Supersymmetry: Example Sheet 1

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1. Show that:

i) $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}} = \left(1, -\sigma^i\right)$

ii) $(\sigma^\mu)_{\alpha\dot{\beta}}\,(\bar{\sigma}_\mu)^{\dot{\gamma}\delta} = 2\,\delta^\delta_\alpha\,\delta^{\dot{\gamma}}_{\dot{\beta}}$

iii) $(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha^{\ \beta} = 2\,\eta^{\mu\nu}\delta^\beta_\alpha$

iv) $\mathrm{tr}(\sigma^\mu\bar{\sigma}^\nu) = 2\,\eta^{\mu\nu}$

v) if V_μ is a vector and $V_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}V_\mu$, then $V^\mu = \frac{1}{2}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\,V_{\alpha\dot{\alpha}}$

vi) $\bar{\chi}_{\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\psi_\alpha = -\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}$

vii) $\psi_\alpha\bar{\chi}_{\dot{\alpha}} = \frac{1}{2}(\sigma^\mu)_{\alpha\dot{\alpha}}\,(\psi\sigma_\mu\bar{\chi})$

Note: this last part makes explicit the decomposition $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$.

2. Under a Lorentz transformation, a vector X^μ transforms as $X^\mu \to \Lambda^\mu_{\ \nu}X^\nu$. The corresponding bi-spinor $X_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}X_\mu$ transforms as $X \to SXS^\dagger$. Show that

$$\Lambda^\mu_{\ \nu}[S] = \frac{1}{2}\,\mathrm{tr}\left(\bar{\sigma}^\mu S\sigma_\nu S^\dagger\right)$$

Note: this provides an explicit map from $SL(2,\mathbb{C})$ to $SO(1,3)$.

3. Under an $SL(2,\mathbb{C})$ transformation, $\psi_\alpha \to S_\alpha^{\ \beta}\psi_\beta$. Show that:

i) $(S^{-1})_\beta^{\ \alpha} = \epsilon^{\alpha\gamma}S_\gamma^{\ \lambda}\epsilon_{\lambda\beta}$

ii) $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ transforms as $\psi^\alpha \to \psi^\beta(S^{-1})_\beta^{\ \alpha}$

iii) $\psi\chi = \psi^\alpha\chi_\alpha$ is an $SL(2,\mathbb{C})$ scalar.

iv) $\psi X\bar{\chi}$ is an $SL(2,\mathbb{C})$ scalar, where $X_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}X_\mu$ transforms as in Question 2.

4. Show that $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$ satisfies the Lorentz algebra,

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = i\left(\eta^{\nu\rho}\sigma^{\mu\sigma} - \eta^{\nu\sigma}\sigma^{\mu\rho} + \eta^{\mu\sigma}\sigma^{\nu\rho} - \eta^{\mu\rho}\sigma^{\nu\sigma}\right)$$

Hint: you may find it useful to first rewrite: $\sigma^{\mu\nu} = \frac{i}{2}(\eta^{\mu\nu} - \sigma^\nu\bar{\sigma}^\mu)$.

5. Use the fact that

$$(\sigma^{\mu\nu})_\alpha^{\ \beta}(\sigma_{\mu\nu})_\gamma^{\ \delta} = \epsilon_{\alpha\gamma}\epsilon^{\beta\delta} + \delta_\alpha^\delta\delta_\gamma^\beta$$

to show that

$$\psi_\alpha\chi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\psi\chi + \frac{1}{2}(\sigma^{\mu\nu}\epsilon^T)_{\alpha\beta}(\psi\sigma_{\mu\nu}\chi)$$

Note: This provides an explicit decomposition of the tensor product $(\frac{1}{2},0)\otimes(\frac{1}{2},0) = (0,0)\oplus(1,0)$.

6. Prove the *Fierz identities*:

i) $(\theta\psi)(\bar{\chi}\bar{\eta}) = -\frac{1}{2}(\theta\sigma^\mu\bar{\eta})(\bar{\chi}\bar{\sigma}_\mu\psi)$

ii) $(\psi\sigma^\mu\bar{\psi})(\psi\sigma^\nu\bar{\psi}) = \frac{1}{2}\eta^{\mu\nu}(\psi\psi)(\bar{\psi}\bar{\psi})$

7a. Differential operators on superspace are defined by

$$\mathcal{P}_\mu = -i\partial_\mu\ , \quad \mathcal{Q}_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\ , \quad \bar{\mathcal{Q}}_{\dot{\alpha}} = +i\bar{\partial}_{\dot{\alpha}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$$

where $\partial_\mu = \partial/\partial x^\mu$, $\partial_\alpha = \partial/\partial\theta^\alpha$ and $\bar{\partial}_{\dot{\alpha}} = \partial/\partial\bar{\theta}^{\dot{\alpha}}$. Show that these provide a representation of the supersymmetry algebra

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{P}_\mu$$

together with $\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \{\bar{\mathcal{Q}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\} = [\mathcal{P}_\mu, \mathcal{P}_\nu] = 0$.

b. Two further differential operators are defined by

$$\mathcal{D}_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\ , \quad \bar{\mathcal{D}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$$

Show that $\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = 0$ and that

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot{\beta}}\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\} = 0$$

Show also that

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{P}_\mu$$

c. If $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, show that $\bar{\mathcal{D}}_{\dot{\alpha}}y^\mu = 0$.

8. A complex, scalar superfield has the component expansion

$$\begin{aligned} Y(x,\theta,\bar\theta) &= \phi(x) + \theta^\alpha\psi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}(x) + \theta^2 M(x) + \bar\theta^2 N(x) \\ &\quad + \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} V_\mu(x) + \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) + \bar\theta^2\theta^\alpha\rho_\alpha(x) + \theta^2\bar\theta^2 D(x) \end{aligned}$$

and transforms as $\delta Y = i(\epsilon\mathcal{Q} + \bar\epsilon\bar{\mathcal{Q}})Y$. Show that the top and bottom component fields transform as

$$\begin{aligned} \delta\phi &= \epsilon\psi + \bar\epsilon\bar\chi \\ \delta D &= \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\bar\lambda - \rho\sigma^\mu\bar\epsilon) \end{aligned}$$

[Optional] If you have the energy, further show that

$$\begin{gathered} \delta\psi = 2\epsilon M + (\sigma^\mu\bar\epsilon)(i\partial_\mu\phi + V_\mu) \quad , \quad \delta\bar\chi = 2\bar\epsilon N - (\epsilon\sigma^\mu)(i\partial_\mu\phi - V_\mu) \\ \delta M = \bar\epsilon\bar\lambda - \tfrac{i}{2}\partial_\mu\psi\sigma^\mu\bar\epsilon \quad , \quad \delta N = \epsilon\rho + \tfrac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar\chi \\ \delta V_\mu = \epsilon\sigma_\mu\bar\lambda + \rho\sigma_\mu\bar\epsilon + \tfrac{i}{2}\left(\partial^\nu\psi\sigma_\mu\bar\sigma_\nu\epsilon - \bar\epsilon\bar\sigma_\nu\sigma_\mu\partial^\nu\bar\chi\right) \\ \delta\bar\lambda = 2\bar\epsilon D + \tfrac{i}{2}\bar\sigma^\nu\sigma^\mu\bar\epsilon\,\partial_\mu V_\nu + i\bar\sigma^\mu\epsilon\,\partial_\mu M \quad , \quad \delta\rho = 2\epsilon D - \tfrac{i}{2}\sigma^\nu\bar\sigma^\mu\epsilon\,\partial_\mu V_\nu + i\sigma^\mu\bar\epsilon\,\partial_\mu N \end{gathered}$$

Warning: this calculation is somewhat laborious. You will need to use the Fierz identity, the fact that $\theta^\alpha\theta^\beta = -\frac{1}{2}\theta^2\epsilon^{\alpha\beta}$, and the identity from part vii) of Question 1.

9*. For a chiral superfield Φ, show that

$$\int d^4x\, d^4\theta\ \Phi^\dagger\Phi = \int d^4x\ \left[\partial_\mu\phi^\dagger\partial^\mu\phi - i\bar\psi\bar\sigma^\mu\partial_\mu\psi + F^\dagger F\right]$$

Show that

$$\int d^4x\, d^2\theta\ W(\Phi) = \int d^4x\ \left(F\frac{\partial W}{\partial\phi} - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi\right)$$

10*. Show explicitly that the Wess-Zumino action

$$S = \int d^4x\ \left[\partial_\mu\phi^\dagger\partial^\mu\phi - i\bar\psi\bar\sigma^\mu\partial_\mu\psi - \left|\frac{\partial W}{\partial\phi}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi - \frac{1}{2}\frac{\partial^2 W^\dagger}{\partial\phi^{\dagger\,2}}\bar\psi\bar\psi\right]$$

is invariant under the supersymmetry transformations

$$\delta\phi = \sqrt{2}\epsilon\psi \quad \text{and} \quad \delta\psi = \sqrt{2}i\sigma^\mu\bar\epsilon\,\partial_\mu\phi - \sqrt{2}\epsilon\frac{\partial W^\dagger}{\partial\phi^\dagger}$$