

# Supersymmetry: Example Sheet 1

David Tong, January 2023

1. Show that:

$$\text{i)} \quad (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \equiv \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma^\mu_{\beta\dot{\beta}} = (1, -\sigma^i)$$

$$\text{ii)} \quad (\sigma^\mu)_{\alpha\dot{\beta}} (\bar{\sigma}_\mu)^{\dot{\gamma}\delta} = 2 \delta_\alpha^\delta \delta_{\dot{\beta}}^{\dot{\gamma}}$$

$$\text{iii)} \quad (\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu)_\alpha^\beta = 2 \eta^{\mu\nu} \delta_\alpha^\beta$$

$$\text{iv)} \quad \text{tr}(\sigma^\mu \bar{\sigma}^\nu) = 2 \eta^{\mu\nu}$$

$$\text{v)} \quad \text{if } V_\mu \text{ is a vector and } V_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} V_\mu, \text{ then } V^\mu = \frac{1}{2} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}}$$

$$\text{vi)} \quad \bar{\chi}_{\dot{\alpha}} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \psi_\alpha = -\psi^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$$

$$\text{vii)} \quad \psi_\alpha \bar{\chi}_{\dot{\alpha}} = \frac{1}{2} (\sigma^\mu)_{\alpha\dot{\alpha}} (\psi \sigma_\mu \bar{\chi})$$

Note: this last part makes explicit the decomposition  $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$ .

2. Under a Lorentz transformation, a vector  $X^\mu$  transforms as  $X^\mu \rightarrow \Lambda^\mu_\nu X^\nu$ . The corresponding bi-spinor  $X_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} X_\mu$  transforms as  $X \rightarrow S X S^\dagger$ . Show that

$$\Lambda^\mu_\nu[S] = \frac{1}{2} \text{tr}(\bar{\sigma}^\mu S \sigma_\nu S^\dagger)$$

Note: this provides an explicit map from  $SL(2, \mathbb{C})$  to  $SO(1, 3)$ .

3. Under an  $SL(2, \mathbb{C})$  transformation,  $\psi_\alpha \rightarrow S_\alpha^\beta \psi_\beta$ . Show that:

$$\text{i)} \quad (S^{-1})_\beta^\alpha = \epsilon^{\alpha\gamma} S_\gamma^\lambda \epsilon_{\lambda\beta}$$

$$\text{ii)} \quad \psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta \text{ transforms as } \psi^\alpha \rightarrow \psi^\beta (S^{-1})_\beta^\alpha$$

$$\text{iii)} \quad \psi \chi = \psi^\alpha \chi_\alpha \text{ is an } SL(2, \mathbb{C}) \text{ scalar.}$$

$$\text{iv)} \quad \psi X \bar{\chi} \text{ is an } SL(2, \mathbb{C}) \text{ scalar, where } X_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} X_\mu \text{ transforms as in Question 2.}$$

4. Show that  $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$  satisfies the Lorentz algebra,

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = i(\eta^{\nu\rho} \sigma^{\mu\sigma} - \eta^{\nu\sigma} \sigma^{\mu\rho} + \eta^{\mu\sigma} \sigma^{\nu\rho} - \eta^{\mu\rho} \sigma^{\nu\sigma})$$

Hint: you may find it useful to first rewrite:  $\sigma^{\mu\nu} = \frac{i}{2}(\eta^{\mu\nu} - \sigma^\nu \bar{\sigma}^\mu)$ .

5. Use the fact that

$$(\sigma^{\mu\nu})_\alpha^\beta (\sigma_{\mu\nu})_\gamma^\delta = \epsilon_{\alpha\gamma} \epsilon^{\beta\delta} + \delta_\alpha^\delta \delta_\gamma^\beta$$

to show that

$$\psi_\alpha \chi_\beta = \frac{1}{2} \epsilon_{\alpha\beta} \psi \chi + \frac{1}{2} (\sigma^{\mu\nu} \epsilon^T)_{\alpha\beta} (\psi \sigma_{\mu\nu} \chi)$$

Note: This provides an explicit decomposition of the tensor product  $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (0, 0) \oplus (1, 0)$ .

6. Prove the *Fierz identities*:

$$\text{i) } (\theta\psi)(\bar{\chi}\bar{\eta}) = -\frac{1}{2}(\theta\sigma^\mu\bar{\eta})(\bar{\chi}\bar{\sigma}_\mu\psi)$$

$$\text{ii) } (\psi\sigma^\mu\bar{\psi})(\psi\sigma^\nu\bar{\psi}) = \frac{1}{2}\eta^{\mu\nu}(\psi\psi)(\bar{\psi}\bar{\psi})$$

7a. Differential operators on superspace are defined by

$$\mathcal{P}_\mu = -i\partial_\mu, \quad \mathcal{Q}_\alpha = -i\partial_\alpha - \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu, \quad \bar{\mathcal{Q}}_{\dot{\alpha}} = +i\bar{\partial}_{\dot{\alpha}} + \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$$

where  $\partial_\mu = \partial/\partial x^\mu$ ,  $\partial_\alpha = \partial/\partial\theta^\alpha$  and  $\bar{\partial}_{\dot{\alpha}} = \partial/\partial\bar{\theta}^{\dot{\alpha}}$ . Show that these provide a representation of the supersymmetry algebra

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu \mathcal{P}_\mu$$

together with  $\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \{\bar{\mathcal{Q}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\} = [\mathcal{P}_\mu, \mathcal{P}_\nu] = 0$ .

b. Two further differential operators are defined by

$$\mathcal{D}_\alpha = \partial_\alpha + i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu, \quad \bar{\mathcal{D}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$$

Show that  $\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = 0$  and that

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot{\beta}}\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{Q}_\beta\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\} = 0$$

Show also that

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu \mathcal{P}_\mu$$

c. If  $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ , show that  $\bar{\mathcal{D}}_{\dot{\alpha}}y^\mu = 0$ .

8. A complex, scalar superfield has the component expansion

$$Y(x, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) \\ + \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} V_\mu(x) + \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}^2 \theta^\alpha \rho_\alpha(x) + \theta^2 \bar{\theta}^2 D(x)$$

and transforms as  $\delta Y = i(\epsilon \mathcal{Q} + \bar{\epsilon} \bar{\mathcal{Q}})Y$ . Show that the top and bottom component fields transform as

$$\delta \phi = \epsilon \psi + \bar{\epsilon} \bar{\chi} \\ \delta D = \frac{i}{2} \partial_\mu (\epsilon \sigma^\mu \bar{\lambda} - \rho \sigma^\mu \bar{\epsilon})$$

[Optional] If you have the energy, further show that

$$\delta \psi = 2\epsilon M + (\sigma^\mu \bar{\epsilon})(i\partial_\mu \phi + V_\mu) \quad , \quad \delta \bar{\chi} = 2\bar{\epsilon} N - (\epsilon \sigma^\mu)(i\partial_\mu \phi - V_\mu) \\ \delta M = \bar{\epsilon} \bar{\lambda} - \frac{i}{2} \partial_\mu \psi \sigma^\mu \bar{\epsilon} \quad , \quad \delta N = \epsilon \rho + \frac{i}{2} \epsilon \sigma^\mu \partial_\mu \bar{\chi} \\ \delta V_\mu = \epsilon \sigma_\mu \bar{\lambda} + \rho \sigma_\mu \bar{\epsilon} + \frac{i}{2} (\partial^\nu \psi \sigma_\mu \bar{\sigma}_\nu \epsilon - \bar{\epsilon} \bar{\sigma}_\nu \sigma_\mu \partial^\nu \bar{\chi}) \\ \delta \bar{\lambda} = 2\bar{\epsilon} D + \frac{i}{2} \bar{\sigma}^\nu \sigma^\mu \bar{\epsilon} \partial_\mu V_\nu + i \bar{\sigma}^\mu \epsilon \partial_\mu M \quad , \quad \delta \rho = 2\epsilon D - \frac{i}{2} \sigma^\nu \bar{\sigma}^\mu \epsilon \partial_\mu V_\nu + i \sigma^\mu \bar{\epsilon} \partial_\mu N$$

Warning: this calculation is somewhat laborious. You will need to use the Fierz identity, the fact that  $\theta^\alpha \theta^\beta = -\frac{1}{2} \theta^2 \epsilon^{\alpha\beta}$ , and the identity from part vii) of Question 1.

9\*. For a chiral superfield  $\Phi$ , show that

$$\int d^4x d^4\theta \Phi^\dagger \Phi = \int d^4x [\partial_\mu \phi^\dagger \partial^\mu \phi - i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi + F^\dagger F]$$

Show that

$$\int d^4x d^2\theta W(\Phi) = \int d^4x \left( F \frac{\partial W}{\partial \phi} - \frac{1}{2} \frac{\partial^2 W}{\partial \phi^2} \psi \psi \right)$$

10\*. Show explicitly that the Wess-Zumino action

$$S = \int d^4x \left[ \partial_\mu \phi^\dagger \partial^\mu \phi - i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - \left| \frac{\partial W}{\partial \phi} \right|^2 - \frac{1}{2} \frac{\partial^2 W}{\partial \phi^2} \psi \psi - \frac{1}{2} \frac{\partial^2 W^\dagger}{\partial \phi^{\dagger 2}} \bar{\psi} \bar{\psi} \right]$$

is invariant under the supersymmetry transformations

$$\delta \phi = \sqrt{2} \epsilon \psi \quad \text{and} \quad \delta \psi = \sqrt{2} i \sigma^\mu \bar{\epsilon} \partial_\mu \phi - \sqrt{2} \epsilon \frac{\partial W^\dagger}{\partial \phi^\dagger}$$