear that we should be worried about this because, again, the factor of $e^{-2\Phi}$ sits out front meaning that the kinetic terms are not canonically normalized anyway.

To put the action in more familiar form, we can make a field redefinition. Firstly, it's useful to distinguish between the constant part of the dilaton, Φ_0, and the part that varies which we call $\tilde{\Phi}$. We defined the constant part in (7.13); it is related to the string coupling constant. The varying part is simply given by

$$\tilde{\Phi} = \Phi - \Phi_0 \tag{7.17}$$

In D dimensions, we define a new metric $\tilde{G}_{\mu\nu}$ as a combination of the old metric and the dilaton,

$$\tilde{G}_{\mu\nu}(X) = e^{-4\tilde{\Phi}/(D-2)}\, G_{\mu\nu}(X) \tag{7.18}$$

Note that this isn't to be thought of as a coordinate transformation or symmetry of the action. It's merely a relabeling, a mixing-up, of the fields in the theory. We could make such redefinitions in any field theory. Typically, we choose not to because the fields already have canonical kinetic terms. The point of the transformation (7.18) is to get the fields in (7.16) to have canonical kinetic terms as well.

The new metric (7.18) is related to the old by a conformal rescaling. One can check that two metrics related by a general conformal transformation $\tilde{G}_{\mu\nu} = e^{2\omega}G_{\mu\nu}$, have Ricci scalars related by

$$\tilde{\mathcal{R}} = e^{-2\omega}\left(\mathcal{R} - 2(D-1)\nabla^2\omega - (D-2)(D-1)\partial_\mu\omega\,\partial^\mu\omega\right)$$

(We used a particular version of this earlier in the course when considering $D = 2$ conformal transformations). With the choice $\omega = -2\tilde{\Phi}/(D-2)$ in (7.18), and restricting back to $D = 26$, the action (7.16) becomes

$$S = \frac{1}{2\kappa^2}\int d^{26}X\,\sqrt{-\tilde{G}}\left(\tilde{\mathcal{R}} - \frac{1}{12}e^{-\tilde{\Phi}/3}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{1}{6}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi}\right) \tag{7.19}$$

The kinetic terms for $\tilde{\Phi}$ are now canonical and come with the right sign. Notice that there is no potential term for the dilaton and therefore nothing that dynamically sets its expectation value in the bosonic string. However, there do exist backgrounds of the superstring in which a potential for the dilaton develops, fixing the string coupling constant.

The gravitational part of the action takes the standard Einstein-Hilbert form. The gravitational coupling is given by

$$\kappa^2 = \kappa_0^2 \, e^{2\Phi_0} \sim l_s^{24} g_s^2 \tag{7.20}$$

The coefficient in front of Einstein-Hilbert term is usually identified with Newton's constant

$$8\pi G_N = \kappa^2$$

Note, however, that this is Newton's constant in $D = 26$ dimensions: it will differ from Newton's constant measured in a four-dimensional world. From Newton's constant, we define the $D = 26$ Planck length $8\pi G_N = l_p^{24}$ and Planck mass $M_p = l_p^{-1}$. (With the factor of 8π sitting there, this is usually called the reduced Planck mass). Comparing to (7.20), we see that weak string coupling, $g_s \ll 1$, provides a parameteric separation between the Planck scale and the string scale,

$$g_s \ll 1 \quad \Rightarrow \quad l_p \ll l_s$$

Often the mysteries of gravitational physics are associated with the length scale l_p. We understand string theory best when $g_s \ll 1$ where much of stringy physics occurs at $l_s \gg l_p$ and can be disentangled from strong coupling effects in gravity.

The original metric $G_{\mu\nu}$ is usually called the *string metric* or *sigma-model metric*. It is the metric that strings see, as reflected in the action (7.1). In contrast, $\tilde{G}_{\mu\nu}$ is called the *Einstein metric*. Of course, the two actions (7.16) and (7.19) describe the same physics: we have simply chosen to package the fields in a different way in each. The choice of metric — $G_{\mu\nu}$ or $\tilde{G}_{\mu\nu}$ — is usually referred to as a choice of *frame*: string frame, or Einstein frame.

The possibility of defining two metrics really arises because we have a massless scalar field Φ in the game. Whenever such a field exists, there's nothing to stop us measuring distances in different ways by including Φ in our ruler. Said another way, massless scalar fields give rise to long range attractive forces which can mix with gravitational forces and violate the principle of equivalence. Ultimately, if we want to connect to Nature, we need to find a way to make Φ massive. Such mechanisms exist in the context of the superstring.

7.3.2 Corrections to Einstein's Equations

Now that we know how Einstein's equations arise from string theory, we can start to try to understand new physics. For example, what are the quantum corrections to Einstein's equations?

On general grounds, we expect these corrections to kick in when the curvature r_c of spacetime becomes comparable to the string length scale $\sqrt{\alpha'}$. But that dovetails very nicely with the discussion above where we saw that the perturbative expansion parameter for the non-linear sigma model is α'/r_c^2. Computing the next loop correction to the beta function will result in corrections to Einstein's equations!

If we ignore H and Φ , the 2-loop sigma-model beta function can be easily computed and results in the α' correction to Einstein's equations:

$$\beta_{\mu\nu} = \alpha' \mathcal{R}_{\mu\nu} + \frac{1}{2}\alpha'^2 \mathcal{R}_{\mu\lambda\rho\sigma}\mathcal{R}_\nu{}^{\lambda\rho\sigma} + \ldots = 0$$

Such two loop corrections also appear in the heterotic superstring. However, they are absent for the type II string theories, with the first corrections appearing at 4-loops from the perspective of the sigma-model.

String Loop Corrections

Perturbative string theory has an α' expansion and g_s expansion. We still have to discuss the latter. Here an interesting subtlety arises. The sigma-model beta functions arise from regulating the UV divergences of the worldsheet. Yet the g_s expansion cares only about the topology of the string. How can the UV divergences care about the global nature of the worldsheet. Or, equivalently, how can the higher-loop corrections to the beta-functions give anything interesting?

The resolution to this puzzle is to remember that, when computing higher g_s corrections, we have to integrate over the moduli space of Riemann surfaces. But this moduli space will include some tricky points where the Riemann surface degenerates. (For example, one cycle of the torus may pinch off). At these points, the UV divergences suddenly do care about global topology and this results in the g_s corrections to the low-energy effective action.

7.3.3 Nodding Once More to the Superstring

In section 2.5, we described the massless bosonic content for the four superstring theories: Heterotic $SO(32)$, Heterotic $E_8 \times E_8$, Type IIA and Type IIB. Each of them contains the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ that appear in the bosonic string, together with a collection of further massless fields. For each, the low-energy effective action describes the dynamics of these fields in $D = 10$ dimensional spacetime. It naturally splits up into three pieces,

$$S_{\text{superstring}} = S_1 + S_2 + S_{\text{fermi}}$$

Here $S_{\rm fermi}$ describes the interactions of the spacetime fermions. We won't describe these here. But we will briefly describe the low-energy bosonic action $S_1 + S_2$ for each of these four superstring theories.

S_1 is essentially the same for all theories and is given by the action we found for the bosonic string in string frame (7.16). We'll start to use form notation and denote $H_{\mu\nu\lambda}$ simply as H_3, where the subscript tells us the degree of the form. Then the action reads

$$S_1 = \frac{1}{2\kappa_0^2}\int d^{10}X\sqrt{-G}\,e^{-2\Phi}\,\left(\mathcal{R} - \frac{1}{2}|\tilde{H}_3|^2 + 4\partial_\mu\Phi\,\partial^\mu\Phi\right) \tag{7.21}$$

There is one small difference, which is that the field $\tilde{H}_3$ that appears here for the heterotic string is not quite the same as the original H_3; we'll explain this further shortly.

The second part of the action, S_2, describes the dynamics of the extra fields which are specific to each different theory. We'll now go through the four theories in turn, explaining S_2 in each case.

- **Type IIA**: For this theory, $\tilde{H}_3$ appearing in (7.21) is $H_3 = dB_2$, just as we saw in the bosonic string. In Section 2.5, we described the extra bosonic fields of the Type IIA theory: they consist of a 1-form C_1 and a 3-form C_3. The dynamics of these fields is governed by the so-called Ramond-Ramond part of the action and is written in form notation as,

$$S_2 = -\frac{1}{4\kappa_0^2}\int d^{10}X\;\left[\sqrt{-G}\;\left(|F_2|^2 + |\tilde{F}_4|^2\right) + B_2\wedge F_4\wedge F_4\right]$$

  Here the field strengths are given by $F_2 = dC_1$ and $F_4 = dC_3$, while the object that appears in the kinetic terms is $\tilde{F}_4 = F_4 - C_1\wedge H_3$. Notice that the final term in the action does not depend on the metric: it is referred to as a *Chern-Simons* term.

- **Type IIB**: Again, $\tilde{H}_3 \equiv H_3$. The extra bosonic fields are now a scalar C_0, a 2-form C_2 and a 4-form C_4. Their action is given by

$$S_2 = -\frac{1}{4\kappa_0^2}\int d^{10}X\;\left[\sqrt{-G}\;\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) + C_4\wedge H_3\wedge F_3\right]$$

  where $F_1 = dC_0$, $F_3 = dC_2$ and $F_5 = dC_4$. Once again, the kinetic terms involve more complicated combinations of the forms: they are $\tilde{F}_3 = F_3 - C_0\wedge H_3$ and

$\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. However, for type IIB string theory, there is one extra requirement on these fields that cannot be implemented in any simple way in terms of a Lagrangian: $\tilde{F}_5$ must be self-dual

$$\tilde{F}_5 = {}^\star\tilde{F}_5$$

Strictly speaking, one should say that the low-energy dynamics of type IIB theory is governed by the equations of motion that we get from the action, supplemented with this self-duality requirement.

- **Heterotic**: Both heterotic theories have just one further massless bosonic ingredient: a non-Abelian gauge field strength F_2, with gauge group $SO(32)$ or $E_8 \times E_8$. The dynamics of this field is simply the Yang-Mills action in ten dimensions,

$$S_2 = \frac{\alpha'}{8\kappa_0^2}\int d^{10}X\,\sqrt{-G}\,\mathrm{Tr}\,|F_2|^2$$

The one remaining subtlety is to explain what $\tilde{H}_3$ means in (7.21): it is defined as $\tilde{H}_3 = dB_2 - \alpha'\omega_3/4$ where ω_3 is the Chern-Simons three form constructed from the non-Abelian gauge field A_1

$$\omega_3 = \mathrm{Tr}\left(A_1 \wedge dA_1 + \frac{2}{3}A_1 \wedge A_1 \wedge A_1\right)$$

The presence of this strange looking combination of forms sitting in the kinetic terms is tied up with one of the most intricate and interesting aspects of the heterotic string, known as anomaly cancelation.

The actions that we have written down here probably look a little arbitrary. But they have very important properties. In particular, the full action $S_{\text{superstring}}$ of each of the Type II theories is invariant under $\mathcal{N} = 2$ spacetime supersymmetry. (That means 32 supercharges). They are the unique actions with this property. Similarly, the heterotic superstring actions are invariant under $\mathcal{N} = 1$ supersymmetry and, crucially, do not suffer from anomalies. The second book by Polchinski is a good place to start learning more about these ideas.

7.4 Some Simple Solutions

The spacetime equations of motion,

$$\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$$

have many solutions. This is part of the story of vacuum selection in string theory. What solution, if any, describes the world we see around us? Do we expect this putative solution to have other special properties, or is it just a random choice from the many possibilities? The answer is that we don't really know, but there is currently no known principle which uniquely selects a solution which looks like our world — with the gauge groups, matter content and values of fundamental constants that we observe — from the many other possibilities. Of course, these questions should really be asked in the context of the superstring where a greater understanding of various non-perturbative effects such as D-branes and fluxes leads to an even greater array of possible solutions.

Here we won't discuss these problems. Instead, we'll just discuss a few simple solutions that are well known. The first plays a role when trying to make contact with the real world, while the value of the others lies mostly in trying to better understand the structure of string theory.

7.4.1 Compactifications

We've seen that the bosonic string likes to live in $D = 26$ dimensions. But we don't. Or, more precisely, we only observe three macroscopically large spatial dimensions. How do we reconcile these statements?

Since string theory is a theory of gravity, there's nothing to stop extra dimensions of the universe from curling up. Indeed, under certain circumstances, this may be required dynamically. Here we exhibit some simple solutions of the low-energy effective action which have this property. We set $H_{\mu\nu\rho} = 0$ and Φ to a constant. Then we are simply searching for Ricci flat backgrounds obeying $\mathcal{R}_{\mu\nu} = 0$. There are solutions where the metric is a direct product of metrics on the space

$$\mathbf{R}^{1,3} \times \mathbf{X} \tag{7.22}$$

where $\mathbf{X}$ is a compact 22-dimensional Ricci-flat manifold.

The simplest such manifold is just $\mathbf{X} = \mathbf{T}^{22}$, the torus endowed with a flat metric. But there are a whole host of other possibilities. Compact, complex manifolds that admit such Ricci-flat metrics are called *Calabi-Yau* manifolds. (Strictly speaking, Calabi-Yau manifolds are complex manifolds with vanishing first Chern class. Yau's theorem guarantees the existence of a unique Ricci flat metric on these spaces).

The idea that there may be extra, compact directions in the universe was considered long before string theory and goes by the name of *Kaluza-Klein compactification.* If the characteristic length scale L of the space $\mathbf{X}$ is small enough then the presence of these extra dimensions would not have been observed in experiment. The standard model of particle physics has been accurately tested to energies of a TeV or so, meaning that if the standard model particles can roam around $\mathbf{X}$, then the length scale must be $L \lesssim (\text{TeV})^{-1} \sim 10^{-16}$ cm.

However, one can cook up scenarios in which the standard model is stuck somewhere in these extra dimensions (for example, it may be localized on a D-brane). Under these circumstances, the constraints become much weaker because we would rely on gravitational experiments to detect extra dimensions. Present bounds require only $L \lesssim 10^{-5}$ cm.

Consider the Einstein-Hilbert term in the low-energy effective action. If we are interested only in the dynamics of the 4d metric on $\mathbf{R}^{1,3}$, this is given by

$$S_{EH} = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-\tilde{G}}\;\tilde{\mathcal{R}} = \frac{\text{Vol}(\mathbf{X})}{2\kappa^2}\int d^4X\sqrt{-G_{4d}}\;\mathcal{R}_{4d}$$

(There are various moduli of the internal manifold $\mathbf{X}$ that are being neglected here). From this equation, we learn that effective 4d Newton constant is given in terms of $26d$ Newton constant by,

$$8\pi G_N^{4d} = \frac{\kappa^2}{\text{Vol}(\mathbf{X})}$$

Rewriting this in terms of the 4d Planck scale, we have $l_p^{(4d)} \sim g_s l_s^{12}/\sqrt{\text{Vol}(\mathbf{X})}$. To trust this whole analysis, we require $g_s \ll 1$ and all length scales of the internal space to be bigger than l_s. This ensures that $l_p^{(4d)} < l_s$. Although the 4d Planck length is ludicrously small, $l_p^{(4d)} \sim 10^{-33}$ cm, it may be that we don't have to probe to this distance to uncover UV gravitational physics. The back-of-the-envelope calculation above shows that the string scale l_s could be much larger, enhanced by the volume of extra dimensions.

7.4.2 The String Itself

We've seen that quantizing small loops of string gives rise to the graviton and $B_{\mu\nu}$ field. Yet, from the sigma model action (7.12), we also know that the string is charged under the $B_{\mu\nu}$. Moreover, the string has tension, which ensures that it also acts as a source for the metric $G_{\mu\nu}$. So what does the back-reaction of the string look like? Or, said another way: what is the sigma-model describing a string moving in the background of another string?

Consider an infinite, static, straight string stretched in the X^1 direction. We can solve for the background fields by coupling the equations of motion to a delta-function string source. This is the same kind of calculation that we're used to in electromagnetism. The resulting spacetime fields are given by

$$\begin{aligned} ds^2 &= f(r)^{-1}\,(-dt^2 + dX_1^2) + \textstyle\sum_{i=2}^{25} dX_i^2 \\ B &= (f(r)^{-1} - 1)\, dt \wedge dX_1 \quad , \quad e^{2\Phi} = f(r)^{-1} \end{aligned} \tag{7.23}$$

The function $f(r)$ depends only on the transverse direction $r^2 = \sum_{i=2}^{25} X_i^2$ and is given by

$$f(r) = 1 + \frac{g_s^2 N l_s^{22}}{r^{22}}$$

Here N is some constant which we will shortly demonstrate counts the number of strings which source the background. The string length scale in the solutions is $l_s = \sqrt{\alpha'}$. The function $f(r)$ has the property that it is harmonic in the space transverse to the string, meaning that it satisfies $\nabla^2_{\mathbf{R}^{24}} f(r) = 0$ except at $r = 0$.

Let's compute the B-field charge of this solution. We do exactly what we do in electromagnetism: we integrate the total flux through a sphere which surrounds the object. The string lies along the X^1 direction so the transverse space is $\mathbf{R}^{24}$. We can consider a sphere $\mathbf{S}^{23}$ at the boundary of this transverse space. We should be integrating the flux over this sphere. But what is the expression for the flux?

To see what we should do, let's look at the action for $H_{\mu\nu\rho}$ in the presence of a string source. We will use form notation since this is much cleaner and refer to $H_{\mu\nu\rho}$ simply as H_3. Schematically, the action takes the form

$$\frac{1}{g_s^2}\int_{\mathbf{R}^{26}} H_3 \wedge {}^\star H_3 + \int_{\mathbf{R}^2} B_2 = \frac{1}{g_s^2}\int_{\mathbf{R}^{26}} H_3 \wedge {}^\star H_3 + g_s^2 B_2 \wedge \delta(\omega)$$

Here $\delta(\omega)$ is a delta-function source with support on the 2d worldsheet of the string. The equation of motion is

$$d^\star H_3 \sim g_s^2 \delta(\omega)$$

From this we learn that to compute the charge of a single string we need to integrate

$$\frac{1}{g_s^2}\int_{\mathbf{S}^{23}} {}^\star H_3 = 1$$

After these general comments, we now return to our solution (7.23). The above discussion was schematic and no attention was paid to factors of 2 and π. Keeping in this spirit, the flux of the solution (7.23) can be checked to be

$$\frac{1}{g_s^2}\int_{\mathbf{S}^{23}} {}^\star H_3 = N$$

This is telling us that the solution (7.23) describes the background sourced by N coincident, parallel fundamental strings. Another way to check this is to compute the ADM mass per unit length of the solution: it is $NT \sim N/\alpha'$ as expected.

Note as far as the low-energy effective action is concerned, there is nothing that insists $N \in \mathbf{Z}$. This is analogous to the statement that nothing in classical Maxwell theory requires e to be quantized. However, in string theory, as in QED, we know the underlying sources of the microscopic theory and N must indeed take integer values.

Finally, notice that as $r \to 0$, the solution becomes singular. It is not to be trusted in this regime where higher order α' corrections become important.

7.4.3 Magnetic Branes

We've already seen that string theory is not just a theory of strings; there are also D-branes, defined as surfaces on which strings can end. We'll have much more to say about D-branes in Section 7.5. Here, we will consider a third kind of object that exists in string theory. It is again a brane – meaning that it is extended in some number of spacetime directions — but it is not a D-brane because the open string cannot end there. In these lectures we will call it the *magnetic brane*.

Electric and Magnetic Charges

You're probably not used to talking about magnetically charged objects in electromagnetism. Indeed, in undergraduate courses we usually don't get much further than pointing out that $\nabla \cdot B = 0$ does not allow point-like magnetic charges. However, in the context of quantum field theory, much of the interesting behaviour often boils down to understanding how magnetic charges behave. And the same is true of string theory. Because this may be unfamiliar, let's take a minute to discuss the basics.

In electromagnetism in $d = 3 + 1$ dimensions, we measure electric charge q by integrating the electric field $\vec{E}$ over a sphere $\mathbf{S}^2$ that surrounds the particle,

$$q = \int_{\mathbf{S}^2} \vec{E} \cdot d\vec{S} = \int_{\mathbf{S}^2} {}^\star F_2 \tag{7.24}$$

In the second equality we have introduced the notation of differential forms that we also used in the previous example to discuss the string solutions.

Suppose now that a particle carries magnetic charge g. This can be measured by integrating the magnetic field $\vec{B}$ over the same sphere. This means

$$g = \int_{\mathbf{S}^2} \vec{B} \cdot d\vec{S} = \int_{\mathbf{S}^2} F_2 \tag{7.25}$$

In $d = 3+1$ dimensions, both electrically and magnetically charged objects are particles. But this is not always true in any dimension! The reason that it holds in $4d$ is because both the field strength F_2 and the dual field strength ${}^\star F_2$ are 2-forms. Clearly, this is rather special to four dimensions.

In general, suppose that we have a p-brane that is electrically charged under a suitable gauge field. As we discussed in Section 7.2.1, a $(p+1)$-dimensional object naturally couples to a $(p+1)$-form gauge potential C_{p+1} through,

$$\mu \int_W C_{p+1}$$

where μ is the charge of the object, while W is the worldvolume of the brane. The $(p+1)$-form gauge potential has a $(p+2)$-form field strength

$$G_{p+2} = dC_{p+1}$$

To measure the electric charge of the p-brane, we need to integrate the field strength over a sphere that completely surrounds the object. A p-brane in D-dimensions has a transverse space $\mathbf{R}^{D-p-1}$. We can integrate the flux over the sphere at infinity, which is $\mathbf{S}^{D-p-2}$. And, indeed, the counting works out nicely because, in D dimensions, the dual field strength is a $(D-p-2)$-form, ${}^\star G_{p+2} = \tilde{G}_{D-p-2}$, which we can happily integrate over the sphere to find the charge sitting inside,

$$q = \int_{\mathbf{S}^{D-p-2}} {}^\star G_{p+2}$$

This equation is the generalized version of (7.24)

Now let's think about magnetic charges. The generalized version of (7.25) suggest that we should compute the magnetic charge by integrating G_{p+2} over a sphere $\mathbf{S}^{p+2}$. What kind of object sits inside this sphere to emit the magnetic charge? Doing the sums backwards, we see that it should be a $(D-p-4)$-brane.

We can write down the coupling between the $(D-p-4)$-brane and the field strength. To do so, we first need to introduce the magnetic gauge potential defined by

$${}^\star G_{p+2} = \tilde{G}_{D-p-2} = d\tilde{C}_{D-p-3} \tag{7.26}$$

We can then add the magnetic coupling to the worldvolume $\tilde{W}$ of a $(D-p-4)$-brane simply by writing

$$\tilde{\mu} \int_{\tilde{W}} \tilde{C}_{D-p-3}$$

where $\tilde{\mu}$ is the magnetic charge. Note that it's typically not possible to write down a Lagrangian that includes both magnetically charged object and electrically charged objects at the same time. This would need us to include both C_{p+1} and $\tilde{C}_{D-p-3}$ in the Lagrangian, but these are not independent fields: they're related by the rather complicated differential equations (7.26).

The Magnetic Brane in Bosonic String Theory

After these generalities, let's see what it means for the bosonic string. The fundamental string is a 1-brane and, as we saw in Section 7.2.1, carries electric charge under the 2-form B. The appropriate object carrying magnetic charge under B is therefore a $(D-p-4) = (26-1-4) = 21$-brane.

To stress a point: neither the fundamental string, nor the magnetic 21-brane are D-branes. They are not surfaces where strings can end. We are calling them *branes* only because they are extended objects.

The magnetic 21-brane of the bosonic string can be found as a solution to the low-energy equations of motion. The solution can be written in terms of the dual potential $\tilde{B}_{22}$ such that $d\tilde{B}_{22} = {}^\star dB_2$. It is

$$\begin{aligned} ds^2 &= \left(-dt^2 + \sum_{i=1}^{21} dX_i^2\right) + h(r)\left(dX_{22}^2 + \ldots + dX_{25}^2\right) \\ \tilde{B}_{22} &= (1 - h(r)^{-2})\, dt \wedge dX_1 \wedge \ldots \wedge dX_{21} \\ e^{2\Phi} &= h(r) \end{aligned} \tag{7.27}$$

The function $h(r)$ depends only on the radial direction in $\mathbf{R}^4$ transverse to the brane: $r^2 = \sum_{i=22}^{25} X_i^2$. It is a harmonic function in $\mathbf{R}^4$, given by

$$h(r) = 1 + \frac{Nl_s^2}{r^2}$$

The role of this function in the metric (7.27) is to warp the transverse $\mathbf{R}^4$ directions. Distances get larger as you approach the brane and the origin, $r = 0$, is at infinite distance.

It can be checked that the solution carried N units of magnetic charge and has tension

$$T \sim \frac{N}{l_s^{22}} \frac{1}{g_s^2}$$

Let's summarize how the tension of different objects scale in string theory. The powers of $\alpha' = l_s^2$ are entirely fixed on dimensional grounds. (Recall that the tension is mass per spatial volume, so the tension of a p-brane has $[T_p] = p+1$). More interesting is the dependence on the string coupling g_s. The tension of the fundamental string does not depend on g_s, while the magnetic brane scales as $1/g_s^2$. This kind of $1/g^2$ behaviour is typical of solitons in field theories. The D-branes sit between the two: their tension scales as $1/g_s$. Objects with this behaviour are somewhat rarer (although not unheard of) in field theory.

In the perturbative limit, $g_s \to 0$, both D-branes and magnetic branes are heavy. The coupling of an object with tension T to gravity is governed by $T\kappa^2$ where the gravitational coupling scales as $\kappa \sim g_s^2$ (7.20). This means that in the weak coupling limit, the gravitational backreaction of the string and D-branes can be neglected. However, the coupling of the magnetic brane to gravity is always of order one.

The Magnetic Brane in Superstring Theory

Superstring theories also have a brane magnetically charged under B. It is a $(D-p-4) = (10-1-4) = 5$-brane and is usually referred to as the NS5-brane. The solution in the transverse $\mathbf{R}^4$ again takes the form (7.27).

The NS5-brane exists in both type II and heterotic string. In many ways it is more mysterious than D-branes and its low-energy effective dynamics is still poorly understood. It is closely related to the 5-brane of M-theory.

7.4.4 Moving Away from the Critical Dimension

The beta function equations provide a new view on the critical dimension $D = 26$ of the bosonic string. To see this, let's look more closely at the dilaton beta function $\beta(\Phi)$ defined in (7.15): it takes the same form as the Weyl anomaly that we discussed back in Section 4.4.2. This means that if we consider a string propagating in $D \neq 26$ then the Weyl anomaly simply arises as the leading order term in the dilaton beta function. So let's relax the requirement of the critical dimension. The equations of motion arising from $\beta_{\mu\nu}(G)$ and $\beta_{\mu\nu}(B)$ are unchanged, while the dilaton beta function equation becomes

$$\beta(\Phi) = \frac{D-26}{6} - \frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\mu\Phi\,\nabla^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} = 0 \qquad (7.28)$$

The low-energy effective action in string frame picks up an extra term which looks like a run-away potential for Φ,

$$S = \frac{1}{2\kappa_0^2}\int d^DX\sqrt{-G}\,e^{-2\Phi}\left(\mathcal{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\,\partial^\mu\Phi - \frac{2(D-26)}{3\alpha'}\right)$$

This sounds quite exciting. Can we really get string theory living in $D = 4$ dimensions so easily? Well, yes and no. Firstly, with this extra potential term, flat D-dimensional Minkowski space no longer solves the equations of motion. This is in agreement with the analysis in Section 2 where we showed that full Lorentz invariance was preserved only in $D = 26$.

Another, technical, problem with solving the string equations of motion this way is that we're playing tree-level term off against a one-loop term. But if tree-level and one-loop terms are comparable, then typically all higher loop contributions will be as well and it is likely that we can't trust our analysis.

The Linear Dilaton CFT

In fact, there is one simple solution to (7.28) which we can trust. It is the solution to

$$\partial_\mu \Phi \, \partial^\mu \Phi = \frac{26 - D}{6\alpha'}$$

Recall that we're working in signature $(-,+,+,\ldots)$, meaning that Φ takes a spacelike profile if $D < 26$ and a timelike profile if $D > 26$,

$$\Phi = \sqrt{\frac{26 - D}{6\alpha'}}\, X^1 \qquad D < 26$$
$$\Phi = \sqrt{\frac{D - 26}{6\alpha'}}\, X^0 \qquad D > 26$$

This gives a dilaton which is linear in one direction. This can be compared to the study of the path integral for non-critical strings that we saw in 5.3.2. There are two ways of seeing the same physics.

The reason that we can trust this solution is that there is an exact CFT underlying it which we can analyze to all orders in α'. It's called, for obvious reasons, the *linear dilaton CFT*. Let's now look at this in more detail.

Firstly, consider the worldsheet action associated to the dilaton coupling. For now we'll consider an arbitrary dilaton profile $\Phi(X)$,

$$S_{\text{dilaton}} = \frac{1}{4\pi} \int d^2\sigma \sqrt{g}\; \Phi(X) R^{(2)} \tag{7.29}$$

Although this term vanishes on a flat worldsheet, it nonetheless changes the stress-energy tensor $T_{\alpha\beta}$ because this is defined as

$$T_{\alpha\beta} = -4\pi \; \frac{\partial S}{\partial g^{\alpha\beta}}\bigg|_{g_{\alpha\beta} = \delta_{\alpha\beta}}$$

The variation of (7.29) is straightforward. Indeed, the term is akin to the Einstein-Hilbert term in general relativity but things are simpler in 2d because, for example $R_{\alpha\beta} = \frac{1}{2}\, g_{\alpha\beta} R$. We have

$$\delta(\sqrt{g} g^{\alpha\beta} R_{\alpha\beta}) = \sqrt{g} g^{\alpha\beta}\, \delta R_{\alpha\beta} = \sqrt{g}\, \nabla^\alpha v_\alpha$$

where

$$v_\alpha = \nabla^\beta \delta g_{\alpha\beta} - g^{\gamma\delta} \nabla_\alpha \delta g_{\gamma\delta}$$

Using this, the variation of the dilaton term in the action is given by

$$\delta S_{\text{dilaton}} = \frac{1}{4\pi} \int d^2\sigma \sqrt{g}\, \left(\nabla^\alpha \nabla^\beta \Phi - \nabla^2 \Phi\, g^{\alpha\beta}\right)\, \delta g_{\alpha\beta}$$

which, restricting to flat space $g_{\alpha\beta} = \delta_{\alpha\beta}$, finally gives us the stress-energy tensor of a theory with dilaton coupling

$$T^{\text{dilaton}}_{\alpha\beta} = -\partial_\alpha \partial_\beta \Phi + \partial^2 \Phi\, \delta_{\alpha\beta}$$

Note that this stress tensor is not traceless. This is to be expected because, as we described above, the dilaton coupling is not Weyl invariant at tree-level. In complex coordinates, the stress tensor is

$$T^{\text{dilaton}} = -\partial^2 \Phi \quad , \quad \bar{T}^{\text{dilaton}} = -\bar{\partial}^2 \Phi$$

Linear Dilaton OPE

The stress tensor above holds for any dilaton profile $\Phi(X)$. Let's now restrict to a linear dilaton profile for a single scalar field X,

$$\Phi = QX$$

where Q is some constant. We also include the standard kinetic terms for D scalar fields, of which X is a chosen one, giving the stress tensor

$$T = -\frac{1}{\alpha'} : \partial X\, \partial X : -Q\, \partial^2 X$$

It is a simple matter to compute the TT OPE using the techniques described in Section 4. We find,

$$T(z)\, T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

where the central charge of the theory is given by

$$c = D + 6\alpha' Q^2$$

Note that Q^2 can be positive or negative depending on the whether we have a timelike or spacelike linear dilaton. In this way, we see explicitly how a linear dilaton gradient can absorb central charge.

7.4.5 The Elephant in the Room: The Tachyon

We've been waxing lyrical about the details of solutions to the low-energy effective action, all the while ignoring the most important, relevant field of them all: the tachyon. Since our vacuum is unstable, this is a little like describing all the beautiful pictures we could paint if only that damn paintbrush would balance, unaided, on its tip.

Of course, the main reason for discussing these solutions is that they all carry directly over to the superstring where the tachyon is absent. Nonetheless, it's interesting to ask what happens if the tachyon is turned on. Its vertex operator is simply

$$V_{\text{tachyon}} \sim \int d^2\sigma \sqrt{g}\, e^{ip\cdot X}$$

where $p^2 = 4/\alpha'$. Piecing together a general tachyon profile $V(X)$ from these Fourier modes and exponentiating, results in a potential on the worldsheet of the string

$$S_{\text{potential}} = \int d^2\sigma \sqrt{g}\ \alpha' \, V(X)$$

This is a relevant operator for the worldsheet CFT. Whenever such a relevant operator turns on, we should follow the RG flow to the infra-red until we land on another CFT. The c-theorem tells us that $c_{IR} < c_{UV}$, but in string theory we always require $c = 26$. The deficit, at least initially, is soaked up by the dilaton in the manner described above. The end point of the tachyon RG flow for the bosonic string is not understood. It may be that there is no end point and the bosonic string simply doesn't make sense once the tachyon is turned on. Or perhaps we haven't yet understood the true ground state of the bosonic string.

7.5 D-Branes Revisited: Background Gauge Fields

Understanding the constraints of conformal invariance on the closed string backgrounds led us to Einstein's equations and the low-energy effective action in spacetime. Now we would like to do the same for the open string. We want to understand the restrictions that consistency places on the dynamics of D-branes.

We saw in Section 3 that there are two types of massless modes that arise from the quantization of an open string: scalars, corresponding to the fluctuation of the D-brane, and a $U(1)$ gauge field. We will ignore the scalar fluctuations for now, but will return to them later. We focus initially on the dynamics of a gauge field A_a, $a = 0, \dots, p$ living on a Dp-brane

The first question that we ask is: how does the end of the string react to a background gauge field? To answer this, we need to look at the vertex operator associated to the photon. It was given in (5.10)

$$V_{\text{photon}} \sim \int_{\partial\mathcal{M}} d\tau\ \zeta_a\, \partial^\tau X^a\, e^{ip\cdot X}$$

which is Weyl invariant and primary only if $p^2 = 0$ and $p^a\zeta_a = 0$. Exponentiating this vertex operator, as described at the beginning of Section 7, gives the coupling of the open string to a general background gauge field $A_a(X)$,

$$S_{\text{end-point}} = \int_{\partial\mathcal{M}} d\tau\ A_a(X)\,\frac{dX^a}{d\tau}$$

But this is a very familiar coupling — we've already mentioned it in (7.9). It is telling us that the end of the string is charged under the background gauge field A_a on the brane.

7.5.1 The Beta Function

We can now perform the same type of beta function calculation that we saw for the closed string[9]. To do this, it's useful to first use conformal invariance to map the open string worldsheet to the Euclidean upper-half plane as we described in Section 4.7. The action describing an open string propagating in flat space, with its ends subject to a background gauge field on the D-brane splits up into two pieces

$$S = S_{\text{Neumann}} + S_{\text{Dirichlet}}$$

where S_{Neumann} describes the fluctuations parallel to the Dp-brane and is given by

$$S_{\text{Neumann}} = \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\ \partial^\alpha X^a\,\partial_\alpha X^b\,\delta_{ab} + i\int_{\partial\mathcal{M}} d\tau\ A_a(X)\dot{X}^a \qquad (7.30)$$

Here $a, b = 0, \ldots, p$. The extra factor of i arises because we are in Euclidean space. Meanwhile, the fields transverse to the brane have Dirichlet boundary conditions and take range $I = p+1, \ldots, D-1$. Their dynamics is given by

$$S_{\text{Dirichlet}} = \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\ \partial^\alpha X^I\,\partial_\alpha X^J\,\delta_{IJ}$$

[9]We'll be fairly explicit here, but if you want to see more details then the best place to look is the original paper by Abouelsaood, Callan, Nappi and Yost, "*Open Strings in Background Gauge Fields*", Nucl. Phys. B280 (1987) 599.

The action $S_{\rm Dirichlet}$ describes free fields and doesn't play any role in the computation of the beta-function. The interesting part is $S_{\rm Neumann}$ which, for non-zero $A_a(X)$, is an interacting quantum field theory with boundary. Our task is to compute the beta function associated to the coupling $A_a(X)$. We use the same kind of technique that we earlier applied to the closed string. We expand the fields $X^a(\sigma)$ as

$$X^a(\sigma) = \bar{x}^a(\sigma) + \sqrt{\alpha'}\, Y^a(\sigma)$$

where $\bar{x}^a(\sigma)$ is taken to be some fixed background which obeys the classical equations of motion,

$$\partial^2 \bar{x}^a = 0$$

(In the analogous calculation for the closed string we chose the special case of $\bar{x}^a$ constant. Here we are more general). However, we also need to impose boundary conditions for this classical solution. In the absence of the gauge field A_a, we require Neumann boundary conditions $\partial_\sigma X^a = 0$ at $\sigma = 0$. However, the presence of the gauge field changes this. Varying the full action (7.30) shows that the relevant boundary condition is supplemented by an extra term,

$$\partial_\sigma \bar{x}^a + 2\pi\alpha' i\, F^{ab}\, \partial_\tau \bar{x}_b = 0 \qquad \text{at } \sigma = 0 \tag{7.31}$$

where the F_{ab} is the field strength

$$F_{ab}(X) = \frac{\partial A_b}{\partial X^a} - \frac{\partial A_a}{\partial X^b} \equiv \partial_a A_b - \partial_b A_a$$

The fields $Y^a(\sigma)$ are the fluctuations which are taken to be small. Again, the presence of $\sqrt{\alpha'}$ in the expansion ensures that Y^a are dimensionless. Expanding the action $S_{\rm Neumann}$ (which we'll just call S from now on) to second order in fluctuations gives,

$$\begin{aligned} S[\bar{x} + \sqrt{\alpha'} Y] = S[\bar{x}] &+ \frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma\ \partial Y^a\, \partial Y^b \delta_{ab} \\ &+ i\alpha' \int_{\partial\mathcal{M}} d\tau\ \left(\partial_a A_b\, Y^a\, \dot{Y}^b + \frac{1}{2}\partial_a \partial_b A_c\, Y^a\, Y^b\, \dot{\bar{x}}^c \right) + \ldots \end{aligned}$$

where all expressions involving the background gauge fields are now evaluated on the classical solution $\bar{x}$. We can rearrange the boundary terms by splitting the first term up into two halves and integrating one of these pieces by parts,

$$\int d\tau\ (\partial_a A_b) Y^a \dot{Y}^b = \frac{1}{2}\int d\tau\ \partial_a A_b\, Y^a\, \dot{Y}^b - \partial_a A_b\, \dot{Y}^a Y^b - \partial_c \partial_a A_b\, Y^a Y^b \dot{\bar{x}}^c$$

Combining this with the second term means that we can write all interactions in terms of the gauge invariant field strength F_{ab},

$$S[\bar{x}+\sqrt{\alpha'}Y] = S[\bar{x}] + \frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma\ \partial Y^a\,\partial Y^b \delta_{ab} + \frac{i\alpha'}{2}\int_{\partial\mathcal{M}} d\tau\ \left(F_{ab}\,Y^a\dot{Y}^b + \partial_b F_{ac}\,Y^a Y^b \dot{\bar{x}}^c\right) + \ldots \tag{7.32}$$

where the $+\ldots$ refer to the higher terms in the expansion which come with higher derivatives of F_{ab}, accompanied by powers of α'. We can neglect them for the purposes of computing the one-loop beta function.

The Propagator

This Lagrangian describes our interacting boundary theory to leading order. We can now use this to compute the beta function. Firstly, we should determine where possible divergences arise. The offending term is the last one in (7.32). This will lead to a divergence when the fluctuation fields Y^a are contracted with their propagator

$$\langle Y^a(z,\bar{z})Y^b(w,\bar{w})\rangle = G^{ab}(z,\bar{z};w,\bar{w})$$

We should be used to these free field Green's functions by now. The propagator satisfies

$$\partial\bar{\partial}\, G^{ab}(z,\bar{z}) = -2\pi\delta^{ab}\delta(z,\bar{z}) \tag{7.33}$$

in the upper half plane. But now there's a subtlety. The Y^a fields need to satisfy a boundary condition at $\mathrm{Im}\, z = 0$ and this should be reflected in the boundary condition for the propagator. We discussed this briefly for Neumann boundary conditions in Section 4.7. But we've also seen that the background field strength shifts the Neumann boundary conditions to (7.31). Correspondingly, the propagator $G(z,\bar{z};w,\bar{w})$ must now satisfy

$$\partial_\sigma G^{ab}(z,\bar{z};w,\bar{w}) + 2\pi\alpha' i\, F^a_{\ c}\,\partial_\tau G^{cb}(z,\bar{z};w,\bar{w}) = 0 \qquad \text{at } \sigma = 0 \tag{7.34}$$

In Section 4.7, we showed how Neumann boundary conditions could be imposed by considering an image charge in the lower half plane. A similar method works here. We extend $G^{ab} \equiv G^{ab}(z,\bar{z};w,\bar{w})$ to the entire complex plane. The solution to (7.33) subject to (7.34) is given by

$$G^{ab} = -\delta^{ab}\ln|z-w| - \frac{1}{2}\left(\frac{1-2\pi\alpha' F}{1+2\pi\alpha' F}\right)^{ab}\ln(z-\bar{w}) - \frac{1}{2}\left(\frac{1+2\pi\alpha' F}{1-2\pi\alpha' F}\right)^{ab}\ln(\bar{z}-w)$$

The Counterterm and Beta Function

Let's now return to the interacting theory (7.32) and see what counterterm is needed to remove the divergence. Since all interactions take place on the boundary, we should evaluate our propagator on the boundary, which means $z = \bar{z}$ and $w = \bar{w}$. In this case, all the logarithms become the same and, in the limit that $z \to w$, gives the leading divergence $\ln |z - w| \to \epsilon^{-1}$. We learn that the UV divergence takes the form,

$$-\frac{1}{\epsilon}\left[\delta^{ab} + \frac{1}{2}\left(\frac{1-2\pi\alpha' F}{1+2\pi\alpha' F}\right)^{ab} + \frac{1}{2}\left(\frac{1+2\pi\alpha' F}{1-2\pi\alpha' F}\right)^{ab}\right] = -\frac{2}{\epsilon}\left(\frac{1}{1-4\pi^2\alpha'^2 F^2}\right)^{ab}$$

It's now easy to determine the necessary counterterm. We simply replace $Y^a Y^b$ in the final term with $\langle Y^a Y^b \rangle$. This yields

$$-\frac{i2\pi\alpha'^2}{\epsilon}\int_{\partial\mathcal{M}} d\tau\ \partial_b F_{ac}\left[\frac{1}{1-4\pi^2\alpha'^2\, F^2}\right]^{ab}\ \dot{x}^c$$

For the open string theory to retain conformal invariance, we need the associated beta function to vanish. This gives us the condition on the field strength F_{ab}: it must satisfy the equation

$$\partial_b F_{ac}\left[\frac{1}{1-4\pi^2\alpha'^2 F^2}\right]^{ab} = 0 \tag{7.35}$$

This is our final equation governing the equations of motion that F_{ab} must satisfy to provide a consistent background for open string propagation.

7.5.2 The Born-Infeld Action

Equation (7.35) probably doesn't look too familiar! Following the path we took for the closed string, we wish to write down an action whose equations of motion coincide with (7.35). The relevant action was actually constructed many decades ago as a non-linear alternative to Maxwell theory: it goes by the name of the *Born-Infeld action*:

$$S = -T_p \int d^{p+1}\xi\ \sqrt{-\det\left(\eta_{ab} + 2\pi\alpha'\, F_{ab}\right)} \tag{7.36}$$

Here ξ are the worldvolume coordinates on the brane and T_p is the tension of the Dp-brane (which, since it multiplies the action, doesn't affect the equations of motion). The gauge potential is to be thought of as a function of the worldvolume coordinates: $A_a = A_a(\xi)$. It actually takes a little work to show that the equations of motion that we derive from this action coincide with the vanishing of the beta function (7.35). Some hints on how to proceed are provided on Example Sheet 4.

For small field strengths, $F_{ab} \ll 1/\alpha'$, the action (7.36) coincides with Maxwell's action. To see this, we need simply expand to get

$$S = -T_p \int d^{p+1}\xi \; \left(1 + \frac{(2\pi\alpha')^2}{4}\, F_{ab}F^{ab} + \ldots\right)$$

The leading order term, quadratic in field strengths, is the Maxwell action. Terms with higher powers of F_{ab} are suppressed by powers of α'.

So, for small field strengths, the dynamics of the gauge field on a D-brane is governed by Maxwell's equations. However, as the electric and magnetic field strengths increase and become of order $1/\alpha'$, non-linear corrections to the dynamics kick in and are captured by the Born-Infeld action.

The Born-Infeld action arises from the one-loop beta function. It is the exact result for constant field strengths. If we want to understand the dynamics of gauge fields with large gradients, ∂F, then we will have determine the higher loop contributions to the beta function.

7.6 The DBI Action

We've understood that the dynamics of gauge fields on the brane is governed by the Born-Infeld action. But what about the fluctuations of the brane itself. We looked at this briefly in Section 3.2 and suggested, on general grounds, that the action should take the Dirac form (3.6). It would be nice to show this directly by considering the beta function equations for the scalar fields ϕ^I on the brane. Turning these on corresponds to considering boundary conditions where the brane is bent. It is indeed possible to compute something along the lines of beta-function equations and to show directly that the fluctuations of the brane are governed by the Dirac action[10].

More generally, one could consider both the dynamics of the gauge field and the fluctuation of the brane. This is governed by a mixture of the Dirac action and the Born-Infeld action which is usually referred to as the *DBI action*,

$$S_{DBI} = -T_p \int d^{p+1}\xi \; \sqrt{-\det(\gamma_{ab} + 2\pi\alpha'\, F_{ab})}$$

As in Section (3.2), γ_{ab} is the pull-back of the the spacetime metric onto the worldvolume,

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\,\eta_{\mu\nu}$$

[10] A readable discussion of this calculation can be found in the original paper by Leigh, *Dirac-Born-Infeld Action from Dirichlet Sigma Model*, Mod. Phys. Lett. A4: 2767 (1989).

The new dynamical fields in this action are the embedding coordinates $X^\mu(\xi)$, with $\mu = 0, \ldots, D-1$. This appears to be D new degrees of freedom while we expect only $D-p-1$ transverse physical degrees of freedom. The resolution to this should be familiar by now: the DBI action enjoys a reparameterization invariance which removes the longitudinal fluctuations of the brane.

We can use this reparameterization invariance to work in static gauge. For an infinite, flat Dp-brane, it is useful to set

$$X^a = \xi^a \qquad a = 0, \ldots, p$$

so that the pull-back metric depends only on the transverse fluctuations X^I,

$$\gamma_{ab} = \eta_{ab} + \frac{\partial X^I}{\partial \xi^a}\frac{\partial X^J}{\partial \xi^b}\,\delta_{IJ}$$

If we are interested in situations with small field strengths F_{ab} and small derivatives $\partial_a X$, then we can expand the DBI action to leading order. We have

$$S = -(2\pi\alpha')^2 T_p \int d^{p+1}\xi \ \left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^I\partial^a\phi^I + \ldots\right)$$

where we have rescaled the positions to define the scalar fields $\phi^I = X^I/2\pi\alpha'$. We have also dropped an overall constant term in the action. This is simply free Maxwell theory coupled to free massless scalar fields ϕ^I. The higher order terms that we have dropped are all suppressed by powers of α'.

7.6.1 Coupling to Closed String Fields

The DBI action describes the low-energy dynamics of a Dp-brane in flat space. We could now ask how the motion of the D-brane is affected if it moves in a background created by closed string modes $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ. Rather than derive this, we'll simply write down the answer and then justify each term in turn. The answer is:

$$S_{DBI} = -T_p \int d^{p+1}\xi \ e^{-\tilde{\Phi}}\,\sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab} + B_{ab})}$$

Let's start with the coupling to the background metric $G_{\mu\nu}$. It's actually hidden in the notation in this expression: it appears in the pull-back metric γ_{ab} which is now given by

$$\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\,G_{\mu\nu}$$

It should be clear that this is indeed the natural place for it to sit.

Next up is the dilaton. As in (7.17), we have decomposed the dilaton into a constant piece and a varying piece: $\Phi = \Phi_0 + \tilde{\Phi}$. The constant piece governs the asymptotic string coupling, $g_s = e^{\Phi_0}$, and is implicitly sitting in front of the action because the tension of the D-brane scales as

$$T_p \sim 1/g_s$$

This, then, explains the factor of $e^{-\tilde{\Phi}}$ in front of the action: it simply reunites the varying part of the dilaton with the constant piece. Physically, it's telling us that the tension of the D-brane depends on the local value of the dilaton field, rather than its asymptotic value. If the dilaton varies, the effective string coupling at a point X in spacetime is given by $g_s^{eff} = e^{\Phi(X)} = g_s\, e^{\tilde{\Phi}(X)}$. This, in turn, changes the tension of the D-brane. It can lower its tension by moving to regions with larger g_s^{eff}.

Finally, let's turn to the $B_{\mu\nu}$ field. This is a 2-form in spacetime. The function B_{ab} appearing in the DBI action is the pull-back to the worldvolume

$$B_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\, B_{\mu\nu}$$

Its appearance in the DBI action is actually required on grounds of gauge invariance alone. This can be seen by considering an open string, moving in the presence of both a background $B_{\mu\nu}(X)$ in spacetime and a background $A_a(X)$ on the worldvolume of a brane. The relevant terms on the string worldsheet are

$$\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\ \epsilon^{\alpha\beta}\partial_\alpha X^\mu\, \partial_\beta X^\nu B_{\mu\nu} + \int_{\partial\mathcal{M}} d\tau\ A_a \dot{X}^a$$

Under a spacetime gauge transformation

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu \tag{7.37}$$

the first term changes by a total derivative. This is fine for a closed string, but it doesn't leave the action invariant for an open string because we pick up the boundary term. Let's quickly look at what we get in more detail. Under the gauge transformation (7.37), we have

$$\begin{aligned} S_B &= \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\ \epsilon^{\alpha\beta}\partial_\alpha X^\mu\, \partial_\beta X^\nu B_{\mu\nu} \\ &\longrightarrow S_B + \frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d\sigma d\tau\ \epsilon^{\alpha\beta}\partial_\alpha X^\mu\, \partial_\beta X^\nu\, \partial_\mu C_\nu \\ &= S_B + \frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d\sigma d\tau\ \epsilon^{\alpha\beta}\partial_\alpha\ (\partial_\beta X^\nu C_\nu) \\ &= S_B + \frac{1}{2\pi\alpha'}\int_{\partial\mathcal{M}} d\tau \dot{X}^\nu C_\nu = S_B + \frac{1}{2\pi\alpha'}\int_{\partial\mathcal{M}} d\tau \dot{X}^a C_a \end{aligned}$$

where, in the last line, we have replaced the sum over all directions X^ν with the sum over those directions obeying Neumann boundary conditions X^a, since $\dot{X}^I = 0$ at the end-points for any directions with Dirichlet boundary conditions.

The result of this short calculation is to see that the string action is not invariant under (7.37). To restore this spacetime gauge invariance, this boundary contribution must be canceled by an appropriate shift of A_a in the second term,

$$A_a \rightarrow A_a - \frac{1}{2\pi\alpha'} C_a \tag{7.38}$$

Note that this is not the usual kind of gauge transformation that we consider in electrodynamics. In particular, the field strength F_{ab} is not invariant. Rather, the gauge invariant combination under (7.37) and (7.38) is

$$B_{ab} + 2\pi\alpha' F_{ab}$$

This is the reason that this combination must appear in the DBI action. This is also related to an important physical effect. We have already seen that the string in spacetime is charged under $B_{\mu\nu}$. But we've also seen that the end of the string is charged under the gauge field A_a on the D-brane. This means that the open string deposits B charge on the brane, where it is converted into A charge. The fact that the gauge invariant field strength involves a combination of both F_{ab} and B_{ab} is related to this interplay of charges.

7.7 The Yang-Mills Action

Finally, let's consider the case of N coincident D-branes. We discussed this in Section 3.3 where we showed that the massless fields on the brane could be naturally packaged as $N \times N$ Hermitian matrices, with the element of the matrix telling us which brane the end points terminate on. The gauge field then takes the form

$$(A_a)^m{}_n$$

with $a = 0, \ldots, p$ and $m, n = 1, \ldots, N$. Written this way, it looks rather like a $U(N)$ gauge connection. Indeed, this is the correct interpretation. But how do we see this? Why is the gauge field describing a $U(N)$ gauge symmetry rather than, say, $U(1)^{N^2}$?

The quickest way to see that coincident branes give rise to a $U(N)$ gauge symmetry is to recall that the end point of the string is charged under the $U(1)$ gauge field that inhabits the brane it's ending on. Let's illustrate this with the simplest example. Suppose that we have two branes. The diagonal components $(A_a)^1{}_1$ and $(A_a)^2{}_2$ arise from strings which begin and end on the same brane. Each is a $U(1)$ gauge field. What about the off-diagonal terms $(A_a)^1{}_2$ and $(A_a)^2{}_1$? These come from strings stretched between the two branes. They are again massless gauge bosons, but they are charged under the two original $U(1)$ symmetries; they carry charge $(+1,-1)$ and $(-1,+1)$ respectively. But this is precisely the structure of a $U(2)$ gauge theory, with the off-diagonal terms playing a role similar to W-bosons. In fact, the only way to make sense of massless, charged spin 1 particles is through non-Abelian gauge symmetry.

So the massless excitations of N coincident branes are a $U(N)$ gauge field $(A_a)^m{}_n$, together with scalars $(\phi^I)^m{}_n$ which transform in the adjoint representation of the $U(N)$ gauge group. We saw in Section 3 that the diagonal components $(\phi^I)^m{}_m$ have the interpretation of the transverse fluctuations of the $m^{\rm th}$ brane. Can we now write down an action describing the interactions of these fields?

In fact, there are several subtleties in writing down a non-Abelian generalization of the DBI action and such an action is not known (if, indeed, it makes sense at all). However, we can make progress by considering the low-energy limit, corresponding to small field strengths. The field strength in question is now the appropriate non-Abelian expression which, neglecting the matrix indices, reads

$$F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$$

The low-energy action describing the dynamics of N coincident Dp-branes can be shown to be (neglecting an overall constant term),

$$S = -(2\pi\alpha')^2 T_p \int d^{p+1}\xi \ \mathrm{Tr}\left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\mathcal{D}_a\phi^I\mathcal{D}^a\phi^I - \frac{1}{4}\sum_{I\neq J}[\phi^I,\phi^J]^2\right) \qquad (7.39)$$

We recognize the first term as the $U(N)$ Yang-Mills action. The coefficient in front of the Yang-Mills action is the coupling constant $1/g_{YM}^2$. For a Dp-brane, this is given by $\alpha'^2 T_p$, or

$$g_{YM}^2 \sim l_s^{p-3} g_s$$

The kinetic term for ϕ^I simply reflects the fact that these fields transform in the adjoint representation of the gauge group,

$$\mathcal{D}_a\phi^I = \partial_a\phi^I + i[A_a, \phi^I]$$

We won't derive this action in these lectures: the first two terms basically follow from gauge invariance alone. The potential term is harder to see directly: the quick ways to derive it use T-duality or, in the case of the superstring, supersymmetry.

A flat, infinite Dp-brane breaks the Lorentz group of spacetime to

$$S(1,D-1)\ \to SO(1,p)\times SO(D-p-1) \tag{7.40}$$

This unbroken group descends to the worldvolume of the D-brane where it classifies all low-energy excitations of the D-brane. The $SO(1,p)$ is simply the Lorentz group of the D-brane worldvolume. The $SO(D-p-1)$ is a global symmetry of the D-brane theory, rotating the scalar fields ϕ^I.

The potential term in (7.39) is particularly interesting,

$$V=-\frac{1}{4}\sum_{I\neq J}\mathrm{Tr}\ [\phi^I,\phi^J]^2$$

The potential is positive semi-definite. We can look at the fields that can be turned on at no cost of energy, $V=0$. This requires that all ϕ^I commute which means that, after a suitable gauge transformation, they take the diagonal form,

$$\phi^I=\begin{pmatrix}\phi^I_1 & & \\ & \ddots & \\ & & \phi^I_N\end{pmatrix} \tag{7.41}$$

The diagonal component ϕ^I_n describes the position of the $n^{\rm th}$ brane in transverse space $\mathbf{R}^{D-p-1}$. We still need to get the dimensions right. The scalar fields have dimension $[\phi]=1$. The relationship to the position in space (which we mentioned before in 3.2) is

$$\vec{X}_n=2\pi\alpha'\vec{\phi}_n \tag{7.42}$$

where we've swapped to vector notation to replace the I index.

The eigenvalues ϕ^I_n are not quite gauge invariant: there is a residual gauge symmetry — the Weyl group of $U(N)$ — which leaves ϕ^I in the form (7.41) but permutes the entries by S_N, the permutation group of N elements. But this has a very natural interpretation: it is simply telling us that the D-branes are indistinguishable objects.

When all branes are separated, the vacuum expectation value (7.41) breaks the gauge group from $U(N)\ \to\ U(1)^N$. The W-bosons gain a mass M_W through the Higgs mechanism. Let's compute this mass. We'll consider a $U(2)$ theory and we'll separate the two D-branes in the direction $X^D \equiv X$. This means that we turn on a vacuum expectation value for $\phi^D = \phi$, which we write as

$$\phi = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \tag{7.43}$$

The values of ϕ_1 and ϕ_2 are the positions of the first and second brane. Or, more precisely, we need to multiply by the conversion factor $2\pi\alpha'$ as in (7.42) to get the position X_m of the $m = 1^{\text{st}}, 2^{\text{nd}}$ brane,

Let's compute the mass of the W-boson from the Yang-Mills action (7.39). It comes from the covariant derivative terms $\mathcal{D}\phi$. We expand out the gauge field as

$$A_a = \begin{pmatrix} A_a^{11} & W_a \\ W_a^\dagger & A_a^{22} \end{pmatrix}$$

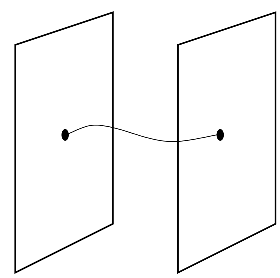


Figure 44:

with A^{11} and A^{22} describing the two $U(1)$ gauge fields and W the W-boson. The mass of the W-boson comes from the $[A_a, \phi]$ term inside the covariant derivative which, using the expectation value (7.43), is given by

$$\frac{1}{2}\text{Tr}\,[A_a, \phi]^2 = -(\phi_2 - \phi_1)^2 |W_a|^2$$

This gives us the mass of the W-boson: it is

$$M_W^2 = (\phi_2 - \phi_1)^2 = T^2 |X_2 - X_1|^2$$

where $T = 1/2\pi\alpha'$ is the tension of the string. But this has a very natural interpretation. It is precisely the mass of a string stretched between the two D-branes as shown in the figure above. We see that D-branes provide a natural geometric interpretation of the Higgs mechanism using adjoint scalars.

Notice that when branes are well separated, and the strings that stretch between them are heavy, their positions are described by the diagonal elements of the matrix given in (7.41). However, as the branes come closer together, these stretched strings become light and are important for the dynamics of the branes. Now the positions of the branes should be described by the full $N \times N$ matrices, including the off-diagonal elements. In this manner, D-branes begin to see space as something non-commutative at short distances.

In general, we can consider N D-branes located at positions $\vec{X}_m$, $m = 1, \ldots, N$ in transverse space. The string stretched between the m^{th} and n^{th} brane has mass

$$M_W = |\vec{\phi}_n - \vec{\phi}_m| = T|\vec{X}_n - \vec{X}_m|$$

which again coincides with the mass of the appropriate W-boson computed using (7.39).

7.7.1 D-Branes in Type II Superstring Theories

As we mentioned previously, D-branes are ingredients of the Type II superstring theories. Type IIA has Dp-branes with p even, while Type IIB is home to Dp-branes with p odd. The D-branes have a very important property in these theories: they preserve half the supersymmetries.

Let's take a moment to explain what this means. We'll start by returning to the Lorentz group $SO(1, D-1)$ now, of course, with $D = 10$. We've already seen that an infinite, flat Dp-brane is not invariant under the full Lorentz group, but only the subgroup (7.40). If we act with either $SO(1, p)$ or $SO(D-p-1)$ then the D-brane solution remains invariant. We say that these symmetries are preserved by the solution.

However, the role of the preserved symmetries doesn't stop there. The next step is to consider small excitations of the D-brane. These must fit into representations of the preserved symmetry group (7.40). This ensures that the low-energy dynamics of the D-brane must be governed by a theory which is invariant under (7.40) and we have indeed seen that the Lagrangian (7.39) has $SO(1, p)$ as a Lorentz group and $SO(D-p-1)$ as a global symmetry group which rotates the scalar fields.

Now let's return to supersymmetry. The Type II string theories enjoy a lot of supersymmetry: 32 supercharges in total. The infinite, flat D-branes are invariant under half of these; if we act with one half of the supersymmetry generators, the D-brane solutions don't change. Objects that have this property are often referred to as *BPS* states. Just as with the Lorentz group, these unbroken symmetries descend to the worldvolume of the D-brane. This means that the low-energy dynamics of the D-branes is described by a theory which is itself invariant under 16 supersymmetries.

There is a unique class of theories with 16 supersymmetries and a non-Abelian gauge field and matter in the adjoint representation. This class is known as maximally supersymmetric Yang-Mills theory and the bosonic part of the action is given by (7.39). Supersymmetry is realized only after the addition of fermionic fields which also live on the brane. These theories describe the low-energy dynamics of multiple D-branes.

As an illustrative example, consider D3-branes in the Type IIB theory. The theory describing N D-branes is $U(N)$ Yang-Mills with 16 supercharges, usually referred to as $U(N)$ $\mathcal{N} = 4$ super-Yang-Mills. The bosonic part of the action is given by (7.39), where there are $D - p - 1 = 6$ scalar fields ϕ^I in the adjoint representation of the gauge group. These are augmented with four Weyl fermions, also in the adjoint representation.