Supersymmetry: Example Sheet 2

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1. Determine the (possibly spurious) symmetries of the superpotential

$$W = \mu_2 \Phi^2 + \mu_3 \Phi^3 + \ldots + \mu_n \Phi^n$$

Argue that the superpotential is not renormalised at any order in perturbation theory.

2*. Find the vacuum structure, including the vacuum energy, of the following theories:

- a) A chiral multiplet Z with $W = \alpha Z + \beta/Z$, with $\alpha, \beta \neq 0$.
- b) Three chiral multiplets X, Y and Z with W = XYZ.
- c) Three chiral multiplets X, Y and Z with $W = \alpha Y + \beta Y X^2 + \gamma X Z$ with $\alpha, \beta, \gamma \neq 0$ and $|\gamma|^2 > 2|\alpha\beta|$.

(You may assume a canonical Kähler potential for all fields.)

3a. For an abelian vector superfield V, show that the field strength superfield

$$W_{\alpha} = -\frac{1}{4}\bar{\mathcal{D}}^2 \mathcal{D}_{\alpha} V$$

is chiral, i.e. $\bar{\mathcal{D}}_{\dot{\alpha}}W_{\alpha} = 0$. Show further that it is invariant under extended gauge transformations $V \to V + i(\Omega - \Omega^{\dagger})$.

b. Show that the components of W_{α} are

$$W_{\alpha}(x,\theta) = \lambda_{\alpha}(x) + \theta_{\alpha}D(x) + (\sigma^{\mu\nu}\theta_{\alpha})F_{\mu\nu}(x) - i\theta^{2}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}\bar{\lambda}^{\dot{\alpha}}(x) + \dots$$

Hint: you will be well served to work in Wess-Zumino gauge and, at an appropriate time, to use the fact that W_{α} is a chiral superfield.

c. Show that the F-term integral gives

$$\int d^2\theta \ W^{\alpha}W_{\alpha} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}F_{\mu\nu}{}^{\star}F^{\mu\nu} - 2i\lambda\sigma^{\mu}\partial_{\mu}\bar{\lambda} + D^2$$

Note: You will need the identity

$$\operatorname{tr}(\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\sigma}\bar{\sigma}^{\rho}) = 2i\epsilon^{\mu\nu\sigma\rho} + 2\eta^{\mu\nu}\eta^{\sigma\rho} - 2\eta^{\mu\sigma}\eta^{\nu\rho} + 2\eta^{\mu\rho}\eta^{\nu\sigma}$$

4*. Super Yang-Mills (with vanishing theta angle) has action

$$S = \frac{1}{g^2} \int d^4x \, \operatorname{Tr} \left[-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2i\lambda \sigma^{\mu} \mathcal{D}_{\mu} \bar{\lambda} \right]$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - i[A_{\mu}, A_{\nu}]$ and λ is an adjoint Weyl fermion with $\mathcal{D}_{\mu}\lambda = \partial_{\mu}\lambda - i[A_{\mu}, \lambda]$. Show that the action is invariant under the supersymmetry transformations

$$\delta A_{\mu} = \epsilon \sigma_{\mu} \bar{\lambda} + \lambda \sigma_{\mu} \bar{\epsilon} \quad \text{and} \quad \delta \lambda = (\sigma^{\mu\nu} \epsilon) F_{\mu\nu}$$

Hint: You can check the supersymmetry transformation just for ϵ with the $\bar{\epsilon}$ terms guaranteed to follow suit on the grounds that the action is real. To do the calculation, you will need to invoke the Bianchi identity $\mathcal{D}_{\mu}^{*}F^{\mu\nu} = 0$ together with the sigma-matrix identity

$$\sigma^{\nu}\bar{\sigma}^{\mu}\sigma^{\rho} = \eta^{\mu\nu}\sigma^{\rho} + \eta^{\mu\rho}\sigma^{\nu} - \eta^{\nu\rho}\sigma^{\mu} + i\epsilon^{\nu\mu\rho\kappa}\sigma_{\kappa}$$

5. A $U(N_c)$ supersymmetric gauge theory is coupled to N_f flavours, comprising of chiral multiplets Φ_i in the fundamental representation and $\tilde{\Phi}_i$ in the anti-fundamental with $i = 1, \ldots, N_f$. The D-terms conditions are

$$D^A = \phi_i^{\dagger} T^A \phi^i - \tilde{\phi}_i T^A \tilde{\phi}^{\dagger i} = 0 \qquad A = 1, \dots N_c^2$$

By constructing an explicit set of Hermitian generators $(T^A)_b{}^a$, with $a, b = 1, \ldots, N_c$, show that the conditions $D^A = 0$ are equivalent to

$$\phi_i^{\dagger a} \phi_b^i - \tilde{\phi}_i^a \tilde{\phi}_b^{\dagger i} = 0 \quad a, b = 1, \dots N_c$$

6. An SU(2) gauge theory is coupled to three chiral multiplets, each in the adjoint representation. The theory has a superpotential given by

$$W = \text{Tr}\left(\Phi_1[\Phi_2, \Phi_3] - \frac{m}{2}\sum_{i=1}^{3}\Phi_i^2\right)$$

Write down the D-term and F-term contributions to the potential energy. Show that the zero energy ground states obey

$$[\phi_i, \phi_j] = m\epsilon_{ijk}\phi_k$$
 and $\sum_{i=1}^3 [\phi_i, \phi_i^{\dagger}] = 0$

What are the solutions to these equations? What is the surviving gauge symmetry in each ground state?