Supersymmetry: Example Sheet 2

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1. Determine the (possibly spurious) symmetries of the superpotential

$$W = \mu_2 \Phi^2 + \mu_3 \Phi^3 + \ldots + \mu_n \Phi^n$$

Argue that the superpotential is not renormalised at any order in perturbation theory.

2*. Find the vacuum structure, including the vacuum energy, of the following theories:

a) A chiral multiplet Z with $W = \alpha Z + \beta/Z$, with $\alpha, \beta \neq 0$.

b) Three chiral multiplets X, Y and Z with $W = XYZ$.

c) Three chiral multiplets X, Y and Z with $W = \alpha Y + \beta Y X^2 + \gamma XZ$ with $\alpha, \beta, \gamma \neq 0$ and $|\gamma|^2 > 2|\alpha\beta|$.

(You may assume a canonical Kähler potential for all fields.)

3a. For an abelian vector superfield V, show that the field strength superfield

$$W_\alpha = -\frac{1}{4}\bar{\mathcal{D}}^2 \mathcal{D}_\alpha V$$

is chiral, i.e. $\bar{\mathcal{D}}_{\dot{\alpha}} W_\alpha = 0$. Show further that it is invariant under extended gauge transformations $V \to V + i(\Omega - \Omega^\dagger)$.

b. Show that the components of W_α are

$$W_\alpha(x,\theta) = \lambda_\alpha(x) + \theta_\alpha D(x) + (\sigma^{\mu\nu}\theta_\alpha)F_{\mu\nu}(x) - i\theta^2 \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\lambda}^{\dot{\alpha}}(x) + \ldots$$

Hint: you will be well served to work in Wess-Zumino gauge and, at an appropriate time, to use the fact that W_α is a chiral superfield.

c. Show that the F-term integral gives

$$\int d^2\theta\; W^\alpha W_\alpha = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}F_{\mu\nu}{}^\star F^{\mu\nu} - 2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + D^2$$

Note: You will need the identity

$$\mathrm{tr}(\sigma^\mu\bar{\sigma}^\nu\sigma^\sigma\bar{\sigma}^\rho) = 2i\epsilon^{\mu\nu\sigma\rho} + 2\eta^{\mu\nu}\eta^{\sigma\rho} - 2\eta^{\mu\sigma}\eta^{\nu\rho} + 2\eta^{\mu\rho}\eta^{\nu\sigma}$$

4*. Super Yang-Mills (with vanishing theta angle) has action

$$S = \frac{1}{g^2}\int d^4x \ \mathrm{Tr}\left[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\lambda\sigma^\mu\mathcal{D}_\mu\bar{\lambda}\right]$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ and λ is an adjoint Weyl fermion with $\mathcal{D}_\mu\lambda = \partial_\mu\lambda - i[A_\mu, \lambda]$. Show that the action is invariant under the supersymmetry transformations

$$\delta A_\mu = \epsilon\sigma_\mu\bar{\lambda} + \lambda\sigma_\mu\bar{\epsilon} \quad \text{and} \quad \delta\lambda = (\sigma^{\mu\nu}\epsilon)F_{\mu\nu}$$

Hint: You can check the supersymmetry transformation just for ϵ with the $\bar{\epsilon}$ terms guaranteed to follow suit on the grounds that the action is real. To do the calculation, you will need to invoke the Bianchi identity $\mathcal{D}_\mu{}^\star F^{\mu\nu} = 0$ together with the sigma-matrix identity

$$\sigma^\nu\bar{\sigma}^\mu\sigma^\rho = \eta^{\mu\nu}\sigma^\rho + \eta^{\mu\rho}\sigma^\nu - \eta^{\nu\rho}\sigma^\mu + i\epsilon^{\nu\mu\rho\kappa}\sigma_\kappa$$

5. A $U(N_c)$ supersymmetric gauge theory is coupled to N_f flavours, comprising of chiral multiplets Φ_i in the fundamental representation and $\tilde{\Phi}_i$ in the anti-fundamental with $i = 1, \ldots, N_f$. The D-terms conditions are

$$D^A = \phi_i^\dagger T^A \phi^i - \tilde{\phi}_i T^A \tilde{\phi}^{\dagger\, i} = 0 \qquad A = 1, \ldots N_c^2$$

By constructing an explicit set of Hermitian generators $(T^A)_b{}^a$, with $a, b = 1, \ldots, N_c$, show that the conditions $D^A = 0$ are equivalent to

$$\phi_i^{\dagger\, a}\phi_b^i - \tilde{\phi}_i^a\tilde{\phi}_b^{\dagger\, i} = 0 \qquad a, b = 1, \ldots N_c$$

6. An $SU(2)$ gauge theory is coupled to three chiral multiplets, each in the adjoint representation. The theory has a superpotential given by

$$W = \mathrm{Tr}\left(\Phi_1[\Phi_2, \Phi_3] - \frac{m}{2}\sum_{i=1}^{3}\Phi_i^2\right)$$

Write down the D-term and F-term contributions to the potential energy. Show that the zero energy ground states obey

$$[\phi_i, \phi_j] = m\epsilon_{ijk}\phi_k \quad \text{and} \quad \sum_{i=1}^{3}[\phi_i, \phi_i^\dagger] = 0$$

What are the solutions to these equations? What is the surviving gauge symmetry in each ground state?